Chapter
Vector Algebra

Topic-1: Algebra of Vectors, Linear Dependence & Independence of Vectors, Vector Inequality

1 MCQs with One Correct Answer

1. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}, 4\hat{i}, 3\hat{i}+3\hat{j}$ and $-3\hat{i}+2\hat{j}$ respectively. The quadrilateral PQRS must be a **[2010]**
   (a) parallelogram, which is neither a rhombus nor a rectangle
   (b) square
   (c) rectangle, but not a square
   (d) rhombus, but not a square
2. Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}, \beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}, \gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
   (a) are collinear **[1994]**
   (b) form an equilateral triangle
   (c) form a scalene triangle
   (d) form a right angled triangle
3. The points with position vectors $60i+3j, 40i-8j$, $ai-52j$ are collinear if **[1983 - 1 Mark]**
   (a) $a=-40$
   (b) $a=40$
   (c) $a=20$
   (d) none of these

2 Integer Value Answer/ Non-Negative Integer

4. Consider the set of eight vectors $V=\left\{a\hat{i}+b\hat{j}+c\hat{k}: a, b, c \in \{-1, 1\}\right\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[Adv. 2013]**

5 True / False

5. The points with position vectors $a+b$, $a-b$, and $a+kb$ are collinear for all real values of k. **[1984 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

6. If $a=i+j+k, \vec{b}=4i+3j+4k$ and $c=i+\alpha j+\beta k$ are linearly dependent vectors and $|c|=\sqrt{3}$, then **[1998 - 2 Marks]**
   (a) $\alpha=1, \beta=-1$
   (b) $\alpha=1, \beta=\pm 1$
   (c) $\alpha=-1, \beta=\pm 1$
   (d) $\alpha=\pm 1, \beta=1$

7 Match the Following

7. Match the following : [Adv. 2015]

Column I	Column II
(A) In a triangle ΔXYZ, let a, b, and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(p) 1
(B) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y, and Z respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value (s) of $\frac{a}{b}$ is (are)	(q) 2
(C) In R^2, let $\sqrt{3}i + \hat{j}, \hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overline{OX}$ with $\overline{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(r) 3
(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2, y^2 = 4x$ and $y = \lvert \alpha x - 1 \rvert + \lvert \alpha x - 2 \rvert + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(s) 5
	(t) 6

8. Match the following : [Adv. 2015]

Column I	Column II
(A) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value of $\lvert\alpha\rvert$ is/are	(p)
(B) Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ if differentiable for all $x \in R$. Then possible value of a is (are)	(q) 2
(C) Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value (s) of n is (are)	(r) 3
(D) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q - a \rvert$ is (are)	(s) 4
	(t) 5

10 Subjective Problems

9. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. **[2001 - 5 Marks]**

10. Prove, by vector methods or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides. (You may assume that the trapezium is not a parallelogram.) **[1998 - 8 Marks]**

11. In a triangle ABC, D and E are points on BC and AC respectively, such that $BD = 2\,DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods. **[1993 - 5 Marks]**

12. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that $AD : DB = 2 : 1$. If OD and AE intersect at P, determine the ratio $OP : PD$ using vector methods. **[1989 - 4 Marks]**

13. Let $OA\,CB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio. **[1988 - 3 Marks]**

14. A vector $\vec{A}$ has components A_1, A_2, A_3 in a right -handed rectangular Cartesian coordinate system $oxyz$. The coordinate system is rotated about the x-axis through an angle $\frac{\pi}{2}$. Find the components of A in the new coordinate system, in terms of A_1, A_2, A_3. **1983 - 2 Marks]**

Topic-2: Scalar or Dot Product of two Vectors

1 MCQs with One Correct Answer

1. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by **[2011]**
   (a) $\hat{i}-3\hat{j}+3\hat{k}$ (b) $-3\hat{i}-3\hat{j}-\hat{k}$
   (c) $3\hat{i}-\hat{j}+3\hat{k}$ (d) $\hat{i}+3\hat{j}-3\hat{k}$

2. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cos t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Then, **[2008]**
   (a) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$
   (b) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$
   (c) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$
   (d) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

3. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors and

$$\vec{b_1}=\vec{b}-\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a},\ \vec{b_2}=\vec{b}+\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a},$$

$$\vec{c_1}=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}+\frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b_1},\ \vec{c_2}=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}-\frac{\vec{b_1}\cdot\vec{c}}{|\vec{b_1}|^2}\vec{b_1},$$

$$\vec{c_3}=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}+\frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b_1},\ \vec{c_4}=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}=\frac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b_1},$$

then the set of orthogonal vectors is **[2005S]**
   (a) $(\vec{a},\vec{b_1},\vec{c_3})$ (b) $(\vec{a},\vec{b_1},\vec{c_2})$
   (c) $(\vec{a},\vec{b_1},\vec{c_1})$ (d) $(\vec{a},\vec{b_2},\vec{c_2})$

4. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then the angle between $\vec{a}$ and $\vec{b}$ is **[2002S]**
   (a) $45°$ (b) $60°$
   (c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

5. If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors, then
$\left|\vec{a}-\vec{b}\right|^2+\left|\vec{b}-\vec{c}\right|^2+\left|\vec{c}-\vec{a}\right|^2$ does NOT exceed **[2001S]**
   (a) 4 (b) 9 (c) 8 (d) 6

6. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors such that $\vec{u}+\vec{v}+\vec{w}=0$. If $|\vec{u}|=3, |\vec{v}|=4$ and $|\vec{w}|=5$, then $\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}$ is **[1995S]**
   (a) 47 (b) -25 (c) 0 (d) 25

7. Let $\vec{p}$ and $\vec{q}$ be the position vectors of P and Q respectively, with respect to O and $|\vec{p}|=p, |\vec{q}|=q$. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular then **[1994]**
   (a) $9q^2=4q^2$ (b) $4p^2=9q^2$ (c) $9p=4q$ (d) $4p=9q$

2 Integer Value Answer/ Non-Negative Integer

8. Let $\vec{p}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{q}=\hat{i}-\hat{j}+\hat{k}$. If for some real numbers α, β and γ, we have
$15\hat{i}+10\hat{j}+6\hat{k}=\alpha(2\vec{p}+\vec{q})+\beta(\vec{p}-2\vec{q})+\gamma(\vec{p}\times\vec{q})$,
then the value of γ is ________. **[Adv. 2024]**

9. If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors satisfying **[2012]**
$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is

4 Fill in the Blanks

10. The components of a vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are.......and ....respectively. **[1988 - 2 Marks]**

11. Let $b=4\hat{i}+3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively, are given by .... **[1987 - 2 Marks]**

12. A, B, C and D, are four points in a plane with position vectors ***a***, ***b***, ***c*** and ***d*** respectively such that
$(\vec{a}-\vec{d})(\vec{b}-\vec{c})=(\vec{b}-\vec{d})(\vec{c}-\vec{a})=0$ **[1984 - 2 Marks]**
The point D, then, is the .... of the triangle ABC.

13. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ be vectors of length 3, 4, 5 respectively. Let $\vec{A}$ be perpendicular to $\vec{B}+\vec{C}$, $\vec{B}$ to $\vec{C}+\vec{A}$ and $\vec{C}$ *to* $\vec{A}+\vec{B}$. Then the length of vector $\vec{A}+\vec{B}+\vec{C}$ is .... **[1981 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

14. The vector $\frac{1}{3}(2\hat{i}-2\hat{j}+\hat{k})$ is **[1994]**

(a) a unit vecotr

(b) makes an angle $\frac{\pi}{3}$ with the vector $(2\hat{i}-4\hat{j}+3\hat{k})$

(c) parallel to the vector $\left(-\hat{i}+\hat{j}-\frac{1}{2}\hat{k}\right)$

(d) perpendicular to the vector $3\hat{i}+2\hat{j}-2\hat{k}$

15. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{2/3}$, is : **[1993 - 2 Marks]**

(a) $2\hat{i}+3\hat{j}-3\hat{k}$ (b) $2\hat{i}+3\hat{j}+3\hat{k}$

(c) $-2\hat{i}-\hat{j}+5\hat{k}$ (d) $2\hat{i}+\hat{j}+5\hat{k}$

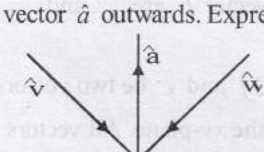
7 Match the Following

16. Match the statements given in Column-I with the values given in Column-II. **[2011]**

Column-I	Column-II
(A) If $\vec{a}=\hat{j}+\sqrt{3}\hat{k}, \vec{b}=-\hat{j}+\sqrt{3}\hat{k}$ and $\vec{c}=2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x)-3x)\,dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6}\sec(\pi x)\,dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left\lvert Arg\left(\frac{1}{1-z}\right)\right\rvert$ for $\lvert z\rvert=1, z\neq 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

10 Subjective Problems

17. If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$. **[2005 - 4 Marks]**



18. Let $\vec{A}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}$ and

$$\vec{B}(t)=g_1(t)\hat{i}+g_2(t)\hat{j}, t\in[0,1],$$

where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are nonzero vectors for all t and $\vec{A}(0)=2\hat{i}+3\hat{j}$, $\vec{A}(1)=6\hat{i}+2\hat{j}$, $\vec{B}(0)=3\hat{i}+2\hat{j}$ and $\vec{B}(1)=2\hat{i}+6\hat{j}$. Then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t. **[2001 - 5 Marks]**

19. Find 3–dimensional vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ satisfying

$\vec{v}_1.\vec{v}_1=4, \vec{v}_1.\vec{v}_2=-2, \vec{v}_1.\vec{v}_3=6, \vec{v}_2.\vec{v}_2=2, \vec{v}_2.\vec{v}_3=-5, \vec{v}_3.\vec{v}_3=29$ **[2001 - 5 Marks]**

20. Determine the value of 'c' so that for all real x, the vector $cx\hat{i}-6\hat{j}-3\hat{k}$ and $x\hat{i}+2\hat{j}+2cx\hat{k}$ make an obtuse angle with each other. **[1991 - 4 Marks]**

21. From a point O inside a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB respectively. Prove that the perpendiculars from A, B, C to the sides EF, FD, DE are concurrent. **[1978]**

Topic-3: Vector or Cross Product of two vectors, Scalar & Vector Triple Product

1 MCQs with One Correct Answer

1. Let the position vectors of the points P, Q, R and S be $\vec{a}=\hat{i}+2\hat{j}-5\hat{k}, \vec{b}=3\hat{i}+6\hat{j}+3\hat{k}$, $\vec{c}=\frac{17}{5}\hat{i}+\frac{16}{5}\hat{j}+7\hat{k}$ and $\vec{d}=2\hat{i}+\hat{j}+\hat{k}$, respectively. Then which of the following statements is true? [Adv. 2023]
   (a) The points P,Q,R and S are NOT coplanar.
   (b) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio 5 : 4.
   (c) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio 5 : 4.
   (d) The square of the magnitude of the vector $\vec{b}\times\vec{d}$ is 95.

2. If $\vec{a}$ and $\vec{b}$ are vectors such that $\left|\vec{a}+\vec{b}\right|=\sqrt{29}$ and $\vec{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\vec{b}$, then a possible value of $\left(\vec{a}+\vec{b}\right).\left(-7\hat{i}+2\hat{j}+3\hat{k}\right)$ is [2012]
   (a) 0 (b) 3 (c) 4 (d) 8

3. Two adjacent sides of a parallelogram ABCD are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$
   The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by [2010]
   (a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$ (c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

4. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are unit vectors such that $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=1$ and $\vec{a}\cdot\vec{c}=\frac{1}{2}$, then [2009]
   (a) $\vec{a},\vec{b},\vec{c}$ are non-coplanar
   (b) $\vec{b},\vec{c},\vec{d}$ are non-coplanar
   (c) $\vec{b},\vec{d}$ are non-parallel
   (d) $\vec{a},\vec{d}$ are parallel and $\vec{b},\vec{c}$ are parallel

5. The edges of a parallelpiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a},\hat{b},\hat{c}$ such that $\hat{a}.\hat{b}=\hat{b}.\hat{c}=\hat{c}.\hat{a}=\frac{1}{2}$. Then, the volume of the parallelpiped is [2008]
   (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{3}}$

6. Let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Which one of the following is correct ? [2007 -3 marks]
   (a) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{0}$
   (b) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq\vec{0}$
   (c) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{a}\times\vec{c}\neq\vec{0}$
   (d) $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are muturally perpendicular

7. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar, is [2007 -3 marks]
   (a) zero (b) one (c) two (d) three

8. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-\hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is [2006 - 3M, –1]
   (a) $4\hat{i}-\hat{j}+4\hat{k}$ (b) $3\hat{i}+\hat{j}-3\hat{k}$
   (c) $2\hat{i}+\hat{j}-2\hat{k}$ (d) $4\hat{i}+\hat{j}-4\hat{k}$

9. The unit vector which is orthogonal to the vector $5\hat{i}+2\hat{j}+6\hat{k}$ and is coplanar with the vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is [2004S]
   (a) $\frac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ (b) $\frac{2\hat{i}-3\hat{j}}{\sqrt{13}}$
   (c) $\frac{3\hat{j}-\hat{k}}{\sqrt{10}}$ (d) $\frac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

10. If $\vec{a}=(\hat{i}+\hat{j}+\hat{k}), \vec{a}.\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}$ is [2004S]
   (a) $\hat{i}-\hat{j}+\hat{k}$ (b) $2\hat{j}-\hat{k}$
   (c) $\hat{i}$ (d) $2\hat{i}$

11. The value of 'a' so that the volume of parallelopiped formed by $\hat{i}+a\hat{j}+\hat{k}$, $\hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum is [2003S]
   (a) -3 (b) 3 (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

12. Let $\vec{V}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{W}=\hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the maximum value of the scalar triple product $|\vec{U}\vec{V}\vec{W}|$ is [2002S]
   (a) -1 (b) $\sqrt{10}+\sqrt{6}$
   (c) $\sqrt{59}$ (d) $\sqrt{60}$

13. Let $\vec{a}=\vec{i}-\vec{k}$, $\vec{b}=x\vec{i}+\vec{j}+(1-x)\vec{k}$ and $\vec{c}=y\vec{i}+x\vec{j}+(1+x-y)\vec{k}$. Then $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ depends on [2001S]
   (a) only x (b) only y
   (c) Neither x Nor y (d) both x and y

14. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product $\left[2\vec{a}-\vec{b}, 2\vec{b}-\vec{c}, 2\vec{c}-\vec{a}\right]=$ **[2000S]**

(a) 0 (b) 1 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

15. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=0$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is **[2000S]**

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

16. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the sides BC, CA and AB respectively of a triangle ABC, then **[2000S]**

(a) $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=0$

(b) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

(c) $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}$

(d) $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$

17. Let $a=2i+j+k, b=i+2j-k$ and a unit vector c be coplanar. If c is perpendicular to a, then $c=$ **[1999 - 2 Marks]**

(a) $\frac{1}{\sqrt{2}}(-j+k)$ (b) $\frac{1}{\sqrt{3}}(-i-j-k)$

(c) $\frac{1}{\sqrt{5}}(i-2j)$ (d) $\frac{1}{\sqrt{3}}(i-j-k)$

18. Let $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}+\boldsymbol{j}$. If c is a vector such that $\boldsymbol{a}.\boldsymbol{c}=|\boldsymbol{c}|, |\boldsymbol{c}-\boldsymbol{a}|=2\sqrt{2}$ and the angle between $(\boldsymbol{a}\times\boldsymbol{b})$ and $\boldsymbol{c}$ is 30°, then $|(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}|=$ **[1999 - 2 Marks]**

(a) 2/3 (b) 3/2 (c) 2 (d) 3

19. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non coplanar vectors, then **[1995S]**

$(\vec{a}+\vec{b}+\vec{c}).[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$ equals

(a) 0 (b) $[\vec{a}\ \vec{b}\ \vec{c}]$

(c) $2[\vec{a}\ \vec{b}\ \vec{c}]$ (d) $-[\vec{a}\ \vec{b}\ \vec{c}]$

20. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{(\vec{b}+\vec{c})}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is **[1995S]**

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\pi/2$ (d) π

21. Let $\vec{a}=\hat{i}-\hat{j}, \vec{b}=\hat{j}-\hat{k}, \vec{c}=\hat{k}-\hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a}.\vec{d}=0=[\vec{b}\ \vec{c}\ \vec{d}]$, then $\vec{d}$ equals **[1995S]**

(a) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ (b) $\pm\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$

(c) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (d) $\pm\hat{k}$

22. Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is

(a) the Arithmetic Mean of a and b **[1993 - 1 Marks]**

(b) the Geometric Mean of a and b

(c) the harmonic Mean of a and b

(d) equal to zero

23. Let $\overline{a}, \overline{b}, \overline{c}$, be three non-coplanar vectors and $\overline{p}, \overline{q}, \overline{r}$, are vectors defined by the relations $\overline{p}=\frac{\overline{b}\times\overline{c}}{[a\,b\,c]}$, $\overline{q}=\frac{\overline{c}\times\overline{a}}{[a\,b\,c]}$, $\overline{r}=\frac{\overline{a}\times\overline{b}}{[a\,b\,c]}$ then the value of the expression $(\overline{a}+\overline{b}).\overline{p}+(\overline{b}+\overline{c}).\overline{q}+(\overline{c}+\overline{a}).\overline{r}$ is equal to **[1988 - 2 Marks]**

(a) 0 (b) 1 (c) 2 (d) 3.

24. The volume of the parallelopiped whose sides are given by $\overline{OA}=2i-2j, \overline{OB}=i+j-k, \overline{OC}=3i-k$, is **[1983 - 1 Mark]**

(a) $\frac{4}{13}$ (b) 4

(c) $\frac{2}{7}$ (d) none of these

25. For non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, $|(\vec{a}\times\vec{b}).\vec{c}|=|\vec{a}||\vec{b}||\vec{c}|$ holds if and only if **[1982 - 2 Marks]**

(a) $\vec{a}.\vec{b}=0,\ \vec{b}.\vec{c}=0$ (b) $\vec{b}.\vec{c}=0,\ \vec{c}.\vec{a}=0$

(c) $\vec{c}.\vec{a}=0\ \vec{a}.\vec{b}=0$ (d) $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$

26. The scalar $\vec{A}.(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals : **[1981 - 2 Marks]**

(a) 0 (b) $[\vec{A}\ \vec{B}\ \vec{C}]+[\vec{B}\ \vec{C}\ \vec{A}]$

(c) $[\vec{A}\ \vec{B}\ \vec{C}]$ (d) None of these

2 Integer Value Answer/Non-Negative Integer

27. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and $\vec{u}.\vec{w}=1,\ \vec{v}.\vec{w}=1,\ \vec{w}.\vec{w}=4$

If the volume of the parallelopipede, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u}+5\vec{v}|$ is _____. **[Adv. 2021]**

28. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}\cdot\vec{b}=0$. For

some $x, y \in \mathbb{R}$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ______. **[Adv. 2018]**

29. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is **[Adv. 2015]**

30. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p, q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is **[Adv. 2014]**

31. Let $\vec{a} = -\hat{i} - \hat{k}, \vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$, then the value of $\vec{r}.\vec{b}$ is **[2011]**

32. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$, then find the value of $\left(2\vec{a} + \vec{b}\right).\left[\left(\vec{a} \times \vec{b}\right) \times \left(\vec{a} - 2\vec{b}\right)\right]$. **[2010]**

3 Numeric/ New Stem Based Questions

33. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}, \alpha, \beta \in R$. If the projection of $\vec{c}$ on the vector $\left(\vec{a} + \vec{b}\right)$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c} - \left(\vec{a} \times \vec{b}\right)\right).\vec{c}$ equals______. **[Adv. 2019]**

4 Fill in the Blanks

34. Let $OA = a$, $OB = 10a + 2b$ and $OC = b$ where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$, then $k =$ .... **[1997 - 2 Marks]**

35. If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector, then $(\vec{a}.\vec{b})\vec{b} + (\vec{a}.\vec{c})\vec{c} + \frac{\vec{a}.(\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}(\vec{b} \times \vec{c}) =$ .... **[1996 - 2 Marks]**

36. A unit vector coplanar with $\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$ and perpendicular to $\vec{i} + \vec{j} + \vec{k}$ is .... **[1992 - 2 Marks]**

37. If the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ $(a \neq b \neq c \neq 1)$ are coplanar, then the value of $\frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)} =$....... **[1987 - 2 Marks]**

38. If $\overline{A} = (1, 1, 1)$, $\overline{C} = (0, 1, -1)$ are given vectors, then a vector B satisfying the equations $\overline{A} \times \overline{B} = \overline{C}$ and $\overline{A} \cdot \overline{B} = 3$ ...... **[1985 - 2 Marks]**

39. If $\overline{A}\,\overline{B}\,\overline{C}$ are three non-coplanar vectors, then –

$$\frac{\overline{A} \cdot \overline{B} \times \overline{C}}{\overline{C} \times \overline{A} \cdot \overline{B}} + \frac{\overline{B} \cdot \overline{A} \times \overline{C}}{\overline{C} \cdot \overline{A} \times \overline{B}} = \text{.......}$$ **[1985 - 2 Marks]**

40. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\overline{A} = (1, a, a^2)$, $\overline{B} = (1, b, b^2)$, $\overline{C} = (1, c, c^2)$, are non-coplanar, then the product $abc =$ .... **[1985 - 2 Marks]**

41. The area of the triangle whose vertices are A (1, –1, 2), B (2, 1, –1), C (3, –1, 2) is .... **[1983 - 1 Mark]**

5 True / False

42. For any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$,

$(\vec{a} - \vec{b}) \cdot (\vec{b} - \vec{c}) \times (\vec{c} - \vec{a}) = 2\vec{a} \cdot \vec{b} \times \vec{c}$. **[1989 - 1 Mark]**

43. If $X.A = 0, X.B = 0, X.C = 0$ for some non-zero vector X, then $[A\ B\ C] = 0$ **[1983 - 1 Mark]**

44. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be unit vectors suppose that $\vec{A}.\vec{B} = \vec{A}.\vec{C} = 0$, and that the angle between $\vec{B}$ and $\vec{C}$ is $\pi/6$. Then $\vec{A} = \pm 2(\vec{B} \times \vec{C})$. **[1981 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

45. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along the three positive coordinate axes. Let **[Adv. 2022]**

$\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$,

$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}, \quad b_2, b_3 \in \mathbb{R}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \quad c_1, c_2, c_3 \in \mathbb{R}$

be three vectors such that $b_2 b_3 > 0$, $\vec{a} \cdot \vec{b} = 0$ and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}.$$

Then, which of the following is/are TRUE ?

(a) $\vec{a}.\vec{c} = 0$ (b) $\vec{b}.\vec{c} = 0$

(c) $|\vec{b}| > \sqrt{10}$ (d) $|\vec{c}| \le \sqrt{11}$

46. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $\left|\overrightarrow{OB} \times \overrightarrow{OA}\right| = \frac{9}{2}$, then which of the following statements is (are) **TRUE**? **[Adv. 2021]**

(a) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(b) Area of the triangle OAB is $\frac{9}{2}$

(c) Area of the triangle ABC is $\frac{9}{2}$

(d) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

47. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram $PQRS$. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively.

If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram $PQRS$ is 8, then which of the following statements is/are TRUE? **[Adv. 2020]**

(a) $a + b = 4$

(b) $a - b = 2$

(c) The length of the diagonal PR of the parallelogram $PQRS$ is 4

(d) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

48. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$.

If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$, $\vec{b}.\vec{c} = 24$, then which of the following is (are) true? **[JEE Adv. 2015]**

(a) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ (b) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$ (d) $\vec{a}.\vec{b} = -72$

49. Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then **[Adv. 2014]**

(a) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$

(b) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$

(c) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$

(d) $\vec{a} = -(\vec{a}.\vec{y})(\vec{z} - \vec{y})$

50. The vector (s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$ is/are **[2011]**

(a) $\hat{j} - \hat{k}$ (b) $-\hat{i} + \hat{j}$

(c) $\hat{i} - \hat{j}$ (d) $-\hat{j} + \hat{k}$

51. Let a and b be two non-collinear unit vectors. If $u = a - (a.b)b$ and $v = a \times b$, then $|v|$ is **[1999 - 3 Marks]**

(a) $|u|$ (b) $|u| + |u.a|$

(c) $|u| + |u.b|$ (d) $|u| + u.(a+b)$

52. Which of the following expressions are meaningful? **[1998 - 2 Marks]**

(a) $u \cdot (v \times w)$ (b) $(u \cdot v) \cdot w$

(c) $(u \cdot v)\, w$ (d) $u \times (v \cdot w)$

53. For three vectors u, v, w which of the following expression is not equal to any of the remaining three? **[1998 - 2 Marks]**

(a) $u \cdot (v \times w)$ (b) $(v \times w) \cdot u$

(c) $v \cdot (u \times w)$ (d) $(u \times v) \cdot w$

54. The number of vectors of unit length perpendicular to vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (0, 1, 1)$ is **[1987 - 2 Marks]**

(a) one (b) two

(c) three (d) infinite

55. Let $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$ and $\vec{c} = c_1 i + c_2 j + c_3 k$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$ is equal to **[1986 - 2 Marks]**

(a) 0

(b) 1

(c) $\frac{1}{4}(a_1^2 + a_2^2 + a_2^3)(b_1^2 + b_2^2 + b_3^2)$

(d) $\frac{3}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)$

7 Match the Following

56. Match List I with List II and select the correct answer using the code given below the lists : [Adv. 2014]

List - I	List - II
P. Let $y(x)=\cos\left(3\cos^{-1}x\right)$, $x\in[-1,1]$, $x\neq\pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{\left(x^2-1\right)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals	1. 1
Q. Let $A_1, A_2, ..., A_n$ $(n>2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k=1,2, ..., n$. If $\left\|\sum_{k=1}^{n-1}\left(\vec{a_k}\times\vec{a_{k+1}}\right)\right\|=\left\|\sum_{k=1}^{n-1}\left(\vec{a_k}.\vec{a_{k+1}}\right)\right\|$, then the minimum value of n is	2. 2
R. If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line $x+y=8$, then the value of h is	3. 8
S. Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is	4. 9

	P	Q	R	S
(a)	4	3	2	1
(b)	2	4	3	1
(c)	4	3	1	2
(d)	2	4	1	3

57. Match List I with List II and select the correct answer using the code given below the lists : [Adv. 2013]

List I	List II
P. Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $2(\vec{c}\times\vec{a})$ is	1. 100
Q. Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), 3(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2. 30
R. Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	3. 24
S. Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4. 60

Codes:

	P	Q	R	S
(a)	4	2	3	1
(b)	2	3	1	4
(c)	3	4	1	2
(d)	1	4	3	2

58. Match the statements / expressions given in **Column-I** with the values given in **Column-II**. [2009]

Column-I	Column-II
(A) Root(s) of the equation $2\sin^2\theta+\sin^2 2\theta=2$	(p) $\frac{\pi}{6}$
(B) Points of discontinuity of the unction $f(x)=\left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, f where $[y]$ denotes the largest integer less than or equal to y	(q) $\frac{\pi}{4}$

(C) Volume of the parallellpiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$ (r) $\frac{\pi}{3}$

(D) Angle between vector $\vec{a}$ and $\vec{b}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\,\vec{c}=\vec{0}$ (s) $\frac{\pi}{2}$

8 Comprehension/Passage Based Questions

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$ respectively, of a triangle PQR. [Adv. 2017]

59. $|\overrightarrow{OX}\times\overrightarrow{OY}| =$

(a) $\sin(P+Q)$ (b) $\sin 2R$

(c) $\sin(P+R)$ (d) $\sin(Q+R)$

60. If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

(a) $-\frac{5}{3}$ (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$

9 Assertion and Reason/Statements Type Questions

61. Let the vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RS}, \overrightarrow{ST}, \overrightarrow{TU}$ and $\overrightarrow{UP}$ represent the sides of a regular hexagon.

STATEMENT-1 : $\overrightarrow{PQ}\times\left(\overrightarrow{RS}+\overrightarrow{ST}\right)\neq\overrightarrow{0}$. **because**

STATEMENT-2 : $\overrightarrow{PQ}\times\overrightarrow{RS}=\overrightarrow{0}$ and $\overrightarrow{PQ}\times\overrightarrow{ST}\neq\overrightarrow{0}$.

[2007 -3 marks]

(a) Statement-1 is True, statement-2 is True; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True.

10 Subjective Problems

62. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ and $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$. Prove that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c})\neq 0$ i.e. $\vec{a}.\vec{b}+\vec{d}.\vec{c}\neq\vec{d}.\vec{b}+\vec{a}.\vec{c}$

[2004 - 2 Marks]

63. If $\vec{u}, \vec{v}, \vec{w}$, are three non-coplanar unit vectors and α, β, γ are the angles between $\vec{u}$ and $\vec{v}$ and $\vec{w}$, $\vec{w}$ and $\vec{u}$ respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisectors of the angles α, β, γ respectively. Prove that

$$[\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}]=\frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}.$$

[2003 - 4 Marks]

64. Let V be the volume of the parallelopiped formed by the vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. If a_r, b_r, c_r, where $r=1,2,3$, are non-negative real numbers and $\sum_{r=1}^{3}(a_r+b_r+c_r)=3L$, show that $V\le L^3$. **[2002 - 5 Marks]**

65. Let u and v be unit vectors. If w is a vector such that $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$, then prove that $|(\vec{u}\times\vec{v})\cdot\vec{w}|\le 1/2$ and that the equality holds if and only if u is perpendicular to v. **[1999 - 10 Marks]**

66. For any two vectors u and v, prove that **[1998 - 8 Marks]**

(a) $(\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2$ and

(b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)=(1-\vec{u}\cdot\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$.

67. If A, B and C are vectors such that $|\vec{B}|=|\vec{C}|$. Prove that $\left[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})\right]\times(\vec{B}\times\vec{C})(\vec{B}+\vec{C})=0$.

[1997 - 5 Marks]

68. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, find the position vector of the point E for all its possible positions. **[1996 - 5 Marks]**

69. If the vectors $\vec{b},\vec{c},\vec{d}$, are not coplanar, then prove that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$. **[1994 - 4 Marks]**

70. Let $\vec{A}=2\vec{i}+\vec{k}$, $\vec{B}=\vec{i}+\vec{j}+\vec{k}$, and $\vec{C}=4\vec{i}-3\vec{j}+7\vec{k}$. Determine a vector $\vec{R}$. Satisfying $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A}=0$ **[1990 - 3 Marks]**

71. If vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, show that

$$\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\end{vmatrix}=\vec{0}$$

[1989 - 2 Marks]

72. If A, B, C, D are any four points in space, prove that – [1987 - 2 Marks]

$$\left|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}\right|$$
$= 4$ (area of triangle ABC)

73. The position vectors of the points A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively. If the points A, B, C and D lie on a plane, find the value of λ. [1986 - 2½ Marks]

74. Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and $(\bar{i}+\bar{j}+3\bar{k})x+(3\bar{i}-3\bar{j}+\bar{k})y+(-4\bar{i}+5\bar{j})z$ $=\lambda(x\bar{i}\times\bar{j}\ y+\bar{k}\ z)$ where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along the coordinate axes. [1982 - 3 Marks]

75. $A_1, A_2, \ldots\ldots A_n$ are the vertices of a regular plane polygon with n sides and O is its centre. Show that

$$\sum_{i=1}^{n-1}(\overline{OA}_i\times\overline{OA}_{i+1})=(1-n)(\overline{OA}_2\times\overline{OA}_1)$$ [1982 - 2 Marks]

Answer Key

Topic-1 : Algebra of Vectors, Linear Dependence & Independence of Vectors, Vector Inequality

1. (a) **2.** (b) **3.** (a) **4.** (5) **5.** True **6.** (d) **7.** (A)-p, r, s; (B)-p; (C)-p, q; (D)-s, t
8. (A)-q; (B)-p, q; (C)-p, q, s, t; (D)-q, t

Topic-2 : Scalar or Dot Product of two Vectors

1. (c) **2.** (a) **3.** (b) **4.** (b) **5.** (b) **6.** (b) **7.** (a) **8.** (2) **9.** (3)
10. $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}, \vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ **11.** $(2i-j)$ **12.** orthocentre **13.** $5\sqrt{2}$ **14.** (a, c, d)
15. (a, c) **16.** A-q, B-p, C-s, D-t

Topic-3 : Vector or Cross Product of two vectors, Scalar & Vector Triple Product

1. (b) **2.** (c) **3.** (b) **4.** (c) **5.** (a) **6.** (b) **7.** (c) **8.** (a) **9.** (c) **10.** (c)
11. (c)
12. (c) **13.** (c) **14.** (a) **15.** (a) **16.** (b) **17.** (a) **18.** (b) **19.** (d) **20.** (a) **21.** (a)
22. (b) **23.** (d) **24.** (d) **25.** (d) **26.** (a) **27.** (7) **28.** (3) **29.** (9) **30.** (4) **31.** (9)
32. (5) **33.** (18) **34.** (6) **35.** $(\hat{a})$ **36.** $\frac{\hat{j}-\hat{k}}{\sqrt{2}}, \frac{-\hat{j}+\hat{k}}{\sqrt{2}}$ **37.** (1) **38.** $\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$
39. (0) **40.** (–1) **41.** $\sqrt{13}$ **42.** False **43.** True **44.** True **45.** (b,c,d) **46.** (a,b,c) **47.** (b, c, d)
48. (a,c,d) **49.** (a,b,c) **50.** (a,d) **51.** (a, c) **52.** (a, c) **53.** (c) **54.** (b) **55.** (c) **56.** (a) **57.** (c)
58. (A)-q,s; (B)-p,r,s,t; (C)-t; (D)-r **59.** (a) **60.** (b) **61.** (c)

Hints & Solutions

Topic-1: Algebra of Vectors, Linear Dependence & Independence of Vectors, Vector Inequality

1. (a) Since $\overrightarrow{PQ} = 6\hat{i} + \hat{j}, \overrightarrow{QR} = -\hat{i} + 3\hat{j}, \overrightarrow{SR} = 6\hat{i} + \hat{j},$

$\overrightarrow{PS} = -\hat{i} + 3\hat{j}$.

Here $\overrightarrow{PQ} = \overrightarrow{SR}$; $\overrightarrow{QR} = \overrightarrow{PS}$ and $\overrightarrow{PQ}.\overrightarrow{PS} \neq 0$

Also $\left|\overrightarrow{PQ}\right| \neq \left|\overrightarrow{QR}\right|$

$\Rightarrow$ PQRS is a parallelogram but neither a rhombus nor a rectangle.

2. (b) Let $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}, \beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$ and $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$ be position vectors of points *A*, *B* and *C* respectively, then

$$|\overrightarrow{AB}| = \sqrt{(\beta-\alpha)^2 + (\gamma-\beta)^2 + (\alpha-\gamma)^2}$$

$$|\overrightarrow{BC}| = \sqrt{(\gamma-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\beta)^2}$$

$$|\overrightarrow{CA}| = \sqrt{(\alpha-\gamma)^2 + (\beta-\alpha)^2 + (\gamma-\beta)^2}$$

$\because \quad |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

$\Rightarrow \quad \Delta ABC$ is an equilateral triangle.

3. (a) Let $60\hat{i} + 3\hat{j}, 40\hat{i} - 8\hat{j}$ and $a\hat{i} - 52\hat{j}$ be position vector of points A, B and C respectively.

$\therefore \quad \overrightarrow{AB} = 40\hat{i} - 8\hat{j} - 60\hat{i} - 3\hat{j} = -20\hat{i} - 11\hat{j}$

and $\overrightarrow{AC} = a\hat{i} - 52\hat{j} - 60\hat{i} - 3\hat{j} = (a-60)\hat{i} - 55\hat{j}$

Given that A, B and are collinear

$$\therefore \quad \overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow \frac{a-60}{-20} = \frac{-55}{-11} \Rightarrow a = -40$$

4. (5) Given 8 vectors are

(1, 1, 1), (–1, –1, –1); (–1, 1, 1), (1, –1, –1); (1, –1, 1), (–1, 1, –1); (1, 1, –1), (–1, –1, 1)

These are 4 diagonals of a cube and their opposites.

For 3 non coplanar vectors first we select 3 groups of diagonals and its opposite in $^4C_3$ ways. Then one vector from each group can be selected in 2 × 2 × 2 ways.

$\therefore$ Total ways = $^4C_3 \times 2 \times 2 \times 2 = 32 = 2^5$

$\therefore \quad p = 5$

5. **True:** Let $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $\vec{a} + k\vec{b}$ be position vectors of points *A*, *B* and C respectively.

Then, $\overrightarrow{AB} = (\vec{a} - \vec{b}) - (\vec{a} + \vec{b}) = -2\vec{b}$

and $\overrightarrow{BC} = \vec{a} + k\vec{b} - \vec{a} + \vec{b} = (k+1)\vec{b}$

$\Rightarrow \overrightarrow{AB} \parallel \overrightarrow{BC} \quad \forall k \in R$

$\Rightarrow$ *A*,*B*, *C* are collinear $\forall k \in R$

$\therefore$ Statement is true.

6. (d) Given that, $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent,

$\therefore \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & \alpha-1 & \beta-1 \end{vmatrix} = 0$$

$\Rightarrow \beta - 1 = 0 \Rightarrow \beta = 1$

Also given that $|\vec{c}| = \sqrt{3} \Rightarrow 1 + \alpha^2 + \beta^2 = 3$

Substituting the value of β we get $\alpha^2 = 1$

$\Rightarrow \quad \alpha = \pm 1$

7. **(A)→p, r, s; (B) → p; (C) → p, q; (D)→s, t**

(A) Since, $2(a^2 - b^2) = c^2$

$\Rightarrow \quad 2(\sin^2 x - \sin^2 y) = \sin^2 z$

$\Rightarrow \quad 2\sin(x+y)\sin(x-y) = \sin^2 z$

$\Rightarrow \quad 2\sin(x-y) = \sin z \qquad (\because \sin(x+y) = \sin z)$

$$\Rightarrow \quad \frac{\sin(x-y)}{\sin z} = \frac{1}{2} = \lambda \qquad \text{(Given)}$$

$\therefore \quad \cos(n\pi\lambda) = 0 \Rightarrow \cos\dfrac{n\pi}{2} = 0 \Rightarrow n = 1, 3, 5$

(B) $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$

$\Rightarrow \quad 2\cos^2 X - 2(1 - 2\sin^2 Y) = 2\sin X \sin Y$

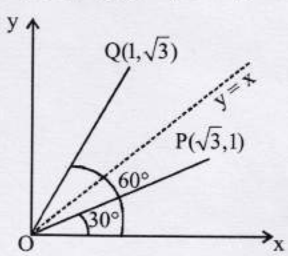
$\Rightarrow \quad 1 - \sin^2 X - 1 + 2\sin^2 Y = \sin X \sin Y$

$\Rightarrow \quad \sin^2 X + \sin X \sin Y - 2\sin^2 Y = 0$

$\Rightarrow \quad (\sin X - \sin Y)(\sin X + 2\sin Y) = 0$

$\Rightarrow \quad \dfrac{\sin X}{\sin Y} = 1$ or $-2 \therefore \dfrac{a}{b} = 1$.

(C) $P(\sqrt{3}, 1), Q(1, \sqrt{3}), R(\beta, 1-\beta)$



From figure, acute angle bisector of $\angle XOY$ is $y = x$.

$\therefore \quad$ Distance of $R(\beta, 1-\beta)$ from bisector

$$= \left|\frac{\beta - 1 + \beta}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

$\Rightarrow 2\beta - 1 = \pm 3$ or $\beta = 2$ or -1

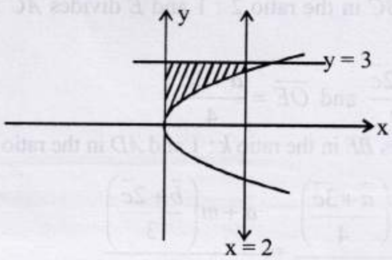
$\therefore \quad |\beta| = 1, 2$

(D) For $\alpha = 0, y = 3$
For $\alpha = 1, y = |x-1| + |x-2| + x$
$\alpha = 0$

Case I

$F(\alpha)$ is the area bounded by $x = 0, x = 2, y^2 = 4x$ and $y = 3$

$$\therefore \quad F(\alpha) = \int_0^2 (3 - 2\sqrt{x})dx$$



$$= \left|3x - \frac{4x\sqrt{x}}{3}\right|_0^2 = 6 - \frac{8\sqrt{2}}{3}$$

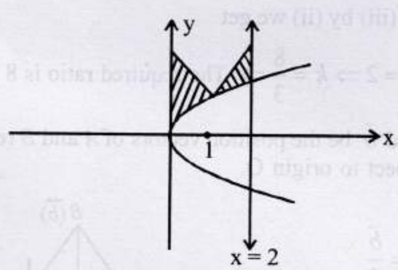
$$\therefore \quad F(\alpha) + \frac{8}{3}\sqrt{2} = 6$$

Case II

$F(\alpha)$ is the area bounded by $x = 0, x = 2, y^2 = 4x$ and $y = |x-1| + |x-2| + x$

$$= \begin{cases} 3 - x, 0 \le x < 1 \\ x + 1, 1 \le x < 2 \\ 3x - 3, x \ge 2 \end{cases}$$

$$\therefore \quad F(\alpha) = \int_0^1 (3 - x - 2\sqrt{x})dx + \int_1^2 (x + 1 - 2\sqrt{x})dx$$



$$= \left(3x - \frac{x^2}{2} - \frac{4x}{3}\sqrt{x}\right)_0^1 + \left(\frac{x^2}{2} + x - \frac{4}{3}x\sqrt{x}\right)_1^2$$

$$= 3 - \frac{1}{2} - \frac{4}{3} + 2 + 2 - \frac{8\sqrt{2}}{3} - \frac{1}{2} - 1 + \frac{4}{3} = 5 - \frac{8\sqrt{2}}{3}$$

$$F(\alpha) + \frac{8\sqrt{2}}{3} = 5$$

8. **(A) → q; (B) → p, q; (C) → p, q, s, t; (D) → q, t**

(A) Projection of $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ $\dfrac{\sqrt{3}\alpha + \beta}{2} = \sqrt{3}$

$$\Rightarrow \quad \alpha = \frac{2\sqrt{3} - \beta}{\sqrt{3}}$$

$$\therefore \quad \frac{2\sqrt{3} - \beta}{\sqrt{3}} = 2 + \sqrt{3}\beta \Rightarrow \beta = 0 \Rightarrow \alpha = 2$$

(B) LHD $= f'(1) = -6a$ and RHD $= f'(1) = b$
$-6a = b$ ...(i)
Also f is continuous at $x = 1$,
$\therefore \quad -3a - 2 = b + a^2$
$\Rightarrow \quad a^2 - 3a + 2 = 0$ (from (i))
$\Rightarrow \quad a = 1, 2$

(C) $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$ $[\because \omega^3 = 1]$

$$\Rightarrow \quad (3 - 3\omega + 2\omega^2)^{4n+3} + \left(\frac{2\omega^2 + 3 - 3\omega}{\omega^2}\right)^{4n+3} + \left(\frac{-3\omega + 2\omega^2 + 3}{\omega}\right)^{4n+3} = 0$$

$\Rightarrow \quad (3 - 3\omega + 2\omega^2)^{4n+3}[1 + \omega^{4n+3} + (\omega^2)^{4n+3}] = 0$
$\Rightarrow \quad 4n + 3$ should be an integer other than multiple of 3.
$\therefore \quad n = 1, 2, 4, 5$

(D) $\because$ H.M of a and b be 4.

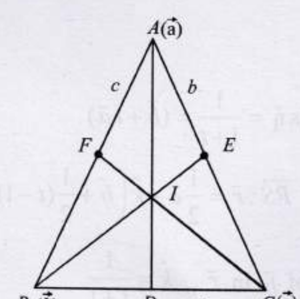
$$\therefore \quad \frac{2ab}{a+b} = 4 \Rightarrow ab = 2a + 2b \quad ...(i)$$

Also $a + q = 10$ or $a = 10 - q$
and $b + 5 = 2q$ or $b = 2q - 5$
Putting values of a and b in eq^n(i), we get

$$q = 4 \text{ or } \frac{15}{2} \Rightarrow a = 6 \text{ or } \frac{5}{2}$$

$\therefore \quad |q - a| = 2$ or 5.

9. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of vertices A, B and C respectively with respect to origin,
Let AD, BE and CF be the bisectors of $\angle A, \angle B$, and $\angle C$ respectively.



Let a, b, c are the lengths of sides BC, CA and AB respectively, we know that by angle bisector theorem $BD : DC = AB : AC = c : b$.

$\therefore$ The position vector of D is $\vec{d} = \dfrac{b\vec{b} + c\vec{c}}{b + c}$

Let I be the point of intersection of BE and AD. Then in ΔABD, BI is bisector of $\angle B$.

$\therefore DI : IA = BD : BA$

But $\dfrac{BD}{DC} = \dfrac{c}{b} \Rightarrow \dfrac{BD}{BD+DC} = \dfrac{c}{c+b}$

$\Rightarrow \dfrac{BD}{BC} = \dfrac{c}{c+b} \Rightarrow BD = \dfrac{ac}{b+c}$

$\therefore DI : IA = \dfrac{ac}{b+c} : c = a : (b+c)$

$\therefore$ P.V. of $I = \dfrac{\vec{a}.\, a + \vec{d}(b+c)}{a+b+c}$

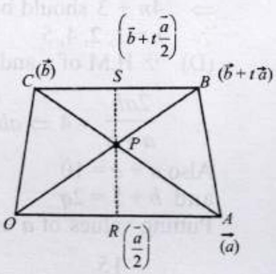
$$= \frac{a\vec{a} + \left(\dfrac{b\vec{b} + c\vec{c}}{b+c}\right)(b+c)}{a+b+c} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$$

Similarly p.v. of intersection of

AD and CF is also $\dfrac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$

Hence all the $\angle$ bisectors passes through I, i.e., these are concurrent.

10. Let OABC is trapezium and position vector of A,B,C with respect to origin O are $A(\vec{a}), C(\vec{b}), B(\vec{b} + t\vec{a})$



Equation of $\overline{OB}$

$\vec{r} = \lambda(\vec{b} + t\vec{a})$

Equation of AC :

$\vec{r} = \vec{a} + \mu(\vec{b} - \vec{a})$

Let P be the point of intersection $\overline{OB}$ and $\overline{AC}$

$\therefore \lambda(\vec{b} + t\vec{a}) = \vec{a} + \mu(\vec{b} - \vec{a})$

On comparing both sides

$\lambda = \mu$ ... (i)

$\lambda t = 1 - \mu$ ... (ii)

from (i) and (ii)

$\lambda = \dfrac{1}{1+t}$

$\therefore P.V$ of P is $\vec{r}_1 = \dfrac{1}{1+t} = (\vec{b} + t\vec{a})$

Equation of $\overline{RS} : \vec{r} = \dfrac{1}{2}\vec{a} + \hat{k}\left(\vec{b} + \dfrac{1}{2}(t-1)\vec{a}\right)$

Coefficient of $\vec{b}$ in $\vec{r}_1$, $\hat{k} = \dfrac{1}{t+1}$

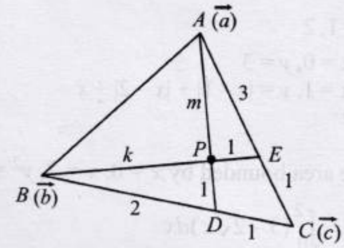
$\therefore \vec{r} = \dfrac{1}{2}\vec{a} + \dfrac{1}{t+1}\left[\vec{b} + \dfrac{1}{2}(t-1)\vec{a}\right]$

$= \dfrac{1}{t+1}\vec{b} + \dfrac{1}{2(t+1)}[t-1+t+1]\vec{a} = \dfrac{1}{t+1}\left[\vec{b} + t\vec{a}\right] = r_1$

Hence p lies on RS.

Hence proved.

11. Let $\vec{a}, \vec{b}, \vec{c}$, be the position vectors of points A, B and C respectively with respect to origin O.



$\because D$ divides BC in the ratio 2 : 1 and E divides AC in the ratio 3 : 1.

$\therefore \overline{OD} = \dfrac{\vec{b} + 2\vec{c}}{3}$ and $\overline{OE} = \dfrac{\vec{a} + 3\vec{c}}{4}$

Let pt. P divides BE in the ratio $k : 1$ and AD in the ratio $m : 1$,

$$\therefore \overline{OP} = \frac{\vec{b} + k\left(\dfrac{\vec{a} + 3\vec{c}}{4}\right)}{k+1} = \frac{\vec{a} + m\left(\dfrac{\vec{b} + 2\vec{c}}{3}\right)}{m+1}$$

$\Rightarrow \dfrac{k}{4(k+1)}\vec{a} + \dfrac{1}{k+1}\vec{b} + \dfrac{3k}{4(k+1)}\vec{c}$

$= \dfrac{1}{m+1}\vec{a} + \dfrac{m}{3(m+1)}\vec{b} + \dfrac{2m}{3(m+1)}\vec{c}$

On comparing both sides, we get

$\Rightarrow \dfrac{k}{4(k+1)} = \dfrac{1}{m+1}$ ... (i)

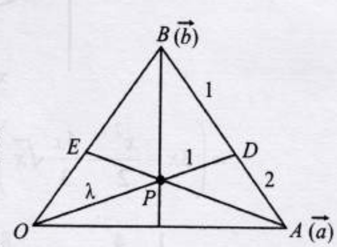
$\dfrac{1}{k+1} = \dfrac{m}{3(m+1)}$ ... (ii)

$\dfrac{3k}{4(k+1)} = \dfrac{2m}{3(m+1)}$ ... (iii)

Dividing (iii) by (ii) we get

$\dfrac{3k}{4} = 2 \Rightarrow k = \dfrac{8}{3} \Rightarrow$ The required ratio is 8 : 3.

12. Let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B respectively with respect to origin O.



$\Rightarrow \overline{OE} = \dfrac{\vec{b}}{2}$

$\overline{OD} = \dfrac{1.\vec{a} + 2\vec{b}}{1+2} = \dfrac{\vec{a} + 2\vec{b}}{3}$

$\therefore$ Equation of OD is

$\vec{r} = t\left(\dfrac{\vec{a} + 2\vec{b}}{3}\right)$ ...(i)

and Equation of AE is

$\vec{r} = \vec{a} + s\left(\dfrac{\vec{b}}{2} - \vec{a}\right)$ ...(ii)

Since OD and AE intersect at P, then comparing the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{t}{3}=1-s \text{ and } \frac{2t}{3}=\frac{s}{2} \Rightarrow t=\frac{3}{5} \text{ and } s=\frac{4}{5}$$

Putting value of t in eq. (i) we get position vector of point of intersection P is $\frac{\vec{a}+2\vec{b}}{5}$ ... (iii)

Let P divides OD in the ratio $\lambda : 1$, then position vector of P is

$$\frac{\lambda\left(\frac{\vec{a}+2\vec{b}}{3}\right)+1.0}{\lambda+1}=\frac{\lambda}{3(\lambda+1)}(\vec{a}+2\vec{b}) \quad \text{... (iv)}$$

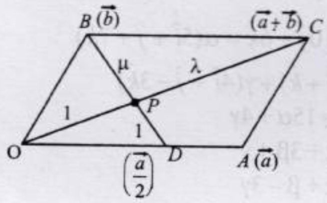
From (iii) and (iv) we get

$$\frac{\lambda}{3(\lambda+1)}=\frac{1}{5} \Rightarrow 5\lambda=3\lambda+3 \Rightarrow \lambda=3/2$$

$\therefore \quad OP:PD=3:2$

13. Given $OACB$ is a parallelogram with O as origin.

Let $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b} \Rightarrow \overline{OC}=\vec{a}+\vec{b}$ and $\overline{OD}=\frac{\vec{a}}{2}$.



Let $\overline{CO}$ and $\overline{BD}$ intersect each other at P.

Let P dividing CO in ratio $\lambda:1$

$$\overline{OP}=\frac{\lambda\times 0+1\times(\vec{a}+\vec{b})}{\lambda+1}=\frac{(\vec{a}+\vec{b})}{\lambda+1} \quad \text{...(i)}$$

And P dividing BD in the ratio $\mu : 1$

$$\overline{OP}=\frac{\mu\left(\frac{\vec{a}}{2}\right)+1(\vec{b})}{\mu+1}=\frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)} \quad \text{...(ii)}$$

From (i) and (ii)

$$\frac{\vec{a}+\vec{b}}{\lambda+1}=\frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)}$$

Equating the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{1}{\lambda+1}=\frac{\mu}{2(\mu+1)} \quad \text{...(iii)}$$

$$\frac{1}{\lambda+1}=\frac{1}{\mu+1} \quad \text{...(iv)}$$

From (iv) we get $\lambda=\mu \Rightarrow P$ divides CO and BD in the same ratio.

Putting $\lambda=\mu$ in eq. (iii) we get $\mu=2$

Thus required ratio is 2 : 1.

14. Since vector $\vec{A}$ has components A_1, A_2, A_3, in the coordinate system $OXYZ$,

$$\therefore \quad \vec{A}=\hat{i}A_1+\hat{j}A_2+\hat{k}A_3$$

When given system is rotated through $\frac{\pi}{2}$, the new x-axis is along old y-axis and new y-axis is along the old negative x-axis and z remains same as before.

Hence the components of A in the new system are $A_2, -A_1, A_3$.

$\therefore \vec{A}$ becomes $A_2\hat{i}-A_1\hat{j}+A_3\hat{k}$.

Topic-2: Scalar or Dot Product of two Vectors

1. **(c)** We know that vector in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{v}=\vec{a}+\lambda\vec{b}$

$$\Rightarrow \quad \vec{v}=(1+\lambda)\hat{i}+(1-\lambda)\hat{j}+(1+\lambda)\hat{k}$$

$\because$ Projection of $\vec{v}$ on $\vec{c}$ is $\frac{1}{\sqrt{3}}$

$$\therefore \quad \frac{\vec{v}.\vec{c}}{|\vec{c}|}=\frac{1}{\sqrt{3}} \Rightarrow \frac{(1+\lambda)-(1-\lambda)-(1+\lambda)}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$\Rightarrow 1-\lambda=-1$

$\Rightarrow \lambda=2$

$\vec{v}=3\hat{i}-\hat{j}+3\hat{k}$

2. **(a)** Given, $\overline{OP}=\hat{a}\cos t+\hat{b}\sin t$

$$|\overline{OP}|^2=|\hat{a}|^2\cos^2 t+|\hat{b}|^2\sin^2 t+2\hat{a}\cdot\hat{b}\sin t\cos t$$

$$\Rightarrow |\overrightarrow{OP}|^2=\cos^2 t+\sin^2 t+2\hat{a}\hat{b}\sin t\cos t$$

$$\Rightarrow |\overrightarrow{OP}|^2=1+\hat{a}\hat{b}\sin 2t$$

$$|\overline{OP}|=\sqrt{1+\hat{a}\cdot\hat{b}\sin 2t}$$

$$\therefore |\overline{OP}|_{\max}=\sqrt{1+\hat{a}.\hat{b}}=M \text{ at } \sin 2t=1 \Rightarrow t=\frac{\pi}{4}$$

$$\therefore (\overline{OP})_{\max}=\frac{\hat{a}+\hat{b}}{\sqrt{2}} \therefore (\widehat{OP})_{\max}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

Hence $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=\sqrt{1+\hat{a}.\hat{b}}$

3. **(b)** We observe that

$$\vec{a}.\vec{b}_1=\vec{a}.\vec{b}-\left(\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}\right)|\vec{a}|^2=\vec{a}.\vec{b}-\vec{a}.\vec{b}=0 \quad \text{...(i)}$$

$$\vec{a}.\vec{c}_2=\vec{a}.\left(\vec{c}-\frac{\vec{c}.\vec{a}}{|\vec{a}|^2}\vec{a}-\frac{\vec{c}.\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$$

$$=\vec{a}.\vec{c}-\frac{\vec{a}.\vec{c}}{|\vec{a}|^2}|\vec{a}|^2-\frac{\vec{c}.\vec{b}_1}{|\vec{b}_1|^2}(\vec{a}.\vec{b}_1)$$

$= \vec{a}.\vec{c} - \vec{a}.\vec{c} - 0 = 0$ [from (i)]

And $\vec{b}_1.\vec{c}_2 = \vec{b}_1.\left(\vec{c} - \frac{\vec{c}.\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}.\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$

$= \vec{b}_1.\vec{c} - \frac{(\vec{c}.\vec{a})(\vec{b}_1.\vec{a})}{|\vec{a}|^2} - \frac{\vec{c}.\vec{b}_1}{|\vec{b}_1|^2}|\vec{b}_1|^2$

$= \vec{b}_1.\vec{c} - 0 - \vec{b}_1.\vec{c} = 0$ (from (i))

Hence $\vec{a}.\vec{b}_1 = \vec{a}.\vec{c}_2 = \vec{b}_1.\vec{c}_2 = 0$

$\Rightarrow$ $(\vec{a}, \vec{b}_1, \vec{c}_2)$ is a set of orthogonal vectors.

4. **(b)** Given that $\vec{a}$ and $\vec{b}$ are two unit vectors

$\therefore$ $|\vec{a}| = |\vec{b}| = 1$

Also, given that $(\vec{a} + 2\vec{b})$ is perpendicular to $(5\vec{a} - 4\vec{b})$

$\therefore$ $(\vec{a} + 2\vec{b}).(5\vec{a} - 4\vec{b}) = 0$

$\Rightarrow$ $5|\vec{a}|^2 - 8|\vec{b}|^2 - 4\vec{a}.\vec{b} + 10\vec{b}.\vec{a} = 0$

$\Rightarrow$ $5 - 8 + 6\vec{a}.\vec{b} = 0 \Rightarrow 6|\vec{a}||\vec{b}|\cos\theta = 3$

[where θ is the angle between $\vec{a}$ and $\vec{b}$]

$\Rightarrow$ $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$

5. **(b)** Since $\vec{a}, \vec{b}$ and $\vec{c}$ are units vectors.

$\therefore |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Let $x = \left|\vec{a} - \vec{b}\right|^2 + \left|\vec{b} - \vec{c}\right|^2 + \left|\vec{c} - \vec{a}\right|^2$

$= |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2 + |\vec{c}|^2 - 2\vec{b}.\vec{c} + |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a}$

$= 6 - 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right)$ ...(i)

We have

$|\vec{a} + \vec{b} + \vec{c}| \geq 0 \Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 \geq 0$

$\Rightarrow$ $|\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) \geq 0$

$\Rightarrow$ $3 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) \geq 0$

$\Rightarrow$ $2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) \geq -3$

$\Rightarrow$ $-2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) \leq 3$

$\Rightarrow$ $6 - 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) \leq 9$

$x \leq 9$ (From (i)) $\therefore$ x does not exceed 9

6. **(b)** Since $\vec{u} + \vec{v} + \vec{w} = 0$

$\Rightarrow |\vec{u} + \vec{v} + \vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u})$

$\Rightarrow 0 = 9 + 16 + 25 + 2(\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u})$

$\Rightarrow (\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u}) = -25$

7. **(a)** Since R divide PQ internally in ratio 2 : 3

$\therefore \overrightarrow{OR} = \frac{3\vec{p} + 2\vec{q}}{3 + 2} = \frac{1}{5}(3\vec{p} + 2\vec{q})$

S divide PQ externally in ratio 2 : 3

$\therefore \overrightarrow{OS} = \frac{3\vec{p} - 2\vec{q}}{3 - 2} = 3\vec{p} - 2\vec{q}$

Given that $\overrightarrow{OR} \perp \overrightarrow{OS} \Rightarrow \overrightarrow{OR}.\overrightarrow{OS} = 0$

$\Rightarrow$ $\frac{1}{5}[3\vec{p} + 2\vec{q}].(3\vec{p} - 2\vec{q}) = 0$

$\Rightarrow$ $9|\vec{p}|^2 = 4|\vec{q}|^2 \Rightarrow 9p^2 = 4q^2$

8. **(2)** $2\vec{p} + \vec{q} = 5\hat{i} + \hat{j} + 7\hat{k}$

$\vec{p} - 2\vec{q} = 0\hat{i} + 3\hat{j} + \hat{k}$

$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(4) - \hat{j}(-1) + \hat{k}(-3)$

$= 4\hat{i} + \hat{j} - 3\hat{k}$

$15\hat{i} + 10\hat{j} + 6\hat{k} = \alpha(5\hat{i} + \hat{j} + 7\hat{k})$

$+\beta(3\hat{j} + \hat{k}) + \gamma(4\hat{i} + \hat{j} - 3\hat{k})$

$\therefore 15 = 15\alpha + 4\gamma$ ...(i)

$10 = \alpha + 3\beta + \gamma$ ...(ii)

$6 = 7\alpha + \beta - 3\gamma$ ...(iii)

On solving (i), (ii) and (iii), we get

$\therefore \alpha = \frac{7}{5}, \beta = \frac{11}{5}, \gamma = 2 \quad \therefore \gamma = 2$

9. **(3)** $\because |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$

$\Rightarrow 2\left(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2\right) - 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 9$

$\Rightarrow \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = \frac{-3}{2}$

Also,

$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

$= 1 + 1 + 1 + 2 \times \left(-\frac{3}{2}\right) = 0$

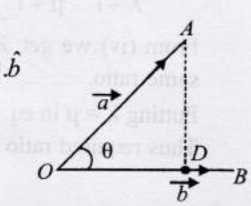
$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow (\vec{b} + \vec{c}) = -\vec{a}$

$\therefore \left|2\vec{a} + 5(\vec{b} + \vec{c})\right| = |2\vec{a} - 5\vec{a}| = |-3\vec{a}| = 3$

10. Component of $\vec{a}$ along

$\vec{b} = \overrightarrow{OD} = \left(\text{projection of } \vec{a} \text{ on } \vec{b}\right).\hat{b}$

$= \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$



Component of $\vec{a}$ perpendicular to $\vec{b}$

$$= \overrightarrow{DA} = \vec{a} - \overrightarrow{OD} = \vec{a} - \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

11. Let $\vec{c} = a\hat{i} + b\hat{j}$

$\because \hat{b} \perp \hat{c} \Rightarrow \vec{b}.\vec{c} = 0$

$\Rightarrow (4\hat{i} + 3\hat{j}).(a\hat{i} + b\hat{j}) = 0 \Rightarrow 4a + 3b = 0$

$\Rightarrow a = -\frac{3b}{4} \Rightarrow \frac{a}{+3} = \frac{b}{-4} = \lambda$

$\Rightarrow a = +3\lambda, b = -4\lambda$ ...(i)

$\therefore \vec{c} = \lambda\left(3\hat{i} - 4\hat{j}\right)$

Now, let $\vec{a} = x\hat{i} + y\hat{j}$ be the required vectors.

Projection of $\vec{a}$ along $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} \Rightarrow \frac{4x+3y}{\sqrt{4^2+3^2}} = 1$

$\Rightarrow 4x + 3y = 5$ ...(ii)

Also, projection of $\vec{a}$ along $\vec{c} = 2$

$\Rightarrow \frac{\vec{a}.\vec{c}}{|\vec{c}|} = 2 \Rightarrow \frac{3\lambda x - 4\lambda y}{\sqrt{(3\lambda)^2 + (-4\lambda)^2}} = 2$

$\Rightarrow 3\lambda x - 4\lambda y = 10\lambda \Rightarrow 3x - 4y = 10$ ...(iii)

Solving (ii) and (iii), we get $x = 2, y = -1$

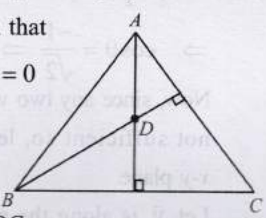
$\therefore$ The required vector is $2\hat{i} - \hat{j}$

12. Given that the position vectors of points A, B, C and D are $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ respectively, such that

$\left(\vec{a} - \vec{d}\right).(\vec{b} - \vec{c}) = (\vec{b} - \vec{d}).(\vec{c} - \vec{a}) = 0$

$\Rightarrow \overrightarrow{DA}.\overrightarrow{CB} = \overrightarrow{DB}.\overrightarrow{AC} = 0$

$\Rightarrow \overrightarrow{DA} \perp \overrightarrow{CB}$ and $\overrightarrow{DB} \perp \overrightarrow{AC}$



Clearly D is orthocentre of $\Delta\ ABC$

13. Given that $|\vec{A}| = 3$; $|\vec{B}| = 4$; $|\vec{C}| = 5$

$\because \vec{A} \perp (\vec{B} + \vec{C}) \Rightarrow \vec{A}.(\vec{B} + \vec{C}) = 0$

$\vec{B} \perp (\vec{C} + \vec{A}) \Rightarrow \vec{B}.(\vec{C} + \vec{A}) = 0$

$\vec{C} \perp (\vec{A} + \vec{B}) \Rightarrow \vec{C}.(\vec{A} + \vec{B}) = 0$

Now, $|\vec{A} + \vec{B} + \vec{C}|^2 = \left(\vec{A} + \vec{B} + \vec{C}\right).\left(\vec{A} + \vec{B} + \vec{C}\right)$

$= |\vec{A}|^2 + \vec{A}\cdot\left(\vec{B} + \vec{C}\right) + |\vec{B}|^2 + \vec{B}\cdot\left(\vec{C} + \vec{A}\right) + |\vec{C}|^2 + \vec{C}\cdot\left(\vec{A} + \vec{B}\right)$

$= |\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2 + 0$

$= 9 + 16 + 25 = 50 \therefore |\vec{A} + \vec{B} + \vec{C}| = 5\sqrt{2}$

14. **(a, c, d)** $|\vec{a}| = \frac{1}{3}\sqrt{(4+4+1)} = 1 \Rightarrow |\vec{a}| = 1$

$\therefore$ It is unit vector

Let $\vec{b} = 2\hat{i} - 4\hat{j} + 3\hat{k}$ then

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{29}} \Rightarrow \theta \neq \frac{\pi}{3}$

Let $\vec{c} = -\hat{i} + \hat{j} - \frac{1}{2}\hat{k} = \frac{-3}{2}\vec{a} \Rightarrow \vec{c} \parallel \vec{a}$

Let $\vec{d} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ then $\vec{a}.\vec{d} = 0 \Rightarrow \vec{a} \perp \vec{d}$

15. **(a, c)** Any vector in the plane of $\vec{b}$ and $\vec{c}$ is $\vec{u} = \vec{b} + \lambda\vec{c}$

$= (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

$= (1+\lambda)\hat{i} + (2+\lambda)\hat{j} - (1+2\lambda)\hat{k}$

Given that magnitude of projection of $\vec{u}$ on $\vec{a}$ is $\sqrt{\frac{2}{3}}$

$\Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{\vec{u}.\vec{a}}{|\vec{a}|}\right| \Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{2(1+\lambda) - (2+\lambda) - (1+2\lambda)}{\sqrt{6}}\right|$

$\Rightarrow |-\lambda - 1| = 2 \Rightarrow \lambda + 1 = 2$ or $\lambda + 1 = -2$

$\Rightarrow \lambda = 1$ or $\lambda = -3$

$\therefore$ The required vector is either,

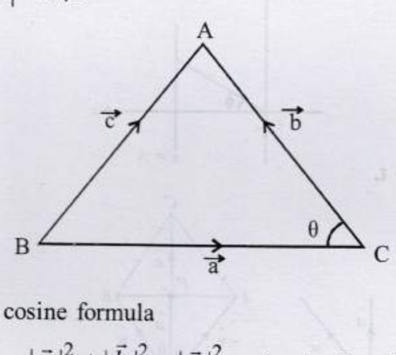
$2\hat{i} + 3\hat{j} - 3\hat{k}$ or $-2\hat{i} - \hat{j} + 5\hat{k}$

16. **A→q, B→p, C→s, D→t**

A. $|\vec{a}| = \sqrt{1+3} = 2$

$|\vec{b}| = \sqrt{1+3} = 2$

$|\vec{c}| = 2\sqrt{3}$



Using cosine formula

$$\text{Cos C} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{a}|\,|\vec{b}|} = \frac{4+4-12}{2\times2\times2} = \frac{1}{2}$$

$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

$\Rightarrow$ **A → q**

B. $\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$

$\Rightarrow \int_a^b f(x)\,dx + \frac{3}{2}[-b^2 + a^2] = a^2 - b^2$

$\Rightarrow \int_a^b f(x)\,dx = -\frac{1}{2}(a^2 - b^2) = \int_a^b x\,dx$

$\Rightarrow f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\Rightarrow$ **B $\rightarrow$ p**

C. $\frac{\pi^2}{\ell n3}\int_{\frac{7}{6}}^{\frac{5}{6}} \sec(\pi x)dx = \frac{\pi^2}{\pi \ell n3}\left[\ell n|\sec \pi x + \tan \pi x|\right]_{\frac{7}{6}}^{\frac{5}{6}}$

$= \frac{\pi}{\ell n3}\left[\ell n\left|\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}\right| - \ell n\left|\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}\right|\right]$

$= \frac{\pi}{\ell n3}\left[\ell n\left|-\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ell n\left|-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right|\right] = \frac{\pi}{\ell n3}\ell n3 = \pi$

$\therefore$ **C $\rightarrow$ s**

D. For $|z| = 1$ and $z \neq 1$. Let $z = \cos\theta + i \sin\theta$

Then $1 - z = 1 - \cos\theta - i\sin\theta = 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

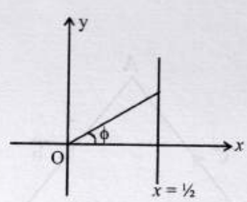
or $1 - z = 2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]$

$\therefore \frac{1}{1-z} = \frac{1}{2}\left[1 + i\cot\frac{\theta}{2}\right]$

Clear that real part of $\frac{1}{1-z}$ is always $\frac{1}{2}$

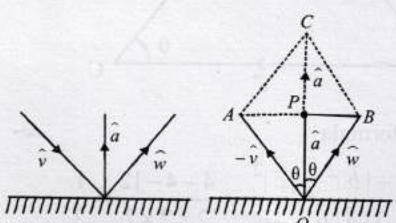
$\therefore$ Locus of $\frac{1}{1-z}$ is $x = \frac{1}{2}$

$\left|Arg\left(\frac{1}{1-z}\right)\right|$ is maximum when value of ϕ approaches to $\frac{\pi}{2}$ but will not be attained.



$\therefore$ **D $\rightarrow$ t.**

17.



We know that incident ray, reflected ray and normal lie in a same plane, and angle of incidence = angle of reflection.

Therefore unit vector $\hat{a}$ will be along the angle bisector of $\hat{w}$ and $-\hat{v}$,

i.e., $\hat{a} = \frac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|}$ ...(1)

[$\because$ Angle bisector will along a vector dividing in same ratio as the ratio of the sides forming that angle.]

Since $|\hat{w} - \hat{v}| = OC = 2OP = 2|\hat{w}|\cos\theta = 2\cos\theta$

Substituting this value in equation (1) we get

$\hat{a} = \frac{\hat{w} - \hat{v}}{2\cos\theta}$

$\therefore \hat{w} = \hat{v} + (2\cos\theta)\hat{a} = \hat{v} - 2(\hat{a}.\hat{v})\hat{a}$ $[\because \hat{a}.\hat{v} = -\cos\theta]$

18. We have $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0,1]$ if and only if

$\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)}$ for some $t \in [0,1]$

$\Rightarrow f_1(t).g_2(t) = f_2(t).g_1(t)$ for some $t \in [0,1]$

Let $h(t) = f_1(t).g_2(t) - f_2(t).g_1(t)$

$h(0) = f_1(0).g_2(0) - f_2(0).g_1(0)$

$= 2 \times 2 - 3 \times 3 = -5 < 0$

$h(1) = f_1(1).g_2(1) - f_2(1).g_1(1)$

$= 6 \times 6 - 2 \times 2 = 32 > 0$

Since h (t) is a continuous function, and $h(0).h(1) < 0$

$\Rightarrow$ There is some $t \in [0,1]$ for which $h(t) = 0$ i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

19. From the given data, we get

$\vec{v}_1 . \vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$

$\vec{v}_2 . \vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$

$\vec{v}_3 . \vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$

Let θ is the angle between $\vec{v}_1$ and $\vec{v}_2$

$\therefore \vec{v}_1.\vec{v}_2 = -2 \Rightarrow |\vec{v}_1|\,|\vec{v}_2|\cos\theta = -2$

$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = 135°$

Now, since any two vectors are always coplanar and data is not sufficient so, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in *x-y* plane.

Let $\vec{v}_1$ is along the positive direction of *x*-axis

then $\vec{v}_1 = 2\hat{i}$. $[\because |\vec{v}_1| = 2]$

$\because \vec{v}_2$ makes an angle 135° with $\vec{v}_1$ and $|\vec{v}_2| = \sqrt{2}$,

$\therefore \vec{v}_2 = -\hat{i} \pm \hat{j}$

Let $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\because \vec{v}_3.\vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \Rightarrow \alpha = 3$

and $\vec{v}_3.\vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5 \Rightarrow \beta = \pm 2$

Also $|\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29 \Rightarrow \gamma = \pm 4$

Hence $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

Thus, $\vec{v}_1 = 2\hat{i}$; $\vec{v}_2 = -\hat{i} \pm \hat{j}$; $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$ are some possible answers.

20. $\vec{a} \cdot \vec{b} = cx^2 - 12 - 6cx$

Since $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Given that angle between $\vec{a}$ and $\vec{b}$ is obtuse, therefore

$\cos\theta < 0 \Rightarrow \vec{a}.\vec{b} < 0$

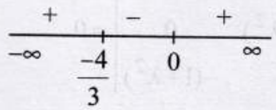
$\Rightarrow cx^2 - 12 - 6cx < 0$

$\Rightarrow cx^2 - 6cx - 12 < 0, \forall x \in R$ ...(i)

Clearly above condition is satisfied if $c = 0$

And $c < 0$ and $D < 0$

$\Rightarrow 36c^2 + 48c < 0 \Rightarrow 12c(3c+4) < 0$



$\therefore \frac{-4}{3} < c \le 0.$

21. Let with respect to *O*, position vectors of points *A, B, C, D, E, F* be $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$.

Let perpendiculars from *A* to *EF* and from *B* to *DF* intersect each other at *H*. Let position vector of H be $\vec{r}$. We join *CH*. Know we have to prove that *CH* is perpendicular to *DE*.

Given that $OD \perp \overline{BC} \Rightarrow \vec{d}.(\vec{c} - \vec{b}) = 0$

$\Rightarrow \vec{d}.\vec{b} = \vec{d}.\vec{c}$ ...(i)

$OE \perp AC \Rightarrow \vec{e}.(\vec{c} - \vec{a}) = 0 \Rightarrow \vec{e}.\vec{c} = \vec{e}.\vec{a}$ ...(ii)

And $OF \perp AB \Rightarrow \vec{f}.(\vec{b} - \vec{a}) = 0 \Rightarrow \vec{f}.\vec{a} = \vec{f}.\vec{b}$ ...(iii)

Also given that $AH \perp EF \Rightarrow (\vec{r} - \vec{a}).(\vec{f} - \vec{e}) = 0$

$\Rightarrow \vec{r}.\vec{f} - \vec{r}.\vec{e} - \vec{a}.\vec{f} + \vec{a}.\vec{e}$ ...(iv)

and $BH \perp FD \Rightarrow (\vec{r} - \vec{b}).(\vec{d} - \vec{f}) = 0$

$\Rightarrow \vec{r}.\vec{d} - \vec{r}.\vec{f} - \vec{b}.\vec{d} + \vec{b}.\vec{f} = 0$ ...(v)

Adding (iv) and (v), we get

$\vec{r}.\vec{d} - \vec{r}.\vec{e} - \vec{a}.\vec{f} + \vec{a}.\vec{e} - \vec{b}.\vec{d} + \vec{b}.\vec{f} = 0$

$\Rightarrow \vec{r}.\vec{d} - \vec{r}.\vec{e} + \vec{e}.\vec{c} - \vec{d}.\vec{c} = 0$

(using (i), (ii) and (iii))

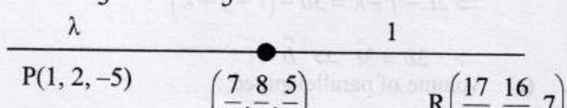
$\Rightarrow \vec{r}.(\vec{d} - \vec{e}) - \vec{c}.(\vec{d} - \vec{e}) = 0 \Rightarrow (\vec{r} - \vec{c}).(\vec{d} - \vec{e}) = 0$

$\Rightarrow \overline{CH}.\overline{ED} = 0 \Rightarrow CH \perp ED$ Hence Proved.

Topic-3: Vector or Cross Product of two vectors, Scalar & Vector Triple Product

1. **(b)** P (1, 2, –5), Q (3, 6, 3), R $\left(\frac{17}{5}, \frac{16}{5}, 7\right)$, S (2, 1, 1)

Now, $\frac{\vec{b} + 2\vec{d}}{3} = \frac{7\hat{i} + 8\hat{j} + 5\hat{k}}{3}$



$\frac{7}{3} = \frac{\frac{17\lambda}{5} + 1}{\lambda + 1}; \frac{8}{3} = \frac{\frac{16\lambda}{5} + 2}{\lambda + 1}; \frac{5}{3} = \frac{7\lambda - 5}{\lambda + 1}$

$\Rightarrow \lambda = \frac{5}{4}; \lambda = \frac{5}{4}; \lambda = \frac{5}{4}$

2. **(c)** Given that $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$

$\Rightarrow (\vec{a} + \vec{b}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = \vec{0}$

But $\vec{a} + \vec{b} \neq 0$ and $2\hat{i} + 3\hat{j} + 4\hat{k} \neq 0$

$\therefore (\vec{a} + \vec{b}) \,||\, (2\hat{i} + 3\hat{j} + 4\hat{k})$.

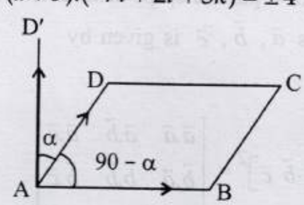
Let $\vec{a} + \vec{b} = \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$

Also given that $|\vec{a} + \vec{b}| = \sqrt{29} \Rightarrow \lambda = \pm 1$

$\therefore \vec{a} + \vec{b} = \pm(2\hat{i} + 3\hat{j} + 4\hat{k})$

So, $(\vec{a} + \vec{b}).(-7\hat{i} + 2\hat{i} + 3\hat{k}) = \pm 4$

3. **(b)**



$\overline{AB} \times \overline{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 10 & 11 \\ -1 & 2 & 2 \end{vmatrix} = -2\hat{i} - 15\hat{j} + 14\hat{k}$

$\therefore \left|\overline{AB} \times \overline{AD}\right| = \sqrt{4 + 225 + 196} = \sqrt{425}$

$\left|\overline{AB}\right| = \sqrt{4 + 100 + 121} = \sqrt{225} = 15$

$\left|\overline{AD}\right| = \sqrt{1 + 4 + 4} = 3$

$\therefore \sin(90 - \alpha) = \frac{\left|\overline{AB} \times \overline{AD}\right|}{\left|\overline{AB}\right|\left|\overline{AD}\right|}$

$\therefore \sin(90 - \alpha) = \frac{\sqrt{425}}{15 \times 3} = \frac{\sqrt{17}}{9} \Rightarrow \cos\alpha = \frac{\sqrt{17}}{9}$

4. **(c)** Given: $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors,

Let $\vec{a} \times \vec{b} = (\sin\alpha)\overrightarrow{n_1}$ and $\vec{c} \times \vec{d} = (\sin\beta)\overrightarrow{n_2}$

where $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are unit normal vectors

then $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = 1$

$\Rightarrow (\sin\alpha)\overrightarrow{n_1}.(\sin\beta)\overrightarrow{n_2} = 1$

$\Rightarrow \sin\alpha\sin\beta\ \overrightarrow{n_1}.\overrightarrow{n_2} = 1 \Rightarrow \sin\alpha\sin\beta\ \cos\gamma = 1$

where γ is the angle between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$.

$\Rightarrow \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}$ and $\gamma = 0°$

If $\gamma = 0° \Rightarrow \vec{a} \times \vec{b} \,||\, \vec{c} \times \vec{d}$

Let $\vec{a} \times \vec{b} = \lambda(\vec{c} \times \vec{d}) \Rightarrow (\vec{a} \times \vec{b}).\vec{c} = \lambda(\vec{c} \times \vec{d}).\vec{c} = 0$

and $(\vec{a} \times \vec{b}).\vec{d} = \lambda(\vec{c} \times \vec{d}).\vec{d} = 0$

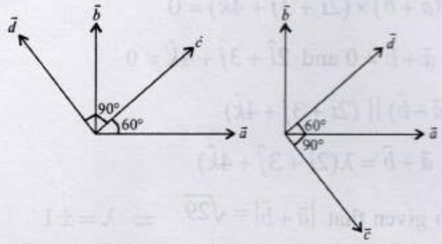
$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{d}] = 0$

$\therefore\quad \vec{a}, \vec{b}, \vec{c}$ are coplanar and $\vec{a}, \vec{b}, \vec{d}$ are coplanar

$\Rightarrow\quad \vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar

and $\alpha = 90° \Rightarrow \vec{a} \perp \vec{b}$ and $\beta = 90° \Rightarrow \vec{c} \perp \vec{d}$

But angle between $\vec{a}$ and $\vec{c}$ is $\pi/3$ $(\because\ \vec{a}.\vec{c} = \frac{1}{2})$



The possible cases are shown in figures and in any case $\vec{b}$ and $\vec{d}$ are non-parallel vectors.

5. **(a)** The volume of a parallelepipe with coterminus edges as the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$V = \left[\vec{a}\vec{b}\vec{c}\right]$

We have $\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

$\Rightarrow V^2 = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = \frac{1}{2} \Rightarrow V = \frac{1}{\sqrt{2}}$

6. **(b)** Since, $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow\quad \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow\quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

$(\because \vec{a} \times \vec{a} = 0$ and $\vec{a} \times \vec{c} = -\vec{c} \times \vec{a})$

Similarly, $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\therefore\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Also since $\vec{a}, \vec{b}, \vec{c}$ are non-parallel and unit vector (these form an equilateral Δ).

$\therefore\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq \vec{0}$

7. **(c)** Since given three vectors are coplanar.

$\therefore\quad \begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$

Applying, $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow\quad \begin{vmatrix} 2-\lambda^2 & 2-\lambda^2 & 2-\lambda^2 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$

$\Rightarrow\quad (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$

$\Rightarrow\quad (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(1+\lambda^2) & 0 \\ 0 & 0 & -(1+\lambda^2) \end{vmatrix} = 0$

$\Rightarrow\quad (2-\lambda^2)(1+\lambda^2)^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}\quad [\because 1+\lambda^2 \neq 0]$

$\therefore$ Two real solutions.

8. **(a)** We know that a vector in the plane of $\vec{a}$ and $\vec{b}$ is

$\vec{u} = \vec{a} + \lambda\vec{b} = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (1+\lambda)\hat{k}$

Given that projection of $\vec{u}$ on $\vec{c} = \frac{1}{\sqrt{3}}$

$\Rightarrow\quad \frac{\vec{u}.\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$

$\Rightarrow\quad \frac{\vec{u}.\vec{c}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow\quad \vec{u}.\vec{c} = 1 \Rightarrow |1+\lambda+2-\lambda-1-\lambda| = 1$

$\Rightarrow\quad |2-\lambda| = 1 \Rightarrow \lambda = 1$ or 3

$\Rightarrow\quad \vec{u} = 2\hat{i} + \hat{j} + 2\hat{k}$ or $4\hat{i} - \hat{j} + 4\hat{k}$

9. **(c)** We know that any vector which is coplanar to $\vec{a}$ and $\vec{b}$ can be written as

$\vec{r} = \vec{a} + \lambda\vec{b} = \left(\hat{i} - \hat{j} + \hat{k}\right) + \lambda\left(2\hat{i} + \hat{j} + \hat{k}\right)$

$\vec{r} = (1+2\lambda)\hat{i} + (-1+\lambda)\hat{j} + (1+\lambda)\hat{k}$

Since $\vec{r}$ is orthogonal to $5\hat{i} + 2\hat{j} + 6\hat{k}$

$\Rightarrow\quad \vec{r}.\left(5\hat{i} + 2\hat{j} + 6\hat{k}\right) = 0$

$\Rightarrow\quad 5(1+2\lambda) + 2(-1+\lambda) + 6(1+\lambda) = 0$

$\Rightarrow\quad 9 + 18\lambda = 0 \Rightarrow \lambda = -\frac{1}{2}$

$\therefore\ \vec{r}$ is $3\hat{j} - \hat{k}$

$\therefore\quad \hat{r} = \frac{3\hat{j} - \hat{k}}{\sqrt{10}}$

10. **(c)** We have $(\vec{a} \times \vec{b}) \times \vec{a} = (\vec{a}.\vec{a})\vec{b} - (\vec{a}.\vec{b})\vec{a}$

$\therefore\quad (\hat{j} - \hat{k}) \times (\hat{i} + \hat{j} + \hat{k}) = (\sqrt{3})^2(\vec{b}) - (\hat{i} + \hat{j} + \hat{k})$

$\Rightarrow 2\hat{i} - \hat{j} - \hat{k} = 3\vec{b} - \left(\hat{i} + \hat{j} + \hat{k}\right)$

$\Rightarrow\quad 3\vec{b} = 3\hat{i} \Rightarrow \vec{b} = \hat{i}$

11. **(c)** Volume of parallelopiped :

$$V=\begin{vmatrix}1 & a & 1\\0 & 1 & a\\a & 0 & 1\end{vmatrix}$$

$=1(1-0)-a(0-a^2)+1(0-a)=1+a^3-a$

$\frac{dV}{da}=3a^2-1$

For max. and min.

$\Rightarrow\ 3a^2-1=0\ \Rightarrow\ a=\pm\frac{1}{\sqrt{3}}$

$\frac{d^2V}{da^2}=6a\Rightarrow\left(\frac{d^2V}{da^2}\right)_{a=\frac{1}{\sqrt{3}}}=\frac{6}{\sqrt{3}}>0$

$\therefore$ V is minimum at $a=\frac{1}{\sqrt{3}}$

12. (c) Given that $\vec{V}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{W}=\hat{i}+3\hat{k}$ and $\vec{U}$ is a unit vector $\therefore\ |\vec{U}|=1$

$$\vec{V}\times\vec{W}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\2 & 1 & -1\\1 & 0 & 3\end{vmatrix}=3\hat{i}-7\hat{j}-\hat{k}$$

Now, $[\vec{U}\vec{V}\vec{W}]=\vec{U}.(\vec{V}\times\vec{W})$

$=\vec{U}.(3\hat{i}-7\hat{j}-\hat{k})=|\vec{U}|\sqrt{3^2+7^2+1^2}\cos\theta$

$=\sqrt{59}\cos\theta$

which is max. when $\cos\theta=1$

$\therefore$ Max. value of $[\vec{U}\vec{V}\vec{W}]=\sqrt{59}$

13. (c) $$[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix}1 & 0 & -1\\x & 1 & 1-x\\y & x & 1+x-y\end{vmatrix}$$

$=1(1+x-y-x+x^2)-1(x^2-y)=1$

$\therefore$ It neither depends on x nor on y.

14. (a) Since $\vec{a},\vec{b},\vec{c}$ are unit coplanar vectors, then their linear combination $\overrightarrow{2a}-\vec{b},\overrightarrow{2b}-\vec{c}$ and $\overrightarrow{2c}-\vec{a}$ are also coplanar vectors.

Thus, $[\overrightarrow{2a}-\vec{b}\ \ \overrightarrow{2b}-\vec{c}\ \ \overrightarrow{2c}-\vec{a}]=0$

15. (a) $\because\ P_1$ is the plane determined by vectors $\vec{a}$ and $\vec{b}$

Let $\overrightarrow{n_1}$ be normal vector of P_1 then $\overrightarrow{n_1}=\vec{a}\times\vec{b}$

Now, P_2 is the plane determined by vectors $\vec{c}$ and $\vec{d}$

Let $\overrightarrow{n_2}$ be normal vector of P_2 then $\overrightarrow{n_2}=\vec{c}\times\vec{d}$

Now given that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$

$\therefore\ \ \overrightarrow{n_1}\times\overrightarrow{n_2}=0\ \Rightarrow\ \overrightarrow{n_1}\parallel\overrightarrow{n_2}$

and hence the planes will also be parallel to each other. Thus angle between the planes = 0.

16. (b) Given that $\vec{a}+\vec{b}+\vec{c}=0$ (by triangle law)

$\therefore\ \ \vec{a}\times(\vec{a}+\vec{b}+\vec{c})=\vec{a}\times\vec{0}=\vec{0}$

$\Rightarrow\ \ \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{0}$

$\Rightarrow\ \ \vec{a}\times\vec{b}=\vec{c}\times\vec{a}\qquad[\because\ \vec{a}\times\vec{a}=0$ and $\vec{a}\times\vec{c}=-\vec{c}\times\vec{a}]$

Similarly, $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$;

Therefore $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

17. (a) Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

Given that $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar

$\therefore\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

$$\Rightarrow\begin{vmatrix}2 & 1 & 1\\1 & 2 & -1\\x & y & z\end{vmatrix}=0\Rightarrow x-y-z=0\qquad ...(i)$$

$\because\ \vec{c}$ is $\perp$ to $\vec{a}$, $\therefore\ \vec{a}\cdot\vec{c}=0$

$2x+y+z=0\qquad ...(ii)$

from (i) and (ii)

$\frac{x}{0}=\frac{y}{3}=\frac{z}{-3}=\lambda\Rightarrow x=0, y=3\lambda$ and $z=-3\lambda$

But $|\vec{c}|=1\Rightarrow x^2+y^2+z^2=1$

$\Rightarrow 9\lambda^2+9\lambda^2=1\Rightarrow\lambda=\pm\frac{1}{3\sqrt{3}}$

$\therefore\vec{c}=\pm\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$

Thus, we have $\vec{c}=\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$

18. (b) $|(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}|\sin 30°$

$=\frac{1}{2}|\vec{a}\times\vec{b}||\vec{c}|\qquad ...(i)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\2 & 1 & -2\\1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}+\hat{k}$$

$\Rightarrow\ \ |\vec{a}\times\vec{b}|=\sqrt{9}=3$

Also given that $|\vec{c}-\vec{a}|=2\sqrt{2}\ \Rightarrow|\vec{c}-\vec{a}|^2=8$

$\Rightarrow\ \ |\vec{c}|^2-\vec{c}.\vec{a}-\vec{a}.\vec{c}+|\vec{a}|^2=8$

$\because\ |\vec{a}|=3$ and $\vec{a}.\vec{c}=|\vec{c}|$, we get

$|\vec{c}|^2-2|\vec{c}|+1=0\ \ (|\vec{c}|-1)^2=0\ \Rightarrow\ |\vec{c}|=1$

Substituting values of $|\vec{a}\times\vec{b}|$ and $|\vec{c}|$ in (i), we get

$|(\vec{a}\times\vec{b})\times\vec{c}|=\frac{1}{2}\times3\times1=\frac{3}{2}$

19. (d) $(\vec{a}+\vec{b}+\vec{c}).[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$

$=(\vec{a}+\vec{b}+\vec{c}).[\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$=(\vec{a}+\vec{b}+\vec{c}).[\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]\qquad[\because\ \vec{a}\times\vec{a}=0]$

$=\vec{a}.\left(\vec{a}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{a}\times\vec{c}\right)$

$+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{c}.(\vec{a}\times\vec{c})+\vec{c}.(\vec{b}\times\vec{a})+\vec{c}.(\vec{b}\times\vec{c})$

$=[\vec{a}\vec{b}\vec{c}]-[\vec{a}\vec{b}\vec{c}]-[\vec{a}\vec{b}\vec{c}]=-[\vec{a}\vec{b}\vec{c}]\ \left[\because [\vec{a}\vec{a}\vec{b}]=0\right]$

20. **(a)** Since $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$

$\therefore (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}=\dfrac{1}{\sqrt{2}}\vec{b}+\dfrac{1}{\sqrt{2}}\vec{c}$

$[\because \vec{b}$ and $\vec{c}$ are non-coplanar$]$

On comparing both sides, we get

$\Rightarrow \vec{a}.\vec{c}=\dfrac{1}{\sqrt{2}}$ and $\vec{a}.\vec{b}=-\dfrac{1}{\sqrt{2}}\Rightarrow \cos\theta=-\dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos\theta=\cos\dfrac{3\pi}{4}\ \Rightarrow \theta=3\pi/4$

21. **(a)** Let $\vec{d}=x\hat{i}+y\hat{j}+z\hat{k}$

Given that $|\vec{d}|=1\Rightarrow\ x^2+y^2+z^2=1$ ....(i)

and $\vec{a}.\vec{d}=0$

$\Rightarrow x-y=0\Rightarrow x=y$ ...(ii)

Now, $\left[\vec{b}\,\vec{c}\,\vec{d}\right]=0\Rightarrow\begin{vmatrix}0&1&-1\\-1&0&1\\x&y&z\end{vmatrix}=0$

$\Rightarrow x+y+z=0$

$\Rightarrow 2x+z=0$ (from (i))

$\Rightarrow z=-2x$ ...(iii)

From (i), (ii) and (iii)

$x^2+x^2+4x^2=1\Rightarrow x=\pm\dfrac{1}{\sqrt{6}}$

$\therefore d=\pm\left(\dfrac{1}{\sqrt{6}}\vec{i}+\dfrac{1}{\sqrt{6}}\vec{j}-\dfrac{2}{\sqrt{6}}\vec{k}\right)=\pm\left(\dfrac{\vec{i}+\vec{j}-2\vec{k}}{\sqrt{6}}\right)$

22. **(b)** Given that a, b, c are distinct non negative numbers and the vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar.

$\therefore\begin{vmatrix}a&a&c\\1&0&1\\c&c&b\end{vmatrix}=0$

Applying $C_3\to C_3-C_1$

$\Rightarrow\begin{vmatrix}a&a&c-a\\1&0&0\\c&c&b-c\end{vmatrix}=0$

Expanding along R_2, we get

$=[c\,(c-a)-a\,(b-c)]=0$

$\Rightarrow c^2-ac-ab+ac=0$

$\Rightarrow c^2=ab\Rightarrow\ a, c, b$ are in G.P.

$\therefore$ c is the G.M. of a and b.

23. **(d)** Given that $\vec{a},\vec{b},\vec{c}$ are non coplanar

$\therefore\left[\vec{a}\,\vec{b}\,\vec{c}\right]\neq 0$

And $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Now, $(\vec{a}+\vec{b}).\vec{p}+(\vec{b}+\vec{c}).\vec{q}+(\vec{c}+\vec{a}).\vec{r}$

$=(\vec{a}+\vec{b}).\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}+(\vec{b}+\vec{c}).\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}+(\vec{c}+\vec{a}).\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$

$=\dfrac{\vec{a}.(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{\vec{b}.(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{\vec{c}.(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$[\because \vec{b}.\vec{b}\times\vec{c}=\vec{c}.\vec{c}\times\vec{a}=\vec{a}.\vec{a}\times\vec{b}=0]$

$=\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=1+1+1=3$

24. **(d)** Volume of parallelopiped $=\left[\overline{OA}\,\overline{OB}\,\overline{OC}\right]$

$=\begin{vmatrix}2&-2&0\\1&1&-1\\3&0&-1\end{vmatrix}=2(-1)+2(-1+3)=2$

25. **(d)** $|(\vec{a}\times\vec{b}).\vec{c}|=|\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow|\hat{a}||\hat{b}||\sin\theta\,\hat{n}.\hat{c}|=|\hat{a}||\hat{b}||\hat{c}|$

where θ is angle between $\vec{a}$ and $\vec{b}$.

$\Rightarrow\quad|\sin\theta\hat{n}\cdot\vec{c}|=|\vec{c}|\ \Rightarrow|\hat{c}|\ |\sin\theta\cos\alpha|=|\hat{c}|$

where α is angle between $\vec{c}$ and $\hat{n}$.

$\Rightarrow\quad|\sin\theta|=1\Rightarrow\theta=\dfrac{\pi}{2}\Rightarrow\vec{a}\cdot\vec{b}=0$

and $|\cos\alpha|=1\Rightarrow\alpha=0\Rightarrow\vec{c}\,||\,\vec{n}\Rightarrow\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

$\Rightarrow\quad\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$

26. **(a)** $\vec{A}.(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$

$=\vec{A}.[\vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B}+\vec{C}\times\vec{C}]$

$(\because\vec{A}\times\vec{A}=0)$

$=\vec{A}\cdot(\vec{B}\times\vec{A}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B})$

$=\vec{A}.\vec{B}\times\vec{A}+\vec{A}.\vec{B}\times\vec{C}+\vec{A}.\vec{C}\times\vec{A}+\vec{A}.\vec{C}\times\vec{B}$

$(\because\vec{A}\cdot(\vec{B}\times\vec{A})=0)$

$=0+[ABC]+0+[ACB]$

$=\left[\overline{ABC}\right]-\left[\overline{ABC}\right]$ $\quad(\because[\vec{a}\vec{b}\vec{c}]=-[\vec{a}\vec{c}\vec{b}])$

$=0$

27. **(7)** Given that

$|\vec{u}|=1;\ |\vec{v}|=1;\ \vec{u}.\vec{v}\neq0;\ \vec{u}.\vec{w}=1;\ \vec{v}.\vec{w}=1$

and $\vec{w}.\vec{w}=|\vec{w}|^2=4\Rightarrow|\vec{w}|=2$

Volume of parallelopiped = $[\vec{u}\ \vec{v}\ \vec{w}] = \sqrt{2}$

$$\Rightarrow [\vec{u}\ \vec{v}\ \vec{w}]^2 = \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{v} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{v} & \vec{v}.\vec{w} \\ \vec{w}.\vec{u} & \vec{w}.\vec{v} & \vec{w}.\vec{w} \end{vmatrix} = 2$$

$$\Rightarrow \begin{vmatrix} 1 & \vec{u}.\vec{v} & 1 \\ \vec{u}.\vec{v} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2 \Rightarrow \vec{u}.\vec{v} = \frac{1}{2}$$

Now, $|3\vec{u}+5\vec{v}|^2 = (3\vec{u}+5\vec{v})(3\vec{u}+5\vec{v})$

$= 9|\vec{u}|^2 + 25|\vec{v}|^2 + 30\vec{u}.\vec{v}$

$= 9 + 25 + 15 = 49$

$\therefore\ |3\vec{u}+5\vec{v}| = \sqrt{49} = 7$

28. (3) Given that $|\vec{a}| = |\vec{b}| = 1$, $\vec{a}.\vec{b} = 0$ and $|\vec{c}| = 2$

$\vec{c}$ makes angle α with both $\vec{a}$ and $\vec{b}$

Also, $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$

$\vec{c}.\vec{a} = |\vec{c}||\vec{a}|\cos\alpha = 2\cos\alpha \Rightarrow x = 2\cos\alpha$

$\vec{c}.\vec{b} = 2\cos\alpha \Rightarrow y = 2\cos\alpha$

$|\vec{c}|^2 = \vec{c}.\vec{c} = \left|(2\cos\alpha)\vec{a} + (2\cos\alpha)\vec{b} + \vec{a}\times\vec{b}\right|^2$

$\Rightarrow (2)^2 = 4\cos^2\alpha + 4\cos^2\alpha + |\vec{a}\times\vec{b}|^2$

$\Rightarrow 4 = 8\cos^2\alpha + 1 \quad (\because |\vec{a}\times\vec{b}| = 1\times1\times\sin 90^\circ = 1)$

$\Rightarrow 8\cos^2\alpha = 3$

29. (9) $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$ ...(i)

$\vec{s} = x(-\vec{p}+\vec{q}+\vec{r}) + y(\vec{p}-\vec{q}+\vec{r}) + z(-\vec{p}-\vec{q}+\vec{r})$ ...(ii)

comparing (i) and (ii) we get

$\Rightarrow -x+y-z=4$

$x-y-z=3$

$x+y+z=5$

Solving above equations, we get $x=4,\ y=\frac{9}{2},\ z=\frac{-7}{2}$

$\therefore\ 2x+y+z=9$

30. (4) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = \cos\frac{\pi}{3} = \frac{1}{2}$

Given $p\vec{a} + q\vec{b} + r\vec{c} = \vec{a}\times\vec{b} + \vec{b}\times\vec{c}$

Taking its dot product with $\vec{a}, \vec{b}, \vec{c}$, we get

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = p|\vec{a}|^2 + q\left(\vec{b}\cdot\vec{a}\right) + r\left(\vec{c}\cdot\vec{a}\right)$

$= p + \frac{1}{2}q + \frac{1}{2}r$ ...(i)

Given that $\frac{1}{2}p + q + \frac{1}{2}r = 0$ ...(ii)

and $\frac{1}{2}p + \frac{1}{2}q + r = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$ ...(iii)

From (i) and (iii), $p = r$ Using (ii) $q = -p$

$\therefore\ \frac{p^2+2q^2+r^2}{q^2} = \frac{p^2+2p^2+p^2}{p^2} = 4$

31. (9) $\because \vec{r}\times\vec{b} = \vec{c}\times\vec{b} \Rightarrow \vec{r}\times\vec{b} - \vec{c}\times\vec{b} = \vec{0}$

$\Rightarrow (\vec{r}-\vec{c})\times\vec{b} = \vec{0} \Rightarrow \vec{r}-\vec{c} \parallel \vec{b}$

Let $\vec{r}-\vec{c} = \lambda\vec{b} \Rightarrow \vec{r} = \vec{c} + \lambda\vec{b}$

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} - \lambda\hat{i} + \lambda\hat{j}$

$= (1-\lambda)i + (2+\lambda)\hat{j} + 3\hat{k}$

$\therefore\ \vec{r}.\vec{a} = 0 \Rightarrow -1+\lambda-3 = 0 \Rightarrow \lambda = 4$

$\therefore\ \vec{r} = -3\hat{i} + 6\hat{j} + 3\hat{k}$

Now, $\vec{r}.\vec{b} = 3 + 6 = 9$

32. (5) We have $|\vec{a}| = 1\ |\vec{b}| = 1$ and $\vec{a}.\vec{b} = 0$

Now, $\left(2\vec{a}+\vec{b}\right).\left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-2\vec{b}\right)\right]$

$= \left(2\vec{a}+\vec{b}\right).\left[\left(\vec{a}\cdot\vec{a}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{a} - 2\left(\vec{b}\cdot\vec{a}\right)\vec{b} + 2\left(\vec{b}\cdot\vec{b}\right)\vec{a}\right]$

$= \left(2\vec{a}+\vec{b}\right).\left[\left(\vec{a}\right)^2\vec{b} + 2\left(\vec{b}\right)^2\vec{a}\right]$

$= \left(2\vec{a}+\vec{b}\right).\left(2\vec{a}+\vec{b}\right) = 4|\vec{a}|^2 + |\vec{b}|^2 \quad \left(\because \vec{a}.\vec{b} = 0\right)$

$= 4 + 1 = 5.$

33. (18) Given that $\vec{a} = 2\hat{i} + \hat{j} - \hat{k},\ \vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

$\Rightarrow \vec{a}+\vec{b} = 3\hat{i} + 3\hat{j}$

Given that projection of $\vec{c}$ on $\vec{a}+\vec{b} = 3\sqrt{2}$

$\Rightarrow \frac{\vec{c}.\left(\vec{a}+\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|} = 3\sqrt{2} \Rightarrow \frac{3(2\alpha+\beta)+3(\alpha+2\beta)}{3\sqrt{2}} = 3\sqrt{2}$

$\Rightarrow \alpha+\beta = 2$ ....(i)

$\vec{c} = \alpha\vec{a} + \beta\vec{b} = (2\alpha+\beta)\hat{i} + (\alpha+2\beta)\hat{j} + (-\alpha+\beta)\hat{k}$

$\vec{c} = (\alpha+2)\hat{i} + (4-\alpha)\hat{j} + (2-2\alpha)\hat{k}$ Using eq^n^ (i)

Now $\vec{c}$ is in the plane of $\vec{a}$ and $\vec{b}$ $\left(\because \vec{c} = \alpha\vec{a} + \beta\vec{b}\right)$

$\therefore \left(\vec{a}\times\vec{b}\right).\vec{c} = 0$

Hence $\left(\vec{c} - \left(\vec{a}\times\vec{b}\right)\right).\vec{c} = \vec{c}.\vec{c}$

$= (\alpha+2)^2 + (4-\alpha)^2 + (2-2\alpha)^2 = 6\left(\alpha^2 - 2\alpha + 4\right)$

$= 6\left((\alpha-1)^2 + 3\right)$

which has minimum value as 18 when $\alpha = 1$

34. Given that q = area of parallelogram with $\overline{OA}$ and $\overline{OC}$ as adjacent sides $= \left|\overline{OA}\times\overline{OC}\right| = \left|\vec{a}\times\vec{b}\right|$

and p = area of quadrilateral $OABC$

$$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| + \frac{1}{2}\left|\overrightarrow{OB} \times \overrightarrow{OC}\right|$$

$$= \frac{1}{2}\left|\vec{a} \times \left(\overrightarrow{10a} + \overrightarrow{2b}\right)\right| + \frac{1}{2}\left|\left(\overrightarrow{10a} + \overrightarrow{2b}\right) \times \vec{b}\right|$$

$$= \left|\vec{a} \times \vec{b}\right| + 5\left|\vec{a} \times \vec{b}\right| = 6\left|\vec{a} \times \vec{b}\right| \quad \therefore p = 6q \Rightarrow k = 6$$

35. Let $\vec{b}$ and $\vec{c}$ be unit vector along $\hat{i}$ and $\hat{j}$ respectively

i.e. $\vec{b} = \hat{i}$ and $\vec{c} = \hat{j}$ then $\vec{b} \times \vec{c} = \hat{k}$

Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \vec{a} \cdot \vec{b} = x, \vec{a} \cdot \vec{c} = y$ and $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = k$. Then,

$$\left(\vec{a}.\vec{b}\right)\vec{b} + (\vec{a}.\vec{c})\vec{c} + \frac{\vec{a}.\left(\vec{b} \times \vec{c}\right)}{|\vec{b} \times \vec{c}|}(\vec{b} \times \vec{c}) = x\hat{i} + y\hat{j} + z\hat{k} = \hat{a}$$

36. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector, coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$.

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow -3x + y + z = 0 \quad .....(i)$$

also $\vec{a}$ is perpendicular to $\hat{i} + \hat{j} + \hat{k}$

$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) = 0$

$\Rightarrow x + y + z = 0$ .....(ii)

Solving the above eqns., we get

$$\frac{x}{0} = \frac{y}{4} = \frac{z}{-4} \text{ or } \frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = \lambda(say)$$

$\Rightarrow x = 0, y = \lambda, z = -\lambda \Rightarrow \vec{a} = \lambda\hat{j} - \lambda\hat{k} = 0$

Given that $\vec{a}$ is a unit vector, therefore

$$0 + \lambda^2 + \lambda^2 = 1 \Rightarrow \lambda^2 = \frac{1}{2} \Rightarrow \lambda = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ The required vector is $\frac{\hat{j} - \hat{k}}{\sqrt{2}}$ or $\frac{-\hat{j} + \hat{k}}{\sqrt{2}}$

37. Given that the vectors $\vec{a} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + c\hat{k}$ where a ≠ b ≠ c ≠1 are coplanar

$$\therefore \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$

$$\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$$

Taking $(1 - a)$, $(1 - b)$, $(1 - c)$ common from R_1, R_2 and R_3 respectively, we get

$$\Rightarrow (1-a)(1-b)(1-c)\begin{vmatrix} -1 & 0 & \frac{1}{1-a} \\ 1 & -1 & \frac{1}{1-b} \\ 0 & 1 & \frac{c}{1-c} \end{vmatrix} = 0$$

Applying $R_2 \to R_2 + R_1$

$$\Rightarrow (1-a)(1-b)(1-c)\begin{bmatrix} -1 & 0 & \frac{1}{1-a} \\ 0 & -1 & \frac{1}{1-b} + \frac{1}{1-a} \\ 0 & 1 & \frac{c}{1-c} \end{bmatrix} = 0$$

$$\Rightarrow (1-a)(1-b)(1-c)(-1)\left[-\frac{c}{1-c} - \frac{1}{1-b} - \frac{1}{1-a}\right] = 0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a} + \frac{1}{1-b} + \frac{c}{1-c}\right] = 0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a} + \frac{1}{1-b} - \frac{(1-c)-1}{1-c}\right] = 0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} - 1\right] = 0$$

But $a \neq b \neq c \neq 1$

$$\therefore \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} - 1 = 0 \Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

38. Position vetor of $\vec{A}$ and $\vec{C}$ are $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\overline{C} = \hat{j} - \hat{k}$ respectively

Let $\overline{B} = x\hat{i} + y\hat{j} + z\hat{k}$

$$ATQ, \quad \overline{A} \times \overline{B} = \overline{C} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$$

$\Rightarrow (z - y)\ \hat{i} - (z - x)\ \hat{j} + (y - x)\ \hat{k} = \hat{j} - \hat{k}$

$z - y = 0. \quad \Rightarrow y = z$ ...(i)

$\Rightarrow x - z = 1 \quad \Rightarrow x = z = 1$ ...(ii)

and $y - x = -1$

Also, $\overline{A}.\overline{B} = 3 \Rightarrow x + y + z = 3$ ...(iii)

From eqs. (i), (ii) and (iii), we get

$1 + z + z + z = 3$

$\Rightarrow z = 2/3 \Rightarrow y = 2/3, x = 5/3$

$$\therefore \overline{B} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

39. Since $\vec{A}, \vec{B}, \vec{C}$ are three non-coplanar vectors, therefore, $\left[\vec{A}\vec{B}\vec{C}\right] \neq 0$

$$\therefore \frac{\vec{A}.(\vec{B} \times \vec{C})}{(\vec{C} \times \vec{A}).\vec{B}} + \frac{\vec{B}.(\vec{A} \times \vec{C})}{(\vec{C} \times \vec{A}).\vec{B}} = \frac{\left[\vec{A}\,\vec{B}\,\vec{C}\right]}{\left[\vec{C}\,\vec{A}\,\vec{B}\right]} + \frac{\left[\vec{B}\,\vec{A}\,\vec{C}\right]}{\left[\vec{C}\,\vec{A}\,\vec{B}\right]}$$

$$= \frac{\left[\vec{A}\,\vec{B}\,\vec{C}\right]}{\left[\vec{A}\,\vec{B}\,\vec{C}\right]} + \frac{-\left[\vec{A}\,\vec{B}\,\vec{C}\right]}{\left[\vec{A}\,\vec{B}\,\vec{C}\right]} = 0$$

40. $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ (Given)

$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Apply $C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$ in first determinant, we get

$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

But given that the vectors $\vec{A}, \vec{B}, \vec{C}$ are non-coplanar

$\therefore 1 + abc = 0 \Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$

$\Rightarrow abc = -1$

41. $\because \overrightarrow{BA} = -\hat{i} - 2\hat{j} + 3\hat{k}$, $\overrightarrow{BC} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$

$\Delta = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = \frac{1}{2}\left|6\hat{j} + 4\hat{k}\right| = |3\hat{j} + 2\hat{k}| = \sqrt{9+4} = \sqrt{13}$

42. **False:** L.H.S. $= (\vec{a} - \vec{b}).(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})$

$= (\vec{a} - \vec{b}).(\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{c} \times \vec{c} + \vec{c} \times \vec{a})$

$= (\vec{a} - \vec{b}).(\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a})$ $\quad [\because \vec{c} \times \vec{c} = 0]$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{a} \times \vec{b}) + \vec{a}.(\vec{c} \times \vec{a})$

$\quad - \vec{b}.(\vec{b} \times \vec{c}) - \vec{b}.(\vec{a} \times \vec{b}) - \vec{b}.(\vec{c} \times \vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}] - \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0 \neq$ R.H.S. $\quad \left[\because \left[\vec{a}.\vec{a}\vec{b}\right] = 0\right]$

$\therefore$ The given statement is false.

43. **True:** $\because \overrightarrow{X}.\overrightarrow{A} = 0 \Rightarrow$ either $\overrightarrow{A} = 0$ or $\overrightarrow{X} \perp \overrightarrow{A}$ ...(i)

$\overrightarrow{X}.\overrightarrow{B} = 0 \Rightarrow$ either $\overrightarrow{B} = 0$ or $\overrightarrow{X} \perp \overrightarrow{B}$ ...(ii)

$\overrightarrow{X}.\overrightarrow{C} = 0 \Rightarrow$ either $\overrightarrow{C} = 0$ or $\overrightarrow{X} \perp \overrightarrow{C}$ ...(iii)

From (i), (ii) and (iii)

if $\overrightarrow{A}$ or $\overrightarrow{B}$ or $\overrightarrow{C} = 0 \Rightarrow \left[\overrightarrow{A}\,\overrightarrow{B}\,\overrightarrow{C}\right] = 0$

Otherwise if $\overrightarrow{X} \perp \overrightarrow{A}, \overrightarrow{X} \perp \overrightarrow{B}, \overrightarrow{X} \perp \overrightarrow{C}$ then $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}$ are coplanar $\Rightarrow \left[\overrightarrow{A}\,\overrightarrow{B}\,\overrightarrow{C}\right] = 0$

$\therefore$ Given statement is true.

44. **True:** Given that $\overrightarrow{A}.\overrightarrow{B} = \overrightarrow{A}.\overrightarrow{C} = 0$

$\therefore \overrightarrow{A}$ is perpendicular to both $\overrightarrow{B}$ and $\overrightarrow{C}$.

$\Rightarrow \overrightarrow{B} \times \overrightarrow{C} = \lambda\overrightarrow{A}$ where λ is any scalar.

$\Rightarrow |\overrightarrow{B} \times \overrightarrow{C}| = |\lambda\overrightarrow{A}| \Rightarrow \sin \pi/6 = \pm\lambda$ $\quad (\because |B| = |C| = |A| = 1)$

($\because \pi/6$ is the angle between $\overrightarrow{B}$ & $\overrightarrow{C}$)

$\Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow \overrightarrow{B} \times \overrightarrow{C} = \pm\frac{1}{2}\overrightarrow{A} \Rightarrow \overrightarrow{A} = \pm 2\left(\overrightarrow{B} \times \overrightarrow{C}\right)$

$\therefore$ Given statement is true.

45. **(b, c, d)** $\begin{bmatrix} 0 & -C_3 & C_2 \\ C_3 & 0 & -C_1 \\ -C_2 & C_1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 - C_1 \\ 1 - C_2 \\ -1 - C_3 \end{bmatrix}$

$\Rightarrow -b_2C_3 + b_3C_2 = 3 - C_1$ .... (i)

$\Rightarrow C_3 - C_1b_3 = 1 - C_2$ ...... (ii)

$\Rightarrow -C_2 + b_2C_1 = -1 - C_3$ ...... (iii)

Applying (i) $\hat{i} - (2)\hat{j} + (3)\hat{k}$, we get

$i(b_2c_3 - b_3c_2) - \hat{j}(c_3 - b_3c_1) + \hat{k}(c_2 - b_2c_1)$

$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} - 3\hat{i} - \hat{j} + \hat{k} \Rightarrow \vec{b} \times \vec{c} = \vec{c} - \vec{a}$

$\Rightarrow \vec{b}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} - \vec{a}) \Rightarrow 0 = \vec{b}.\vec{c} - \vec{b}.\vec{a}$

$\Rightarrow \vec{b}.\vec{c} = 0 \quad [\because \vec{a}.\vec{b} = 0]$

$\because \vec{c}.(\vec{b} \times \vec{c}) = \vec{c}.(\vec{c} - \vec{a})$

$\Rightarrow 0 = |\vec{c}|^2 - \vec{c}.\vec{a} \Rightarrow \vec{c}.\vec{a} = |\vec{c}|^2$

$\because \vec{a}.\vec{c} \neq 0$

$(\vec{b} \times \vec{c})^2 = (\vec{c} - \vec{a})^2$

$|\vec{b}|^2|\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a}\ [\because \vec{b} \perp \vec{c}]$

$|\vec{b}|^2|\vec{c}|^2 = |\vec{c}|^2 + 11 - 2[\vec{c}]^2$

$\Rightarrow |\vec{c}|^2 = \frac{11}{|\vec{b}|^2 + 1} \Rightarrow |\vec{c}| \leq \sqrt{11}$

Given that $\vec{a}.\vec{b} = 0 \Rightarrow b_2 - b_3 + 3 = 0 \Rightarrow b_3 - b_2 = 3$

Also $b_2.b_3 > 0$

Now, $|\vec{b}|^2 = 1 + b_2^2 + b_3^2$

$= 1 + (b_3 - b_2)^2 + 2b_2b_3$

$= 10 + 2\,b_2b_3$

$\Rightarrow |\vec{b}|^2 > 10 \Rightarrow |\vec{b}| > \sqrt{10}$

46. **(a,b,c)** $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{1}{2}\overrightarrow{OB} \times (\overrightarrow{OB} - \lambda\overrightarrow{OA})$

$= \frac{\lambda}{2}(\overrightarrow{OA} \times \overrightarrow{OB})$ ...(i)

Since, $\overline{OA}\cdot\overline{OB}=0$ therfore $\overline{OA}\perp^r\overline{OB}$

$$\therefore \quad \left|\overline{OB}\times\overline{OC}\right|=\left|\frac{\lambda}{2}(\overline{OA}\times\overline{OB})\right| \qquad \text{[from (i)]}$$

$$=\frac{|\lambda|}{2}|\overline{OA}||\overline{OB}|\sin\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{9}{2}=\frac{|\lambda|}{2}\times 3\times 3 \Rightarrow \lambda=1 \text{ (given } \lambda>0)$$

$$\text{So, } \overline{OC}=\frac{\overline{OB}-\overline{OA}}{2}=\frac{1}{2}\left(-\hat{i}-4\hat{j}+\hat{k}\right)$$

(a) Projection $\overline{OC}$ on $\overline{OA}$

$$=\frac{\overline{OC}\cdot\overline{OA}}{|\overline{OA}|}=\frac{\frac{1}{2}(-2-8+1)}{3}=-\frac{3}{2}$$

(b) Area of $\Delta\, OAB=\frac{1}{2}\left|\overline{OA}\times\overline{OB}\right|=\frac{9}{2}$

(c) Area of $\Delta\, ABC$

$$=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|=\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -4 & 1\\ \frac{-5}{2} & -4 & -\frac{1}{2}\end{matrix}\right\|$$

$$=\frac{1}{2}\left|6\hat{i}-3\hat{j}-6\hat{k}\right|=\frac{9}{2}$$

(d) Let acute angle between diagonals is θ then

$$\cos\theta=\frac{\left(\overline{OA}+\overline{OC}\right).\left(\overline{OA}-\overline{OC}\right)}{\left|\overline{OA}+\overline{OC}\right|\left|\overline{OA}-\overline{OC}\right|}=\frac{18}{3\sqrt{2}\sqrt{90}}$$

$$\therefore \quad \theta\neq\frac{\pi}{3}$$

47. (b, c, d) $\begin{bmatrix}0 & -C_3 & C_2\\ C_3 & 0 & -C_1\\ -C_2 & C_1 & 0\end{bmatrix}\begin{bmatrix}1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix}3-C_1\\ 1-C_2\\ -1-C_3\end{bmatrix}$

$\Rightarrow -b_2C_3+b_3C_2=3-C_1$ .... (i)

$\Rightarrow C_3-C_1b_3=1-C_2$ .... (ii)

$\Rightarrow -C_2+b_2C_1=-1-C_3$ .... (iii)

Applying (i) $\hat{i}-(2)\hat{j}+(3)\hat{k}$, we get

$i(b_2c_3-b_3c_2)-\hat{j}(c_3-b_3c_1)+\hat{k}(c_2-b_2c_1)$

$=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}-3\hat{i}-\hat{j}+\hat{k}\Rightarrow\vec{b}\times\vec{c}=\vec{c}-\vec{a}$

$\Rightarrow\vec{b}.(\vec{b}\times\vec{c})=\vec{b}.(\vec{c}-\vec{a})\Rightarrow 0=\vec{b}.\vec{c}-\vec{b}.\vec{a}$

$\Rightarrow\vec{b}.\vec{c}=0 \quad [\because \vec{a}.\vec{b}=0]$

$\because \vec{c}.(\vec{b}\times\vec{c})=\vec{c}.(\vec{c}-\vec{a})\Rightarrow 0=|\vec{c}|^2-\vec{c}.\vec{a}\Rightarrow\vec{c}.\vec{a}=|\vec{c}|^2$

$\because \vec{a}.\vec{c}\neq 0$

$(\vec{b}\times\vec{c})^2=(\vec{c}-\vec{a})^2$

$|\vec{b}|^2|\vec{c}|^2=|\vec{c}|^2+|\vec{a}|^2-2\vec{c}.\vec{a}\,[\because \vec{b}\perp\vec{c}]$

$|\vec{b}|^2|\vec{c}|^2=|\vec{c}|^2+11-2[\vec{c}]^2$

$\Rightarrow|\vec{c}|^2=\frac{11}{|\vec{b}|^2+1}\Rightarrow|\vec{c}|\leq\sqrt{11}$

Given that $\vec{a}.\vec{b}=0\Rightarrow b_2-b_3+3=0\Rightarrow b_3-b_2=3$

Also $b_2.b_3>0$

Now, $|\vec{b}|^2=1+b_2^2+b_3^2$

$=1+(b_3-b_2)^2+2b_2b_3$

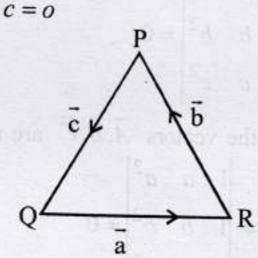
$=10+2\,b_2b_3$

$\Rightarrow|\vec{b}|^2>10\Rightarrow|\vec{b}|>\sqrt{10}$

48. (a, c, d)

From given information

$$\vec{a}+\vec{b}+\vec{c}=\vec{o}$$



$$\Rightarrow \quad \left|\vec{b}+\vec{c}\right|^2=\left|-\vec{a}\right|^2 \Rightarrow \left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\vec{b}.\vec{c}=\left|\vec{a}\right|^2$$

$$\Rightarrow \quad 48+\left|\vec{c}\right|^2+48=144\Rightarrow\left|\vec{c}\right|^2=48\Rightarrow\left|\vec{c}\right|=4\sqrt{3}$$

$$\therefore \quad \frac{\left|\vec{c}\right|^2}{2}-\left|\vec{a}\right|=\frac{48}{2}-12=12$$

∴ (a) is correct

$$\frac{\left|\vec{c}\right|^2}{2}+\left|\vec{a}\right|=24\neq 30$$

∴ (b) is not correct

Also $\left|\vec{b}\right|=\left|\vec{c}\right| \Rightarrow \angle Q=\angle R$

and $\cos(180-P)=\frac{\vec{b}.\vec{c}}{\left|\vec{b}\right|\left|\vec{c}\right|}=\frac{1}{2}$

$\Rightarrow \quad \angle P=120^\circ \therefore \angle Q=\angle R=30^\circ$

And $\vec{a}.\vec{b}=12\times 4\sqrt{3}\times\cos 150=-72$

∴ (d) is correct

Now, $\vec{a}\times(\vec{a}+\vec{b}+\vec{c})=\vec{0}\Rightarrow\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

$\therefore \quad \left|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right|=2\left|\vec{a}\times\vec{b}\right|=2\times 12\times 4\sqrt{3}\times\sin 150=48\sqrt{3}$

∴ (c) is correct.

49. (a, b, c) Given that $|\vec{x}|=|\vec{y}|=|\vec{z}|=\sqrt{2}$

and angle between each pair is $\frac{\pi}{3}$.

Since $\vec{a}$ is perpendicular to both $\vec{x}$ and $\vec{y}\times\vec{z}$.

$$\therefore \vec{a} = \lambda\left[\vec{x}\times\left(\vec{y}\times\vec{z}\right)\right] = \lambda\left[\left(\vec{x}\cdot\vec{z}\right)\vec{y} - \left(\vec{x}\cdot\vec{y}\right)\vec{z}\right]$$

$$= \lambda\left[\left(\sqrt{2}.\sqrt{2}\cos\frac{\pi}{3}\right)\vec{y} - \left(\sqrt{2}.\sqrt{2}\cos\frac{\pi}{3}\right)\vec{z}\right]$$

$$= \lambda\left(\vec{y}-\vec{z}\right)$$

Since $\vec{b}$ is per pendicular to both y and $z \times x$.

$$\vec{b} = \mu\left[\vec{y}\times\left(\vec{z}\times\vec{x}\right)\right] = \mu\left[\left(\vec{y}.\vec{x}\right)\vec{z} - \left(\vec{y}.\vec{z}\right)\vec{x}\right]$$

$$= \mu\left[\left(\sqrt{2}.\sqrt{2}.\cos\frac{\pi}{3}\right)\vec{z} - \left(\sqrt{2}.\sqrt{2}.\cos\frac{\pi}{3}\right)\vec{x}\right]$$

$$= \mu\left(\vec{z}-\vec{x}\right)$$

Now, $\vec{b}.\vec{z} = \mu\left[\vec{z}.\vec{z} - \vec{x}.\vec{z}\right] = \mu(2-1) = \mu$

$$\therefore \vec{b} = \left(\vec{b}.\vec{z}\right)\left(\vec{z}-\vec{x}\right)$$

$\therefore$ (a) is correct

Also $\vec{a}.\vec{y} = \lambda\left(\vec{y}.\vec{y} - \vec{z}.\vec{y}\right) = \lambda(2-1) = \lambda$

$$\therefore \vec{a} = \left(\vec{a}.\vec{y}\right)\left(\vec{y}-\vec{z}\right)$$

$\therefore$ (b) is also correct

$$\vec{a}.\vec{b} = \lambda\mu\left(\vec{y}.\vec{z} - \vec{y}.\vec{x} - \vec{z}.\vec{z} + \vec{z}.\vec{x}\right)$$

$$= \lambda\mu(1-1-2+1) = -\lambda\mu = -\left(\vec{a}.\vec{y}\right)\left(\vec{b}.\vec{z}\right)$$

$\therefore$ (c) is correct.

$$-\left(\vec{a}.\vec{y}\right)\left(\vec{z}-\vec{y}\right) = \lambda\left(\vec{z}-\vec{y}\right) = -\vec{a}$$

(d) is not correct.

50. **(a,d)** Let $\vec{a} = \hat{i}+\hat{j}+2\hat{k},\ \vec{b} = \hat{i}+2\hat{j}+\hat{k},\ \vec{c} = \hat{i}+\hat{j}+\hat{k}$

$\because$ Vector coplanar with $\vec{a}$ and $\vec{b}$ is

$\Rightarrow \vec{r} = \vec{a}+\lambda\vec{b}$

$\Rightarrow \vec{r} = (1+\lambda)\ \hat{i} + (1+2\lambda)\ \hat{j} + (2+\lambda)\ \hat{k}$

$\because \vec{r}\perp\vec{c} \Rightarrow \vec{r}.\vec{c} = 0$

$\Rightarrow 1+\lambda+1+2\lambda+2+\lambda = 0 \Rightarrow \lambda+\mu = 0 \Rightarrow \lambda = -\mu$

$\Rightarrow 4\lambda+4 = 0$

$\Rightarrow \lambda = -1$

Any scalar multiple of $\vec{r}$ is also soluiton.

$\therefore$ a and d are the correct options.

51. **(a, c)** Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\vec{v} = \vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n} = \sin\theta\hat{n}$$

[$\because \vec{a}$ and $\vec{b}$ are unit vectors.]

$\therefore \quad |\vec{v}| = \sin\theta$ ...(i)

Now, $\vec{u} = \vec{a} - (\vec{a}.\vec{b})\vec{b} = \vec{a} - \vec{b}\cos\theta$

(where $\vec{a}.\vec{b} = \cos\theta$)

$\therefore \quad |\vec{u}|^2 = |\vec{a} - \vec{b}\cos\theta|^2 = 1+\cos^2\theta - 2\cos\theta.\cos\theta$

$= 1-\cos^2\theta = \sin^2\theta = |v|^2 \Rightarrow |\vec{u}| = |\vec{v}|$ (from (i))

Also, $\vec{u}.\vec{b} = \vec{a}.\vec{b} - (\vec{a}.\vec{b})(\vec{b}.\vec{b}) = \vec{a}.\vec{b} - \vec{a}.\vec{b} = 0,$

$\therefore |\vec{u}.\vec{b}| = 0$

So, $|\vec{v}| = |\vec{u}| + |\vec{u}.\vec{b}|$ is also correct

Similarly, $|\vec{u}| + |\vec{u}.\vec{a}| \neq |\vec{v}|$

and $|\vec{u}| + |\vec{u}||\vec{a}+\vec{b}| \neq |\vec{v}|$

52. **(a, c)** We know that dot product of two vectors gives a scalar quantity.

53. **(c)** We know that $[\vec{u}\,\vec{v}\,\vec{w}] = [\vec{v}\,\vec{w}\,\vec{u}] = [\vec{w}\,\vec{u}\,\vec{v}]$

but $[\vec{v}\,\vec{u}\,\vec{w}] = -[\vec{u}\,\vec{v}\,\vec{w}]$

54. **(b)** Vector perpendicular to vector $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \lambda\left(\vec{a}\times\vec{b}\right)$$

$$\therefore \hat{n} = \frac{\lambda\left(\vec{a}\times\vec{b}\right)}{|\lambda|\left|\vec{a}\times\vec{b}\right|} = \pm\frac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}$$

$\therefore$ We have two possible values of $\hat{n}$

55. **(c)** Since 'c' is unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. So, $\vec{c} \parallel \vec{a}\times\vec{b}$

$$\text{Then } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\,\vec{b}\,\vec{c}]^2 = (\vec{a}\times\vec{b}.\vec{c})^2$$

$$= (|\vec{a}\times\vec{b}|.1\cos 0°)^2$$

$$= (|\vec{a}\times\vec{b}|)^2 = \left(|\vec{a}||\vec{b}|.\sin\frac{\pi}{6}\right)^2$$

[$\because$ angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$]

$$= \left(\frac{1}{2}\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\right)^2$$

$$= \frac{1}{4}\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)$$

56. **(a)** $P \to 4$: $y = \cos\left(3\cos^{-1}x\right)$

$$y = \cos\left[\cos^{-1}\left(4x^3-3x\right)\right]$$

$$y = 4x^3-3x$$

$$\Rightarrow \frac{dy}{dx} = 12x^2-3 \text{ and } \frac{d^2y}{dx^2} = 24x$$

$\therefore \frac{1}{y}\left\{\left(x^2-1\right)\frac{d^2y}{dx}+x\frac{dy}{dx}\right\}$

$=\frac{1}{4x^3-3x}\left\{\left(x^2-1\right)24x+x\left(12x^2-3\right)\right\}$

$=\frac{1}{4x^3-3x}\left\{36x^3-27x\right\}=\frac{9\left\{4x^3-3x\right\}}{4x^3-3x}=9$

Q→ 3: $\because \vec{a_1},\vec{a_2},\ldots \vec{a_n}$ are position vectors of vertices $A_1, A_2, A_3, \ldots A_n$ of a regular polygon of n sides with its centre at origin.

$\therefore \quad \left|\vec{a_1}\right|=\left|\vec{a_2}\right|=\ldots=\left|\vec{a_n}\right|=\lambda$

Now, $\vec{a_k}\times\vec{a_{k+1}}=\lambda^2\sin\frac{2\pi}{n}\hat{n}$

and $\vec{a_k}\cdot\vec{a_{k+1}}=\lambda^2\cos\frac{2\pi}{n}$

$\therefore\left|\sum_{k=1}^{n-1}(\vec{a}_k\times\vec{a}_{k+1})\right|=\left|\sum_{k=1}^{n-1}(\vec{a}_k\cdot\vec{a}_{k+1})\right|$

$\Rightarrow (n-1)\lambda^2\sin\frac{2\pi}{n}=(n-1)\lambda^2\cos\frac{2\pi}{n}$

$\Rightarrow \tan\frac{2\pi}{n}=1 \Rightarrow \frac{2\pi}{n}=\frac{\pi}{4}\Rightarrow n=8$

R→2: Normal from $P(h, 1)$ on $\frac{x^2}{6}+\frac{y^2}{3}=1$ is

$\frac{x-h}{h/6}=\frac{y-1}{1/3}$

$\Rightarrow 2(x-h)=h(y-1)\Rightarrow 2x-hy-h=0$

Slope of Normal = $\frac{2}{h}$

It is perpendicular to $x + y = 8$

$\therefore \frac{2}{h}\times-1=-1\Rightarrow h=2$

S→1: $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \frac{3x+1}{4x^2+3x}=\frac{2}{x^2}$

$\Rightarrow 3x^2-7x-6=0 \Rightarrow x=3$ or $-\frac{2}{3}$

Since $x > 0$

$\therefore$ Only one +ve solution is there

Hence (a) is the correct option.

57. **(c)** (P) Given that $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=2$

$\therefore \left[2\left(\vec{a}\times\vec{b}\right)\ 3\left(\vec{b}\times\vec{c}\right)\ \vec{c}\times\vec{a}\right]$

$=6\left[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\right]$

$=6\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2=6\times4=24 \therefore$ (P)→(3)

(Q) Given that $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=5$

$\therefore \left[3\left(\vec{a}+\vec{b}\right)\ \ \vec{b}+\vec{c}\ \ 2\left(\vec{c}+\vec{a}\right)\right]$

$=6\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]$

$=6\times2\left[\vec{a}\ \vec{b}\ \vec{c}\right]=6\times2\times5=60$

$\therefore$ (Q)→(4)

(R) Given that $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=20 \Rightarrow \left|\vec{a}\times\vec{b}\right|=40$

$\therefore \frac{1}{2}\left|\left(2\vec{a}+3\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\frac{1}{2}\left|-2\vec{a}\times\vec{b}+3\vec{b}\times\vec{a}\right|$

$=\frac{1}{2}\times5\left|\vec{a}\times\vec{b}\right|=\frac{5}{2}\times40=100$

$\therefore$ (R)→(1)

(S) Given that $\left|\vec{a}\times\vec{b}\right|=30$

$\therefore \left|\left(\vec{a}+\vec{b}\right)\times\vec{a}\right|=\left|\left(\vec{b}\times\vec{a}\right)\right|=30$

$\therefore$ (s)→(2)

58. **A → q,s; B → p,r,s,t; C → t; D → r**

(A) The given equation is

$2\sin^2\theta+\sin^2 2\theta=2$

$\Rightarrow 2\sin^2\theta+4\sin^2\theta\cos^2\theta-2=0$

$\Rightarrow \sin^2\theta+2\sin^2\theta(1-\sin^2\theta)-1=0$

$\Rightarrow 2\sin^4\theta-3\sin^2\theta+1=0$

$\Rightarrow 2\sin^4\theta-2\sin^2\theta-\sin^2\theta+1=0$

$\Rightarrow 2\sin^2\theta(\sin^2\theta-1)-1(\sin^2\theta-1)=0$

$\Rightarrow (\sin^2\theta-1)(2\sin^2\theta-1)=0$

$\Rightarrow \sin^2\theta=1$ or $\sin^2\theta=\frac{1}{2}$

$\Rightarrow \text{Sin}\,\theta=\pm1$ or $\sin\theta=\pm\frac{1}{\sqrt{2}}$

$\Rightarrow \theta=n\pi\pm\frac{\pi}{2}$ or $n\pi\pm\frac{\pi}{4}\Rightarrow\theta=\frac{\pi}{2}$ or $\frac{\pi}{4}$

(B) We know that $[x]$ is discontinuous at all integral values, therefore $\left[\frac{6x}{\pi}\right]$ is discontinuous at $x=\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ and π. Also $\cos\left[\frac{3x}{\pi}\right]\neq0$ for any of these values of x.

$\therefore \quad \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$ is discontinuous at $x=\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ and π.

(C) Let $\vec{a} = \hat{i} + \hat{j}, \vec{b} = \hat{i} + 2\hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + \pi\hat{k}$

$\therefore$ The volume parallelopiped $= \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi$$

(D) $\because \vec{a} + \vec{b} = -\sqrt{3}\vec{c} \Rightarrow |\vec{a} + \vec{b}|^2 = 3|\vec{c}|^2$

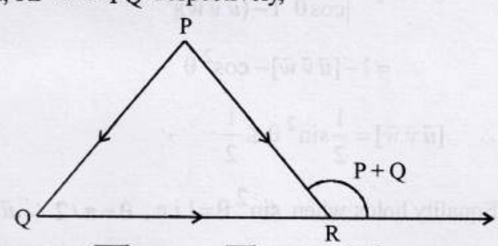
$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 3$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 3 \Rightarrow 1 + 1 + 2\cos\theta = 3$

(where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

59. (a) $\overline{OX}, \overline{OY}, \overline{OZ}$ are unit vectors in the directions of sides $\overline{QR}, \overline{RP}$ and $\overline{PQ}$ respectively,



$\therefore \overline{OX} = \frac{\overline{QR}}{|\overline{QR}|}, \overline{OY} = \frac{\overline{RP}}{|\overline{RP}|}, \overline{OZ} = \frac{\overline{PQ}}{|\overline{PQ}|}$

$\therefore |\overline{OX} \times \overline{OY}| = \frac{|\overline{QR} \times \overline{RP}|}{|\overline{QR}||\overline{RP}|} = \frac{|\overline{QR}||\overline{RP}|\sin(P + Q)}{|\overline{QR}||\overline{RP}|}$

$= \sin(P + Q)$

60. (b) $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$
$= \cos(180 - R) + \cos(180 - P) + \cos(180 - Q)$
$= -[\cos P + \cos Q + \cos R]$

To minimize the expression we need to maximize $\cos P + \cos Q + \cos R$.

We know that $\cos P + \cos Q + \cos R$ will be maximum when $\cos P = \cos Q = \cos R$

$\Rightarrow P = Q = R = \frac{\pi}{3}$

$\therefore$ Minimum value $= -3\cos\frac{\pi}{3} = \frac{-3}{2}$

61. (c) $\overrightarrow{PQ} \times (\overrightarrow{RS} + \overrightarrow{ST}) = \overrightarrow{PQ} \times \overrightarrow{RT}$ (using triangle law)

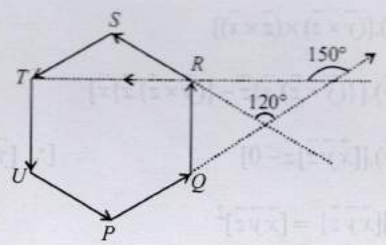
$= |\overrightarrow{PQ}| \times |\overrightarrow{RT}| \sin 150 \hat{n} \neq 0$

Statement-1 is true.

Also, $\overrightarrow{PQ} \times \overrightarrow{RS} = |\overrightarrow{PQ}| \times |\overrightarrow{RS}| \sin 120° \times n_1 \neq 0$

And $\overrightarrow{PQ} \times \overrightarrow{ST} = |\overrightarrow{PQ}| \times |\overrightarrow{ST}| \sin 180° \times \hat{n}_2 = 0$

$\therefore$ Statement-2 is false.



62. Given that

$\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ ...(i)

$\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ ...(ii)

Subtracting eq^n (ii) from (i) we get

$\vec{a} \times (\vec{c} - \vec{b}) = (\vec{b} - \vec{c}) \times \vec{d} \Rightarrow \vec{a} \times (\vec{c} - \vec{b}) = \vec{d} \times (\vec{c} - \vec{b})$

$\Rightarrow \vec{a} \times (\vec{c} - \vec{b}) - \vec{d} \times (\vec{c} - \vec{b}) = \vec{0}$

$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{c} - \vec{b}) = \vec{0} \Rightarrow (\vec{a} - \vec{d}) \| (\vec{c} - \vec{b})$

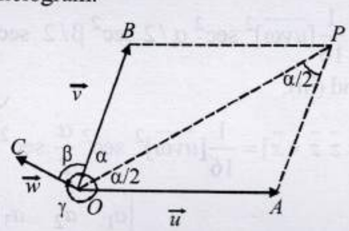
$[\because \vec{a} - \vec{d} \neq 0, \vec{c} - \vec{b} \neq 0$ as $\vec{a} \neq \vec{b} \neq \vec{c} \neq \vec{d}]$

$\Rightarrow$ Angle between $\vec{a} - \vec{d}$ and $\vec{c} - \vec{b}$ is either 0 or 180°.

$\Rightarrow (\vec{a} - \vec{d}).(\vec{c} - \vec{b}) = |\vec{a} - \vec{d}||\vec{c} - \vec{b}|\cos 0 [\text{or} \cos 180°] \neq 0$

as *a, b, c, d* all are different. Hence Proved.

63. $\vec{u}, \vec{v}$ and $\vec{\omega}$ represent in figure as $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ resp. Let P be a pt. on angle bisector of $\angle AOB$ such that $OAPB$ is a parallelogram.



$\because \angle POA = \angle BOP = \alpha/2$

$\therefore \angle APO = \angle BOP = \alpha/2$ (alt. int. $\angle$'s)

$\therefore$ In $\Delta OAP, OA = AP$

$\therefore \overline{OP} = \overline{OA} + \overline{AP} = \vec{u} + \vec{v}$

$\therefore$ A unit vector in the direction of

$\overline{OP} = \frac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|}$. i.e. $\vec{x} = \frac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|}$

But $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}).(\vec{u} + \vec{v}) = 1 + 1 + 2\vec{u}.\vec{v}$

$[\because |\vec{u}| = |\vec{v}| = 1]$

$= 2 + 2\cos\alpha = 4\cos^2\alpha/2$

$\therefore |\vec{u} + \vec{v}| = 2\cos\alpha/2 \Rightarrow \vec{x} = \frac{1}{2}(\sec\alpha/2)(\vec{u} + \vec{v})$

Similarly, $\vec{y} = \frac{1}{2}\sec\frac{\beta}{2}(\vec{v} + \vec{\omega})$ and $\vec{z} = \frac{1}{2}\sec\frac{\gamma}{2}(\vec{\omega} + \vec{u})$

Now $[\vec{x} \times \vec{y}\ \vec{y} \times \vec{z}\ \vec{z} \times \vec{x}]$

$= (\vec{x}\times\vec{y}).[(\vec{y}\times\vec{z})\times(\vec{z}\times\vec{x})]$

$= (\vec{x}\times\vec{y}).[\{(\vec{y}\times\vec{z}).\vec{x}\}\vec{z} - \{(\vec{y}\times\vec{z}).\vec{z}\}\vec{x}]$

$= (\vec{x}\times\vec{y}).[[\vec{x}\,\vec{y}\,\vec{z}]\vec{z} - 0]$ $\quad [\because [\vec{y}\,\vec{z}\,\vec{z}] = 0]$

$= [\vec{x}\,\vec{y}\,\vec{z}][\vec{x}\,\vec{y}\,\vec{z}] = [\vec{x}\,\vec{y}\,\vec{z}]^2$ ...(i)

Also $[\vec{x}\,\vec{y}\,\vec{z}] = \left[\frac{1}{2}\left(\sec\frac{\alpha}{2}\right)(\vec{u}+\vec{v})\ \frac{1}{2}\sec\beta/2(\vec{v}+\vec{\omega})\ \frac{1}{2}\sec\gamma/2(\vec{\omega}+\vec{u})\right]$

$= \frac{1}{8}\sec\alpha/2\sec\beta/2\sec\gamma/2[\vec{u}+\vec{v}\ \vec{v}+\vec{\omega}\ \vec{\omega}+\vec{u}]$

$[(\vec{u}+\vec{v}).\{(\vec{v}+\vec{\omega})\times(\vec{\omega}+\vec{u})\}]$

$[(\vec{u}+\vec{v}).(\vec{v}\times\vec{\omega}+\vec{v}\times\vec{u}+\vec{\omega}\times\vec{u})]$

$[\vec{u}.(\vec{v}\times\vec{\omega})+\vec{v}.(\vec{\omega}\times\vec{u})]$

($\because [\vec{a}\vec{b}\vec{c}] = 0$ when ever any two vectors are same)

$= 2[\vec{u}\vec{v}\vec{\omega}]$

$[\vec{x}\,\vec{y}\,\vec{z}] = \frac{1}{4}(\sec\alpha/2\sec\beta/2\sec\gamma/2)[\vec{u}\vec{v}\vec{\omega}]$ (from (ii))

$\therefore [\vec{x}\vec{y}\vec{z}]^2 = \frac{1}{16}[\vec{u}\vec{v}\vec{\omega}]^2\sec^2\alpha/2\ \sec^2\beta/2\ \sec^2\gamma/2]$ ...(iii)

From (i) and (iii),

$[\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}] = \frac{1}{16}[\vec{u}\vec{v}\vec{\omega}]^2\sec^2\frac{\alpha}{2}.\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$

64. We know that, $V = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$ ....(i)

Now we know that $AM \geq GM$

$\therefore \frac{(a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)}{3}$

$\geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$

$\Rightarrow \frac{3L}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$

$\Rightarrow L^3 \geq (a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$

$\Rightarrow L^3 \geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + 24$ more such terms

$L^3 \geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$

$[\because a_r, b_r, c_r \geq 0 \text{ for } r = 1,2,3]$

$L^3 \geq (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$

[By properties of inquality]

$L^3 \geq V$ (from (i)) Hence Proved.

65. Since, $\vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$

$\Rightarrow \vec{w} = \vec{v} - (\vec{w}\times\vec{u})$

$[\vec{u}\,\vec{v}\,\vec{w}] = (\vec{u}\times\vec{v}).(\vec{v} - \vec{w}\times\vec{u}) = (\vec{u}\times\vec{v}).(\vec{u}\times\vec{w})$

$= \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{w} \end{vmatrix}$

Now, $\vec{u}.\vec{u} = 1$. Let θ is angle between $\vec{u}$ and $\vec{v}$.

$\therefore \vec{u}.\vec{w} = \vec{u}.(\vec{v} - \vec{w}\times\vec{u}) = \vec{u}.\vec{v} - [\vec{u}\,\vec{w}\,\vec{u}] = \vec{u}.\vec{v} = \cos\theta$

$\vec{v}.\vec{w} = \vec{v}.(\vec{v} - \vec{w}\times\vec{u}) = 1 - [\vec{v}\,\vec{w}\,\vec{u}] = 1 - [\vec{u}\,\vec{v}\,\vec{w}]$

$\therefore \quad [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1 - (\vec{u}\,\vec{v}\,\vec{w}) \end{vmatrix}$

$= 1 - [\vec{u}\,\vec{v}\,\vec{w}] - \cos^2\theta$

$\therefore \quad [\vec{u}\,\vec{v}\,\vec{w}] = \frac{1}{2}\sin^2\theta \leq \frac{1}{2}$

Equality holds when $\sin^2\theta = 1$ i.e., $\theta = \pi/2 \therefore \vec{u} \perp \vec{v}$.

66. (a) Since, $\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ and $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

Let θ be the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}, \vec{v}$.

L.H.S $= |\vec{u}.\vec{v}|^2 + |\vec{u}\times\vec{v}|^2$

$= |\vec{u}|^2|\vec{v}|^2\cos^2\theta + |\vec{u}|^2|\vec{v}|^2\sin^2\theta|\hat{n}|^2$

$= |\vec{u}|^2|v|^2(\cos^2\theta + \sin^2\theta)$ $\quad (\because |\hat{n}| = 1)$

$= |\vec{u}|^2|v|^2 = R.H.S$

(b) $|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2 = |\vec{u}+\vec{v}|^2 + |\vec{u}\times\vec{v}|^2 + 2(\vec{u}+\vec{v}).(\vec{u}\times\vec{v})$

$= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}.\vec{v} + |\vec{u}\times\vec{v}|^2 + 0$

$\therefore$ R.H.S $= |\vec{u}+\vec{v}+\vec{u}\times\vec{v}|^2 + |1 - \vec{u}.\vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}.\vec{v} + |\vec{u}\times\vec{v}|^2 + 1 - 2\vec{u}.\vec{v} + |\vec{u} - \vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + 1 + |\vec{u}|^2|\vec{v}|^2$

$= (1 + |\vec{u}|^2)(1 + |\vec{v}|^2) = $ L.H.S.

67. $(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})$

$= \vec{0} + \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}$ $\quad [\because \vec{A}\times\vec{A} = 0]$

$= \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}$

Now, $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$

$=[\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$

$=(\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C})+(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C})$

$=\left\{(\vec{B}\times\vec{A}).\vec{C}\right\}\vec{B}-\left\{(\vec{B}\times\vec{A}).\vec{B}\right\}\vec{C}$

$+\left\{(\vec{A}\times\vec{C}).\vec{C}\right\}\vec{B}-\left\{(\vec{A}\times\vec{C}).\vec{B}\right\}\vec{C}$

$[\because\ (\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}]$

$=[\vec{B}\,\vec{A}\,\vec{C}]\vec{B}-[\vec{A}\,\vec{C}\,\vec{B}]\vec{C}$

$[\because\ [\vec{A}\vec{B}\vec{C}]=0$ if any two of $[\vec{A},\vec{B},\vec{C}]$ are equal.]

$=[\vec{A}\,\vec{C}\,\vec{B}]\left\{\vec{B}-\vec{C}\right\}$

$\text{LHS}=[\vec{A}\,\vec{C}\,\vec{B}]\{(\vec{B}-\vec{C}).(\vec{B}+\vec{C})\}$

$=[\vec{A}\,\vec{C}\,\vec{B}]\{|\vec{B}|^2-|\vec{C}|^2\}=0 \qquad [\because\ |B|=|C|]$

68. Given that $AD = 4$

Let $DE = h$

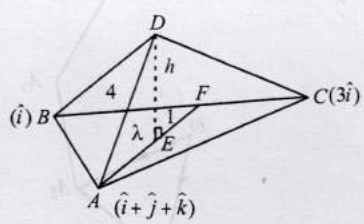
Volume of tetrahedron $=\dfrac{2\sqrt{2}}{3}$

$\Rightarrow\ \dfrac{1}{3}Ar(\Delta ABC)h=\dfrac{2\sqrt{2}}{3}$

$\therefore\ \dfrac{1}{2}|\overline{BA}\times\overline{BC}|\,h=2\sqrt{2}$

$\dfrac{1}{2}|(\hat{j}+\hat{k})\times 2i|\,h=2\sqrt{2}$ or $|\hat{j}-\hat{k}|\,h=2\sqrt{2}$

or $\sqrt{2}h=2\sqrt{2}\quad\therefore h=2$



Let point E divides median AF in the ratio $\lambda : 1$

$\therefore\ \overline{OE}=\dfrac{\lambda.2\hat{i}+(\hat{i}+\hat{j}+\hat{k})}{\lambda+1}$ ...(ii)

$\therefore\ \overline{AE}=\text{P.V. of } E-\text{P.V. of } A=\dfrac{\lambda}{\lambda+1}(\hat{i}-\hat{j}-\hat{k})$

$\therefore\ |\overline{AE}|^2=\overline{AE}^2=\left(\dfrac{\lambda}{\lambda+1}\right)^2.3$ ...(iii)

Now, $h^2+AE^2=AD^2$

$\Rightarrow\ 4+\left(\dfrac{\lambda}{\lambda+1}\right)^2.3=16\ \Rightarrow\ 3\left(\dfrac{\lambda}{\lambda+1}\right)^2=12$

$\Rightarrow\ \left(\dfrac{\lambda}{\lambda+1}\right)=\pm 2\ \Rightarrow\ \lambda=\pm(2\lambda+2)$

$\therefore\ \lambda=-2$ or $-2/3$

Putting the value of λ in (ii) we get the possible positions of E as $(-1, 3, 3)$ or $(3, -1, -1)$

69. Given that $\vec{b},\vec{c},\vec{d}$ are not coplanar $\therefore\ [\vec{b},\vec{c},\vec{d}]\neq 0$

Here, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=-(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})$

$=-(\vec{c}\times\vec{d}.\vec{b})\vec{a}+(\vec{c}\times\vec{d}.\vec{a})\vec{b}$

$=\left[\vec{a}\,\vec{c}\,\vec{d}\right]\vec{b}-\left[\vec{b}\,\vec{c}\,\vec{d}\right]\vec{a}$ ...(i)

$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})=-(\vec{d}\times\vec{b})\times(\vec{a}\times\vec{c})$

$=-(\vec{d}\times\vec{b}.\vec{c})\vec{a}+(\vec{d}\times\vec{b}.\vec{a})\vec{c}$

$=[\vec{a}\,\vec{d}\,\vec{b}]\vec{c}-[\vec{c}\,\vec{d}\,\vec{b}]\vec{a}$ ...(ii)

$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})=\left((\vec{a}\times\vec{d})\vec{c}\right)\vec{b}-(\vec{a}\times\vec{d}.\vec{b})\vec{c}$

$=-[\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{a}\,\vec{d}\,\vec{b}]\vec{c}$ ...(iii)

Adding (i), (ii) and (iii), we get.

given vector $=-2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}=k\vec{a}$

$\Rightarrow$ given vector is parallel to $\vec{a}$.

70. Let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$

Then $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$

$$\Rightarrow\ \begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\1&1&1\end{vmatrix}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&-3&7\\1&1&1\end{vmatrix}$$

$\Rightarrow\ (y-z)\hat{i}-(x-z)\hat{j}+(x-y)\hat{k}=-10\hat{i}+3\hat{j}+7\hat{k}$

$\Rightarrow\ y-z=-10$ ... (i)

$z-x=+3$ ... (ii)

$x-y=7$ ... (iii)

Also $\vec{R}.\vec{A}=0$

$\Rightarrow\ 2x+z=0$ ... (iv)

From (ii) and (iv) we get $x=-1$, from (i) and (iii) we get.

$\Rightarrow\ y=-8$ and $z=2$

$\therefore\ \vec{R}=-\hat{i}-8\hat{j}+2\hat{k}$

71. Given that $\vec{a},\vec{b},\vec{c}$ are three coplanar vectors.

$\therefore$ There exists scalars x, y, z, not all zero, such that

$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$ ... (i)

Taking dot product of eqn. (i) with $\vec{a}$ and $\vec{b}$ respectirely, we get

$x\vec{a}.\vec{a}+y\vec{a}\vec{b}+z\,\vec{a}.\vec{c}=\vec{0}$ ... (ii)

$x\vec{b}.\vec{a}+y\vec{b}\vec{b}+z\,\vec{b}.\vec{c}=\vec{0}$ ... (iii)

Since equations (i), (ii), (iii) form a homogeneous system of equations, where x, y, z are not all zero. Therefore system must have non trivial solution.

$$\therefore \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = 0 \quad \text{Hence Proved.}$$

72. Let the position vectors of points A, B, C, D with respect to origin O be a, b, c, and d respectively.

Then, $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AD} = \vec{d} - \vec{a}$,

$\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{BD} = \vec{d} - \vec{b}$,

$\overrightarrow{CD} = \vec{d} - \vec{c}$, $\overrightarrow{CA} = \vec{a} - \vec{c}$

Now, $|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$

$$= |(\vec{b} - \vec{a}) \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times (\vec{d} - \vec{a}) + (\vec{a} - \vec{c}) \times (\vec{d} - \vec{b})|$$

$$= |\vec{b} \times \vec{d} - \vec{a} \times \vec{d} - \vec{b} \times \vec{c} + \vec{a} \times \vec{c} + \vec{c} \times \vec{d} - \vec{c} \times \vec{a}$$
$$- \vec{b} \times \vec{d} + \vec{b} \times \vec{a} + \vec{a} \times \vec{d} - \vec{a} \times \vec{b} - \vec{c} \times \vec{d} + \vec{c} \times \vec{b}|$$

$$= |-\vec{b} \times \vec{c} + \vec{a} \times \vec{c} - \vec{c} \times \vec{a} + \vec{b} \times \vec{a} - \vec{a} \times \vec{b} + \vec{c} \times \vec{b}|$$

$$= 2|\vec{b} \times \vec{a} + \vec{c} \times \vec{b} + \vec{a} \times \vec{c}| \qquad \text{...(i)}$$

Also Area of ΔABC is

$$= \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}| = \frac{1}{2}|(\vec{c} - \vec{b}) \times (\vec{a} - \vec{b})|$$

$$= \frac{1}{2}|(\vec{c} \times \vec{a} - \vec{c} \times \vec{b} - \vec{b} \times \vec{a} + \vec{b} \times \vec{b}|$$

$$= \frac{1}{2}|-\vec{b} \times \vec{a} - \vec{c} \times \vec{b} - \vec{a} \times \vec{c}| = \frac{1}{2}|\vec{b} \times \vec{a} + \vec{c} \times \vec{b} + \vec{a} \times \vec{c}|$$

$$\Rightarrow 2Ar(\Delta ABC) = |\vec{b} \times \vec{a} + \vec{c} \times \vec{b} + \vec{a} \times \vec{c}| \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$$

$= 2(2Ar(\Delta ABC)) = 4Ar(\Delta ABC)$ Hence Proved.

73. From given position vector $\overrightarrow{AB} = -\hat{i} + 5\hat{j} - 3\hat{k}$,

$\overrightarrow{AC} = -4\hat{i} + 3\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = \hat{i} + 7\hat{j} + (1 + \lambda)\hat{k}$

Given that A, B, C, D lie in a plane then $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar i.e. $\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right] = 0$

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1+\lambda \end{vmatrix} = 0$$

$$\Rightarrow -1(3 + 3\lambda - 21) - 5(-4 - 4\lambda - 3) - 3(-28 - 3) = 0$$

$$\Rightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0 \Rightarrow 17\lambda + 146 = 0$$

$$\Rightarrow \lambda = -\frac{146}{17}$$

74. $\bar{r} = \lambda x\hat{i} + \lambda y\hat{j} + \lambda z\hat{k}$

On comparing both sides, we get

$\Rightarrow x + 3y - 4z = \lambda x \Rightarrow (1 - \lambda)x + 3y - 4z = 0$

$\Rightarrow x - 3y + 5z = \lambda y \Rightarrow x - (3 + \lambda)y + 5z = 0$

$\Rightarrow 3x + y + 0z = \lambda z \Rightarrow 3x + y - \lambda z = 0$

Since $x, y, z \neq (0, 0, 0)$ then all the above three equations non zero solution.

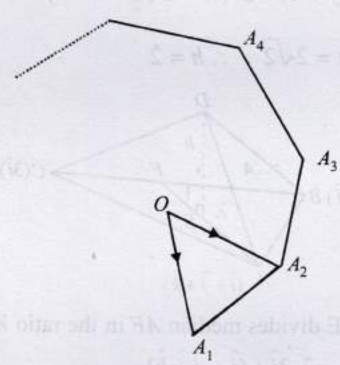
$$\therefore \begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)[3\lambda + \lambda^2 - 5] - 3[-\lambda - 15] - 4[1 + 9 + 3\lambda] = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda + 1)^2 = 0 \Rightarrow \lambda = 0, -1.$$

75. Since $A_1, A_2, \ldots A_n$ are the vertices of a regular plane polygon.

$\therefore \overrightarrow{OA_1}, \overrightarrow{OA_2}, \ldots, \overrightarrow{OA_n}$ all vectors are of same magnitude, say 'a' and angle between any two consecutive vector is same that is $\frac{2\pi}{n}$ radians. Let $\hat{n}$ be the normal unit vectors perpendicular to the plane of the polygon.



$$\therefore \overrightarrow{OA_1} \times \overrightarrow{OA_2} = a^2 \sin\frac{2\pi}{n}\hat{p} \qquad \text{...(i)}$$

Now, $\sum_{i=1}^{n-1} \overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}} = \sum_{i=1}^{n-1} a^2 \sin\frac{2\pi}{n}\hat{p}$

$$= (n-1)a^2 \sin\frac{2\pi}{n}\hat{p} = -(n-1)[\overrightarrow{OA_2} \times \overrightarrow{OA_1}]$$

[using eq^n. (i)]

$$= (1-n)[\overrightarrow{OA_2} \times \overrightarrow{OA_1}] = R.H.S$$