3 Differential Equations

3.1 Introduction

Differential equations are fundamental in engineering mathematics since many of the physical laws and relationships between physical quantities appear mathematically in the form of such equations.

The transition from a given physical problem to its mathematical representation is called modeling. This is of great practical interest to engineer, physicist or computer scientist. Very often, mathematical models consist of a differential equations or system of simultaneous differential equations, which needs to be solved. In this chapter we shall look at classifying differential equations and solving them by various standard methods.

3.2 Differential Equations of First Order

3.2.1 Definitions

A differential equation is an equation which involves derivatives or differential coefficients or differentials. Thus the following are all examples of differential equations.

(a) $x^2dx + y^2\,dy = 0$

(b) $\frac{d^2x}{dt^2} + a^2x = 0$

(c) $y = x\frac{dy}{dx} + \frac{x^2}{dy/dx}$

(d) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-5/3} = a\frac{d^2y}{dx^2}$

(e) $\frac{dx}{dt} - wy = a\cos pt, \frac{dy}{dt} + wx = a\sin pt$

(f) $x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

(g) $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

An **ordinary differential equations** is that in which all the differential coefficients all with respect to a single independent variable. Thus the equations (a) to (d) are all ordinary differential equations. (e) is a **system** of ordinary differential equations.

A **partial differential equations** is that in which there are two or more independent variables and partial differential coefficients with respect to any of them. The equations (f) and (g) are partial differential equations.

The **order** of a differential equation is the order of the highest derivative appearing in it. The **degree** of a differential equation is the degree of the highest derivative occurring in its, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Thus from the examples above,

(a) is of the first order and first degree;

(b) is of the second order and first degree;

(c) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x^2$ is of the first order but of second degree;

(d) After removing radicals is written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-5} = a^3\left(\frac{d^2y}{dx^2}\right)^3$

and is of the second order and third degree.

3.2.2 Solution of a Differential Equation

A solution (or integral) of a differential equation is a relation between the variable which satisfies the given differential equation.

For example, $$y = ce^{\frac{x^3}{3}} \quad \text{... (i)}$$

is a solution of $$\frac{dy}{dx} = x^2y \quad \text{... (ii)}$$

The **general** (or **complete**) **solution** of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (i) is a general solution of (ii) as the number of arbitrary constants (one constant c) is the same as the order of the equations (ii) (first order).

Similarly, in the general solution of a second order differential equation, there will be two arbitrary constants.

A **particular solution** is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example $$y = 4e^{\frac{x^3}{3}}$$

is a particular solution of the equation (ii), as it can be derived from the general solution (i) by putting $c = 4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and usually is not of much practical interest in engineering.

3.2.3 Equations of the First Order and First Degree

It is not possible to analytically solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations:

1. Equations where variables are separable.,
2. Homogenous equations,
3. Linear equations,
4. Exact equations.

In other cases, the particular solution may be determined numerically.

3.2.3.1 Variables Separable

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(y)\,dy = \phi(x)\,dx$.

Integrating both sides, we get $\int f(y)dy = \int \phi(x)dx + c$ as its solution.

Example 1.

Solve $$\frac{dy}{dx} = e^{x+y} + x^2 e^y$$

Solution:

Given equation is $\dfrac{dy}{dx} = e^y(e^x + x^2)$

or $e^{-y}\, dy = (e^x + x^2)dx$

Integrating both sides, $\int e^{-y}dy = \int (e^x + x^2)dx + c$

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

$$3e^{-y} = -3e^x - x^3 + c' \qquad [c' = -3c]$$

NOTE

1. In the above line, we have introduced a new arbitrary constant c' instead of c, in order to put the result in a better form. Such changes are allowed and often made.
2. **Initial value problem:** A differential equation together with an initial condition is called an **initial value problem.** It is of the form given in the next example. The condition $y(0) = 0$ in the example below is called an initial condition. It is used to determine the value of the arbitrary constant in the general solution. In a second order differential equation, two such conditions will be required, since there will be two arbitrary constants which will need to be determined.

Example 2.

Solve $dy/dx = (x + y + 1)^2$, if $y(0) = 0$.

Solution:

Putting $x + y + 1 = t$, we get $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$.

$\therefore$ The given equation becomes $\dfrac{dt}{dx} - 1 = t^2$ or $\dfrac{dt}{dx} = 1 + t^2$

Integrating both sides, we get $\int \dfrac{dt}{1+t^2} = \int dx + c$

or $\tan^{-1} t = x + c$

or $\tan^{-1}(x + y + 1) = x + c$

or $x + y + 1 = \tan(x + c)$

When $x = 0, y = 0$

$\Rightarrow$ $1 = \tan(c)$

$\Rightarrow$ $c = \dfrac{\pi}{4}$

Hence the solution is $x + y + 1 = \tan(x + \pi/4)$.

Note: Equations of the form $\dfrac{dy}{dx} = f(ax + by + c)$ can be reduced to the 'variable separable' form by putting $ax + by + c = t$.

3.2.3.2 Homogeneous Equations

Homogeneous equations are of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x)$ $\phi(x, y)$ homogeneous functions of the same degree in x and y.

Homogeneous Function: An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as $x^n [a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n]$.

Thus any functions $f(x, y)$ which can be expressed in the form $x^n f(y/x)$, is called a homogeneous function of degree n in x and y. For instance $x^3 \cos(y/x)$ is a homogeneous function of degree 3 in x and y.

To solve a homogeneous equation

1. Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dy}{dx}$,
2. Separate the variables v and x, and integrate.

Example:

Solve $\quad (y^2 - x^2)\,dx - 2xy\,dy = 0.$

Solution:

Given equation is $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$ which is homogeneous in x and y. ... (i)

Put $\quad y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

∴ Eq. (i) becomes

$$v + x\frac{dv}{dx} = \frac{1}{2}\left[v - \frac{1}{v}\right]$$

or

$$x\frac{dv}{dx} = \frac{1}{2}\left[\frac{v^2 - 1}{v}\right] - v = \frac{-\left[v^2 + 1\right]}{2v}$$

Separating the variables,

$$\frac{2v}{1 + v^2}dv = -\frac{dx}{x}$$

Integrating both sides,

$$\int \frac{2v\,dv}{1 + v^2} = -\int \frac{dx}{x} + c$$

or

$$\ln(1 + v^2) = -\ln x + c = \ln\frac{1}{x} + \ln c_1$$

or

$$\ln(1 + v^2) = \ln\left(\frac{c_1}{x}\right)$$

$$1 + v^2 = \frac{c_1}{x}$$

replacing v by $\dfrac{y}{x}$, we get

$$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$$

or

$$x^2 + y^2 = cx$$

or

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

This general solution represents a family of circles with centres on the x-axis at $\left(\dfrac{c}{2}, 0\right)$ and radius $= \dfrac{c}{2}$, thus passing through origin as shown.

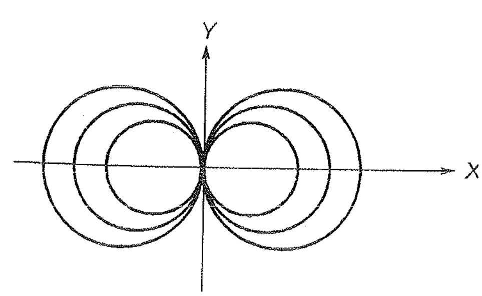

Fig. General Solution (Family of circles)

3.2.3.3 Linear Equations of First Order

A differential equation is said be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the following differential equations are linear

1. $\frac{dy}{dx}+4y=2$ 2. $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+4y=2$

equation (i) is linear first order differential equation while equation (ii) in linear second order differential equation. The following equations are not linear

1. $\left(\frac{dy}{dx}\right)^2+y=5$ 2. $\frac{dy}{dx}+y^{1/2}=2$ 3. $\frac{ydy}{dx}=5$

3.2.3.4 Leibnitze linear equation

The standard form of a linear equation of the first order, commonly known as **Leibnitz's linear equation,** is

$$\frac{dy}{dx}+Py = Q \text{ where } P, Q, \text{ are arbitrary functions of } x. \qquad \text{... (i)}$$

To solve the equation, multiply both sides by $e^{\int Pdx}$ so that we get

$$\frac{dy}{dx}\cdot e^{\int Pdx}+y(e^{\int Pdx}P) = Qe^{\int Pdx} \text{ i.e.} \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}$$

Integrating both sides, we get $ye^{\int Pdx} = \int Qe^{\int Pdx}dx+c$ as the required solution.

NOTE

The factor $e^{\int Pdx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (i).
So remember the following:

$$\text{I.F.} = e^{\int Pdx}$$

and the solution is $\mathbf{y(I.F.)} = \int Q(\text{I.F.})dx+c.$

3.2.3.5 Bernoulli's Equation

The equation $\frac{dy}{dx}+Py = Qy^n$ ... (i)

where P, Q are functions of x, is reducible to the Leibnitz's linear and is usually called the Bernoulli's equation.

To solve (i), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx}+Py^{1-n} = Q$ ... (ii)

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (ii) becomes $\frac{1}{1-n}\frac{dz}{dx}+Pz = Q$

or $$\frac{dz}{dx}+P(1-n)z = Q(1-n),$$

which is Leibnitz's linear in z and can be solved easily.

Example:

Solve $\frac{dy}{dx} + y = 4y^3$

Solution:

Dividing throughout by y^3,

$$y^{-3}\frac{dy}{dx} + y^{-2} = 4 \quad \text{... (i)}$$

Put $y^{-2} = z$, so that $-2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (i) becomes $-\frac{1}{2}\frac{dz}{dx} + z = 4$

or $$\frac{dz}{dx} - 2z = -8 \quad \text{... (ii)}$$

which is Leibnitz's linear in z.

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}$$

$\therefore$ The solution of (ii) is $z\,(\text{I.F.}) = \int (-8)(\text{I.F.})dx + c$

$$ze^{-2x} = \int (-8)e^{-2x}dx + c$$

$$\Rightarrow \quad y^{-2}e^{-2x} = 4e^{-2x} + c \quad (\because z = y^{-2})$$

$$\Rightarrow \quad y^{-2} = 4 + ce^{2x}$$

$$\Rightarrow \quad y = (4 + ce^{2x})^{-1/2}$$

3.2.3.6 Exact Differential Equations

1. **Definition.** A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be **exact** if its left hand member is the exact differential of some function $u(x, y)$ i.e. $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.
2. **Theorem.** The necessary and sufficient condition for the differential equations $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3. **Method of solution.** It can be shown that, the equation $Mdx + Ndy = 0$ becomes

$$d\left[u + \int f(y)dy\right] = 0$$

Integrating $u + \int f(y)dy = 0$.

But $u = \int Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ The solution of $Mdx + Ndy = 0$ is

$$\int Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

(Provides of course that the equation is exact. i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

NOTE: While finding $\int Mdx$, y is treated as constant since we are integrating with respect to x.

Example:

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Solution:

Step 1: Test for exactness

Here $M = x^3 + 3xy^2$ and $N = 3x^2y + y^3$

$$\therefore \quad \frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int M dx + \int (\text{terms of N not containing } x)dy = c$$

which is $\int (x^3 + 3xy^2)dx + \int y^3 dy = c$

$$\Rightarrow \quad \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$$

$$\Rightarrow \quad \frac{1}{4}(x^4 + 6x^2y^2 + y^4) = c$$

3.2.3.7 Equations Reducible To Exact Equations

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an integrating factor. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as given in theorem 1 and 2 below:

In the equation $Mdx + Ndy = 0$

Theorem 1: if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ be a function of x only $= f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.

Theorem 2: if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ be a function of y only $= f(y)$ say, then $e^{\int f(y)\,dy}$ is an integrating factor.

Example 1.

Solve $2\sin(y^2)\,dx + xy\cos y^2\,dy = 0, \qquad y(2) = \sqrt{\frac{\pi}{2}}$

Solution:

Step 1: Here, $M = 2\sin(y^2)$ and $N = xy\cos(y^2)$

Step 2: Test for exactness $\dfrac{\partial M}{\partial y} = 4y\cos(y^2)$ and $\dfrac{\partial N}{\partial x} = y\cos(y^2)$

So $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

and hence, equation is not exact. So we have to find integrating factor by using either theorem 1 or theorem 2.

Step 3: Find an integrating factor: try theorem 1

Here, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y\cos y^2 - y\cos y^2}{xy\cos y^2} = \frac{3}{x}$$

Which is function of x only. So theorem 1 can be used.

$$\therefore \quad \text{I.F.} = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiplying throughout by I.F., we get

$$2x^3 \sin(y^2)\, dx + x^4\, y \cos y^2\, dy = 0$$

This equation will surely be an exact equation. No need to check that.

Step 4: General solution:

$$\int M dx + \int (\text{terms of N containing } x) dy = c$$

Which is $\int 2x^3 \sin(y^2) dx + \int 0 dy = c$

$$\frac{1}{2}x^4 \sin(y^2) = c$$

Step 5: Now to find the particular solution of the initial value problem:

Since $\quad y(2) = \sqrt{\frac{\pi}{2}}$

$$\Rightarrow \quad \frac{1}{2} \cdot 2^4 \sin\frac{\pi}{2} = c$$

$$\Rightarrow \quad c = 8$$

So particular solution is $\frac{1}{2}x^4 \sin(y^2) = 8$

or $\quad x^4 \sin(y^2) = 16$

Example 2.

Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

Solution:

Here $\quad M = xy^3 + y,\ N = 2(x^2y^2 + x + y^4)$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{dx} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2 + 2 - 3xy^2 - 1)$$

$$= \frac{1}{y}, \text{ which is a function of y alone.}$$

$$\therefore \quad \text{I.F.} = e^{\int 1/y dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$, which is an exact equation.

$\therefore$ The solution is $\frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = c$.

3.2.4 Orthogonal Trajectories

3.2.4.1 Definitions

Two families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal trajectories of each other.

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems.

For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of equipotential curves and vice versa.

In fluid flow, the stream lines and the equipotential lines are orthogonal trajectories.

Example 1.

Find the orthogonal trajectory of family of curves xy = Constant.

Solution:

Given family of curves $\quad xy = c \quad$ ...(i)

Differentiate w.r.t. 'x'

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \quad x\frac{dx}{dy} = y$$

By variable separable, $\int x dx = \int y dy$

$$\frac{x^2}{2} = \frac{y^2}{2} + k$$

$\Rightarrow$ $x^2 - y^2 = k_1$ is the orthogonal tragectory of given family of curves

3.2.4.2 Orthogonal trajectory of polar curves

Example 2.

Find the orthogonal trajectory of family of curves $r^n = a^n \sin n\theta$

Solution:

Given family of curves $\quad r^n = a^n \sin \theta \quad$ ..(i)

Differentiate w.r.t. 'θ' and eliminate 'a'

$$nr^{n-1}\frac{dr}{d\theta} = a^n \cos nq \times n \quad \text{...(ii)}$$

Divided equation (ii) by equation (i)

$$\frac{nr^{n-1}\frac{dr}{d\theta}}{r^n} = \frac{a^n \cos n\theta \times n}{a^n \sin n\theta}$$

$$\frac{dr}{d\theta}\frac{1}{r} = \cot n\theta \quad \text{...(iii)}$$

Differential equation represents given family of curves.

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta$$

$$-r\frac{d\theta}{dr} = \cot n\theta$$

$$\int \frac{1}{r} dr = -\int \tan n\theta \, d\theta$$

$$\log r = -\frac{\log \sec n\theta}{n} + \log c$$

$$\log r^n = \log[c^n \cos n\theta]$$

$$\Rightarrow \quad r^n = c^n \cos n\theta$$

is the required orthogonal trajectory.

3.2.4.3 Newton's Law of Cooling

Definitions

The temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

The differential equation is $\frac{d\theta}{dt} = -k(\theta - \theta_s)$

by variable separable $\int \frac{d\theta}{\theta - \theta_s} = \int -kdt$

$\Rightarrow \quad \log(\theta - \theta_s) = -kt + \log c$

$\Rightarrow \quad \theta - \theta_s = ce^{-kt}$

is the solution of Newton's law of cooling.

Example 3.

A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of air being 40°C. What will be the temperature of body after 40 minutes from the original?

Solution:

According to Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - 40)$$

$$\int \frac{d\theta}{\theta - 40} = -\int kdt$$

$\Rightarrow \quad \log(\theta - 40) = -kt + \log c$

$\Rightarrow \quad \theta - 40 = ce^{-kt}$ ...(i)

Put $t = 0,\ \theta = 80°$ in equation (i)

We get, $c = 40$

Put, $t = 20$ min, $\theta = 60°$

Then, $k = \frac{1}{20}\log 2$

By equation (i), $\theta = 40 + 40e^{\left(-\frac{1}{2}\log 2\right)t}$

Put, $t = 40$ min, then $\theta = 50°C$

3.2.4.4 Law of Growth

The rate of change amount of a substance with respect to time is directly proportional to the amount of substance present.

i.e. $\frac{dx}{dt} \propto x$

$$\frac{dx}{dt} = kx \ (k > 0)$$

$$\int \frac{dx}{x} = \int kdt$$

$\Rightarrow \quad \log x = kt + \log c$

$\Rightarrow \quad x = ce^{kt}$ is solution of law of growth

Example 4.

The number N of a bacteria in a culture of grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours?

Solution:

According to law of growth, $\frac{dN}{dt} \propto N$

Solution is $N = ce^{kt}$ ...(i)

Put N = 100 and t = 0 in equation (i)

We get, $c = 100$

Then, $N = 100\, e^{kt}$ ...(ii)

Put N = 332, t = 1 in equation (ii)

$$332 = 100e^{k}$$
$$e^{k} = 3.32$$

Put $t = \frac{3}{2}$ in equation (ii)

Then, $N = 100e^{\frac{3}{2}k} = 100(3.32)^{3/2} \simeq 605$

1.2.4.5 Law of Decay

Definitions

The rate of change of amount of substance is directly proportional to the amount of substance present.

i.e. $\frac{dx}{dt} \propto x$

The differential equation is

$$\frac{dx}{dt} = -kx \ (k > 0)$$

$$\int \frac{dx}{x} = -\int kdt$$

$\Rightarrow$ $\log x = -kt + \log c$

$\Rightarrow$ $x = ce^{-kt}$ is solution of law of decay.

Example 5.

If 30% of radio active substance dissappeared in 10 days. How long will take for 90% of it to disappear?

Solution:

According to law of decay

$$x = ce^{-kt} \quad ...(i)$$

Put $x = 100$, $t = 0$

$$100 = ce^{-k(0)}$$

We get, $c = 100$

Then, $x = 100e^{-kt}$ ...(ii)

Put $x = 70$, $t = 10$ in equation (ii)

Then, $70 = 100e^{10k}$

$$k = \frac{1}{10} ln\left[\frac{7}{10}\right]$$

$\therefore$ Equation (ii) becomes $\qquad x = 100\, e^{\frac{1}{10} ln\left(\frac{7}{10}\right)t}$ ...(iii)

Put, $x = 10$ in equation (iii)

$$10 = 100\, e^{\frac{1}{10} ln\left(\frac{7}{10}\right)t}$$

$$\frac{t}{10} ln(0.7) = ln\left(\frac{1}{10}\right)$$

$$t = \frac{-10 ln 10}{ln(0.7)} = 64.5 \text{ days}$$

3.3 Linear Differential Equations (Of n[th] Order)

3.3.1 Definitions

Linear differential equations are those in which the dependent variable its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of the n[th] order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = X$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear Differential Equations with Constant Coefficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants and X is a function of x only. Such equations are most important in the study of electromechanical vibrations and other engineering problems.

1. **Theorem:** If y_1, y_2 are only two solutions of the equations

$$\frac{d^n y}{dx^n} + \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2}}{dx^{n-2}} + \ldots + k_n y = 0 \qquad \ldots \text{(i)}$$

Then $c_1 y_1 + c_{21} y_2 (= u)$ is also its solution,

since it can be easily shown by differentiating is that $\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots\ldots + k_n u = 0$ ... (ii)

2. Since the general solution of a differential equation of the nth order contains n arbitrary constants, if follows, from above, that if $y_1, y_2, y_3, \ldots, y_n$, are n independent solutions of (1), then $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n (= u)$ is its complete solution.

3. If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots \text{(iii)}$$

then

$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \ldots + k_n v = X \qquad \ldots \text{(iv)}$$

Adding (ii) and (iv), we have $\frac{d^n (u+v)}{dx^n} + k_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + \ldots + k_n (u+v) = X$

This shows that $y = u + v$ is the complete solution of (iii).

The part u is called the **complementary function (C.F.)** and the part v is called the **particular integral (P.I.)** of (iii).

∴ The complete solution (C.S.) of (iii) is y = C.F. + P.I.

Thus in order to solve the equation (iii), we have to first find the C.F. i.e., the complementary function of (i), and then the P.I., i.e. a particular solution of (iii).

Operator D Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc., so that

$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, \frac{d^3x}{dx^3} = D^3y$ etc., the equation (iii) above can be written in the symbolic form

$$(D^n + k_1 D^{n-1} + \dots + k_n)y = X,$$

i.e.
$$f(D)y = X,$$

where $f(D) = D^n + k_1 D^{n-1} + \dots + k_n$, i.e. a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e. $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y$$
$$= (D + 3)(D - 1)y \text{ or } (D - 1)(D + 3)y.$$

3.3.2 Rules for Finding The Complementary Function

To solve the equation $\frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}}{dx^{n-2}} + \dots + k_n y = 0$ ... (i)

where k's are constants.

The equation (i) in symbolic form is

$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n)y = 0$ ... (ii)

Its symbolic co-efficient equated to zero i.e.

$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n = 0$

is called the auxiliary equation (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots. Now 4 cases arise.

Case I. If all the **roots be real and different**, then (ii) is equivalent to

$(D - m_1)(D - m_2) \dots (D - m_n)y = 0$ ... (iii)

Now (iii) will be satisfied by the solution of $(D - m_n)y = 0$, i.e. by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

∴ Its solution is $y e^{-m_n x} = c_n$, i.e. $y = c_n e^{m_n x}$

Similarly, since the factors in (iii) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0$, $(D - m_2) = 0$ etc., i.e. by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (i) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ ... (iv)

Case II. If **two roots are equal** (i.e. $m_1 = m_2$), then (iv) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$
$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

[∵ $c_1 + c_2$ = one arbitrary constant C]

It has only $n-1$ arbitrary constants and is, therefore, not the complete solution of (i). In this case, we proceed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D-m_1)(D-m_1)y=0$

Putting $(D-m_1)y=z$, it becomes $(D-m_1)z=0$ or $\frac{dz}{dx}-m_1z=0$

This is Leibnitz's linear in z and I.F. $=e^{-m_1x}$

$\therefore$ Its solution is $ze^{-m_1x}=c_1$ or $z=c_1e^{m_1x}$

Thus $$(D-m_1)y = z = c_1e^{m_1x} \text{ or } \frac{dy}{dx}-m_1y = c_1e^{m_1x} \quad \text{... (v)}$$

Its I.F. being e^{-m_1x}, the solution of (v) is

$$ye^{-m_1x} = \int c_1e^{m_1x}e^{-m_1x}dx+c_2$$

$$\Rightarrow \quad y = (c_1x+c_2)e^{m_1x}$$

Thus the complete solution of (i) is $y=(c_1x+c_2)e^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$

If, however, the A.E. has three equal roots (i.e. $m_1=m_2=m_3$), then the complete solution is

$$y = (c_1x^2+c_2x+c_3)e^{m_1x}+c_4e^{m_4x}+\ldots+c_ne^{m_nx}$$

Case III. If **one pair of roots be imaginary,** i.e.

$$m_1 = \alpha+i\beta,$$
$$m_2 = \alpha-i\beta,$$

then the complete solution is

$$y = c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

$$= e^{\alpha x}(c_1e^{i\beta x}+c_2e^{-i\beta x})+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

$$= e^{\alpha x}[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)]+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

[$\because$ by Euler's Theorem, $e^{i\theta}=\cos\theta+i\sin\theta$]

$$= e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

where $C_1 = c_1+c_2$

and $C_2 = i(c_1-c_2)$.

Case IV. If **two pair of imaginary roots be equal** i.e.

$$m_1 = m_2 = \alpha+i\beta,$$
$$m_3 = m_4 = \alpha-i\beta,$$

then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1x+c_2)\cos\beta x+(c_3x+c_4)\sin\beta x]+\ldots+c_ne^{m_nx}$$

3.3.3 Inverse Operator

1. Definition, $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants, which when operated upon by $f(D)$ gives X.

   i.e. $$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is, therefore, its particular integral.

Obviously. $f(D)$ and $1/f(D)$ are inverse operators.

2. $$\frac{1}{D}X = \int X dx$$

Let $$\frac{1}{D}X = y$$

Operating by D, $$D\frac{1}{D}X = Dy$$

i.e. $$X = \frac{dy}{dx}$$

integrating w.r.t. x, $$y = \int X dx$$

Thus $$\frac{1}{D}X = \int X dx$$

3. $$\frac{1}{D-a}X = e^{ax}\int X e^{-ax} dx$$

Let $$\frac{1}{D-a}X = y \qquad \text{... (ii)}$$

Operating by $D - a$,

$$(D-a)\cdot\frac{1}{D-a}X = (D-a)y$$

or $$X = \frac{dy}{dx} - ay, \text{ i.e. } \frac{dy}{dx} - ay = X$$

which is a Leibnitz's linear equations.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int X e^{-ax} dx$$

no constant being added as (ii) doesn't contain any constant.

Thus, $$\frac{1}{D-a}X = y = e^{ax}\int X e^{-ax} dx$$

3.3.4 Rules For Finding The Particular Integral

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$

which in symbolic form is $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n)y = X$

$$\therefore \quad \text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n}X$$

Case I. When $X = e^{ax}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
....
$$D^n e^{ax} = a^n e^{ax}$$

$$(D^n + k_1 D^{n-1} ... + k_n)e^{ax} = (a^n + k_1 a^{n-1} ... + k_n)e^{ax}$$

i.e. $$f(D)e^{ax} = f(a)\, e^{ax}$$

Operating on both sides by

$$\frac{1}{f(D)} \cdot f(D)e^{ax} = \frac{1}{f(D)} \cdot f(a)e^{ax}$$

or $$e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$\therefore$ by $\div f(a)$

$\therefore$ $$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \text{ provided } f(a) \neq 0 \qquad \text{... (i)}$$

If f(a) = 0, the above rule fails and we proceed further.

It can be proved that in that case,

$$\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax} \qquad \text{... (ii)}$$

If $f'(a) = 0$, then applying (2) again, we get $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$ ... (iii)

and so on.

Example 1. Solve

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

Solution:

$$(D^2 + 6D + 9)y = 5e^{3x}$$

Auxiliary equation is $D^2 + 6D + 9 = 0$ or $D = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36}$$

The complete solution is $$y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}$$

Example 2. Solve

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x}$$

Solution:

$$(D^2 - 6D + 9)y = 6e^{3x}$$

A.E. is $(D^2 - 6D + 9) = 0$ or $(D - 3)^2 = 0$, or $D = 3, 3$

$$\text{C.F.} = (C_1 + C_2x)e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 9}6e^{3x} = x\frac{1}{2D - 6}6e^{3x} = x^2\frac{1}{2} \cdot 6 \cdot e^{3x} = 3x^2\, e^{3x}$$

Complete solution is $$y = (C_1 + C_2x)\, e^{3x} + 3x^2\, e^{3x}$$

Case II. When $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b) \text{ provided } f(-a^2) \neq 0 \qquad \text{... (iv)}$$

If $f(-a^2) = 0$, the above rule fails and we can prove that,

$$\therefore \quad \frac{1}{f(D^2)}\cdot\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b) \text{ provided } f'(-a^2) \neq 0 \qquad \text{... (v)}$$

If $\quad f'(-a^2) = 0, \frac{1}{f(D^2)}\cdot\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$, provided $f''(-a^2) \neq 0$ and so on...

Similarly, $\quad \frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$, provided $f(-a^2) \neq 0$,

If $\quad f(-a^2) = 0, \frac{1}{f(D^2)}\cos(ax+b) = x\cdot\frac{1}{f'(-a^2)}\cos(ax+b)$, provided $f'(-a^2) \neq 0$.

If $\quad f'(-a^2) = 0, \frac{1}{f(D^2)}\cos(ax+b) = x^2\cdot\frac{1}{f''(-a^2)}\cos(ax+b)$ provided $f''(-a^2) \neq 0$ and so on....

Example 1. Solve

$$(D^2 + 4)y = \sin 3x.$$

Solution:

$$(D^2 + 4)y = \sin 3x$$

Auxiliary equation is $\quad D^2 + 4 = 0$ or $D = \pm 2i$.

$$\text{C.F.} = A\cos 2x + B\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cdot\sin 3x = \frac{\sin 3x}{(-3)^2+4} = \frac{1}{5}\sin 3x$$

Complete solution is $\quad y = A\cos 2x + B\sin 2x - \frac{1}{5}\sin 3x$

Example 2. Solve

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y = \cos 2x$$

Solution:

$$(D^2 + D + 1)y = \cos 2x$$

Auxiliary equation is $\quad D^2 + D + 1 = 0$

$$D = \frac{-1\pm\sqrt{-3}}{2}, \quad \text{C.F.} = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}x}{2}\right]$$

$$\text{P.I.} = \frac{1}{D^2+D+1}\cdot\cos 2x$$

$$= \frac{1}{(-2^2)+D+1}\cdot\cos 2x = \frac{1}{D-3}\cdot\cos 2x$$

$$= \frac{D+3}{D^2-9}\cdot\cos 2x = \frac{D+3}{(-2^2)-9}\cos 2x$$

$$\therefore \quad = -\frac{1}{13}(D+3)\cos 2x = -\frac{1}{13}(-2\sin 2x + 3\cos 2x)$$

Complete solution is

$$y = e^{-x/2}\left[A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right] + \frac{1}{13}[2\sin 2x - 3\cos 2x]$$

Example 3. Solve

$$(D^2 + 4)y = \cos 2x$$

Solution:

$$(D^2 + 4)y = \cos 2x$$

Auxiliary equation is

$$D^2 + 4 = 0$$

$$D = \pm 2i, \text{ C.F.} = A\text{ Cos } 2x + B\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x\cdot\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution is

$$y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$$

Case III. When $X = x^m$.

Here

$$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)^{th}$ and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 1. Solve

~~Find the P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$~~

Solution:

Given equation in symbolic form is $(D^2 + D)y = x^2 + 2x + 4$.

$$\therefore \quad \text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1 - D + D^2 - \ldots)(x^2+2x+4)$$

$$= \frac{1}{D}[x^2 + 2x + 4 - (2x+2) + 2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3} + 4x$$

Case IV. When $X = e^{ax}V$, V being a function of x.

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general,

$$D^n(e^{ax}u) = e^{ax}(D+a)^n u$$

$$\therefore \quad f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

Now put $f(D+a)u = V,$

i.e. $u = \dfrac{1}{f(D+a)}V,$

so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e. $\dfrac{1}{f(D)}(e^{ax}V) = e^{ax}\dfrac{1}{f(D+a)}V$

Example 1. Solve

$$(D^2 - 4\,D + 4)y = x^3\,e^{2x}$$

Solution:

$$(D^2 - 4D + 4)y = x^3\,e^{2x}$$

A.E. $D^2 - 4D + 4 = 0,\ (D-2)^2 = 0 \text{ or } D = 2, 2$

$$\text{C.F.} = (C_1 + C_2 x)e^{2x}$$

$$\text{PI} = \frac{1}{D^2-4D+4}x^3\cdot e^{2x} = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}x^3$$

$$= e^{2x}\frac{1}{D^2}x^3 = e^{2x}\cdot\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}\cdot\frac{x^5}{20}$$

$$y = (C_1+C_2x)\ e_2x + e^{2x}\cdot\frac{x^5}{20}$$

Example 2. Solve

$$(D^2 - 5D + 6)y = e^x\cos 2x$$

Solution:

$$(D^2 - 5D + 6)y = e^x\cos 2x$$

$$D^2 - 5D + 6 = 0$$

$$(D-2), (D-3) = 0, \text{ or } D = 2, 3$$

$$\text{C.F.} = C_1\,e^{2x} + C_2\,e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-5D+6}e^x\cos 2x$$

$$= e^x\cdot\frac{1}{(D+1)^2-5(D+1)+6}\cos 2x$$

$$= e^x\cdot\frac{1}{D^2-3D+2}\cos 2x = e^x\cdot\frac{1}{-4-3D+2}\cos 2x$$

$$= -e^x\frac{1}{3D+2}\cos 2x = -e^x\frac{3D-2}{9D^2-4}\cos 2x$$

$$= -e^x\frac{3D-2}{9(-4)-4}\cos 2x = \frac{e^x}{40}(3D-2)\cos 2x$$

$$= \frac{e^x}{40}(-6\sin 2x - 2\cos 2x) = -\frac{e^x}{20}(3\sin 2x + \cos 2x)$$

$$y = C_1e^2 + C_2e^3x - \frac{e^x}{20}(3\sin 2x + \cos 2x)$$

Case V. When X is any other function of x.

Here $\quad \text{P.I.} = \frac{1}{f(D)}X$

If $\quad f(D) = (D-m_1)(D-m_2)\ldots D(D-m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ldots + \frac{A_n}{D-m_n}$$

$$\therefore \quad \text{P.I.} = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ldots + \frac{A_n}{D-m_n}\right]X$$

$$= A_1\frac{1}{D-m_1}X + A_2\frac{1}{D-m_2}X + \ldots + A_n\frac{1}{D-m_n}X$$

$$= A_1 \cdot e^{m_1x}\int Xe^{-m_1x}dx + A_2 \cdot e^{m_2x}\int Xe^{-m_2x}dx + \ldots + A_n \cdot e^{m_nx}\int Xe^{-m_nx}dx$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

3.3.5 Summary: Working Procedure to Solve The Equation

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1}\frac{dy}{dx} + k_ny = X$$

of which the symbolic form is

$$(D^n + k_1D^{n-1} + \ldots + k_{n-1}D + k_n)y = X$$

Step I. To Find the Complementary Function

1. Write the A.E.

   i.e. $D^n + k_1D^{n-1} + \ldots + k_{n-1}D + k_n = 0$ and

2. Write the C.F. as follows:

Roots of A.E.	C.F.
1. $m_1, m_2, m_3\ldots$(real and different roots)	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots$
2. $m_1, m_1, m_3\ldots$ (two real and equal roots)	$(c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots$
3. $m_1, m_1, m_1, m_4\ldots$ (three real and equal roots)	$(c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3\ldots$ (a pair of imaginary roots)	$e^{ax}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5\ldots$ (2 pairs of equal imaginary roots)	$e^{ax}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + c_5e^{m_5x} + \ldots$

Step II. To Find the Particular Integral

From symbolic form $$\text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n} X$$

$$= \frac{1}{f(D)} \text{ or } \frac{1}{\phi(D^2)} X$$

1. When $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

2. When $X = \sin(ax + b)$ or $\cos(ax + b)$

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2$ $[\phi(-a^2) \neq 0]$

$$= x \frac{1}{\phi'(D^2)} \sin(ax + b) \quad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2$ $[\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2 [\phi'(-a^2) = 0, \phi''(-a^2) \neq 0.$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D)^2$ w.r.t. D, etc.

3. When $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

4. When $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$$

and then evaluate $\frac{1}{f(D + a)} V$ as in (i), (ii), and (iii).

5. When X is any function of x.

$$\text{P.I.} = \frac{1}{f(D)}X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$$

Step III. To find the complete solution:
Then the C.S. is y = C.F. + P.I.

3.4 Two Other Methods of Finding P.I.

3.4.1 Method of Variation of Parameters

This method is quite general and applies to equations of the from

$$y'' + py' + qy = X \quad \text{... (i)}$$

where p, q, and X are functions of x. It gives

$$\text{P.I.} = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx \quad \text{... (ii)}$$

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ ... (iii)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Example 1.

Using the method of variation of parameters, solve

$$y'' + y = \sec x$$

Solution:

Given equation in symbolic form is $(D^2 + 1)y = \sec x$.

(a) To find C.F.

Its A.E. is $D^2 + 1 = 0$,

$\therefore$ $D = \pm i$

Thus C.F. is $y = c_1 \cos x + c_2 \sin x$

(b) To find P.I.

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \sec x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Thus,

$$\begin{aligned}\text{P.I.} &= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx \\ &= -\cos x\int\frac{\sin x \sec x\, dx}{1} + \sin x\int\frac{\cos x \sec x\, dx}{1} \\ &= -\cos x\int \tan x\, dx + \sin x\int 1.dx \\ &= \cos x \ln \cos x + x \sin x\end{aligned}$$

Hence the C.S. is

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + \cos x \ln \cos x + x \sin x \\ &= (c_1 + \ln \cos x)\cos x + (c_2 + x)\sin x\end{aligned}$$

.5 Equations Reducible to Linear Equation with Constant Coefficient

Definitions

Now, we shall study linear differential equation with variable coefficients, which can be reduced to linear 'ferential equations with constant coefficients by suitable substitutions.

Euler-Cauchy differential equation.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} x \frac{dy}{dx} + k_n y = Q(x)$$

It can be reduced into linear differential equations with constant coefficients.

By taking $x = e^t$ (or) $t = \log x$

Let, $$\theta = \frac{d}{dt}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$\Rightarrow$ $$x\frac{dy}{dx} = \theta y$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dt}\right] = -\frac{1}{x^2}\frac{dx}{dt} + \frac{1}{x}\frac{d}{dt}\left[\frac{dy}{dt}\right]\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$$

Similarly, $$x^3\frac{d^3y}{dx^3} = \theta(\theta - 1)(\theta - 2)y \text{ and so on.}$$

Substitute all these values in given differential equation, it results in a linear equation with constant coefficients. ıich can be solved as above methods.

Example 1.

Consider the differential equation $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

with the boundary conditions of $y(0) = 0$, $y(1) = 1$, the complete solution of the differential equation is

(a) x^2

(b) $\sin\frac{\pi x}{2}$

(c) $e^x \sin\frac{\pi x}{2}$

(d) $e^{-x} \sin\frac{\pi x}{2}$

Solution: (a)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0 \text{ and } y(0) = 0,\ y(1) = 1$$

Choice (a) satisfies the initial condition as well as equation as shown in below

if $$y = x^2$$

$\Rightarrow$ $$y(0) = 0, \quad y(1) = 1^2 = 1$$

Substitution in differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2 \times 2 + x \times 2x - 4x^2 = 0$$

$\therefore$ $4 = x^2$ is complete solution

Alternate Solution:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

$$(x^2 D^2 + xD - 4)y = 0$$

$$[\theta(\theta-1) + \theta - 4]y = 0$$

$$(\theta^2 - \theta + \theta - 4) = 0$$

$$(\theta^2 - 4)y = 0$$

Auxilliary equation is $m^2 - 4 = 0$

$$m = \pm 2$$

CF is $C_1 e^{-2z} + C_2 e^{2z}$

Solution is $y = C_1 e^{-2z} + C_2 e^{2z} = C_1 x^{-2} + C_2 x^2 = C_1 \frac{1}{x^2} + C_2 x^2$

One of the independent solution is x^2.

■■■■

Previous GATE and ESE Questions

Q.1 The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$

(b) $y = \frac{-x^3}{3} + c$

(c) ce^x

(d) unsolvable as equation is non-linear

[ME, GATE-2003, 2 marks]

Q.2 Biotransformation of an organic compound having concentration (x) can be modeled using an ordinary differential equation $\frac{dx}{dt} + kx^2 = 0$, where k is the reaction rate constant. If $x = a$ at $t = 0$, the solution of the equation is

(a) $x = ae^{-kt}$ (b) $\frac{1}{x} = \frac{1}{a} + kt$

(c) $x = a(1 - e^{-kt})$ (d) $x = a + kt$

[CE, GATE-2004, 2 marks]

Q.3 The following differential equation has

$$3\left(\frac{d^2y}{dt^2}\right) + 4\left(\frac{dy}{dt}\right)^3 + y^2 + 2 = x$$

(a) degree = 2, order = 1

(b) degree = 1, order = 2

(c) degree = 4, order = 3

(d) degree = 2, order = 3

[EC, GATE-2005, 1 mark]

Q.4 The solution of the first order differential equation $x'(t) = -3x(t)$, $x(0) = x_0$ is

(a) $x(t) = x_0 e^{-3t}$ (b) $x(t) = x_0 e^{-3}$

(c) $x(t) = x_0 e^{-1/3}$ (d) $x(t) = x_0 e^{-1}$

[EE, GATE-2005, 1 mark]

Q.5 Transformation to linear form by substituting $v = y^{1-n}$ of the equation $\frac{dy}{dt} + p(t)y = q(t)y^n$; $n > 0$ will be

(a) $\frac{dv}{dt} + (1-n)pv = (1-n)q$

(b) $\frac{dv}{dt} + (1-n)pv = (1+n)q$

(c) $\frac{dv}{dt} + (1+n)pv = (1-n)q$

(d) $\frac{dv}{dt} + (1+n)pv = (1+n)q$

[CE, GATE-2005, 2 marks

Q.6 The solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$; $y(0) = 1$, $\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$ in the range $0 < x < \frac{\pi}{4}$ is given by

(a) $e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$

(b) $e^{x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

(c) $e^{-4x}\left(\cos x - \frac{1}{4}\sin x\right)$

(d) $e^{-4x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

[CE, GATE-2005, 2 marks]

Statement for Linked Answer Questions 7 and 8.
The complete solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \text{ is } y = c_1e^{-x} + c_2\, e^{-3x}.$$

Q.7 Then, p and q are

(a) $p = 3,\ q = 3$ (b) $p = 3,\ q = 4$

(c) $p = 4,\ q = 3$ (d) $p = 4,\ q = 4$

[ME, GATE-2005, 2 marks]

Q.8 Which of the following is a solution of the differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1) = 0$?

(a) e^{-3x} (b) xe^{-x}

(c) xe^{-2x} (d) x^2e^{-2x}

[ME, GATE-2005, 2 marks]

Q.9 A solution of the following differential equation is given by $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(a) $y = e^{2x} + e^{-3x}$ (b) $y = e^{2x} + e^{3x}$

(c) $y = e^{-2x} + e^{3x}$ (d) $y = e^{-2x} + e^{-3x}$

[EC, GATE-2005, 1 mark]

Q.10 A spherical naphthalene ball exposed to the atmosphere loses volume at a rate proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm and the diameter reduces to 1 cm after 3 months, the ball completely evaporates in

(a) 6 months (b) 9 months

(c) 12 months (d) infinite time

[CE, GATE-2006, 2 marks]

Q.11 The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{+x^2}$ (b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{+x^2}$ (d) $(1-x)e^{-x^2}$

[ME, GATE-2006, 1 mark]

Q.12 For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$

(c) $3e^{2x}$ (d) $C_1e^{-x} + C_2e^{-3x}$

[ME, GATE-2006, 2 marks]

Q.13 The degree of the differential equation $\frac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

[CE, GATE-2007, 1 mark]

Q.14 The solution for the differential equation $\frac{dy}{dx} = x^2y$ with the condition that $y = 1$ at $x = 0$ is

(a) $y = e^{\frac{1}{2x}}$ (b) $\ln(y) = \frac{x^3}{3} + 4$

(c) $\ln(y) = \frac{x^2}{2}$ (d) $y = e^{\frac{x^3}{3}}$

[CE, GATE-2007, 1 mark]

Q.15 The solution of $dy/dx = y^2$ with initial value $y(0) = 1$ bounded in the interval

(a) $-\infty \le x \le \infty$ (b) $-\infty \le x \le 1$

(c) $x < 1,\ x > 1$ (d) $-2 \le x \le 2$

[ME, GATE-2007, 2 marks]

Q.16 A body originally at 60°C cools down to 40°C in 15 minutes when kept in air at a temperature of 25°C. What will be the temperature of the body at the end of 30 minutes?

(a) 35.2°C (b) 31.5°C

(c) 28.7°C (d) 15°C

[CE, GATE-2007, 2 marks]

Q.17 Solution of $\frac{dy}{dx} = -\frac{x}{y}$ at $x = 1$ and $y = \sqrt{3}$ is

(a) $x - y^2 = -2$ (b) $x + y^2 = 4$

(c) $x^2 - y^2 = -2$ (d) $x^2 + y^2 = 4$

[CE, GATE-2008, 2 marks]

Q.18 Which of the following is a solution to the differential equation $\frac{dx(t)}{dt} + 3x(t) = 0$?

(a) $x(t) = 3e^{-t}$ (b) $x(t) = 2e^{-3t}$

(c) $x(t) = -\frac{3}{2}t^2$ (d) $x(t) = 3t^2$

[EC, GATE-2008, 1 mark]

Q.19 The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

(a) $y = P\cos x + Q\sin x$

(b) $y = P\cos x$

(c) $y = P\sin x$

(d) $y = P\sin^2 x$

[CE, GATE-2008, 1 mark]

Q.20 Given that $\ddot{x} + 3x = 0$, and $x(0) = 1$, $\dot{x}(0) = 0$, what is $x(1)$?

(a) –0.99 (b) –0.16

(c) 0.16 (d) 0.99

[ME, GATE-2008, 1 mark]

Q.21 It is given that $y'' + 2y' + y = 0$, $y(0) = 0$, $y(1)=0$. What is $y(0.5)$?

(a) 0 (b) 0.37

(c) 0.62 (d) 1.13

[ME, GATE-2008, 2 marks]

Q.22 The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1 (b) 2

(c) 3 (d) 4

[EC, GATE-2009, 1 mark]

Q.23 Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses (b) circles

(c) parabolas (d) hyperbolas

[CE, GATE-2009, 2 marks]

Q.24 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I	List-II
A. $\frac{dy}{dx} = \frac{y}{x}$	1. Circles
B. $\frac{dy}{dx} = -\frac{y}{x}$	2. Straight lines
C. $\frac{dy}{dx} = \frac{x}{y}$	3. Hyperbolas
D. $\frac{dy}{dx} = -\frac{x}{y}$	

Codes:

	A	B	C	D
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

[EC, GATE-2009, 2 marks]

Q.25 The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$ (b) $y = \frac{4x^4}{5} + \frac{4}{5x}$

(c) $y = \frac{x^4}{5} + 1$ (d) $y = \frac{x^5}{5} + 1$

[ME, GATE-2009, 2 marks]

Q.26 The order and degree of the differential equation $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are respectively

(a) 3 and 2 (b) 2 and 3

(c) 3 and 3 (d) 3 and 1

[CE, GATE-2010, 1 mark]

Q.27 The Blasius equation, $\frac{d^3f}{d\eta^3} + \frac{f}{2}\frac{d^2f}{d\eta^2} = 0$, is a

(a) second order nonlinear ordinary differential equation

(b) third order nonlinear ordinary differential equation

(c) third order linear ordinary differential equation

(d) mixed order nonlinear ordinary differential equation

[ME, GATE-2010, 1 mark]

Q.28 The solution to the ordinary differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ is

(a) $y = c_1e^{3x} + c_2e^{-2x}$

(b) $y = c_1e^{3x} + c_2e^{2x}$

(c) $y = c_1e^{-3x} + c_2e^{2x}$

(d) $y = c_1e^{-3x} + c_2e^{-2x}$

[CE, GATE-2010, 2 marks]

.29 For the differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$ with initial conditions $x(0)=1$ and $\left.\frac{dx}{dt}\right|_{t=0}=0$, the solution is

(a) $x(t)=2e^{-6t}-e^{-2t}$ (b) $x(t)=2e^{-2t}-e^{-4t}$
(c) $x(t)=-e^{-6t}+2e^{-4t}$ (d) $x(t)=e^{-2t}+2e^{-4t}$

[EE, GATE-2010, 2 marks]

.30 A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2}-\frac{n(x)}{L^2}=0$ where L is a constant. The boundary conditions are : $n(0)=K$ and $n(\infty)=0$. The solution to this equation is

(a) $n(x)=K\exp(x/L)$
(b) $n(x)=K\exp\left(-x/\sqrt{L}\right)$
(c) $n(x)=K^2\exp(-x/L)$
(d) $n(x)=K\exp(-x/L)$

[EC, GATE-2010, 1 mark]

.31 Consider the differential equation $\frac{dy}{dx}=\left(1+y^2\right)x$. The general solution with constant c is

(a) $y=\tan\frac{x^2}{2}+\tan c$
(b) $y=\tan^2\left(\frac{x}{2}+c\right)$
(c) $y=\tan^2\left(\frac{x}{2}\right)+c$
(d) $y=\tan\left(\frac{x^2}{2}+c\right)$

[ME, GATE-2011, 2 marks]

32 With K as a constant, the solution possible for the first order differential equation $\frac{dy}{dx}=e^{-3x}$ is

(a) $-\frac{1}{3}e^{-3x}+K$ (b) $-\frac{1}{3}e^{3x}+K$
(c) $-3e^{-3x}+K$ (d) $-3e^{-x}+K$

[EE, GATE-2011, 1 mark]

33 The solution of the differential equation $\frac{dy}{dx}=ky$, $y(0)=c$ is

(a) $x=ce^{-ky}$ (b) $x=ke^{cy}$
(c) $y=ce^{kx}$ (d) $y=ce^{-kx}$

[EC, GATE-2011, 1 mark]

Q.34 The solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=x$, with the condition that $y=1$ at $x=1$, is

(a) $y=\frac{2}{3x^2}+\frac{x}{3}$ (b) $y=\frac{x}{2}+\frac{1}{2x}$
(c) $y=\frac{2}{3}+\frac{x}{3}$ (d) $y=\frac{2}{3x}+\frac{x^2}{3}$

[CE, GATE-2011, 2 marks]

Q.35 The solution of the ordinary differential equation $\frac{dy}{dx}+2y=0$ for the boundary condition, $y=5$ at $x=1$ is

(a) $y=e^{-2x}$ (b) $y=2e^{-2x}$
(c) $y=10.95\,e^{-2x}$ (d) $y=36.95\,e^{-2x}$

[CE, GATE-2012, 2 marks]

Q.36 With initial condition $x(1)=0.5$, the solution of the differential equation, $t\frac{dx}{dt}+x=t$ is

(a) $x=t-\frac{1}{2}$ (b) $x=t^2-\frac{1}{2}$
(c) $x=\frac{t^2}{2}$ (d) $x=\frac{t}{2}$

[EC, EE, IN, GATE-2012, 1 mark]

Q.37 The partial differential equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}$$ is a

(a) linear equation of order 2
(b) non-linear equation of order 1
(c) linear equation of order 1
(d) non-linear equation of order 2

[ME, GATE-2013, 1 mark]

Q.38 The type of the partial differential equation $\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}$ is

(a) Parabolic (b) Elliptic
(c) Hyperbolic (d) Non-linear

[IN, GATE-2013 : 1 mark]

Q.39 The solution to the differential equation $\frac{d^2u}{dx^2}-k\frac{du}{dx}=0$ is where k is constant, subjected to the boundary conditions $u(0)=0$ and $u(L)=U$, is

(a) $u = U\frac{x}{L}$ (b) $u = U\left(\frac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u = U\left(\frac{1-e^{-kx}}{1-e^{-kL}}\right)$ (d) $u = U\left(\frac{1+e^{kx}}{1+e^{kL}}\right)$

[ME, GATE-2013, 2 marks]

Q.40 The maximum value of the solution $y(t)$ of the differential equation $y(t) + \ddot{y}(t) = 0$ with initial conditions $\dot{y}(0) = 1$ and $y(0) = 1$, for $t \geq 0$ is

(a) 1 (b) 2

(c) π (d) $\sqrt{2}$

[IN, GATE-2013 : 2 marks]

Q.41 A system described by a linear, constant coefficient, ordinary, first order differential equation has an exact solution given by $y(t)$ for $t > 0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t > 0$, we need to

(a) change the initial condition to $-y(0)$ and the forcing function to $2x(t)$

(b) change the initial condition to $2y(0)$ and the forcing function to $-x(t)$

(c) change the initial condition to $j\sqrt{2}\,y(0)$ and the forcing function to $j\sqrt{2}\,x(t)$

(d) change the initial condition to $-2y(0)$ and the forcing function to $-2x(t)$

[EC, GATE-2013, 2 marks]

Q.42 The matrix form of the linear system $\frac{dx}{dt} = 3x - 5y$ and $\frac{dy}{dt} = 4x + 8y$ is

(a) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(b) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(c) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(d) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

[ME, GATE-2014 : 1 mark]

Q.43 The general solution of the differential equation $\frac{dy}{dx} = \cos(x + y)$, with c as a constant, is

(a) $y + \sin(x + y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

(c) $\cos\left(\frac{x+y}{2}\right) = x + c$

(d) $\tan\left(\frac{x+y}{2}\right) = x + c$

[ME, GATE-2014 : 2 marks]

Q.44 The solution of the initial value problem $\frac{dy}{dx} = -2xy;\ y(0) = 2$ is

(a) $1 + e^{-x^2}$ (b) $2e^{-x^2}$

(c) $1 + e^{x^2}$ (d) $2e^{x^2}$

[ME, GATE-2014 : 1 mark]

Q.45 Which ONE of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables respectively?

(a) $\frac{dy}{dx} + xy = e^{-x}$ (b) $\frac{dy}{dx} + xy = 0$

(c) $\frac{dy}{dx} + xy = e^{-y}$ (d) $\frac{dy}{dx} + e^{-y} = 0$

[EC, GATE-2014 : 1 mark]

Q.46 The solution for the differential equation $\frac{d^2x}{dt^2} = -9x$ with initial conditions $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 1$, is

(a) $t^2 + t + 1$ (b) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$

(c) $\frac{1}{3}\sin 3t + \cos 3t$ (d) $\cos 3t + t$

[EE, GATE-2014 : 1 mark]

Q.47 Consider two solutions $x(t) = x_1(t)$ and $x(t) = x_2(t)$ of the differential equation $\frac{d^2x(t)}{dt^2} + x(t) = 0,\ t > 0,$ such that $x_2 = 0,\ \left.\frac{dx_2(t)}{dt}\right|_{t=0} = 1$. The Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \dfrac{dx_1(t)}{dt} & \dfrac{dx_2(t)}{dt} \end{vmatrix} \text{ at } t = \pi/2 \text{ is}$$

(a) 1 (b) –1

(c) 0 (d) $\pi/2$

[ME, GATE-2014 : 2 Marks]

Q.48 If the characteristic equation of the differential equation $\dfrac{d^2y}{dx^2} + 2\alpha\dfrac{dy}{dx} + y = 0$ has two equal roots, then the values of α are

(a) ± 1 (b) 0, 0

(c) $\pm j$ (d) $\pm 1/2$

[EC, GATE-2014 : 1 Mark]

Q.49 If a and b are constants, the most general solution of the differential equation $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$ is

(a) ae^{-t} (b) $ae^{-t} + bte^{-t}$

(c) $ae^{t} + bte^{-t}$ (d) ae^{-2t}

[EC, GATE-2014 : 1 Mark]

Q.50 Consider the differential equation $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$. Which of the following is a solution to this differential equation for $x > 0$?

(a) e^x (b) x^2

(c) $1/x$ (d) $\ln x$

[EE, GATE-2014 : 1 Mark]

Q.51 Consider the following differential equation:

$\dfrac{dy}{dt} = -5y$; initial condition: $y = 2$ at $t = 0$

The value of y at $t = 3$ is

(a) $-5e^{-10}$ (b) $2e^{-10}$

(c) $2e^{-15}$ (d) $-15e^{2}$

[ME, GATE-2015 : 2 Marks]

Q.52 Consider the following difference equation

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(a) $\dfrac{x}{y}\cos\dfrac{y}{x} = c$ (b) $\dfrac{x}{y}\sin\dfrac{y}{x} = c$

(c) $xy\cos\dfrac{y}{x} = c$ (d) $xy\sin\dfrac{y}{x} = c$

[CE, GATE-2015 : 2 Marks]

Q.53 Consider the following second order linear differential equation

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

The boundary conditions are: at $x = 0$, $y = 5$ and $x = 2$, $y = 21$

The value of y at $x = 1$ is ________.

[CE, GATE-2015 : 2 Marks]

Q.54 A differential equation $\dfrac{di}{dt} - 0.2i = 0$ is applicable over $-10 < t < 10$. If $i(4) = 10$, then $i(-5)$ is _____.

[EE, GATE-2015 : 2 Marks]

Q.55 The general solution of the differential equation

$\dfrac{dy}{dx} = \dfrac{1+\cos 2y}{1-\cos 2x}$ is

(a) $\tan y - \cot x = c$ (c is a constant)

(b) $\tan x - \cot y = c$ (c is a constant)

(c) $\tan y + \cot x = c$ (c is a constant)

(d) $\tan x + \cot y = c$ (c is a constant)

[EC, GATE-2015 : 1 Mark]

Q.56 A solution of the ordinary differential equation $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $y(1) = -\dfrac{1-3e}{e^3}$. The value of $\dfrac{dy}{dt}(0)$ is _______.

[EE, GATE-2015 : 2 Marks]

Q.57 The solution of the differential equation $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$ with $y(0) = y'(0) = 1$ is

(a) $(2-t)e^{t}$ (b) $(1+2t)e^{-t}$

(c) $(2+t)e^{-t}$ (d) $(1-2t)e^{t}$

[EC, GATE-2015 : 2 Marks]

Q.58 Consider the differential equation $\dfrac{d^2x(t)}{dt^2} + 3\dfrac{dx(t)}{dt} + 2x(t) = 0$. Given $x(0) = 20$ and $x(1) = 10/e$, where $e = 2.718$, the value of $x(2)$ is_____.

[EC, GATE-2015 : 2 Marks]

Q.59 Find the solution of $\frac{d^2y}{dx^2} = y$ which passes through the origin and the point $\left(In\, 2, \frac{3}{4}\right)$.

(a) $y = \frac{1}{2}e^x - e^{-x}$ (b) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

(c) $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ (d) $y = \frac{1}{2}e^x + e^{-x}$

[ME, GATE-2015 : 2 Marks]

Q.60 A function $y(t)$, such that $y(0) = 1$ and $y(1) = 3e^{-1}$, is a solution of the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$. Then $y(2)$ is

(a) $5e^{-1}$ (b) $5e^{-2}$

(c) $7e^{-1}$ (d) $7e^{-2}$

[EE, GATE-2016 : 1 Mark]

Q.61 The solution of the differential equation, for $t > 0$, $y''(t) + 2y'(t) + y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$, is ($u(t)$ denotes the unit step function),

(a) $te^{-t}u(t)$ (b) $(e^{-t} - te^{-t})u(t)$

(c) $(-e^{-t} + te^{-t})u(t)$ (d) $e^{-t}u(t)$

[EE, GATE-2016 : 1 Mark]

Q.62 Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with initial conditions $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$. Then the value of $y(1)$ is ____.

[EE, GATE-2016 : 2 Marks]

Q.63 The particular solution of the initial value problem given below is

$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$ with $y(0) = 3$ and $\left.\frac{dy}{dx}\right|_{x=0} = -36$

(a) $(3 - 18x)e^{-6x}$ (b) $(3 + 25x)e^{-6x}$

(c) $(3 + 20x)e^{-6x}$ (d) $(3 - 12x)e^{-6x}$

[EC, GATE-2016 : 2 Marks]

Q.64 If $y = f(x)$ satisfies the boundary value problem $y'' + 9y = 0$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = \sqrt{2}$, then $y\left(\frac{\pi}{4}\right)$ is ____.

[ME, GATE-2016 : 2 Marks]

Q.65 The respective expressions for complimentary function and particular integral part of the solution of the differential equation

$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$ are

(a) $\left[c_1 + c_2x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(b) $\left[c_2 + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

(c) $\left[c_1 + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(d) $\left[c_1 + c_2x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

[CE, GATE-2016 : 2 Marks]

Q.66 Consider the differential equation $3y''(x) + 27y(x) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 2000$. The value of y at $x = 1$ is ____.

[ME, GATE-2017 : 2 Marks]

Q.67 The differential equation $\frac{d^2y}{dx^2} + 16y = 0$ for $y(x)$ with the two boundary conditions $\left.\frac{dy}{dx}\right|_{x=0} = 1$ and $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -1$ has

(a) no solution

(b) exactly two solutions

(c) exactly one solution

(d) infinitely many solutions

[ME, GATE-2017 : 1 Mark]

Q.68 Consider the differential equation $(t^2 - 81)\frac{dy}{dt} + 5t\, y = \sin(t)$ with $y(1) = 2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(a) $(-2, 2)$ (b) $(-10, 10)$

(c) $(-10, 2)$ (d) $(0, 10)$

[EE, GATE-2017 : 2 Marks]

Q.69 The general solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-5y=0$$

in terms of arbitrary constants K_1 and K_2 is

(a) $K_1e^{(-1+\sqrt{6})x}+K_2e^{(-1-\sqrt{6})x}$

(b) $K_1e^{(-1+\sqrt{8})x}+K_2e^{(-1-\sqrt{8})x}$

(c) $K_1e^{(-2+\sqrt{6})x}+K_2e^{(-2-\sqrt{6})x}$

(d) $K_1e^{(-2+\sqrt{8})x}+K_2e^{(-2-\sqrt{8})x}$

[EC, GATE-2017 : 1 Mark]

Q.70 Which one of the following is the general solution of the first order differential equation

$$\frac{dy}{dx}=(x+y-1)^2,$$

where x, y are real?

(a) $y = 1 + x + \tan^{-1}(x + c)$, where c is a constant.
(b) $y = 1 + x + \tan(x + c)$, where c is a constant.
(c) $y = 1 - x + \tan^{-1}(x + c)$, where c is a constant.
(d) $y = 1 - x + \tan(x + c)$, where c is a constant.

[EC, GATE-2017 : 2 Marks]

Q.71 Consider the following second-order differential equation:

$$y'' - 4y' + 3y = 2t - 3t^2$$

The particular solution of the differential equation is

(a) $-2-2t-t^2$ (b) $-2t-t^2$
(c) $2t-t^2$ (d) $-2-2t-3t^2$

[CE, GATE-2017 : 2 Marks]

Q.72 The solution of the equation $\frac{dQ}{dt}+Q=1$ with $Q = 0$ at $t = 0$ is

(a) $Q(t) = e^{-t} - 1$ (b) $Q(t) = 1 + e^{-t}$
(c) $Q(t) = 1 - e^{t}$ (d) $Q(t) = 1 - e^{-t}$

[CE, GATE-2017 : 2 Marks]

Q.73 A particle of mass 2 kg is travelling at a velocity of 1.5 m/s. A force $f(t) = 3t^2$ (in N) is applied to it in the direction of motion for a duration of 2 seconds, where t denotes time in seconds. The velocity (in m/s, up to one decimal place) of the particle immediately after the removal of the force is ________.

[CE, GATE-2017 : 2 Marks]

Q.74 The complete integral of $(z - px - qy)^3 = pq + 2(p^2 + q)^2$ is

(a) $z=ax+by+\sqrt[3]{pq+2(p^2+q)^2}$

(b) $z=ax+by+\sqrt[3]{ab+2(a^2+b)^2}$

(c) $z=ax+by+\sqrt[3]{ab}+\sqrt[3]{2(a^2+b)^2}$

(d) $z = ax + by + c$

[ESE Prelims-2017]

Q.75 If a clock loses 5 seconds per day, what is the alteration required in the length of the pendulum in order that the clock keeps correct time?

(a) $\frac{4}{86400}$ times its original length be shortened

(b) $\frac{1}{86400}$ times its original length be shortened

(c) $\frac{1}{8640}$ times its original length be shortened

(d) $\frac{4}{8640}$ times its original length be shortened

[ESE Prelims-2017]

Q.76 The solution of the differential equation, $y\sqrt{1-x^2}\,dy+x\sqrt{1-y^2}\,dx=0$ is

(a) $\sqrt{1-x^2}=c$

(b) $\sqrt{1-y^2}=c$

(c) $\sqrt{1-x^2}+\sqrt{1-y^2}=c$

(d) $\sqrt{1+x^2}+\sqrt{1+y^2}=c$

[EE, ESE-2017]

Q.77 The general solution of the differential equation,

$$\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0 \text{ is}$$

(a) $y = (C_1 - C_2x)\,e^x + C_3\cos x + C_4\sin x$
(b) $y = (C_1 + C_2x)\,e^x - C_2\cos x + C_4\sin x$
(c) $y = (C_1 + C_2x)\,e^x + C_3\cos x + C_4\sin x$
(d) $y = (C_1 + C_2x)\,e^x + C_3\cos x - C_4\sin x$

[EE, ESE-2017]

Q.78 The solution (up to three decimal places) at $x = 1$ of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$ subject to boundary conditions y(0) = 1 and $\frac{dy}{dx}=(0)=-1$ is ____.

[CE, GATE-2018 : 2 Marks]

Q.79 If y is the solution of the differential equation

$$y^3\frac{dy}{dx} + x^3 = 0,$$
$$y(0) = 1$$

the value of $y(-1)$ is

(a) –2 (b) –1
(c) 0 (d) 1

[ME, GATE-2018 : 1 Mark]

Q.80 Given the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

with $y(0) = 0$ and $\frac{dy}{dx}(0) = 1$, the value of $y(1)$ is ________ (correct to two decimal places).

[ME, GATE-2018 : 2 Marks]

Q.81 A curve passes through the point ($x = 1$, $y = 0$) and satisfies the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2y} + \frac{y}{x}$. The equation that describes the curve is

(a) $ln\left(1+\frac{y^2}{x^2}\right) = x - 1$ (b) $\frac{1}{2}ln\left(1+\frac{y^2}{x^2}\right) = x - 1$

(c) $ln\left(1+\frac{y}{x}\right) = x - 1$ (d) $\frac{1}{2}ln\left(1+\frac{y}{x}\right) = x - 1$

[EC, GATE-2018 : 2 Marks]

Q.82 Consider a system governed by the following equations:

$$\frac{dx_1(t)}{dt} = x_2(t) - x_1(t) \quad ; \quad \frac{dx_2(t)}{dt} = x_1(t) - x_2(t)$$

The initial conditions are such that $x_1(0) < x_2(0) < \infty$. Let $x_{1f} = \lim_{t\to\infty} x_1(t)$ and $x_{2f} = \lim_{t\to\infty} x_2(t)$. Which one of the following is true?

(a) $x_{1f} < x_{2f} < \infty$ (b) $x_{2f} < x_{1f} < \infty$
(c) $x_{1f} = x_{2f} < \infty$ (d) $x_{1f} = x_{2f} = \infty$

[EE, GATE-2018 : 2 Marks]

Q.83 The solution of the differential equation, $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$, where, $y(0) = 0$ and $y'(0) = -2$ is

(a) $y = e^{-x} - e^{2x} + xe^{2x}$
(b) $y = e^{x} - e^{-2x} - xe^{2x}$
(c) $y = e^{-x} + e^{2x} + xe^{2x}$
(d) $y = e^{x} - e^{-2x} + xe^{2x}$

[EE, ESE-2018]

Q.84 If $\frac{d^2y}{dt^2} + y = 0$ under the conditions $y = 1$, $\frac{dy}{dt} = 0$, when $t = 0$, then y is equal to

(a) $\sin t$ (b) $\cos t$
(c) $\tan t$ (d) $\cot t$

[EE, ESE-2018]

■■■■

Answers Differential Equations

1.	(a)	2.	(b)	3.	(b)	4.	(a)	5.	(a)	6.	(a)	7.	(c)	8.	(c)	9.	(b)
10.	(a)	11.	(b)	12.	(b)	13.	(b)	14.	(d)	15.	(c)	16.	(b)	17.	(d)	18.	(b)
19.	(a)	20.	(b)	21.	(a)	22.	(b)	23.	(a)	24.	(a)	25.	(a)	26.	(a)	27.	(b)
28.	(c)	29.	(b)	30.	(d)	31.	(d)	32.	(a)	33.	(c)	34.	(d)	35.	(d)	36.	(d)
37.	(d)	38.	(a)	39.	(b)	40.	(d)	41.	(d)	42.	(a)	43.	(d)	44.	(b)	45.	(a)
46.	(c)	47.	(a)	48.	(a)	49.	(b)	50.	(c)	51.	(c)	52.	(c)	53.	(18)	54.	(1.65)
55.	(c)	56.	(–3)	57.	(b)	58.	(0.86)	59.	(c)	60.	(b)	61.	(a)	62.	(7.38)	63.	(a)
64.	(–1)	65.	(a)	66.	(94.08)	67.	(a)	68.	(a)	69.	(a)	70.	(d)	71.	(a)	72.	(d)
73.	(5.5)	74.	(b)	75.	(c)	76.	(c)	77.	(c)	78.	(0.37)	79.	(c)	80.	(1.47)	81.	(a)
82.	(c)	83.	(a)	84.	(b)												

1. (a)

Given diffenential equation

$$\frac{dy}{dx} + y^2 = 0$$

$$\Rightarrow \quad -\frac{dy}{y^2} = dx$$

On integrating, we get

$$-\int \frac{dy}{y^2} = \int dx$$

$$\frac{1}{y} = x + c$$

$$\therefore \quad y = \frac{1}{x+c}$$

2. (b)

$$\frac{dx}{dt} = -kx^2$$

(Note: This is in variable separable form)

$$\Rightarrow \quad \frac{dx}{x^2} = -kdt$$

Integrating both sides,

$$\int \frac{dx}{x^2} = -\int kdt$$

$$-\frac{1}{x} = -kt + C$$

$$\Rightarrow \quad \frac{1}{x} = kt + C'$$

at $\quad t = 0, x = a$

$$\Rightarrow \quad \frac{1}{a} = k \times 0 + C'$$

$$\Rightarrow \quad C' = \frac{1}{a}$$

$$\therefore \quad \frac{1}{x} = kt + \frac{1}{a}$$

3. (b)

Order is highest derivative term, so order = 2.
Degree is power of highest derivative term.
So, degree = 1.

4. (a)

Given, $\quad \dot{x}(t) = -3\,x(t)$

i.e. $\quad \frac{dx}{dt} = -3\,x$

$$\frac{dx}{x} = -3\,dt$$

$$\int \frac{dx}{x} = \int -3\,dt$$

$$\Rightarrow \quad \ln x = -3t + C$$

$$\Rightarrow \quad x = e^{-3t+C} = e^C \times e^{-3t}$$

putting $\quad e^C = C_1$

$$x = C_1 \times e^{-3t}$$

Now putting initial condition $x(0) = x_0$

$$x_0 = C_1\,e^0 = C_1$$

$$\therefore \quad C_1 = x_0$$

$\therefore$ Solution is $x = x_0\,e^{-3t}$

i.e. $\quad x(t) = x_0\,e^{-3t}$

5. (a)

Given, $\quad \frac{dy}{dt} + p(t)\,y = q(t)\,y^n\,;\, n > 0$

putting $\quad v = y^{1-n}$

$$\frac{dv}{dt} = (1-n)\,y^{-n}\,\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{(1-n)y^{-n}}\,\frac{dv}{dt}$$

Substituting in the given differential equation, we get,

$$\frac{1}{(1-n)y^{-n}}\,\frac{dv}{dt} + p(t)y = q(t)y^n$$

Multiplying by $(1-n)\,y^{-n}$, we get

$$\frac{dv}{dt} + p(t)\,(1-n)y^{1-n} = q(t)\,(1-n)$$

now since $y^{1-n} = v$, we get

$$\frac{dv}{dt} + (1-n)\,pv = (1-n)q$$

(which is linear with v as dependent variable and t as independent variable)

6. (a)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$$

$$y(0) = 1$$

$$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$$

This is a linear differential equation

$$D^2 + 2D + 17 = 0$$

$$D = -1 \pm 4i$$

$$\therefore \quad y = C_1 e^{(-1+4i)x} + C_2 e^{(-1-4i)x}$$

$$= e^{-x} C_1 e^{4xi} + C_2 e^{-4xi}$$

$$= e^{-x}[C_1(\cos 4x + i\sin 4x)] + C_2[\cos(-4x) + i\sin(-4x)]$$

$$= e^{-x}[(C_1 + C_2)\cos 4x + (C_1 - C_2) i\sin 4x)]$$

Let $C_1 + C_2 = C_3$ and $(C_1 - C_2)\, i = C_4$

$$y = e^{-x}(C_3 \cos 4x + C_4 \sin 4x)$$

since $y(0) = 1$

$\Rightarrow$ $1 = e^{-0}(C_3 \cos 0 + C_4 \sin 0)$

$\Rightarrow$ $C_3 = 1$

$$\frac{dy}{dx} = e^{-x}(-4C_3 \sin 4x + 4C_4 \cos 4x) - e^{-x}[C_3 \cos 4x + C_4 \sin 4x]$$

$$= e^{-x}[(-4C_3 - C_4)\sin 4x + (4C_4 - C_3)\cos 4x]$$

$\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is 0

$\therefore -(-4C_4 - C_3)\, e^{-\pi/4} = 0$

$$4C_4 = C_3$$

$$C_4 = \frac{C_3}{4} = \frac{1}{4}$$

$\therefore$ $C_3 = 1$ and $C_4 = \frac{1}{4}$

$$y = e^{-x}(\cos 4x + \frac{1}{4}\sin 4x)$$

7. (c)

Given equation is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

$$(D^2 + pD + q)y = 0$$

$\therefore$ $D^2 + pD + q = 0$

Its solution is $y = C_1 e^{-x} + C_2 e^{-3x}$

So the roots of

$D^2 + pD + q = 0$ are $\alpha = -1$ and $\beta = -3$

Sum of roots $= -p = -1 - 3 \Rightarrow p = 4$

Product of roots $= q = (-1)(-3) \Rightarrow q = 3$

8. (c)

Given equation is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1) = 0$$

$\Rightarrow$ $[D^2 + PD + (q+1)]y = 0$

Put $p = 4$

and $q = 3$

$\therefore$ $(D^2 + 4D + 4)y = 0$

$$D^2 + 4D + 4 = 0$$

$$(D+2)^2 = 0$$

$\therefore$ $D = -2, -2$

$\Rightarrow$ $y = (c_1 x + c_2)\, e^{-2x}$

out of choices given, $y = x\, e^{-2x}$

is the only answer in the required form (i.e. $(c_1 x + c_2)e^{-2x}$ putting $c_1 = 1$ and $c_2 = 0$)

9. (b)

A.E. $\Rightarrow D^2 - 5D + 6 = 0$

$$(D-2)(D-3) = 0$$

$$D = 2, 3$$

$\therefore$ $y = e^{2x} + e^{3x}$

10. (a)

$$\frac{dV}{dt} = -kA \quad \text{... (i)}$$

where $V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$

$\Rightarrow$ $\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substituting these in (i) we get,

$$4\pi r^2 \frac{dr}{dt} = -k(4\pi r^2)$$

$$\frac{dr}{dt} = -k$$

$\Rightarrow$ $dr = -kdt$

Integrating we get

$$r = -kt + C$$

at $t = 0,\ r = 1$

$\Rightarrow$ $1 = -k \times 0 + C$

$\Rightarrow$ $C = 1$

$\therefore$ $r = -kt + 1$ ...(ii)

Now at $t = 3$ months

$r = 0.5$ cm

$\therefore$ $0.5 = -k \times 3 + 1$

$\Rightarrow$ $k = \frac{0.5}{3}$

Now substituting this value of R in equation (ii) we get,

$$r = -\frac{0.5}{3}t + 1$$

putting $r = 0$ (ball completely evaporates)

in above and solving for t gives $0 = -\frac{0.5}{3}t + 1$

$\Rightarrow$ $t = 6$ months

11. (b)

Given equation

$$\frac{dy}{dx} + 2xy = e^{-x2}$$

This is a leibnitz + z linear equation (i.e. a first order linear differential equation)

Integrating factor

$$\text{I.F.} = e^{\int 2x dx} = e^{x2}$$

Solution is $y(\text{I.F.}) = \int Q(\text{IF}) dx + c$

$$ye^{x^2} = \int e^{-x^2} e^{x^2} dx + c$$

$$ye^{x^2} = x + c$$

at $x = 0,\ y = 1$ (given)

$\therefore\quad 1e^0 = 0 + c$

$\Rightarrow\quad c = 1$

So, the solution is

$$ye^{-x2} = x + 1$$

$\Rightarrow\quad y = e^{-x2}(x + 1)$

12. (b)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$$

$\Rightarrow (D^2 + 4D + 3)y = 3e^{2x}$

Particular integral

$$\text{P.I.} = \frac{1}{D^2 + 4D + 3} 3e^{2x}$$

Now since, $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

$$\text{P.I.} = 3\frac{e^{2x}}{(2)^2 + 4(2) + 3} = \frac{3e^{2x}}{15} = \frac{e^{2x}}{5}$$

13. (b)

Degree of a differential equation is the power of its highest order derivative after the differential equation is made free of radicals and fractions if any, in derivative power.

Hence, here the degree is 1, which is power of $\frac{d^2x}{dt^2}$

14. (d)

$$\frac{dy}{dx} = x^2 y$$

This is variable separable form

$$\frac{dy}{y} = x^2 dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$\Rightarrow\quad \log_e y = \frac{x^3}{3} + C_1$

$\Rightarrow\quad y = e^{\frac{x^3}{3} + C_1} = e^{C_1} \times e^{\frac{x^3}{3}}$

$$y = C \times e^{\frac{x^3}{3}}$$

Now at $x = 0,\ y = 1$

$$1 = C \times e^{\frac{0}{3}}$$

$\Rightarrow\quad C = 1$

$\therefore\quad y = e^{\frac{x^3}{3}}$ is the solution

15. (c)

Given $\frac{dy}{dx} = y^2$

$\Rightarrow\quad \int \frac{dy}{y^2} = \int dx$

$\Rightarrow\quad -\frac{1}{y} = x + c$

$\therefore\quad y = -\frac{1}{x + c}$

When $x = 0$

$y = 1$

$\therefore\quad C = -1$

$\therefore\quad y = -\frac{1}{x - 1}$

y is bounded when

$$x - 1 \neq 0$$

i.e. $x \neq 1$

i.e. $x < 1$ or $x > 1$

16. (b)

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

(Newton's law of cooling)

This is in variable seperable form separating the variables, we get,

$$\frac{d\theta}{\theta - \theta_0} = -kdt$$

$$\int \frac{d\theta}{\theta - \theta_0} = \int -kdt$$

$\Rightarrow \ln(\theta - \theta_0) = -kt + C_1$

$\Rightarrow\quad \theta - \theta_0 = C.e^{-kt}$ (where $C = e^{c_1}$)

$$\theta = \theta_0 + C.e^{-kt}$$

given, $\theta_0 = 25°C$

Now at $t = 0, \theta = 60°$

$$60 = 25 + C.e^0$$

$\Rightarrow$ $C = 35$

$\therefore$ $\theta = 25 + 35\, e^{-kt}$

at $t = 15$ minutes

$$\theta = 40°C$$

$\therefore$ $40 = 25 + 35e^{(-k \times 15)}$

$\Rightarrow$ $e^{-15k} = \frac{3}{7}$ ... (i)

Now at $t = 30$ minutes

$$\theta = 25 + 35\, e^{-30k} = 25 + 35\, (e^{-15k})^2$$

Now substituting $e^{-15\,k} = \frac{3}{7}$ from (i), we get,

$$\theta = 25 + 35 \times \left(\frac{3}{7}\right)^2$$
$$= 31.428°C \approx 31.5°C$$

17. **(d)**

$$\frac{dy}{dx} = \frac{-x}{y}$$

$\Rightarrow$ $y\, dy = -x\, dx$

$\Rightarrow$ $\int y\, dy = \int -x\, dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

at $x = 1,$

$$y = \sqrt{3}$$

$\therefore$ $\frac{(\sqrt{3})^2}{2} = \frac{-1^2}{2} + C$

$\Rightarrow$ $C = 2$

$\therefore$ Solution is $\frac{y^2}{2} = \frac{-x^2}{2} + 2$

$\Rightarrow$ $x^2 + y^2 = 4$

18. **(b)**

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{x} = -3dt$$

$$\int \frac{dx}{x} = \int -3dt$$

in $x = -3t + c$

$\Rightarrow$ $x = e^{-3t + c}$

$\Rightarrow$ $x = e^c \,.\, e^{-3t} = c_1\, e^{-3t}$ $(c_1 = e^c)$

$\Rightarrow$ $x = c_1\, e^{-3t}$

19. **(a)**

$$\frac{d^2y}{dx^2} + y = 0$$

$$D^2 + 1 = 0$$

$$D = \pm\, i = 0 \pm 1i$$

$\therefore$ General solution is

$$y = e^{0x}[C_1 \cos(1 \times x) + C_2 \sin(1 \times x)]$$
$$= C_1 \cos x + C_2 \sin x$$
$$= P \cos x + Q \sin x$$

where P and Q are some constants.

20. **(b)**

$$\ddot{x} + 3x = 0$$

Auxiliary equation is

$$D^2 + 3 = 0$$

i.e. $D = \pm\sqrt{3}\, i$

$\therefore$ $x = A\cos\sqrt{3}t + B\sin\sqrt{3}t$

at $t = 0, x = 1$

$\Rightarrow$ $A = 1$

Now, $\dot{x} = \sqrt{3}(B\cos\sqrt{3}t - A\sin\sqrt{3}t)$

At $t = 0,\ \dot{x} = 0$

$\Rightarrow$ $B = 0$

So, $x = \cos\sqrt{3}t$

$$x(1) = \cos\sqrt{3} = 0.99$$

21. **(a)**

$$y'' + 2y' + y = 0$$

$$(D^2 + 2D + 1)y = 0$$

$\Rightarrow$ $D^2 + 2D + 1 = 0$

$\Rightarrow$ $(D + 1)^2 = 0$

$\Rightarrow$ $D = -1, -1$

$\therefore$ $y = (C_1 + C_2x)e^{-x}$

$y(0) = 0 \Rightarrow 0 = (C_1 + C_2(0))e^{-0}$

$\Rightarrow$ $C_1 = 0$

$y(1) = 0 \Rightarrow 0 = (C_1 + C_2)\, e^{-1}$

$\Rightarrow$ $C_1 + C_2 = 0$

$\Rightarrow$ $C_2 = 0$

$\therefore$ $y = (0 + 0x)e^{-x} = 0$ is the solution

$\therefore$ $y(0.5) = 0$

22. **(b)**

Highest derivative of differential equation is 2.

23. **(a)**

$$3y\frac{dy}{dx} + 2x = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-2x}{3y}$

$$\Rightarrow \quad 3y\,dy = -2x\,dx$$

$$\Rightarrow \quad \int 3y\,dy = \int -2x\,dx$$

$$\Rightarrow \quad \frac{3}{2}y^2 = -2\times\frac{x^2}{2} + C$$

$$\Rightarrow \quad 3y^2 + 2x^2 = C$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}\right)} + \frac{y^2}{\left(\frac{1}{3}\right)} = C$$

$$\Rightarrow \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$$

which is the equation of a family of ellipses.

24. (a)

A. $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log c = \log cx$$

$y = cx$ ... Equation of straight line.

B. $\dfrac{dy}{dx} = \dfrac{-y}{x}$

$$\frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\log y = -\log x + \log c$$

$$\log y + \log x = \log c$$

$$\log yx = \log c$$

$$yx = c$$

$y = c/x$ ... Equation of hyperbola.

C. $\dfrac{dy}{dx} = \dfrac{x}{y}$, $y\,dy = x\,dx$

$$\Rightarrow \int y\,dy = \int x\,dx$$

$\dfrac{y^2}{2} - \dfrac{x^2}{2} = \dfrac{c^2}{2} \rightarrow$ const

$$y^2 - x^2 = c^2$$

$\dfrac{y^2}{c^2} - \dfrac{x^2}{c^2} = 1$ ... Equation of hyperbola.

D. $\dfrac{dy}{dx} = \dfrac{-x}{y} \Rightarrow \int y\,dy = -\int x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c^2}{2}$$

$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$

$x^2 + y^2 = c^2$ ... Equation of a circle

25. (a)

Given differential equation is

$$x\frac{dy}{dx} + y = x^4$$

$\Rightarrow \quad \dfrac{dy}{dx} + \left(\dfrac{y}{x}\right) = x^3$ ... (i)

Standard form of leibnitz linear equation is

$\dfrac{dy}{dx} + Py = Q$ ... (ii)

where P and Q function of x only and solution is given by

$$y(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + C$$

where, integrating factor (I.F.) = $e^{\int P dx}$

Here in equation (i),

$$P = \frac{1}{x} \text{ and } Q = x^3$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Solution $y(x) = \int x^3.x\,dx + c$

$$yx = \frac{x^5}{5} + c$$

given condition

$$y(1) = \frac{6}{5}$$

means at $x = 1; y = \dfrac{6}{5}$

$$\Rightarrow \quad \frac{6}{5}\times 1 = \frac{1}{5} + c$$

$$\Rightarrow \quad c = \frac{6}{5} - \frac{1}{5} = 1$$

Therefore $yx = \dfrac{x^5}{5} + 1$

$$\Rightarrow \quad y = \frac{x^4}{5} + \frac{1}{x}$$

26. (a)

$$\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$$

Removing radicals we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = 16\left[\left(\frac{dy}{dx}\right)^3 + y^2\right]$$

$$U = C_1 + C_2 e^{kL} = -C_2 + C_2 e^{kL}$$

$$C_2 = \frac{U}{(e^{kL}-1)} \Rightarrow C_1 = \frac{U}{1-e^{kL}}$$

$$\therefore \quad u = \frac{U}{1-e^{kL}} - \frac{U}{1-e^{kL}} e^{kx}$$

$$u = U\left[\frac{1-e^{kx}}{1-e^{kL}}\right]$$

40. (d)

$$y(t) + \ddot{y}(t) = 0$$

$$1 + D^2 = 0$$

$$\therefore \quad D = \pm i$$

$$\therefore \quad y = C_1 e^{ix} + C_2 e^{-ix}$$

$$= A \cos x + B \sin x$$

$$y(0) = 1$$

$$\therefore \quad 1 = A \times 1 + B \times 0$$

$$A = 1$$

$$\dot{y} = -A \sin x + B \cos x$$

$$\dot{y}(0) = 1$$

$$\therefore \quad 1 = -A \times 0 + B \times 1$$

$$\therefore \quad B = 1$$

So, $y = \cos x + \sin x$

for maxima,

$$y' = -\sin x + \cos x = 0$$

$$\therefore \quad \sin x = \cos x$$

$$\therefore \quad x = 45^\circ$$

$$y'' = -\cos x - \sin x$$

$\therefore$ $y'' < 0$ for $x = 45°$

$\therefore$ maxima $y(\max) = \cos 45° + \sin 45°$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

41. (d)

$$\frac{dy(t)}{dt} + ky(t) = x(t)$$

Taking Laplace transform of both sides, we have

$$sY(s) - y(0) + kY(s) = X(s)$$

$$Y(s)[s + k] = X(s) + y(0)$$

$$\Rightarrow \quad Y(s) = \frac{X(s)}{s+k} + \frac{y(0)}{s+k}$$

Taking inverse Laplace transform, we have

$$y(t) = e^{-kt}x(t) + y(0)e^{-kt}$$

So if we want $-2y(t)$ as a solution both $x(t)$ and $y(0)$ has to be multiplied by -2; hence change $x(t)$ by $-2x(t)$ and $y(0)$ by $-2y(0)$.

42. (a)

$$\frac{dx}{dt} = 3x - 5y$$

$$\frac{dy}{dt} = 4x + 8y$$

$$\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3x & -5y \\ 4x & +8y \end{bmatrix}$$

43. (d)

Let $z = x + y$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad \frac{dz}{dx} - 1 = \cos z$$

or $\int \frac{dz}{1+\cos z} = \int dx$

or $\frac{1}{2}\int \sec^2\left(\frac{z}{2}\right) dz = x + c$

or $\tan\left(\frac{z}{2}\right) = x + c$

or $\tan\left(\frac{x+y}{2}\right) = x + c$

44. (b)

$$\frac{dy}{dx} = 2xy = 0 \qquad \text{...(1)}$$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying I.F. to both side of equation (1)

$$e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{x^2} y\right) = 0$$

$$e^{x^2} y = C$$

from the given boundary condition, $C = 2$

$$\therefore \quad e^{x^2} y = 2$$

$$y = 2e^{-x^2}$$

45. (a)

General form of linear differential equation

$\frac{dy}{dx} + py = \theta$ when P and θ can be function of x

Only option (a) is in this form.

46. (c)

$$\frac{d^2x}{dt^2} = -9x \qquad \frac{d}{dt} = D$$

$$\frac{d^2x}{dt^2} + 9x = 0 \qquad (D^2 + 9)x = 0$$

Auxiliary equation is $m^2 + 9 = 0$

$$m = \pm 3i$$

$$x = C_1 \cos 3t + C_2 \sin 3t \qquad \text{...(i)}$$

$x(0) = 1$ i.e. $x \to 1$ when $t \to 0$

$$\boxed{1 = C_1}$$

$$\frac{dx}{dt} = -3C_1 \sin 3t + 3C_2 \cos 3t \qquad \text{...(ii)}$$

$x'(0) = 1$ i.e. $x' \to 1$ when $t \to 0$

$$1 = 3C_2 \quad \boxed{C_2 = \frac{1}{3}}$$

$$\therefore \quad x = \cos 3t + \frac{1}{3}\sin 3t$$

47. (a)

Given differential equation in symbolic form is

$(D^2 + 1)\, x(t) = 0$

Its A.E. is

$$D^2 + 1 = 0,$$

$$\therefore \quad D = \pm i$$

So, C.F. is

$$x_1(t) = C_1 \cos t,$$

$$x_2(t) = C_2 \sin t$$

$$\therefore \quad x_1(0) = C_1 = 1$$

$$\Rightarrow \quad x_1(t) = \cos t \left[\text{it satisfies } \left.\frac{dx_1(t)}{dt}\right|_{t=0} = 0\right]$$

$$x_2(0) = 0 = C_2 \sin(0) \qquad (\because C_2 \neq 0)$$

$$\frac{dx_2(t)}{dt} = C_2 \cos t$$

$$\Rightarrow \left.\frac{dx_2(t)}{dt}\right|_{t=0} = C_2 = 1$$

$$\therefore \quad x_2(t) = \sin t$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \dfrac{dx_1(t)}{dt} & \dfrac{dx_2(t)}{dt} \end{vmatrix}$$

$$= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$W(t) = \cos^2 t + \sin^2 t = 1$$

48. (a)

$$\frac{d^2y}{dx^2} + 2\alpha \frac{dy}{dx} + y = 0$$

The characteristic equation is given as

$$(m^2 + 2\alpha\, m + 1) = 0$$

$$m_1, m_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2}$$

Since both roots are equal i.e.

$$m_1 = m_2$$

$$\frac{-2\alpha + \sqrt{4\alpha^2 - 4}}{2} = \frac{-2\alpha - \sqrt{4\alpha^2 - 4}}{2}$$

$$\sqrt{4\alpha^2 - 4} = -\sqrt{4\alpha^2 - 4}$$

$$2\sqrt{4\alpha^2 - 4} = 0$$

$$4\alpha^2 - 4 = 0$$

$$\alpha^2 = 1$$

$$\alpha = \pm 1$$

49. (b)

The differential equation is given as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

$$y = C \cdot F + P \cdot I$$

Since $Q = 0$, i.e. RHS term is zero, so there will be no particular integral.

$$\therefore \quad y = C \cdot F$$

Let $\quad \dfrac{\partial}{\partial x} = D$

So, $(D^2 + 2D + 1)x = 0$

$$\therefore \quad (D + 1)^2 = 0$$

$$\therefore \quad y = ae^{-t} + bte^{-t}$$

50. (c)

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Let $\quad x = e^z \longleftrightarrow z = \log x$

$$x\frac{d}{dx} = xD = \theta = \frac{d}{dz}$$

$$x^2 D^2 = \theta(\theta - 1)$$

$$(x^2 D^2 + xD - 1)y = 0$$

$$[\theta(\theta - 1) + \theta - 1]\, y = 0$$

$$(\theta^2 - \theta + \theta - 1) = 0$$

$$(\theta^2 - 1)y = 0$$

Auxilliary equation is $m^2 - 1 = 0$

$$m = \pm 1$$

CF is $C_1 e^{-z} + C_2 e^{z}$

Solution is $y = C_1 e^{-z} + C_2 e^{z}$

$= C_1 x^{-1} + C_2 x$

$= C_1 \frac{1}{x} + C_2 x$

One independent solution is $\frac{1}{x}$

Another independent solution is x.

51. (c)

$$\frac{dy}{dt} = -5y$$

$$\int \frac{dy}{y} = -\int 5dt$$

$$\ln y = -5t + C$$

at $t = 0$

$y = 2$

$\ln 2 = C$

So, $\ln y = -5t + \ln 2$

$$\ln \frac{y}{2} = -5t$$

$$\frac{y}{2} = e^{-5t}$$

$$y = 2e^{-5t}$$

at $t = 3$

$$y = 2e^{-15}$$

52. (c)

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

$$\frac{ydx + xdy}{xdy - ydx} = \frac{y}{x}\tan\frac{y}{x}$$

Let $y = v \cdot x$

$dy = vdx + xdv$

$$\frac{vxdx + vxdx + x^2dv}{vxdx + x^2dv - vxdx} = v\tan v$$

$$\frac{xdv + 2vdx}{xdv} = v\tan v$$

$$1 + \frac{2v}{x}\frac{dx}{dv} = v\tan v$$

$$\frac{2v}{x}\frac{dx}{dv} = v\tan v - 1$$

$$2\frac{dx}{x} = \left(\tan v - \frac{1}{v}\right)dv$$

Integrating both sides.

$$2\log x = \log|\sec v| - \log v + \log c$$

$$\Rightarrow \quad x^2 = \frac{c\sec v}{v}$$

$$\Rightarrow \quad x^2\frac{y}{x} = c\sec\frac{y}{x}$$

$$\Rightarrow \quad xy\cos\frac{y}{x} = c$$

53. Sol.

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

Integrating both sides w.r.t. x

$$\frac{dy}{dx} = -4x^3 + 12x^2 - 20x + c_1$$

Integrating both sides w.r.t. x

$$y = -x^4 + 4x^3 - 10x^2 + c_1x + c_2 \quad \text{...(i)}$$

At $x = 0, y = 5$

$\Rightarrow$ $5 = c_2$

At $x = 2, y = 21$

$\Rightarrow$ $21 = -16 + 32 - 40 + 2c_1 + c_2$

$2c_1 = 21 + 16 - 2 + 40 - 5$

$2c_1 = 40$

$c_1 = 20$

$\Rightarrow$ $y = -x^4 + 4x^3 - 10x^2 + 20x + 5$

Put $x = 1$

$\Rightarrow$ $y = -1 + 4 - 10 + 20 + 5 = 18$

~~54. Sol.~~

$$\frac{di}{dt} = 0.2i$$

$$\frac{di}{i} = 0.2\,dt$$

$$\int \frac{di}{i} = \int 0.2\,dt$$

$$\log i = 0.2t + \log C$$

$$\log i - \log C = 0.2t$$

$$\log\left(\frac{i}{C}\right) = 0.2t$$

$$\frac{i}{C} = e^{0.2t}$$

$$i = Ce^{0.2t} \quad \text{...(i)}$$

$i(4) = 10$ i.e. $i = 10$ when $t = 4$

$10 = Ce^{(0.2)4}$

$10 = C(2.225)$

$\therefore$ $C = 4.493$

$$i = (4.493)\,e^{0.2t} \quad \text{...(ii)}$$

when, $t = -5$

$$i = (4.493)\,e^{(0.2)(-5)} = 1.652$$

55. **(c)**

$$\frac{dy}{1+\cos 2y} = \frac{dx}{1-\cos 2x}$$

$$\frac{dy}{2\cos^2 y} = \frac{dx}{2\sin^2 x}$$

$$\sec^2 y\, dy = \text{cosec}^2 x\, dx$$

Integrating both sides, we get

$$\tan y = -\cot x + c$$

$$\tan y + \cot x = c$$

56. **Sol.**

$$D^2 + 5D + 6 = 0$$

$$D = -2, -3$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Given, $y(0) = 2$

$\Rightarrow \quad c_1 + c_2 = 2$ ...(i)

$$y(1) = -\left(\frac{1-3e}{e^3}\right)$$

$$\Rightarrow \frac{c_1}{e^2} + \frac{c_2}{e^3} = -\left(\frac{1-3e}{e^3}\right)$$

$\Rightarrow \quad ec_1 + c_2 = 3e - 1$ ...(ii)

Now solving equation (i) and (ii), we get

$$c_1 = 3$$

$$c_2 = -1$$

Substituting in $y(t)$, we get

$$y(t) = 3e^{-2t} - e^{-3t}$$

Now, $\dfrac{dy}{dt} = -6e^{-2t} + 3e^{-3t}$

$$\left(\frac{dy}{dt}\right)_{t=0} = -6 + 3 = -3$$

57. **(b)**

$$(D^2 + 2D + 1) = 0$$

$$D = -1, -1$$

$$y(t) = (C_1 + C_2 t)\, e^{-t}$$

$$y'(t) = C_2 e^{-t} + (C_1 + C_2 t)(-e^{-t})$$

$$y(0) = y'(0) = 1$$

From here, $C_1 = 1$, $C_2 = 2$

$\Rightarrow \quad y(t) = (1 + 2t)e^{-t}$

58. **Sol.**

$$D^2 + 3D + 2 = 0$$

$$D = -1, -2$$

$$x(t) = C_1 e^{-t} + C_2 e^{-2t}$$

$$x(1) = \frac{10}{e} = C_1 e^{-1} + C_2 e^{-2}$$

$C_1 + C_2 e^{-1} = 10$ ...(i)

$C_1 + C_2 = 20$ ...(ii)

From here, $C_1 = \dfrac{10e - 20}{e - 1}$; $C_2 = \left(\dfrac{10e}{e-1}\right)$

$$x(2) = \left(\frac{10e-20}{e-1}\right)e^{-2} + \left(\frac{10e}{e-1}\right)e^{-4}$$

$$= 0.8566$$

59. **(c)**

$$\frac{d^2y}{dx^2} = y$$

$\Rightarrow \quad D^2 y = y \qquad (\because d/dx = D)$

$$(D^2 - 1)y = 0$$

$$D^2 - 1 = 0$$

$$D = \pm 1$$

$$y = C_1 e^x + C_2 e^{-x}$$

Given point passes through origin

$\Rightarrow \quad 0 = C_1 + C_2$

$C_1 = -C_2$ ...(i)

Also, point passes through (ln 2, 3/4)

$$\Rightarrow \frac{3}{4} = C_1 e^{\ln 2} + C_2 e^{-\ln 2}$$

$$\frac{3}{4} = 2C_1 + \frac{C_2}{2}$$

$\Rightarrow \quad C_2 + 4C_1 = 1.5$ ...(ii)

From (i) $C_1 = -C_2$, putting in (ii), we get

$\Rightarrow \quad -3C_2 = 1.5$

$$C_2 = -0.5$$

$\therefore \quad C_1 = 0.5$

$\Rightarrow \quad y = 0.5\,(e^x - e^{-x})$

$$y = \frac{e^x - e^{-x}}{2}$$

60. **(b)**

Auxiliary equation,

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$y = (c_1 + c_2 t)\, e^{-t}$$

$$y(0) = 1$$

$\Rightarrow \quad c_1 = 1$

$$y = (1 + c_2 t)e^{-t}$$

$$y(1) = 3e^{-1}$$

$\Rightarrow \quad (1 + (c_2)e^{-3} = 3e^{-3}$

$$c_2 = 2$$

$$y = (1 + 2t)e^{-t}$$

$$y(2) = 5e^{-2}$$

61. (a)

The differential equation is

$y''(t) + 2y'(t) + y(t) = 0$

So, $(s^2Y(s) - sy(0) - y'(0)) + 2[sY(s) - y(0)] + Y(s)$

So, $Y(s) = \dfrac{sy(0) + y'(0) + 2y(0)}{(s^2 + 2s + 1)}$

Given that, $y'(0) = 1, y(0) = 0$

So, $Y(s) = \dfrac{1}{(s+1)^2}$

So, $y(t) = te^{-t}u(t)$

62. Sol.

A.E. $m^2 - 4m + 4 = 0$

$$m = 2, 2$$
$$y = (C_1 + C_2x)e^{2x}$$
$$y(0) = 0 \Rightarrow C_1 = 0$$
$$y = C_2 x e^{2x}$$
$$y' = C_2e^{2x} + 2C_2x e^{2x}$$
$$y'(0) = 1$$
$$\Rightarrow C_2 = 1$$
$$y = x e^{2x}$$
$$y(1) = e^2 = 7.38$$

63. (a)

$$(D^2 + 12D + 36)^4 = 0$$
$$(D+6)^2y = 0$$
$$D = -6, -6$$
$$y = (C_1 + xC_2)e^{-6x}$$
$$y = C_1e^{-6x} + C_2xe^{-6x}$$
$$y(0) = 33 = C_1 + 0$$
$$\Rightarrow C_1 = 3$$
$$y' = -6C_1e^{-6x} + C_2e^{-6x} - 6C_2xe^{-6x}$$
$$y'(0) = -36$$
$$-36 = -6C_1 + C_2$$
$$-36 = -18 + C_2$$
$$C_2 = -18$$
$$\therefore y = 3e^{-6x} - 18xe^{-6x}$$
$$y = (3 - 18x)e^{-6x}$$

64. Sol.

$$(D^2 + 9)y = 0$$
$$m^2 + 9 = 0$$
$$m = \pm 3i$$
$$y = C_1 \cos 3x + C_2 \sin 3x \quad \text{...(i)}$$
$$x = 0 \qquad \theta = C_1$$
$$y\left(\frac{\pi}{2}\right) = \sqrt{2}$$
$$\sqrt{2} = C_1 \cos\frac{3\pi}{2} + C_2 \sin\frac{3\pi}{2}$$
$$C_2 = -\sqrt{2}$$
$$\therefore y = \sqrt{2}\sin 3x$$
$$y\left(\frac{\pi}{4}\right) = -\sqrt{2}\sin\frac{3\pi}{4} = \frac{-5}{4} = -1$$

65. (a)

D.E. is $(D^4 + 3D^2)y = 108x^2$, $D = \dfrac{d}{dx}$

A.E. $m^4 + 3m^2 = 0$

$\Rightarrow m^2(m^2 + 3) = 0$

$\Rightarrow m = 0, 0, \pm\pm\sqrt{3}i$

$\therefore$ CF $= (C_1 + C_2x) + C_3 \sin(\sqrt{3}x) + C_4 \cos(\sqrt{3}x)$

and PI $= \dfrac{1}{D^4 + 3D^2}(108x^2)$

$$= \frac{1}{3D^2\left[1 + \frac{D^2}{3}\right]}(108x^2) = \frac{36}{D^2}\left[1 + \frac{D^2}{3}\right]^{-1}(x^2)$$

$$= \frac{36}{D^2}\left[1 - \frac{D^2}{3} + \dots\right](x^2) = \frac{36}{D^2}\left[x^2 - \frac{1}{3}(2) + 0\right]$$

$$= \iint\left(36x^2 - \frac{2}{3}\right)dx\,dx$$

$$= 36\left(\frac{x^4}{(4)(3)} - \frac{2}{3}\frac{x^2}{(2)(1)}\right) = 3x^4 - 12x^2$$

66. Sol.

The differential equation is $3y''(x) + 27y(x) = 0$

The auxillary equation is

$$3m^2 + 27 = 0$$
$$m^2 + 9 = 0$$
$$m = \pm 3i$$

Solution is $y = c_1 \cos 3x + c_2 \sin 3x$

given that $y(0) = 0$

$$\therefore 0 = c_1$$
$$y' = 3c_2 \cos 3x$$
$$y'(0) = 2000$$
$$2000 = 0 + 3c_2$$
$$c_2 = \frac{2000}{3}$$
$$\therefore y = \frac{2000}{3}\sin 3x$$

when $x = 1$ $y = \dfrac{2000}{3}\sin 3 = 94.08$

(a)

$(D^2 + 16)y = 0$

AE is $m^2 + 16 = 0$

$m = \pm 4i$

Solution is $y = c_1 \cos 4x + c_2 \sin 4x$

$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

$y'(0) = 1$

$1 = 4c_2$

$c_2 = 1/4$

$y'(\pi/2) = -1$

$-1 = -4c_1 \sin 2\pi + 4c_2 \cos 2\pi$

$-1 = 0 + 4c_2$

$c_2 = -1/4$

Therefore the given differential equation has no solution.

I. (a)

The differential equation

$$(t^2 - 81)\frac{dy}{dt} + 5 + y = \sin t$$

$$\frac{dy}{dt} + \frac{5t}{t^2-81}y = \frac{\sin t}{t^2-81}$$

$$P = \frac{5t}{t^2-81}$$

$$Q = \frac{\sin t}{t^2-81}$$

$$I.F = e^{\int P dt} = e^{\int \frac{5t}{t^2-81} dt}$$

$$= e^{\frac{5}{2}\int \frac{2t}{t^2-81} dt} = e^{\frac{5}{2} ln(t^2-81)}$$

$$= e^{ln(t^2-81)^{5/2}} = \left(t^2-81\right)^{\frac{5}{2}}$$

Solution is,

$$y\left(t^2-81\right)^{\frac{5}{2}} = \int \frac{\sin t}{t^2-81} \cdot \left(t^2-81\right)^{\frac{5}{2}} dt$$

$$y = \frac{\int (\sin t)(t^2-81)^{3/2} dt}{\left(t^2-81\right)^{5/2}} + \frac{C}{\left(t^2-81\right)^{5/2}}$$

The solution exists for $t \neq -9$.

$t \neq +9$

Hence option (a) is correct.

Because the remaining option are involving either 9 or (–9).

69. (a)

The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$$

$(D^2 + 2D - 5)y = 0$

Auxillary equation is

$m^2 + 2m - 5 = 0$

$m =$

$$\frac{-2 \pm \sqrt{4+20}}{2} = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}$$

Solution is $y = K_1 e^{(-1+\sqrt{6})x} + K_2 e^{(-1-\sqrt{6})x}$

70. (d)

$$\frac{dy}{dx} = (x + y - 1)^2 \quad \text{...(i)}$$

Let $x + y - 1 = t$ ...(ii)

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1 \quad \text{...(iii)}$$

Substituting equations (ii) and (iii) in equation (i)

$$\frac{dt}{dx} - 1 = t^2$$

$$\frac{dt}{t^2+1} = dx$$

Integrating both side

$$\int \frac{1}{t^2+1} dt = \int dx$$

$\tan^{-1} t = x + c$

Since, $t = x + y - 1$

$\therefore \tan^{-1}(x + y - 1) = x + c$

$x + y - 1 = \tan(x + c)$

$y = 1 - x + \tan(x + c)$

71. (a)

$y'' - 4y' + 3y = 2t - 3t^2$

$(D^2 - 4D + 3)y = 2t - 3t^2$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3}\left(2t - 3t^2\right)$$

$$= \frac{1}{(1-D)(3-D)}\left(2t - 3t^2\right)$$

$$= \left(\frac{1/2}{1-D} - \frac{1/2}{3-D}\right)\left(2t - 3t^2\right)$$

$$= \frac{1}{2}(1-D)^{-1}(2t-3t^2) - \frac{1}{6}\left(1-\frac{D}{3}\right)^{-1}(2t-3t^2)$$

$$= \frac{1}{2}(1+D+D^2)(2t-3t^2)\frac{1}{6}\left(1+\frac{D}{3}+\frac{D^2}{9}\right)(2t-3t^2)$$

$$= \frac{1}{2}(2t-3t^2+D(2t-3t^2)+D^2(2t-3t^2))$$

$$-\frac{1}{6}(2t-3t^2)+\frac{D(2t-3t)^2}{3}+\frac{D^2(2t-3t^2)}{9}$$

$$= \frac{1}{2}\left[2t-3t^2+2-6t-6\right]-\frac{1}{6}\left[2t-3t^2+\frac{2-6t}{3}+\frac{-6}{9}\right]$$

$$= \frac{1}{2}(-4-4t-3t^2)-\frac{1}{6}(-3t^2)$$

$$= -2 - 2t - t^2$$

72. (d)

$$\frac{dQ}{dt}+Q = 1$$

Comparing with standard form

$$\text{I.F.} = e^{\int 1dt} = e^t$$

Solution is

$$Q.e^t = \int 1\cdot e^t dt$$

$$= e^t + C$$

$$Q = 1 + Ce^{-t} \quad \text{...(i)}$$

When $t = 0, \quad Q = 0$

$\Rightarrow \quad 0 = 1 + C$

$\Rightarrow \quad C = -1$

Therefore, $Q(t) = 1 - e^{-t}$

73. Sol.

$m = 2$ kg, $V_0 = 1.5$ m/sec

$$\int_0^t F(t)dt = m(V-V_0)$$

$$\int_0^2 3t^2 dt = 2(V-1.5)$$

$$[t^3]_0^2 = 2(V-1.5)$$

$$\Rightarrow \quad (8-0) = 2(V-1.5)$$

$$V = 5.5 \text{ m/s}$$

74. (b)

$$(z-px-qy)^3 = pq+2(p^2+q)^2$$

$$z-px-qy = \sqrt[3]{pq+2(p^2+q)^2}$$

$$z = px+qy+\sqrt[3]{pq+2(p^2+q)^2}$$

which is in Clauriats form.

$\therefore$ solution is, $z = ax+by+\sqrt[3]{ab+2(a^2+b)^2}$

75. (c)

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$dT = \frac{2\pi}{\sqrt{g}}\times\frac{1}{2\sqrt{L}}dL$$

$$dT = \frac{T}{\sqrt{L}}\times\frac{1}{2\sqrt{L}}dL = \frac{dL}{2L}$$

$$\frac{5}{24\times60\times60} = \frac{dL}{2L}$$

$$\frac{5\times2}{24\times60\times60} = \frac{dL}{L}$$

$$\frac{dL}{L} = \frac{1}{8640}$$

$$dL = \frac{1}{8640} \text{ times the original length}$$

76. (c)

$$y\sqrt{1-x^2}dy + x\sqrt{1-y^2}dx = 0$$

$$\frac{x\,dx}{\sqrt{1-x^2}} = \frac{-y}{\sqrt{1-y^2}}dy$$

$$-\int\frac{-2x dx}{\sqrt{1-x^2}} = \int\frac{-2y}{\sqrt{1-y^2}}dy$$

$$-\int\frac{dt}{\sqrt{t}} = \int\frac{ds}{\sqrt{s}}$$

$$-\int t^{-1/2}dt = \int s^{-1/2}ds$$

$$-2\sqrt{t} = -2\sqrt{s}+c$$

$$-2\sqrt{1-x^2} = 2\sqrt{1-y^2}+c$$

$$\sqrt{1-y^2}+\sqrt{1-x^2} = c$$

77. (c)

$$(D^4-2D^3+2D^2+1)y = 0$$

AE is

$$m^4-2m^3+2m^2-2m+1 = 0$$

Roots are: $-1, 1, i, -i$

Solution is,

$$y = (C_1+C_2x)\,e^x + C_3\cos x + C_4\sin x$$

78. Sol.

$(D^2+2D+1)y = 0 \qquad (\therefore$ Roots are $-1, -1)$

$$\text{CF} = (C_1+C_2x)\,e^{-x}$$

$$y = C_1e^{-x}+C_2x\,e^{-x} \quad \text{...(i)}$$

$y(0) = 1 \qquad 1 = C_1$ ...(ii)

$$y' = C_1 e^{-x} + C_2 (e^{-x} - xe^{-x})$$

$y'(0) = -1, \qquad -1 = -C_1 + C_2$ ...(iii)

From eq. (ii) and (iii),

$$C_1 = 1, C_2 = 0$$

$$\therefore \qquad y = e^{-x}$$

At $x = 1$, $y = e^{-1} = \frac{1}{e} = 0.368$

79. (c)

$$y^3 \frac{dy}{dx} = -x^3$$

$$y^3 dy = -x^3\, dx$$

$$\int y^3 dy = -\int x^3 dx$$

$$\frac{y^4}{4} = \frac{-x^4}{4} + C$$

$$\frac{x^4 + y^4}{4} = C$$

$$y(0) = 1,$$

$$\frac{0+1}{4} = C$$

$$C = \frac{1}{4}$$

$$x^4 + y^4 = 1$$

$$y^4 = 1 - x^4$$

$$y = \sqrt[4]{1 - x^4}$$

When, $x = -1$

$y = 0$

80. Sol.

$$(D^2 + D - 6)y = 0$$

$$y(0) = 0,$$

$$y'(0) = 1$$

$$(D + 3)(D - 2)y = 0$$

$$D = 2, -3$$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{-3x}$$

$$y = C_1 e^{2x} + C_2 e^{-3x}$$

$$y(0) = 0$$

So, $0 = C_1 + C_2$ ...(i)

$$\frac{dy}{dx} = 2C_1 e^{2x} - 3C_2 e^{-3x}$$

$$y'(0) = 1,$$

$1 = 2C_1 - 3C_2$ ...(ii)

From equation (i) and (ii),

$$C_1 = \frac{1}{5},$$

$$C_2 = \frac{-1}{5}$$

$$y = \frac{1}{5}e^{2x} - \frac{1}{5}e^{-3x}$$

When, $x = 1$

$$y(1) = \frac{e^2 - e^{-3}}{5} = 1.4678$$

81. (a)

$$\frac{dy}{dx} = \frac{x^2}{2y} + \frac{y}{2} + \frac{y}{x}$$

Put, $\frac{y}{x} = t$

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = \frac{x}{2t} + \frac{tx}{2} + t$$

$$x\frac{dt}{dx} = x\left(\frac{1}{2t} + \frac{t}{2}\right)$$

$$x\frac{dt}{dx} = x\left(\frac{1+t^2}{2t}\right)$$

$$\int \frac{2t}{1+t^2} dt = \int dx + C$$

$$ln(1 + t^2) = x + C$$

$$t = \frac{y}{x}$$

So, $ln\left(1 + \frac{y^2}{x^2}\right) = x + C$

At $x = 1$, $y = 0$

$$ln\left(1 + \frac{0}{1}\right) = ln(1) = 0 = 1 + C$$

$$C = -1$$

So, $ln\left(1 + \frac{y^2}{x^2}\right) = x - 1$

82. (c)

$$(D + 1)x_1 = x_2$$

$$\Rightarrow \quad (D + 1)x_1 - x_2 = 0$$

$$(D + 1)x_2 = x_1$$

$$\Rightarrow \quad -x_1 + (D + 1)x_2 = 0$$

$$x_{1f} = \lim_{t \to \infty} x_1(t)$$

$$x_{2f} = \lim_{t\to\infty} x_2(t)$$

$$(D+1)x_1 - x_2 = 0$$

$$-(D+1)x_1 + (D+1)^2 x_2 = 0$$

$$((D+1)^2 - 1)x_2 = 0$$

$$(D^2 + 2D)x_2 = 0$$

$$D^2 + 2D = 0$$

$$D(D+2) = 0$$

$$\therefore \quad x^2 = C_1 + C_2 e^{-2t} \quad ...(1)$$

$$D = 0, -2$$

$$x_{2f} = \lim_{t\to\infty} C_1 + C_2 e^{-2t} = C_1$$

$$(D+1)x_1 - (D+1)x_2 = 0$$

$$-x_1 + (D+1)x_2 = 0$$

$$((D+1)^2 - 1)x_1 = 0$$

$$(D_2 + 2D)x_1 = 0$$

$$\therefore \quad x_{1f} = \lim_{t\to\infty} C_1 + C_2 e^{-2t} = C_1$$

$$\therefore \quad x_1 = C_1 + C_2 e^{-et}$$

83. (a)

$$(D^2 - D - 2)y = 3e^{2x}$$

$$y(0) = 0$$

A.E is $m^2 - m - 2 = 0$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

$$\text{C.F.} = C_1 e^{-x} + 12 e^{2x}$$

~~$y(0) = -2$~~

$$\text{P.I.} = \frac{1}{D^2 - D - 2} 3e^{2x}$$

$$= x \cdot \frac{1}{2D-1} 3e^{2x}$$

$$= x \frac{1}{4-1} 3e^{2x} = xe^{2x}$$

Solution is,

$$y = C_1 e^{-x} + C_2 e^{2x} + xe^{2x} \quad ...(i)$$

$$y(0) = 0$$

$$\Rightarrow \quad 0 = C_1 + C_2 \quad ...(ii)$$

$$y' = -C_1 e^{-x} + 2C_2 e^{2x} + e^{2x} + 2x\, e^{2x} \quad ...(iii)$$

$$y'(0) = -2$$

$$-2 = -C_1 + 2C_2 + 1$$

$$= -C_1 + 2C_2 = -3 \quad ...(iv)$$

From equation (ii) and (iv),

$$C_1 = 1$$

$$C_2 = -1$$

$$y = e^{-x} - e^{2x} + xe^{2x}$$

84. (b)

$$(D^2 + 1)y = 0$$

A.E is $\quad m^2 + 1 = 0$

$$m = \pm i$$

Solution is, $\quad y = C_1 \cos t + C_2 \sin t$

$$y(0) = 1$$

$$1 = C_1 \cos 0 + C_2 \sin 0$$

$$1 = C_1$$

$$y' = -C_1 \sin t + C_2 \cos t$$

~~$y'(0) = 0$~~

$$0 = -C_1 \sin 0 + C_2 \cos 0$$

$$0 = C_2$$

$$y = \cos t$$

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