1 Determinants

INTRODUCTION

In 1693, Leibnitz developed determinants to solve a system of linear equations quickly. However, the present two vertical line notation for determinants was given by Arthur Cayley in 1841. In the present chapter, we shall learn about determinants, their elementary properties and the applications of these in solving system of linear equations.

1.1 DETERMINANT

Consider the system of two homogeneous linear equations

$$\begin{aligned} a_1x + b_1y &= 0 \\ a_2x + b_2y &= 0 \end{aligned} \qquad \ldots(1)$$

in the two variables x and y.

From these equations, we obtain

$$-\frac{a_1}{b_1} = \frac{y}{x} = -\frac{a_2}{b_2}.$$

On eliminating the variables x and y from the system (1), we get

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \quad i.e. \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} \quad i.e. \; a_1 b_2 - a_2 b_1 = 0.$$

The above eliminant is written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \qquad \ldots(2)$$

The left hand side of (2) is called a **determinant of order 2** or a **determinant of second order** and $a_1b_2 - a_2b_1$ is called its value. This leads to :

*A **determinant of order 2** is an arrangement of 2^2 i.e. 4 numbers (or expressions) in the form of a square along two horizontal lines called **rows** and along two vertical lines called **columns** and these numbers are enclosed within two vertical lines.* Thus,

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of order 2 and its value is $a_1b_2 - a_2 b_1$ *i.e.*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The numbers a_1, b_1, a_2, b_2 are called the **elements** of the determinant and the expression $a_1b_2 - a_2b_1$ on the right hand side is called the **expansion** of the determinant.

For example, $\begin{vmatrix} 2 & -4 \\ 7 & 5 \end{vmatrix} = 2.5 - 7.(-4) = 10 - (-28) = 38.$

The elements a_1, b_1 constitute the **first row** and the elements a_2, b_2 constitute the **second row**. The elements a_1, a_2 and b_1, b_2 constitute the **first** and **second columns** respectively. The elements a_1, b_2 are called the **diagonal elements** and the line along which they lie is called the **principal diagonal** or simply the **diagonal** of the determinant.

A ***determinant of order 3*** *is an arrangement of* 3^2 *i.e. 9 numbers (or expressions) in the form of a square along three horizontal lines called* ***rows*** *and along three vertical lines called* ***columns*** *and these numbers are enclosed within two vertical lines.* Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ is called a determinant of order 3.

The numbers a_1, b_1, c_1 etc. are called the **elements** of the determinant, so a determinant of order 3 contains 9 elements. The elements a_1, b_1, c_1; a_2, b_2, c_2 and a_3, b_3, c_3 constitute the **first, second** and **third rows** respectively and the elements a_1, a_2, a_3; b_1, b_2, b_3 and c_1, c_2, c_3 constitute the **first, second** and **third columns** respectively. The elements a_1, b_2, c_3 are called the **diagonal elements** and the line containing these elements is called the **principal diagonal** of the determinant.

A determinant is usually denoted by the symbol Δ or D.

The first, second, third, … rows and columns of a determinant are respectively denoted by R_1, R_2, R_3, … and C_1, C_2, C_3, …

Value of a determinant of order 3

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

be a determinant of order 3, then the value of the determinant Δ is given by

$$\begin{aligned} \Delta &= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \quad \ldots(1) \end{aligned}$$

The expression on R.H.S. of (1) is called the ***expansion of the determinant*** *by the first row.*

Working rule

(*i*) Write the elements of the first row with alternatively positive and negative sign, the first element always has positive sign before it.

(*ii*) Multiply each signed element by the determinant of second order obtained after deleting the row and the column in which that element occurs.

For example,

$$\begin{aligned} \begin{vmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \\ 0 & 4 & 7 \end{vmatrix} &= 3\begin{vmatrix} 2 & -1 \\ 4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} \\ &= 3(2.7 - 4.(-1)) + 2(1.7 - 0.(-1)) + 5(1.4 - 0.2) \\ &= 3(14 + 4) + 2(7 - 0) + 5(4 - 0) \\ &= 3.18 + 2.7 + 5.4 = 54 + 14 + 20 = 88. \end{aligned}$$

Determinant of order one

Let a be any number (or expression), then $|a|$ is a determinant of order one and its value is the number itself *i.e.* $|a| = a$.

For example, $|5| = 5$, $|-7| = -7$.

Remark. A determinant of order one should not be confused with the absolute value of a real (or complex) number.

Determinants of order four and of higher order

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

then Δ is a determinant of order 4. It consists of 4^2 *i.e.* 16 elements arranged in the form of a square along 4 rows and four columns; and its value can be obtained in a manner similar to that of a determinant of order 3.

Similarly, we can define determinants of order 5 and of higher orders. However, in this chapter, we shall be mainly dealing with determinants of order ≤ 3.

1.1.1 Minors and Cofactors

$$\text{Let} \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

be a determinant of order n, $n \geq 2$, then the determinant of order $n - 1$ obtained from the determinant Δ after deleting the ith row and jth column is called the ***minor of the element a_{ij}*** *and it is, usually, denoted by M_{ij} where $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n$.*

If M_{ij} is the minor of the element a_{ij} in the determinant Δ, then the number $(-1)^{i+j} M_{ij}$ is called the ***cofactor of the element a_{ij}***, *it is usually denoted by A_{ij}.*

Thus, $\quad A_{ij} = (-1)^{i+j} M_{ij}$.

Note that $\quad A_{ij} = M_{ij}$ if $i + j$ is even and

$\qquad A_{ij} = -M_{ij}$ if $i + j$ is odd.

For example,

(1) Let $\quad \Delta = \begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$, then

$$M_{11} = |7| = 7,\ M_{12} = |4| = 4,$$
$$M_{21} = |-3| = -3,\ M_{22} = |2| = 2 \text{ and}$$
$$A_{11} = (-1)^{1+1} M_{11} = 7,\ A_{12} = (-1)^{1+2} M_{12} = -4,$$
$$A_{21} = (-1)^{2+1} M_{21} = -(-3) = 3,\ A_{22} = (-1)^{2+2} M_{22} = 2.$$

(2) Let $\quad \Delta = \begin{vmatrix} 7 & 4 & -1 \\ 2 & 3 & 0 \\ 1 & -5 & 2 \end{vmatrix}$, then

$$M_{11} = \begin{vmatrix} 3 & 0 \\ -5 & 2 \end{vmatrix} = 3.2 - (-5).0 = 6,$$

$$M_{22} = \begin{vmatrix} 7 & -1 \\ 1 & 2 \end{vmatrix} = 7.2 - 1.(-1) = 15,$$

$$M_{32} = \begin{vmatrix} 7 & -1 \\ 2 & 0 \end{vmatrix} = 7.0 - 2.(-1) = 2 \text{ etc.}$$

$$A_{11} = (-1)^{1+1} M_{11} = 6,\ A_{22} = (-1)^{2+2} M_{22} = 15 \text{ and}$$
$$A_{32} = (-1)^{3+2} M_{32} = -2 \text{ etc.}$$

For quick working, the signs of the different cofactors according to the positions of the corresponding elements in determinants of order 2 and 3 are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Expansion of a determinant by any row or any column

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a determinant of order 3, then

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}.$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

and $A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

We know that the value of the determinant Δ is given by

$$\Delta = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(Expansion by first row)

$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

(By using values of the cofactors A_{11}, A_{12}, A_{13})

Similarly, we can show that

$$\Delta = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23},$$

$$\Delta = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31} \text{ etc.}$$

Thus, we have:

The sum of the products of elements of any row (or column) of a determinant with their corresponding cofactors is equal to the value of the determinant.

The above result is true for every determinant of order ≥ 2.

*Also, **it follows that the value** of a determinant can be obtained by expanding it with any row or any column.*

Remark. We can obtain the value of a determinant very quickly if we expand it with the help of a row or a column which contains the maximum number of zeros.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following determinants :*

$$(i) \begin{vmatrix} -3 & 1 \\ 5 & 6 \end{vmatrix} \qquad (ii) \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \qquad (iii) \begin{vmatrix} \cos 15° & \sin 15° \\ \sin 75° & \cos 75° \end{vmatrix}.$$

Solution. (*i*) $\begin{vmatrix} -3 & 1 \\ 5 & 6 \end{vmatrix} = (-3).6 - 5.1 = -18 - 5 = -23.$

(*ii*) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta . \cos\theta - \sin\theta . (-\sin\theta)$

$$= \cos^2\theta + \sin^2\theta = 1.$$

(*iii*) $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 75° & \cos 75° \end{vmatrix} = \cos 15° \cos 75° - \sin 15° \sin 75°$

$$= \cos(15° + 75°) = \cos 90° = 0.$$

Example 2. *If* $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$, *find the integral value of x.*

Solution. Given $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$

$\Rightarrow \quad (x-2).2x - 3x.(-3) = 3$

$\Rightarrow \quad 2x^2 - 4x + 9x - 3 = 0 \Rightarrow 2x^2 + 5x - 3 = 0$

$\Rightarrow \quad (2x-1)(x+3) = 0 \Rightarrow 2x - 1 = 0$ or $x + 3 = 0$

$\Rightarrow \quad x = \frac{1}{2}$ or -3 but x is an integer

$\Rightarrow \quad x = -3.$

Example 3. *Evaluate* $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$.

Solution. $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$

$= 2(1-(-8)) - 3(7-6) - 5(28-(-3))$

$= 2.9 - 3.1 - 5.31 = 18 - 3 - 155 = -140.$

Example 4. *Evaluate* $\begin{vmatrix} 3 & 7 & 13 \\ -5 & 0 & 0 \\ 0 & 11 & -2 \end{vmatrix}$.

Solution. As the second row contains two zeros, expanding the given determinant by 2nd row, we get

$$\begin{vmatrix} 3 & 7 & 13 \\ -5 & 0 & 0 \\ 0 & 11 & -2 \end{vmatrix} = -(-5)\begin{vmatrix} 7 & 13 \\ 11 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 13 \\ 0 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 7 \\ 0 & 11 \end{vmatrix}$$

$$= 5(-14 - 143) + 0 - 0 = -785.$$

Example 5. *Show that the value of the determinant* $\begin{vmatrix} 0 & \tan x & 1 \\ 1 & -\sec x & 0 \\ \sec x & 0 & \tan x \end{vmatrix}$ *is independent of x.*

Solution. Expanding the given determinant by first row, we get

$$\begin{vmatrix} 0 & \tan x & 1 \\ 1 & -\sec x & 0 \\ \sec x & 0 & \tan x \end{vmatrix} = 0(-\sec x \tan x - 0) - \tan x (\tan x - 0) + 1(0 + \sec^2 x)$$

$$= 0 - \tan^2 x + \sec^2 x = \sec^2 x - \tan^2 x$$

$$= 1, \text{ which is independent of } x.$$

Example 6. *Find the minors and cofactors of each element of the second column of the determinant Δ and hence find the value of the determinant Δ where*

$$\Delta = \begin{vmatrix} 3 & -2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & 7 \end{vmatrix}.$$

Solution. $M_{12} = \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} = 28 - 10 = 18$, $M_{22} = \begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix} = 21 - 2 = 19$

and $\quad M_{32} = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = 15 - 4 = 11.$

$\therefore \quad A_{12} = (-1)^{1+2} M_{12} = (-1).18 = -18,$

$A_{22} = (-1)^{2+2} M_{22} = 1.19 = 19$ and

$A_{32} = (-1)^{3+2} M_{32} = (-1).11 = -11.$

Now, expanding the given determinant by 2nd column, we get

$$\Delta = (-2).(-18) + 6.19 + (-1).(-11)$$
$$= 36 + 114 + 11 = 161.$$

Example 7. *There are two values of x which make determinant*

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix} = 86,$$ *find the sum of these numbers.*

Solution. Given $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix} = 86$ (Expanding by C_1)

$\Rightarrow \quad 1\begin{vmatrix} x & -1 \\ 4 & 2x \end{vmatrix} - 2\begin{vmatrix} -2 & 5 \\ 4 & 2x \end{vmatrix} + 0\begin{vmatrix} -2 & 5 \\ x & -1 \end{vmatrix} = 86$

$\Rightarrow \quad 1(2x^2 + 4) - 2(-4x - 20) + 0 = 86$

$\Rightarrow \quad 2x^2 + 8x - 42 = 0 \Rightarrow x^2 + 4x - 21 = 0.$

Let α, β be the roots of this equation, then $\alpha + \beta = \frac{-4}{1} = -4.$

Hence, the sum of two values of $x = -4$.

EXERCISE 1.1

1. Evaluate the following determinants :

(*i*) $\begin{vmatrix} y - x & -x^2 + xy - y^2 \\ x + y & x^2 + xy + y^2 \end{vmatrix}$ (*ii*) $\begin{vmatrix} \cos 80° & -\cos 10° \\ \sin 80° & \sin 10° \end{vmatrix}$.

2. (*i*) If $x \in \mathbf{N}$ and $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix}$, find the value(s) of x.

(*ii*) If $x \in \mathbf{I}$ and $\begin{vmatrix} 2x & 3 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ x & 3 \end{vmatrix}$, find the values of x.

(*iii*) If $x \in \mathbf{R}$, $0 \le x \le \frac{\pi}{2}$ and $\begin{vmatrix} 2\sin x & -1 \\ 1 & \sin x \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -4 & \sin x \end{vmatrix}$, find the values of x.

3. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 2 & 4 & 1 \\ 8 & 5 & 2 \\ -1 & 3 & 7 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 8 & 9 \\ 10 & 11 & 12 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

4. Find the integral value of x if $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28.$

5. Prove that $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$

6. Find the minors and the cofactors of each element of the second row of the determinant D and hence find its value where

$$D = \begin{vmatrix} 2 & 4 & 1 \\ 8 & 5 & 2 \\ -1 & 3 & 7 \end{vmatrix}.$$

7. Find the minors and cofactors of each element of the first column of the following determinants and hence find the value of the determinant in each case :

(*i*) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ (*iv*) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

1.2 PROPERTIES OF DETERMINANTS

The properties of determinants serve the purpose of useful tools for computing the values of the given determinants. Proofs of most of the properties of determinants are beyond the scope of the present book. Therefore, we shall state these properties and verify them by taking some examples.

Property 1. *If each element in a row or in a column of a determinant is zero, then the value of the determinant is zero.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant in which each element in the second row is zero.

Expanding Δ by the second row, we get

$$\Delta = -0\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + 0\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - 0\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0.$$

Property 2. *If each element on one side of the principal diagonal of a determinant is zero, then the value of the determinant is the product of the diagonal elements.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix}$ be a determinant in which all element on one side of the principal diagonal are zero.

Expanding Δ by C_1, we get

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ 0 & c_3 \end{vmatrix} - 0\begin{vmatrix} b_1 & c_1 \\ 0 & c_3 \end{vmatrix} + 0\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - 0.c_2) - 0 + 0 = a_1 b_2 c_3.$$

Property 3. *The value of a determinant remains unchanged if its rows and columns are interchanged.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by interchanging its rows and columns

i.e. $$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Expanding Δ by R_1, we get

$$\begin{aligned}\Delta &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \quad \ldots(i)\end{aligned}$$

Expanding Δ_1 by R_1, we get

$$\begin{aligned}\Delta_1 &= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)\\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \quad \ldots(ii)\end{aligned}$$

From (*i*) and (*ii*), we get $\Delta = \Delta_1$.

Property 4. *If any two rows (or columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by interchanging its first and third columns

i.e. $$\Delta_1 = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$$

Expanding Δ by R_1, we get

$$\begin{aligned}\Delta &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \quad \ldots(i)\end{aligned}$$

Expanding Δ_1 by R_1, we get

$$\begin{aligned}\Delta_1 &= c_1(b_2a_3 - b_3a_2) - b_1(c_2a_3 - c_3a_2) + a_1(c_2b_3 - c_3b_2)\\ &= a_3b_2c_1 - a_2b_3c_1 - a_3b_1c_2 + a_2b_1c_3 + a_1b_3c_2 - a_1b_2c_3\\ &= -(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1) \quad \ldots(ii)\end{aligned}$$

From (*i*) and (*ii*), we get $\Delta_1 = -\Delta$.

Corollary. *If any row (or column) of a determinant Δ be passed over m rows (or columns), then the resulting determinant $\Delta_1 = (-1)^m\,\Delta$.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by passing over its first column over the next two columns

i.e. $$\Delta_1 = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}.$$

Let us interchange 1st and 3rd columns in Δ_1, then by property 4 we get

$$\Delta_1 = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = (-1)\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

Now, on interchanging 2nd and 3rd columns, we get

$$\Delta_1 = (-1)^2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{(using property 4 again)}$$

$$\Rightarrow \qquad \Delta_1 = (-1)^2\,\Delta.$$

Property 5. *If two parallel lines (rows or columns) of a determinant are identical, then the value of the determinant is zero.*

Verification. Let Δ be the given determinant which has two parallel lines identical, say first and third rows, then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}.$$

Expanding Δ by R_1, we get

$$\begin{aligned} \Delta &= a_1(b_2c_1 - b_1c_2) - b_1(a_2c_1 - a_1c_2) + c_1(a_2b_1 - a_1b_2) \\ &= a_1b_2c_1 - a_1b_1c_2 - a_2b_1c_1 + a_1b_1c_2 + a_2b_1c_1 - a_1b_2c_1 \\ &= 0. \end{aligned}$$

Property 6. *If each element of a row (or a column) of a determinant is multiplied by the same number k, then the value of the new determinant is k times the value of the original determinant.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by multiplying every element of second row by the same number *k i.e.*

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding Δ by R_1, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \qquad \ldots(i)$$

Expanding Δ_1 by R_1, we get

$$\begin{aligned} \Delta_1 &= a_1(kb_2c_3 - kb_3c_2) - b_1(ka_2c_3 - ka_3c_2) + c_1(ka_2b_3 - ka_3b_2) \\ &= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] \qquad \ldots(ii) \end{aligned}$$

From (*i*) and (*ii*), we get $\Delta_1 = k\,\Delta$.

Corollary 1. *If two parallel lines (rows or columns) of a determinant are such that the elements of one line are equi-multiples of the elements of the other line, then the value of the determinant is zero.* (Using properties 6 and 5)

Corollary 2. *If each element of a determinant Δ is multiplied by the same number k and Δ_1 is the new determinant, then*

$$\Delta_1 = k\,\Delta \text{ if order of } \Delta = 1$$

$$\Delta_1 = k^2\,\Delta \text{ if order of } \Delta = 2$$

$$\Delta_1 = k^3\,\Delta \text{ if order of } \Delta = 3 \text{ etc.}$$

Property 7. *If each element of a row (or a column) of a determinant consists of sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants whose other rows (or columns) are not altered.*

Verification. Let $\Delta = \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix}$. Here, each element in the first column consists of the sum of two terms.

Expanding Δ by C_1, we get

$$\begin{aligned}\Delta &= (a_1 + d_1)(b_2c_3 - b_3c_2) - (a_2 + d_2)(b_1c_3 - b_3c_1) + (a_3 + d_3)(b_1c_2 - b_2c_1)\\ &= [a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]\\ &\quad + [d_1(b_2c_3 - b_3c_2) - d_2(b_1c_3 - b_3c_1) + d_3(b_1c_2 - b_2c_1)]\\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.\end{aligned}$$

Property 8. *If to each element of a row (or a column) of a determinant be added the equi-multiples of the corresponding elements of one or more rows (or columns), the value of the determinant remains unchanged.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and Δ_1 be the determinant obtained from Δ by adding k times the elements of second column to the corresponding elements of the first column *i.e.*

$$\Delta_1 = \begin{vmatrix} a_1 + k\,a_2 & a_2 & a_3 \\ b_1 + k\,b_2 & b_2 & b_3 \\ c_1 + k\,c_2 & c_2 & c_3 \end{vmatrix}$$

By using property 7, we get

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k\,a_2 & a_2 & a_3 \\ k\,b_2 & b_2 & b_3 \\ k\,c_2 & c_2 & c_3 \end{vmatrix}\\ &= \Delta + k\begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} \qquad \text{(using property 6)}\\ &= \Delta + k\,.\,0 \qquad \text{(using property 5)}\\ &= \Delta.\end{aligned}$$

Property 9. *The sum of the products of elements of any row (or column) with the cofactors of the corresponding elements of some other row (or column) is zero.*

Verification. Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Then the sum of the products of elements of first row with the cofactors of the corresponding elements of the third row

$$\begin{aligned}&= a_{11}\,A_{31} + a_{12}\,A_{32} + a_{13}\,A_{33}\\ &= a_{11}\,(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}(-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\\ &= a_{11}\,(a_{12}\,a_{23} - a_{22}\,a_{13}) - a_{12}\,(a_{11}\,a_{23} - a_{21}\,a_{13}) + a_{13}\,(a_{11}\,a_{22} - a_{21}\,a_{12})\\ &= 0.\end{aligned}$$

1.2.1 Elementary operations

Let Δ be a determinant of order n, $n \geq 2$; R_1, R_2, R_3, … denote its first row, second row, third row, … and C_1, C_2, C_3, … denote its first column, second column, third column, … respectively.

(*i*) The operation of interchanging the *i*th row and *j*th row of Δ will be denoted by $R_i \leftrightarrow R_j$ and the operation of interchanging the *i*th column and *j*th column of Δ will be denoted by $C_i \leftrightarrow C_j$.

(*ii*) The operation of multiplying each element of the *i*th row of Δ by a number k will be denoted by $R_i \to k R_i$ and the operation of multiplying each element of the *i*th column of Δ by a number k will be denoted by $C_i \to k C_i$.

(*iii*) The operation of adding to each element of the *i*th row of Δ, k times the corresponding elements of the *j*th row ($j \neq i$) will be denoted by $R_i \to R_i + k R_j$ and the operation of adding to each element of the *i*th column of Δ, k times the corresponding elements of the *j*th column ($j \neq i$) will be denoted by $C_i \to C_i + k C_j$.

ILLUSTRATIVE EXAMPLES

Example 1. *Without expanding, evaluate the following determinants :*

$$(i)\ \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix} \qquad (ii)\ \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}.$$

Solution. (*i*) Operating $C_1 \to C_1 - 8C_3$ (property 8), we get

$$\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 49-8.6 & 1 & 6 \\ 39-8.4 & 7 & 4 \\ 26-8.3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 0 \qquad \text{(By property 5)}$$

(*ii*) Operating $C_3 \to C_3 + C_2$ (property 8), we get

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \qquad \text{(By property 6)}$$

$$= (a+b+c) \times 0 \qquad \text{(By property 5)}$$

$$= 0.$$

Example 2. *Without expanding, show that*

$$(i)\ \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0 \qquad (ii)\ \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} = 0.$$

Solution. (*i*) Operating $C_1 \to C_1 + C_2 + C_3$ (property 8), we get

$$\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix}$$

$$= 0 \qquad \text{(By property 1)}$$

(*ii*) Taking out (–1) from C_1, (–1) from C_2 and (–1) from C_3 (property 6), we get

$$\Delta = \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} = (-1)\,(-1)\,(-1) \begin{vmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$ (Interchanging rows and columns, property 3)

$$= -\Delta$$

$\Rightarrow \quad 2\Delta = 0 \;\Rightarrow\; \Delta = 0.$

Example 3. *Without expanding, show that*

(*i*) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$ (*ii*) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

Solution. (*i*) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = \dfrac{abc}{abc}\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

(Multiply R_1 by a, R_2 by b and R_3 by c, use property 6)

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ca+bc \end{vmatrix}$$

(Take abc out from C_1 and abc out from C_2, property 6)

$$= \frac{abc.abc}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix}$$ (Operate $C_3 \to C_3 + C_1$, property 8)

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$

$$= abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$ (Property 6)

$= abc\,(ab+bc+ca) \times 0$ (Property 5)

$= 0.$

(*ii*) By using property 7, we get

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$$ (Take (–1) out from C_3)

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

(In second determinant, operate $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$)

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} \quad \text{(Take } abc \text{ out from } C_3)$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

(Pass on C_3 over the first two columns)

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0.$$

Example 4. *Without expanding, show that*

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0, \text{ where } a, b, c \text{ are in A.P.}$$

Solution. Given a, b, c are in A.P. $\Rightarrow a + c = 2b$

$\Rightarrow \quad a + c - 2b = 0$ ...(*i*)

Operating $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} \quad \text{(Using (}i\text{))}$$

$= 0$ (By property 1)

Example 5. *By using properties of determinants, prove that the determinant*

$$\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix} \text{ is independent of } x. \qquad \textit{(I.S.C. 2010)}$$

Solution. Operating $R_1 \to R_1 - \cos x\, R_2$ and $R_3 \to R_3 - a\, R_2$, we get

$$\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix} = \begin{vmatrix} a+\sin x\cos x & \sin x + a\cos x & 0 \\ -\sin x & -a & 1 \\ \cos x + a\sin x & 1+a^2 & 0 \end{vmatrix} \quad \text{(Expand by } C_3)$$

$= -1 \times [a + \sin x \cos x)(1 + a^2) - (\cos x + a \sin x)(\sin x + a \cos x)]$

$= -[a + a^3 + \sin x \cos x + a^2 \sin x \cos x - (\sin x \cos x + a \cos^2 x + a \sin^2 x + a^2 \sin x \cos x)]$

$= -[a + a^3 - a(\cos^2 x + \sin^2 x)]$

$= -(a + a^3 - a \times 1) = -a^3$, which is independent of x.

Note. However, if we expand the given determinant by first row, we get

given determinant $= a(-a^2 - 1) - \sin x(-a \sin x - \cos x) + \cos x(-\sin x + a \cos x)$

$= -a^3 - a + a \sin^2 x + a \cos^2 x$

$= -a^3 - a + a(\sin^2 x + \cos^2 x) = -a^3 - a + a \times 1$

$= -a^3$, which is independent of x.

Thus, the above problem can be solved more conveniently if we do not use the properties of determinants.

23. Find all integers k for which the system of equations

$$x + 2y - 3z = 1$$
$$2x - ky - 3z = 2$$
$$x + 2y + kz = 3$$

has a unique solution. Find the solution for $k = 0$.

24. Examine whether or not the system

$x - y + z = 3$, $2x + y - z = 2$, $x + 2y - 2z = -1$ is consistent. If consistent, then solve it.

ANSWERS

EXERCISE 1.1

1. (*i*) $2y^3$ (*ii*) 1. **2.** (*i*) 4 (*ii*) -2 (*iii*) $\frac{\pi}{6}, \frac{\pi}{2}$.

3. (*i*) -145 (*ii*) -48 (*iii*) 0. **4.** 2

6. $M_{21} = 25$, $M_{22} = 15$, $M_{23} = 10$ and $A_{21} = -25$, $A_{22} = 15$, $A_{23} = -10$; -145.

7. (*i*) $M_{11} = -1$, $M_{21} = 20$, $A_{11} = -1$, $A_{21} = -20$; -5

(*ii*) $M_{11} = 3$, $M_{21} = 4$, $A_{11} = 3$, $A_{21} = -4$; -11

(*iii*) $M_{11} = -12$, $M_{21} = -16$, $M_{31} = -4$;

$A_{11} = -12$, $A_{21} = 16$, $A_{31} = -4$; 40

(*iv*) $M_{11} = bc - f^2$, $M_{21} = hc - fg$, $M_{31} = hf - bg$;

$A_{11} = bc - f^2$, $A_{21} = fg - hc$, $A_{31} = hf - bg$;

$abc - af^2 - bg^2 - ch^2 + 2fgh$.

EXERCISE 1.2

1. (*i*) 0 (*ii*) 0. **2.** (*i*) 0 (*ii*) 0. **3.** (*i*) 0 (*ii*) 0.

4. (*i*) 0 (*ii*) 0. **5.** 0. **6.** (*i*) -676 (*ii*) -8.

9. (*i*) 0 (*ii*) 0. **10.** (*i*) $-\frac{\lambda}{3}$ (*ii*) 2, 3, 6.

11. (*i*) 0, 0, $-(a + b + c)$ (*ii*) 4. **13.** (*i*) 0 (*ii*) 0.

35. (*i*) $\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$ (*ii*) 1, 1, -9 (*iii*) 2, 1, -3 (*iv*) 1, -2.

EXERCISE 1.3

1. (*i*) $x = \frac{7}{5}$, $y = \frac{4}{5}$ (*ii*) $x = 7$, $y = -3$.

2. (*i*) $x = 3$, $y = -1$ (*ii*) $x = 2$, $y = 3$.

3. $x = 2$, $y = -\frac{2}{a}$.

4. (*i*) $x = 1$, $y = -1$, $z = -1$ (*ii*) $x = 1$, $y = -1$, $z = -1$.

5. (*i*) $x = 7$, $y = -3$, $z = -4$ (*ii*) $x = \frac{3}{4}$, $y = \frac{5}{4}$, $z = \frac{1}{2}$.

6. (*i*) $x = -2$, $y = -3$, $z = -4$ (*ii*) $x = 2$, $y = 3$, $z = 5$.

7. $x = \frac{(b-k)(k-c)}{(a-b)(c-a)}$, $y = \frac{(a-k)(k-c)}{(a-b)(b-c)}$, $z = \frac{(b-k)(k-a)}{(b-c)(c-a)}$.

8. 13, 2, 5.

9. (*i*) Inconsistent (*ii*) Consistent; $x = 3 - 2k$, $y = k$, k is any number.

10. (*i*) Inconsistent (*ii*) Inconsistent.