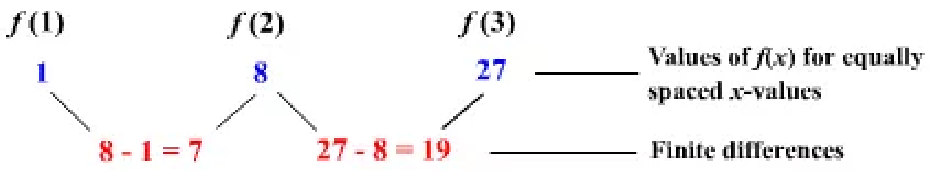
Chapter 5 Quadratic Functions

Ex 5.9

Answer 1e.

Consider a function, say, $f(x) = x^3$.

Let us find the values of $f(x)$ for some equally spaced x-values.



As can be seen from the figure, the differences of consecutive y-values are called finite differences.

Therefore, we can complete the given statement as "When the x-values in a data set are equally spaced, the differences of consecutive y-values are called **finite differences**."

Answer 1gp.

We need to find the cubic function whose graph passes through the points $(-4,0),(0,10),(2,0),(5,0)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y = a(x+4)(x-2)(x-5) \quad \text{...... (1)}$$

To find the value of a by substituting the co-ordinate of the point $(0,10)$ in the equation (1), we have

$$y = a(x+4)(x-2)(x-5)$$
$$10 = a(0+4)(0-2)(0-5) \quad [\text{By putting } x=0 \text{ and } y=10]$$
$$10 = a(4)(-2)(-5)$$
$$10 = 40a \quad [\text{By multiplying}]$$
$$\frac{10}{40} = a \quad [\text{Dividing both sides by } 40]$$
$$a = \frac{1}{4}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

$$y=\frac{1}{4}(x+4)(x-2)(x-5)$$

Answer 1mr.

a. The volume of the rectangular prism is the product of its length, width and height.

We have the volume of the prism as 180 cubic inches, the length as $x+5$ inches, width as $x+1$ inches, and the height as x inches.

$$\begin{array}{ccccccc} \text{Volume} & & \text{length} & & \text{width} & & \text{height} \\ \text{(cubic inches)} & = & \text{(inches)} & \cdot & \text{(inches)} & \cdot & \text{(inches)} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ 180 & = & x+5 & \cdot & x+1 & \cdot & x \end{array}$$

We get the equation as $180=(x+5)(x+1)x$.

Remove the parentheses on the right using the distributive property.

$$180 = \left(x^2+6x+5\right)x$$
$$180 = x^3+6x^2+5x$$

Now, rewrite the equation in the standard form.
$x^3+6x^2+5x-180=0$

Thus, the required polynomial equation is $x^3+6x^2+5x-180=0$.

b. The leading coefficient of the polynomial equation obtained in part **a** is 1, and the constant term is -180.

Divide the factors of the constant term by the factors of the leading coefficient to find the possible rational zeros of the equation.

$$x = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm9, \pm10, \pm12, \pm15, \pm18, \pm20, \pm30, \pm45, \pm60, \pm90, \text{ and } \pm180$$

c. We can test the rational zeros which we obtained in part **b**. Only positive x-values make sense.

Test $x=-4$:

$$\begin{array}{r|rrrr} -4 & 1 & 6 & 5 & -180 \\ & & -4 & -8 & 12 \\ \hline & 1 & 2 & -3 & -168 \end{array}$$

Test $x=4$:

$$\begin{array}{r|rrrr} 4 & 1 & 6 & 5 & -180 \\ & & 4 & 40 & 180 \\ \hline & 1 & 10 & 45 & 0 \end{array}$$

If we test the possible solutions, it can be seen that the remainder is 0 only in case of 4. This means that 4 is a factor of the equation, and thus a real solution.

Use the result obtained to rewrite the equation.
$(x-4)(x^2+10x+45)=0$

We can find the remaining zeros by solving $x^2+10x+45=0$ by using the quadratic formula.

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{100-180}}{2} \\ &= \frac{-10 \pm 4i\sqrt{5}}{2} \\ &= -5 \pm 2i\sqrt{5} \end{aligned}$$

We seek only the real solutions of the equation, and hence $-5 \pm 2i\sqrt{5}$ cannot be considered. Thus, the only real solution to the equation is 4.

d. We have the height of the rectangular prism as 4 inches.

The length of the prism will then be $4+5$ or 9 inches, and the width be $4+1$ or 5 inches.

Therefore, the dimensions of the prism are 9 inches by 5 inches by 4 inches.

Answer 1q.

STEP 1 **Find** the rational zeros of *f*.

Since the degree of *h* is 3, the function will have 3 zeros. The possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15$ and ± 30.

Use synthetic division to divide $f(x)$ by 2.

$$\begin{array}{c|cccc} 2 & 1 & -4 & -11 & 30 \\ & & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

The remainder is 0, which implies that 2 is a zero.

STEP 2 **Write** $f(x)$ in factored form.

The quotient obtained by dividing $f(x)$ by 2 is $x^2-2x-15$.

The function can be thus written as $f(x)=(x-2)(x^2-2x-15)$.

STEP 2 **Find** the complex zeros of *f*.

Factor the trinomial $x^2-2x-15$.
$f(x)=(x-2)(x-5)(x+3)$

Therefore, the zeros of *f* are -3, 2, and 5.

Answer 2e.

We need to explain about first-order differences and second-order differences.

We take a function as:

$$f(x) = 2x^2 + x$$

Then, $f(1) = 3, f(2) = 10, f(3) = 21, f(4) = 36$

First-order differences:
It is a difference which can be obtained by subtracting these consecutive numbers. For example,

$$f(2) - f(1) = 7$$
$$f(3) - f(2) = 11$$
$$f(4) - f(3) = 15$$

Second-order differences:
It is a difference which can be obtained by subtracting consecutive first-order differences. For example,

$$11 - 7 = 4$$
$$15 - 11 = 4$$

Answer 2gp.

We need to find the cubic function whose graph passes through the points $(-1,0), (0,-12), (2,0), (3,0)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y = a(x+1)(x-2)(x-3) \quad \ldots\ldots (1)$$

To find the value of a by substituting the co-ordinate of the point $(0,-12)$ in the equation (1), we have

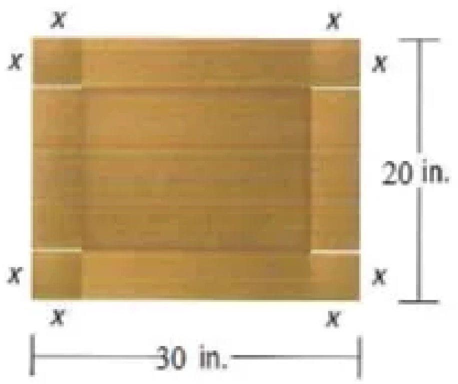
$$y = a(x+1)(x-2)(x-3)$$
$$-12 = a(0+1)(0-2)(0-3) \quad [\text{By putting } x = 0 \text{ and } y = -12]$$
$$-12 = a(1)(-2)(-3)$$
$$-12 = 6a \quad [\text{By multiplying}]$$
$$\frac{-12}{6} = a \quad [\text{Dividing both sides by } 6]$$
$$a = -2$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

$$\boxed{y = -2(x+1)(x-2)(x-3)}$$

Answer 2mr.

The piece of 20 inches by 30 inches cardboard is



(a)

We need to find a polynomial function for the volume of the box.
We write a verbal model for the volume then write a function.

$$\underset{\text{(inches)}}{\text{Volume}} = \underset{\text{(inches)}}{\text{Length}} \cdot \underset{\text{(inches)}}{\text{Width}} \cdot \text{Height}$$

$$\begin{aligned} V(x) &= (30-2x)(20-2x)x \\ &= (600-60x-40x+4x^2)x && \text{[Multiplying the binomials]} \\ &= 600x-60x^2-40x^2+4x^3 && \text{[Multiplying by } x\text{]} \\ &= 4x^3-100x^2+600x && \text{[Writing in standard form]} \end{aligned}$$

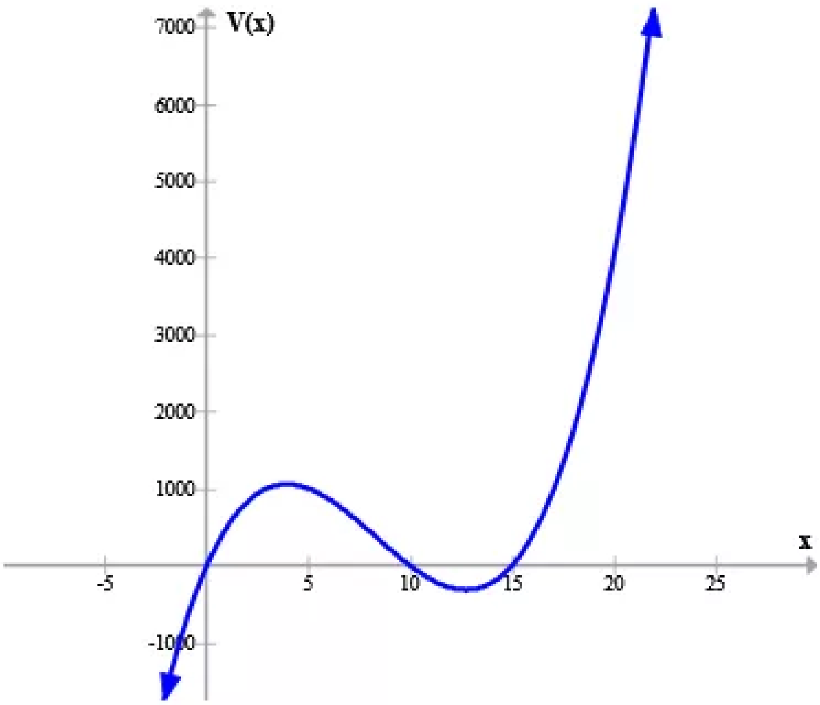
Therefore the function for the volume of the box is

$$\boxed{V(x) = 4x^3-100x^2+600x}$$

(b)

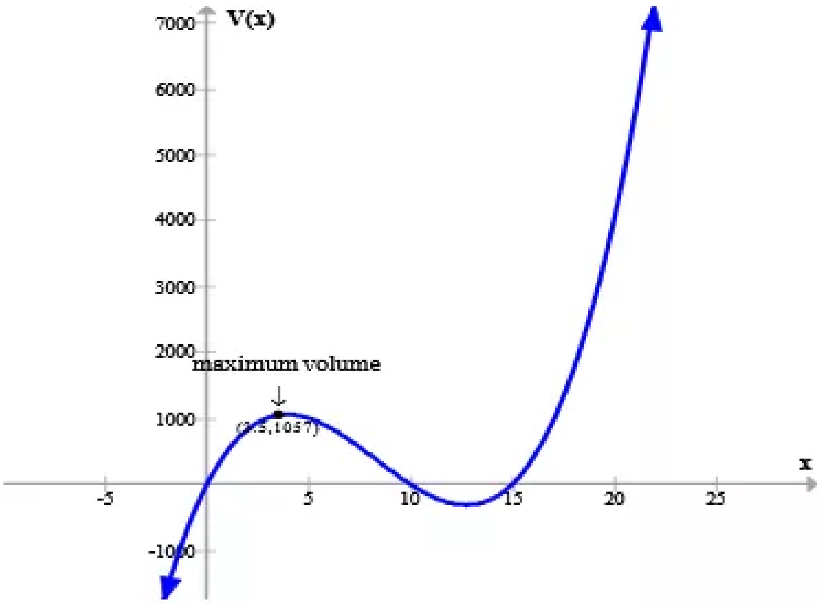
We need to graph the function $V(x) = 4x^3 - 100x^2 + 600x$.

The graph for this function is as follows:



(c)

We need to find the dimension of the box with the maximum volume.



From the graph we see that the maximum volume is about 1057 cubic inches and occurs when $x = 3.5$. Therefore we make the cuts about 3.5 inches long. The dimension of the box with this volume will be about

$$x = 3.5 \text{ inches}$$

by

$$\text{length} = (30 - 2 \times 3.5)$$
$$= 23 \text{ inches}$$

by

$$\text{width} = (20 - 2 \times 3.5)$$
$$= 13 \text{ inches}$$

Therefore the dimension is $\boxed{\text{23 inches by 13 inches 3.5 inches}}$.

(d)

We need to find the maximum volume of the box.

From the graph we see that the maximum volume is about $\boxed{\text{1057 cubic inches}}$ and occurs when $x = 3.5$.

Answer 2q.

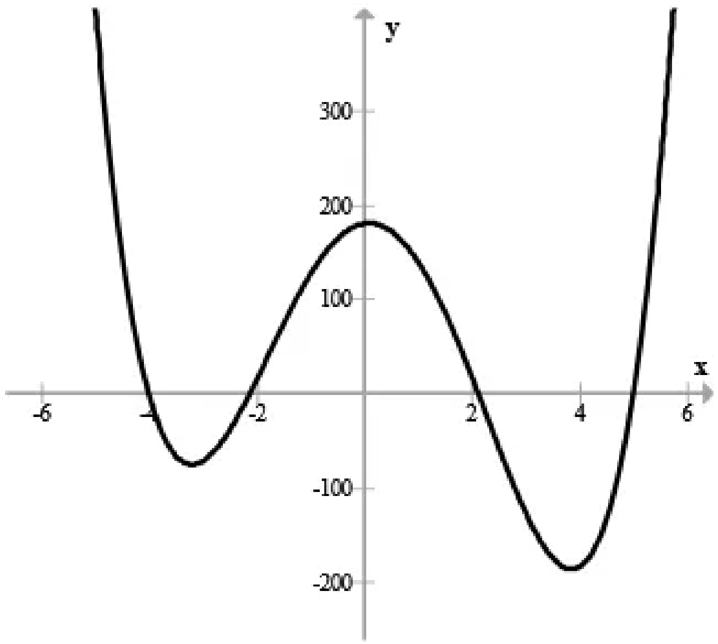
We need to find all zeros of the polynomial function $f(x) = 2x^4 - 2x^3 - 49x^2 + 9x + 180$.

We find the rational zeros of f. Since f is a polynomial function of degree 4, it has 4 zeros. The possible zeros are

$$\pm\frac{1}{1},\pm\frac{2}{1},\pm\frac{3}{1},\pm\frac{4}{1},\pm\frac{5}{1},\pm\frac{6}{1},\pm\frac{9}{1},\pm\frac{10}{1},\pm\frac{12}{1},\pm\frac{15}{1},\pm\frac{18}{1},\pm\frac{20}{1},\pm\frac{30}{1},\pm\frac{45}{1},\pm\frac{60}{1},$$

$$\pm\frac{90}{1},\pm\frac{180}{1},\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\pm\frac{9}{2},\pm\frac{15}{1},\pm\frac{45}{2}$$

The graph of the function is as follows:



We choose reasonable values from the list above to check using the graph of the function. For f the values $x=-4$ and $x=5$ are reasonable and other two zeros are nearest to -2 and 2 based on the graph.

We check the values using synthetic division until a zero is found. First we divide by $x=-4$. Therefore

$$\begin{array}{r|rrrrr} -4 & 2 & -2 & -49 & 9 & 180 \\ & & -8 & 40 & -36 & -180 \\ \hline & 2 & -10 & -9 & 45 & 0 \end{array}$$

Now we divide the result of previous division by $x=5$. Therefore

$$\begin{array}{r|rrrr} 5 & 2 & -10 & -9 & 45 \\ & & 10 & & -45 \\ \hline & 2 & 0 & -9 & 0 \end{array}$$

We factor out a binomial using the result of the synthetic division.

$$\begin{aligned} f(x) &= 2x^4 - 2x^3 - 49x^2 + 9x + 180 \\ &= (x+4)(x-5)(2x^2-9) \quad \text{[Writing as a product of factors]} \\ &= (x+4)(x-5)\left\{\left(\sqrt{2}x\right)^2 - 3^2\right\} \\ &= (x+4)(x-5)\left(\sqrt{2}x-3\right)\left(\sqrt{2}x+3\right) \quad \left[\text{Since } a^2-b^2=(a-b)(a+b)\right] \end{aligned}$$

Therefore the real zeros of f are $\boxed{-4, 5, \frac{3\sqrt{2}}{2} \text{ and } -\frac{3\sqrt{2}}{2}}$.

Answer 3e.

STEP 1 **Use** the three x-intercepts to write the function in factored form.
As can be seen, the given graph has three x-intercepts at -1, 2, and 3, which in turn are the three real zeros of the function. This follows that $x + 1$, $x - 2$ and $x - 3$ are factors of the function.

We can now apply the factor theorem to write the function corresponding to the graph.
$f(x) = a(x + 1)(x - 2)(x - 3)$

STEP 2 **Find** the value of a by substituting the coordinates of the fourth point.

The graph also contains a point (0, 3), which means that the value of the function is 3 when x is 0. Using this result, we can substitute 0 for x and 3 for $f(x)$ in the function.
$3 = a(0 + 1)(0 - 2)(0 - 3)$

Simplify and solve for a.

$$\begin{aligned} 3 &= a(1)(-2)(-3) \\ 3 &= 6a \\ \frac{1}{2} &= a \end{aligned}$$

Now, replace a with $\frac{1}{2}$ in the function.

$$f(x) = \frac{1}{2}(x+1)(x-2)(x-3)$$

Remove the parentheses using the distributive property and simplify.

$$\begin{aligned} f(x) &= 0.5\left(x^3 - 5x^2 + 6x + x^2 - 5x + 6\right) \\ &= 0.5\left(x^3 - 4x^2 + x + 6\right) \\ &= 0.5x^3 - 2x^2 + 0.5x + 3 \end{aligned}$$

Therefore, the cubic function is $f(x) = 0.5x^3 - 2x^2 + 0.5x + 3$.

CHECK Check the end behavior of f.

The degree of f is odd and $a > 0$. As a result, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$, which matches with the graph.

Answer 3gp.

The formula for the nth pentagonal number is

$$f(n) = \frac{1}{2}n(3n-1)$$

We need to show that the function has constant second order differences.

The first five pentagonal numbers are shown below:

$$f(1) = \frac{1}{2}\times 1(3\times 1-1) \quad [\text{By putting } n=1]$$
$$= 1$$

$$f(2) = \frac{1}{2}\times 2(3\times 2-1) \quad [\text{By putting } n=2]$$
$$= 5$$

$$f(3) = \frac{1}{2}\times 3(3\times 3-1) \quad [\text{By putting } n=3]$$
$$= 12$$

$$f(4) = \frac{1}{2}\times 4(3\times 4-1) \quad [\text{By putting } n=4]$$
$$= 22$$

$$f(5) = \frac{1}{2}\times 5(3\times 5-1) \quad [\text{By putting } n=5]$$
$$= 35$$

Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2) - f(1) = 5 - 1$$
$$= 4 \quad [\text{First order difference of } f(2) \text{ and } f(1)]$$

$$f(3) - f(2) = 12 - 5$$
$$= 7 \quad [\text{First order difference of } f(3) \text{ and } f(2)]$$

$$f(4) - f(3) = 22 - 12$$
$$= 10 \quad [\text{First order difference of } f(4) \text{ and } f(3)]$$

$$f(5) - f(4) = 35 - 22$$
$$= 13 \quad [\text{First order difference of } f(5) \text{ and } f(4)]$$

Then we find the second order differences by subtracting consecutive first-order differences.

$7-4=3$ [Second order difference of first two consecutive first order differences]

$10-7=3$ [Second order difference of second two consecutive first order differences]

$13-10=3$ [Second order difference of third two consecutive first order differences]

Thus we found that each second order difference is 3, so the second order differences are $\boxed{\text{constant}}$.

Answer 3mr.

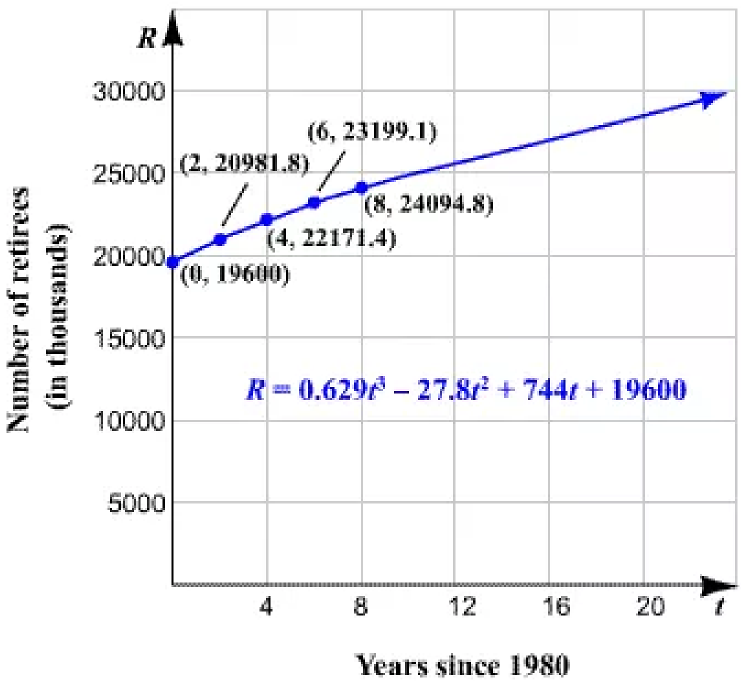
STEP 1 **Make** a table of values.

Since t represents the number of years, the model deals with only positive values of t. Also, as t is the number of years since 1980, $t=0$ corresponds to the year 1980.

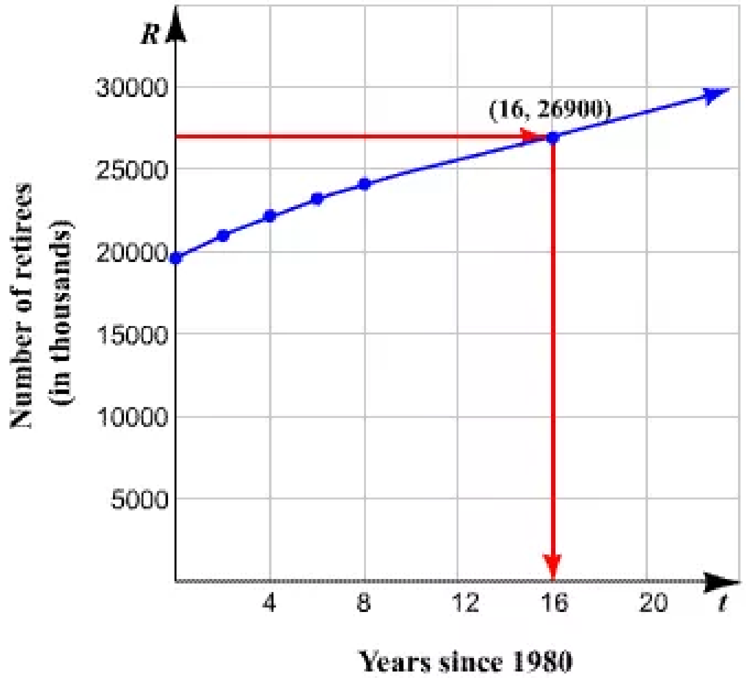
t	0	2	4	6	8	10
S	19,600	20981.8	22171.4	23199.1	24094.8	24889

STEP 2 **Plot** the points and connect them with a smooth curve.

Since the degree is even and the leading coefficient is positive, the graph tends to rise to the right.



STEP 3 Examine the graph to find the year when the number of retirees is about 26,900,000.



From the graph, it is clear that the number of retirees was about 26,900 thousands 16 years after 1980. This value corresponds to the year 1996.

Thus, the number of retirees receiving benefits was about 26,900,000 in the year 1996.

Answer 3q.

Write $f(x)$ as a product of three factors using the three zeros and the factor theorem.
$f(x) = (x + 4)(x + 1)(x - 2)$

Multiply $x + 1$ and $x - 2$ first.
$f(x) = (x + 4)[x^2 - 2x + x - 2]$

Combine the like terms within the brackets.
$f(x) = (x + 4)[x^2 - x - 2]$

Multiply again and combine the like terms.

$$f(x) = x^3 - x^2 - 2x + 4x^2 - 4x - 8$$
$$= x^3 + 3x^2 - 6x - 8$$

Thus, the required polynomial function is $f(x) = x^3 + 3x^2 - 6x - 8$.

Check the result by evaluating $f(x)$ at each of its three zeros. The function should evaluate to 0 each time.

$$f(-4) = (-4)^3 + 3(-4)^2 - 6(-4) - 8 = -64 + 48 + 24 - 8 = 0$$

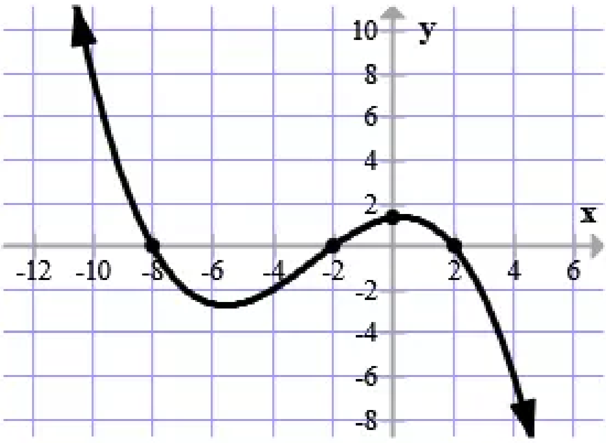
$$f(-1) = (-1)^3 + 3(-1)^2 - 6(-1) - 8 = -1 + 3 + 6 - 8 = 0$$

$$f(2) = (2)^3 + 3(2)^2 - 6(2) - 8 = 8 + 12 - 12 - 8 = 0$$

The result checks.

Answer 4e.

The given graph is



We need to write the cubic function.

By using the three x-intercepts $(-8,0),(-2,0),(2,0)$ to write the function in factorized form, we have

$$f(x) = a(x+8)(x+2)(x-2) \qquad \text{...... (1)}$$

To find the value of a by substituting the co-ordinate of the point $\left(0,\frac{4}{3}\right)$ in the equation (1), we have

$$f(x) = a(x+8)(x+2)(x-2)$$

$$\frac{4}{3} = a(0+8)(0+2)(0-2) \qquad \left[\text{By putting } x=0 \text{ and } y=\frac{4}{3}\right]$$

$$4 = a(8)(2)(-2)(3) \qquad [\text{Multiplying both sides by } 3]$$

$$4 = -96a \qquad [\text{By multiplying right side}]$$

$$\frac{4}{-96} = a \qquad [\text{Dividing both sides by } -96]$$

$$a = -\frac{1}{24}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

$$\boxed{f(x)=-\frac{1}{24}(x+8)(x+2)(x-2)}$$

Answer 4gp.

The given table of data is

x	1	2	3	4	5	6
$f(x)$	6	15	22	21	6	−29

We need to find the polynomial function by using the finite differences.

Now we begin by finding the finite differences.
Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2)-f(1)=15-6$$
$$=9 \quad \left[\text{First order difference of } f(2) \text{ and } f(1)\right]$$
$$f(3)-f(2)=22-15$$
$$=7 \quad \left[\text{First order difference of } f(3) \text{ and } f(2)\right]$$
$$f(4)-f(3)=21-22$$
$$=-1 \quad \left[\text{First order difference of } f(4) \text{ and } f(3)\right]$$
$$f(5)-f(4)=6-21$$
$$=-15 \quad \left[\text{First order difference of } f(5) \text{ and } f(4)\right]$$
$$f(6)-f(5)=-29-6$$
$$=-35 \quad \left[\text{First order difference of } f(6) \text{ and } f(5)\right]$$

Then we find the second order differences by subtracting consecutive first-order differences.

$$7-9=-2$$
$$-1-7=-8$$
$$-15-(-1)=-14$$
$$-35-(-15)=-20$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$-8-(-2)=-6$$

$$-14-(-8)=-6$$

$$-20-(-14)=-6$$

Since the third order difference is constant, the number can be represented by a cubic function of the form $f(n)=an^3+bn^2+cn+d$ ...... (1)

By substituting the first four numbers in the function (1) to obtain a system of four linear equations of four variables, we have

$$a(1)^3+b(1)^2+c(1)+d=6$$

$$a(2)^3+b(2)^2+c(2)+d=15$$

$$a(3)^3+b(3)^2+c(3)+d=22$$

$$a(4)^3+b(4)^2+c(4)+d=21$$

Or,

$$a+b+c+d=6 \quad \text{...... (2)}$$

$$8a+4b+2c+d=15 \quad \text{...... (3)}$$

$$27a+9b+3c+d=22 \quad \text{...... (4)}$$

$$64a+16b+4c+d=21 \quad \text{...... (5)}$$

By performing (3)-(2), we have

$$7a+3b+c=9 \quad \text{...... (6)}$$

By performing (4)-(3), we have

$$19a+5b+c=7 \quad \text{...... (7)}$$

By performing (5)-(4), we have

$$37a+7b+c=-1 \quad \text{...... (8)}$$

By performing (7)-(6), we have

$$12a+2b=-2$$

By performing (8)-(7), we have

$$18a+2b=-8$$

By solving these two equations, we have

$$a=-1, b=5$$

By putting the value of a and b in the above equations, we have

$$c=1, d=1$$

Therefore from the equation (1) the polynomial function is

$$f(n)=-1n^3+5n^2+1n+1$$

$$\boxed{f(n)=-n^3+5n^2+n+1}$$

Answer 4mr.

We need to find a polynomial function with real coefficients and degree 4 and zeros of $-2, 1$ and $4-i$.

Because the coefficients are rational and $4-i$ is zero so $4+i$ must also be zero by the Complex Conjugate Theorem. Now we use the four zeros and the factor theorem to write $f(x)$ as a product of four factors.

$$\begin{aligned}
f(x) &= (x+2)(x-1)\{x-(4-i)\}\{x-(4+i)\} && \text{[Writing } f(x) \text{ in factorized form]} \\
&= (x+2)(x-1)\{(x-4)+i\}\{(x-4)-i\} && \text{[Regroup]} \\
&= (x^2-x+2x-2)\{(x-4)^2-i^2\} && \text{[By multiplying]} \\
&= (x^2+x-2)\{(x-4)^2+1\} && \text{[Simplify]} \\
&= (x^2+x-2)(x^2-8x+17) && \text{[Expanding the binomial]} \\
&= x^4-8x^3+17x^2+x^3-8x^2+17x-2x^2+16x-34 && \\
&= x^4-7x^3+7x^2+33x-34 && \text{[Combining like terms]}
\end{aligned}$$

Therefore the polynomial function is $\boxed{f(x)=x^4-7x^3+7x^2+33x-34}$

Answer 4q.

We need to find a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1 and $-4, -1$ and 2.

Now we use the three zeros and the factor theorem to write $f(x)$ as a product of three factors.

$$\begin{aligned}
f(x) &= (x+4)(x+1)(x-2) && \text{[Writing } f(x) \text{ in factorized form]} \\
&= (x+4)(x^2-2x+x-2) && \text{[By multiplying]} \\
&= (x+4)(x^2-x-2) && \text{[Combining like terms]} \\
&= x^3-x^2-2x+4x^2-4x-8 && \text{[By multiplying]} \\
&= x^3+3x^2-6x-8 && \text{[Combining like terms]}
\end{aligned}$$

Check:
We can check this result by evaluating $f(x)$ at each of its three zeros.

$$\begin{aligned}
f(-4) &= (-4)^3+3(-4)^2-6(-4)-8 \\
&= -64+48+24-8 \\
&= 0 \\
f(-1) &= (-1)^3+3(-1)^2-6(-1)-8 \\
&= -1+3+6-8 \\
&= 0 \\
f(2) &= (2)^3+3(2)^2-6(2)-8 \\
&= 8+12-12-8 \\
&= 0
\end{aligned}$$

Therefore the polynomial function is $\boxed{f(x) = x^3 + 3x^2 - 6x - 8}$.

Answer 5e.

STEP 1 **Use** the three x-intercepts to write the function in factored form.
As can be seen, the given graph has three x-intercepts at -3, 1, and 4, which in turn are the three real zeros of the function. This follows that $x + 3$, $x - 1$ and $x - 4$ are factors of the function.

We can now apply the factor theorem to write the function corresponding to the graph.
$f(x) = a(x + 3)(x - 1)(x - 4)$

STEP 2 **Find** the value of a by substituting the coordinates of the fourth point.

The graph also contains a point (0, 2), which means that the value of the function is 2 when x is 0. Using this result, we can substitute 0 for x and 2 for $f(x)$ in the function.
$2 = a(0 + 3)(0 - 1)(0 - 4)$

Simplify and solve for a.

$$2 = a(3)(-1)(-4)$$
$$2 = 12a$$
$$\frac{1}{6} = a$$

Now, replace a with $\frac{1}{6}$ in the function.

$$f(x) = \frac{1}{6}(x + 3)(x - 1)(x - 4)$$

Remove the parentheses using the distributive property and simplify.

$$\begin{aligned} f(x) &= \frac{1}{6}(x^3 - 5x^2 + 4x + 3x^2 - 15x + 12) \\ &= \frac{1}{6}(x^3 - 2x^2 - 11x + 12) \\ &= \frac{1}{6}x^3 - \frac{1}{3}x^2 - \frac{11}{6}x + 2 \end{aligned}$$

Therefore, the cubic function is $f(x) = \frac{1}{6}x^3 - \frac{1}{3}x^2 - \frac{11}{6}x + 2$.

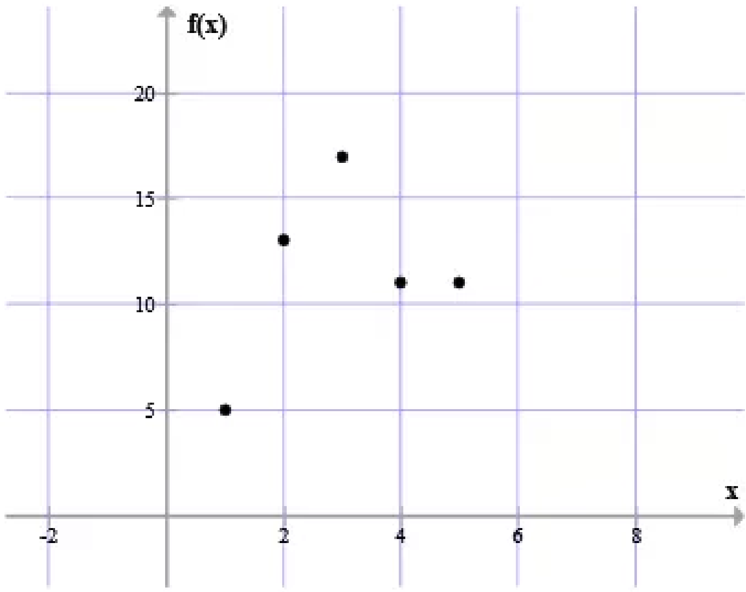
CHECK Check the end behavior of f.
The degree of f is odd and $a > 0$. As a result, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$, which matches with the graph.

Answer 5gp.

We need to find a polynomial function that fits the given table of data:

x	1	2	3	4	5	6
$f(x)$	5	13	17	11	11	56

We enter the data into a graphing calculator and make a scatter plot.

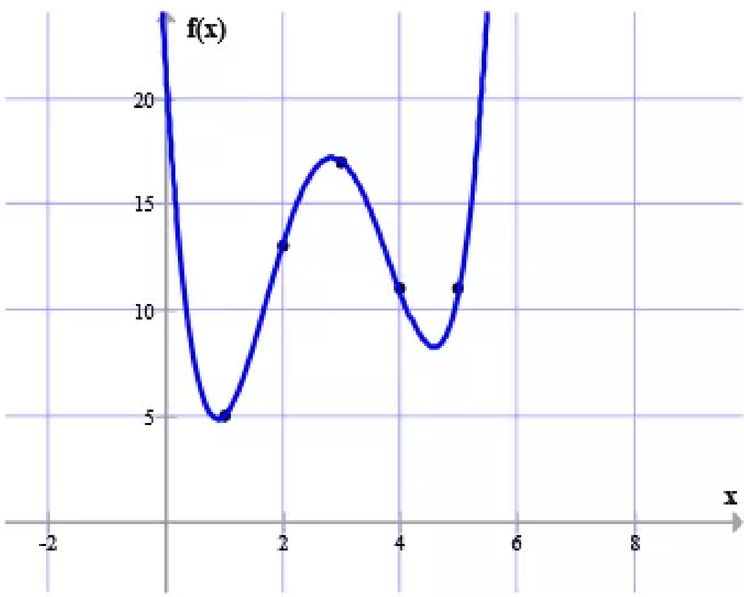


The points suggest a biquadratic model.

From the graph we obtain the polynomial model as:

$$f(x) = 0.9375x^4 - 10.412x^3 + 37.076x^2 - 44.431x + 21.833$$

Now we check the model by graphing it and the data in the same viewing window.



Thus the polynomial function for the given table of data is

$$\boxed{f(x) = 0.9375x^4 - 10.412x^3 + 37.076x^2 - 44.431x + 21.833}$$

Answer 5mr.

We need to write a polynomial function with rational coefficients that has 16 possible rational zeros according to the rational zero theorem but has no actual rational zeros.

The rational zero Theorem states that

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has integer coefficients, then every rational zero of f have the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Suppose we take $\frac{p}{q} = \frac{8}{3}$

Therefore

Factors of the constant term: $\pm1, \pm2, \pm4, \pm8$

Factors of the leading coefficient: $\pm1, \pm3$

Possible rational zeros are: $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

Therefore the polynomial function with these zeros BUT no actual rational zeros can be written as:

$$\boxed{f(x) = 3x^{16} + 8}$$

Answer 5q.

Since the coefficients are rational and $7 + \sqrt{2}$ is a zero of f, $7 - \sqrt{2}$ must also be a zero of f by irrational conjugates theorem.

Write $f(x)$ as a product of four factors using the four zeros and the factor theorem.

$$f(x) = (x+3)(x-5)\left[x - \left(7+\sqrt{2}\right)\right]\left[x - \left(7-\sqrt{2}\right)\right]$$

Regroup the terms within the brackets.

$$f(x) = (x+3)(x-5)\left[(x-7) - \sqrt{2}\right]\left[(x-7) + \sqrt{2}\right]$$

Multiply the expressions of the form $(a+b)(a-b)$.

$f(x) = (x+3)(x-5)[(x-7)^2 - 2]$

Expand the binomial.

$f(x) = (x+3)(x-5)[x^2 - 14x + 49 - 2]$

Simplify.

$f(x) = (x+3)(x-5)[x^2 - 14x + 47]$

Multiply again and combine the like terms.

$$\begin{aligned} f(x) &= (x+3)\left(x^3 - 14x^2 + 47x - 5x^2 + 70x - 235\right) \\ &= (x+3)\left(x^3 - 19x^2 + 117x - 235\right) \\ &= x^4 - 19x^3 + 117x^2 - 235x + 3x^3 - 57x^2 + 351x - 705 \\ &= x^4 - 16x^3 + 60x^2 + 116x - 705 \end{aligned}$$

Thus, the required polynomial function is $f(x) = x^4 - 16x^3 + 60x^2 + 116x - 705$.

Check the result by evaluating $f(x)$ at the zeros -3, 5, and $7+\sqrt{2}$. The function should evaluate to 0 each time.

$$f(-3) = (-3)^4 - 16(-3)^3 + 60(-3)^2 + 116(-3) - 705 = 0$$

$$f(5) = 5^4 - 16(5)^3 + 60(5)^2 + 116(5) - 705 = 0$$

$$f\left(7+\sqrt{2}\right) = \left(7+\sqrt{2}\right)^4 - 16\left(7+\sqrt{2}\right)^3 + 60\left(7+\sqrt{2}\right)^2 + 116\left(7+\sqrt{2}\right) - 705 = 0$$

As $7+\sqrt{2}$ checks, its conjugate $7-\sqrt{2}$ will also check.

Answer 6e.

We need to find the cubic function whose graph passes through the points $(-3,0),(-1,10),(0,0),(4,0)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y = a(x+3)(x-0)(x-4) \qquad \ldots\ldots (1)$$

To find the value of a by substituting the co-ordinate of the point $(-1,10)$ in the equation (1), we have

$$\begin{aligned} y &= a(x+3)(x-0)(x-4) \\ 10 &= a(-1+3)(-1-0)(-1-4) && [\text{By putting } x=-1 \text{ and } y=10] \\ 10 &= a(2)(-1)(-5) \\ 10 &= 10a && [\text{By multiplying}] \\ \frac{10}{10} &= a && [\text{Dividing both sides by } 10] \\ a &= 1 \end{aligned}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

$$y = 1(x+3)(x-0)(x-4)$$

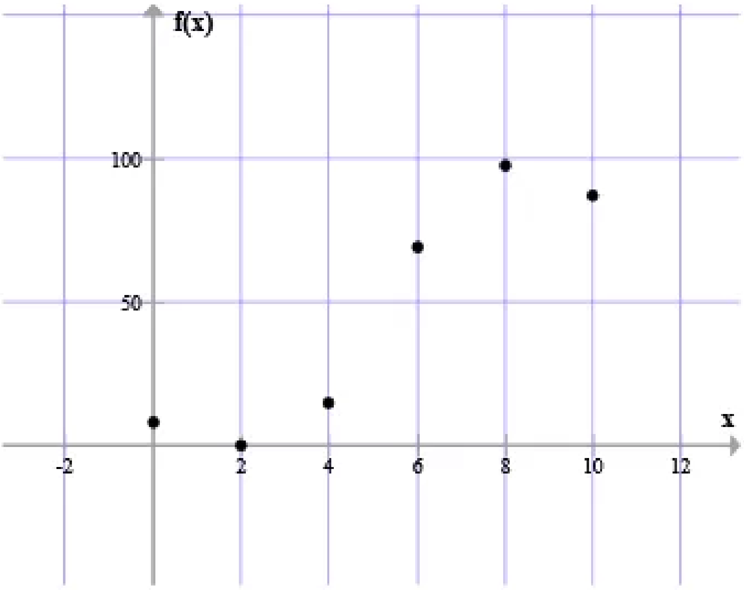
$$\boxed{y = x(x+3)(x-4)}$$

Answer 6gp.

We need to find a polynomial function that fits the given table of data:

x	0	2	4	6	8	10
$f(x)$	8	0	15	69	98	87

We enter the data into a graphing calculator and make a scatter plot.

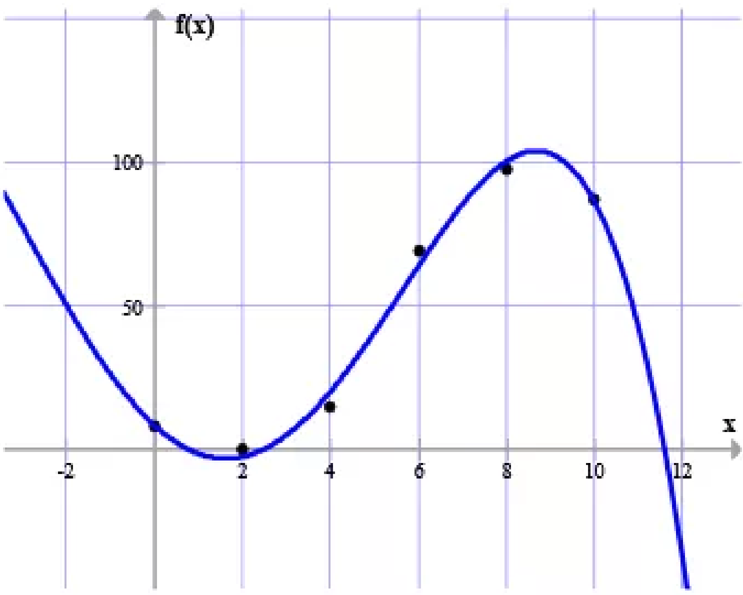


The points suggest a biquadratic model.

From the graph we obtain the polynomial model as:

$$f(x) = -0.04036x^4 + 0.22049x^3 + 4.026x^2 - 14.147x + 8.512$$

Now we check the model by graphing it and the data in the same viewing window.



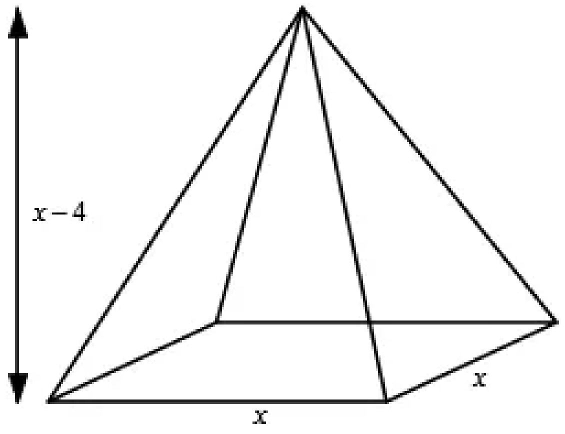
Thus the polynomial function for the given table of data is

$$\boxed{f(x) = -0.04036x^4 + 0.22049x^3 + 4.026x^2 - 14.147x + 8.512}$$

Answer 6mr.

(a)

The sculpture is as follows:



(b)

We need to write a function for the volume V of the sculpture.

We have the volume of pyramid as:

$$\text{Volume} = \frac{1}{3} \cdot \text{Area of base} \cdot \text{Height}$$

Therefore the volume of the sculpture is

$$V(x) = \frac{1}{3} \cdot x^2 \cdot (x-4) \quad [\text{Since length of the base is } x \text{ and height is } x+4]$$

$$V(x) = \frac{1}{3}x^2(x-4) \quad [\text{Writing the equation}]$$

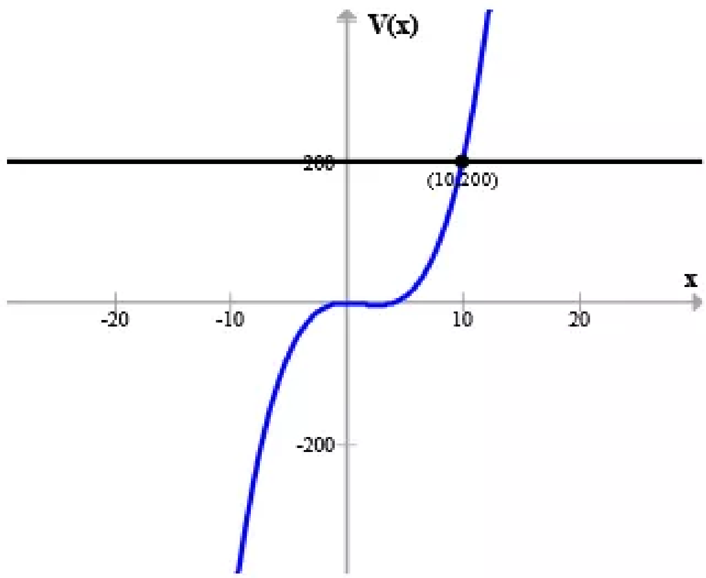
$$V(x) = \frac{1}{3}(x^3 - 4x^2) \quad [\text{Simplifying right side}]$$

Therefore the function for the volume of the sculpture is $\boxed{V(x) = \frac{1}{3}(x^3 - 4x^2)}$.

(c)

We need to draw the graph for the function $V(x)=\frac{1}{3}(x^3-4x^2)$.

The graph for $V(x)=\frac{1}{3}(x^3-4x^2)$ is as follows:



Given that, the volume of the sculpture as 200 cubic inches

From the graph we see that the value of x for the sculpture is $\boxed{10 \text{ inches}}$.

(d)

We have the function for the volume as:

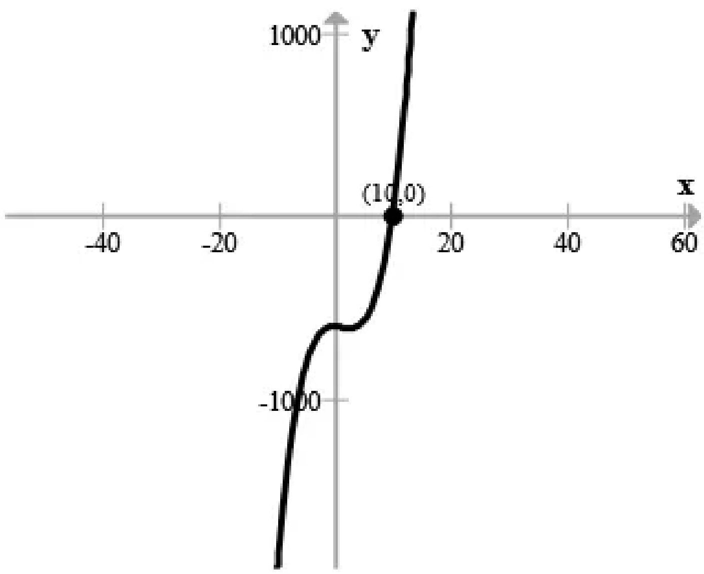
$$V(x)=\frac{1}{3}(x^3-4x^2)$$

$$200=\frac{1}{3}(x^3-4x^2) \quad [\text{Since volume is 200 cubic inches}]$$

$$600=x^3-4x^2 \quad [\text{Multiplying both sides by 3}]$$

$$0=x^3-4x^2-600 \quad [\text{Subtracting 600 from both sides}]$$

Now we draw the graph for $f(x)=x^3-4x^2-600$ as follows:



From the graph we see that 10 is one zero for the equation $x^3-4x^2-600=0$. Therefore

$$x^3-4x^2-600=0$$
$$(x-10)(x^2+6x+60)=0$$

Again solution of $(x^2+6x+60)=0$ is not real. Hence the equation has one single real root that is $x=10$.

Therefore the value of x for the sculpture is $\boxed{\text{10 inches}}$.

Thus we got the same value for x in part (c) and (d).

The dimension of the model is as follows:

The base is $\boxed{\text{10 inches by 10 inches}}$ and height is $10-4=\boxed{\text{6 inches}}$.

Answer 6q.

We need to find a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1 and $1, -2i$ and $3-\sqrt{6}$.

Because the coefficients are rational and $-2i$ and $3-\sqrt{6}$ are zeros so $2i$ and $3+\sqrt{6}$ must also be zeros by the Complex Conjugate Theorem and Irrational Conjugate Theorem. Now we use the five zeros and the factor theorem to write $f(x)$ as a product of five factors.

$$f(x)=(x-1)(x+2i)(x-2i)\{x-(3-\sqrt{6})\}\{x-(3+\sqrt{6})\} \quad \left[\text{Writing } f(x) \text{ in factorized form}\right]$$
$$=(x-1)(x+2i)(x-2i)\{(x-3)+\sqrt{6}\}\{(x-3)-\sqrt{6}\} \quad [\text{Regroup}]$$
$$=(x-1)(x^2+4)\{(x-3)^2-6\} \quad [\text{By multiplying}]$$
$$=(x^3+4x-x^2-4)(x^2-6x+9-6) \quad \left[\text{Multiplying and expanding the binomial}\right]$$
$$=(x^3-x^2+4x-4)(x^2-6x+3) \quad [\text{Simplify}]$$
$$=x^5-6x^4+3x^3-x^4+6x^3-3x^2+4x^3-24x^2+12x-4x^2+24x-12 \quad [\text{By multiplying}]$$
$$=x^5-7x^4+13x^3-31x^2+36x-12 \quad [\text{Combining like terms}]$$

Answer 7e.

STEP 1 **Use** the three x-intercepts to write the function in factored form.

From the given set of points, it can be identified that the graph has three x-intercepts at the points -2, -1 and 2, which in turn are the three real zeros of the function. This follows that $x+2$, $x+1$ and $x-2$ are factors of the function.

We can now apply the factor theorem to write the corresponding function.

$f(x)=a(x+2)(x+1)(x-2)$

STEP 2 **Find** the value of a by substituting the coordinates of the fourth point.

We are also given another point (0, −8), which means that the value of the function is −8 when x is 0. Using this result, we can substitute 0 for x and −8 for $f(x)$ in the function.

$-8 = a(0 + 2)(0 + 1)(0 - 2)$

Simplify and solve for a.

$$-8 = a(2)(1)(-2)$$
$$-8 = -4a$$
$$2 = a$$

Now, replace a with 2 in the function.

$f(x) = 2(x + 2)(x + 1)(x - 2)$

Remove the parentheses using the distributive property and simplify.

$$\begin{aligned} f(x) &= 2\left(x^3 - x^2 - 2x + 2x^2 - 2x - 4\right) \\ &= 2\left(x^3 + x^2 - 4x - 4\right) \\ &= 2x^3 + 2x^2 - 8x - 8 \end{aligned}$$

Therefore, the cubic function is $f(x) = 2x^3 + 2x^2 - 8x - 8$.

Answer 7mr.

The given table of data is

Month, t	1	2	3	4	5
Profit, p	4	2	6	22	56

where p represents the profit in dollars of the business in the first five months. We need to find the polynomial function for these data by using the finite differences.

Now we begin by finding the finite differences.

Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$\begin{aligned} p(2) - p(1) &= 2 - 4 \\ &= -2 \end{aligned}$$

$$\begin{aligned} p(3) - p(2) &= 6 - 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} p(4) - p(3) &= 22 - 6 \\ &= 16 \end{aligned}$$

$$\begin{aligned} p(5) - p(4) &= 56 - 22 \\ &= 34 \end{aligned}$$

Then we find the second order differences by subtracting consecutive first-order differences.

$$4-(-2)=6$$
$$16-4=12$$
$$34-16=18$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$12-6=6$$
$$18-12=6$$

Since the third order difference is constant, the number can be represented by a cubic function of the form $p(t)=at^3+bt^2+ct+d$ ...... (1)

By substituting the first four numbers in the function (1) to obtain a system of four linear equations of four variables, we have

$$a(1)^3+b(1)^2+c(1)+d=4$$
$$a(2)^3+b(2)^2+c(2)+d=2$$
$$a(3)^3+b(3)^2+c(3)+d=6$$
$$a(4)^3+b(4)^2+c(4)+d=22$$

Or,

$$a+b+c+d=4 \quad \text{...... (2)}$$
$$8a+4b+2c+d=2 \quad \text{...... (3)}$$
$$27a+9b+3c+d=6 \quad \text{...... (4)}$$
$$64a+16b+4c+d=22 \quad \text{...... (5)}$$

By performing (3)-(2), we have

$$7a+3b+c=-2 \quad \text{...... (6)}$$

By performing (4)-(3), we have

$$19a+5b+c=4 \quad \text{...... (7)}$$

By performing (5)-(4), we have

$$37a+7b+c=16 \quad \text{...... (8)}$$

By performing (7)-(6), we have

$$12a+2b=6$$

By performing (8)-(7), we have

$$18a+2b=12$$

By solving these two equations, we have

$$a=1, b=-3$$

By putting the value of a and b in the above equations, we have

$$c=0, d=6$$

Therefore from the equation (1) the polynomial function is

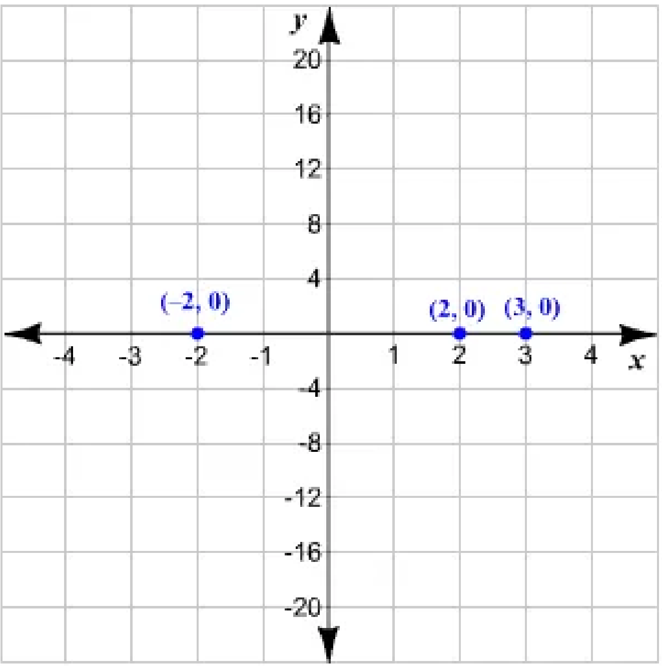
$$p(t)=1t^3+(-3)t^2+0t+6$$

$$\boxed{p(t)=t^3-3t^2+6}$$

Answer 7q.

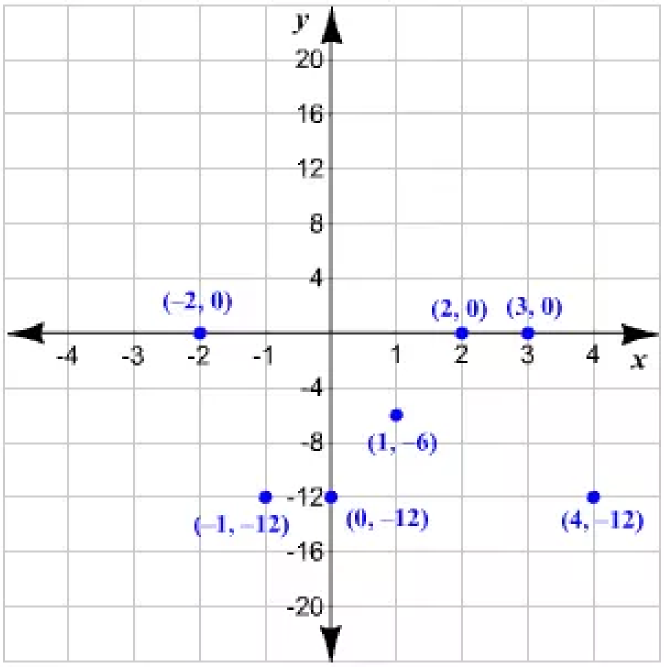
STEP 1 **Plot** the intercepts.

From the given function, we get the zeros, which are the x-intercepts of the graph of the function, as −2, 2 and 3. Plot the points (3, 0), (2, 0), and (−2, 0).



STEP 2 **Plot** points between and beyond the x-intercepts.

x	−1	0	1	4
y	−12	−12	−6	−12

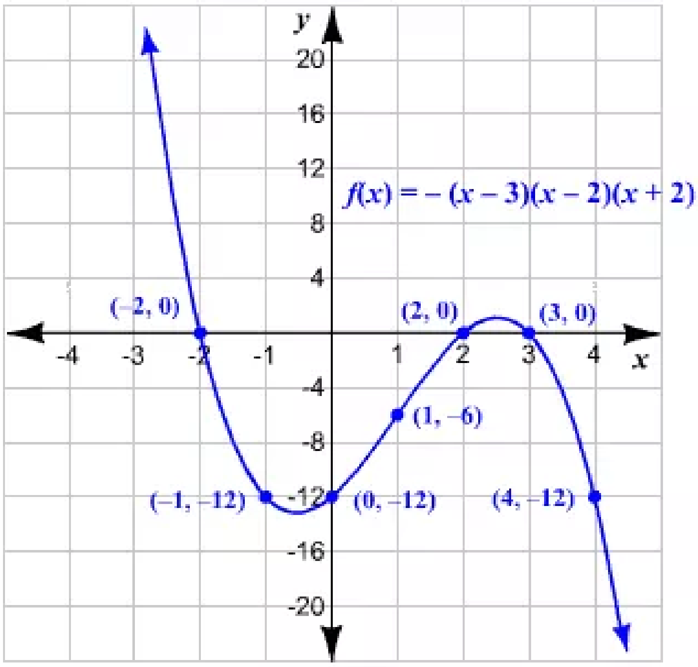


STEP 3 **Determine** end behavior.

Since $f(x)$ has three factors of the form $x - k$, the function is cubic. The leading coefficient is 4, which is positive.

For a function with odd degree and positive leading coefficient, the end behavior is $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

STEP 4 **Draw** the graph so that it passes through the plotted points and has the appropriate end behavior.



Answer 8e.

We need to find the cubic function whose graph passes through the points $(-3,0),(1,0),(3,2),(4,0)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y = a(x+3)(x-1)(x-4) \qquad \ldots\ldots (1)$$

To find the value of a by substituting the co-ordinate of the point $(3,2)$ in the equation (1), we have

$$y = a(x+3)(x-1)(x-4)$$

$$2 = a(3+3)(3-1)(3-4) \qquad [\text{By putting } x = 3 \text{ and } y = 2]$$

$$2 = a(6)(2)(-1)$$

$$2 = -12a \qquad [\text{By multiplying}]$$

$$\frac{2}{-12} = a \qquad [\text{Dividing both sides by } -12]$$

$$a = -\frac{1}{6}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

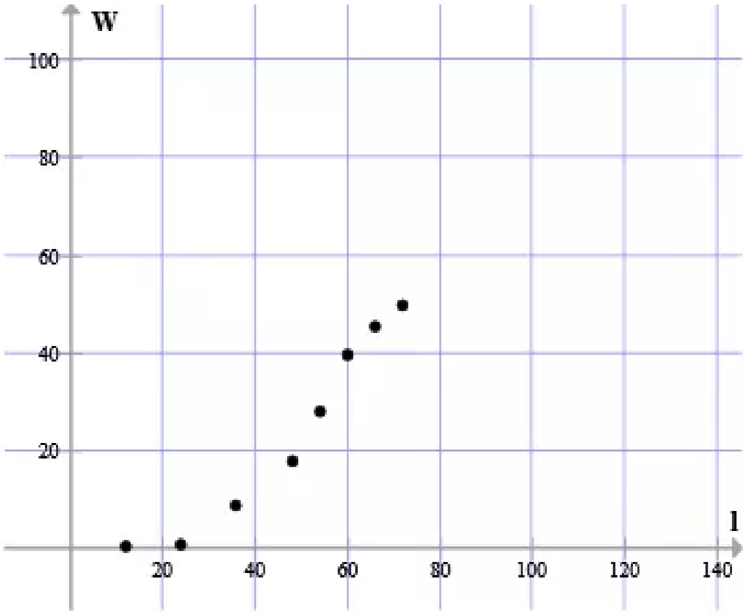
$$\boxed{y=-\frac{1}{6}(x+3)(x-1)(x-4)}$$

Answer 8mr.

We need to find a polynomial function for the average relationship between length (in inches) and weight (in pounds) of an alligator for the given table of data.

Length, l	12	24	36	48	54	60	66	72
Weight, W	0.2	0.7	8.6	17.7	28.0	39.6	45.4	49.6

We enter the data into a graphing calculator and make a scatter plot.

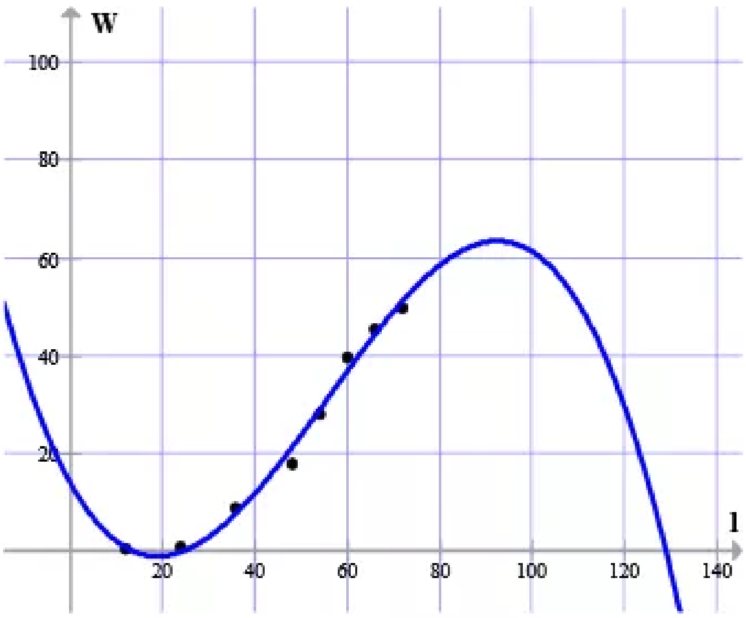


The points suggest a parabolic model.

From the graph we obtain the polynomial model as:

$$W(l)=-0.000324l^3+0.0542l^2-1.707l+13.94$$

Now we check the model by graphing it and the data in the same viewing window.

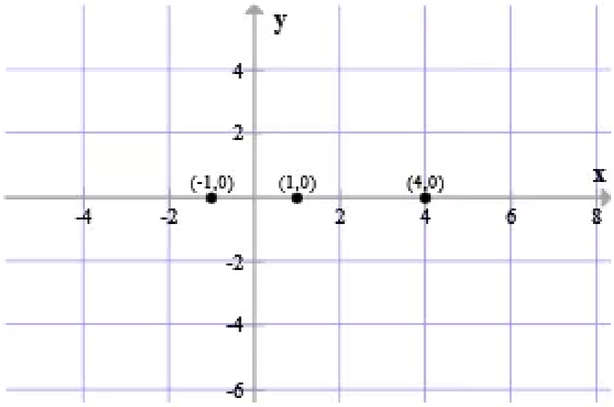


Thus the polynomial function for the average relationship between length and weight of an alligator is $\boxed{W(l) = -0.000324l^3 + 0.0542l^2 - 1.707l + 13.94}$.

Answer 8q.

We need to graph the function $f(x) = 3(x-1)(x+1)(x-4)$.

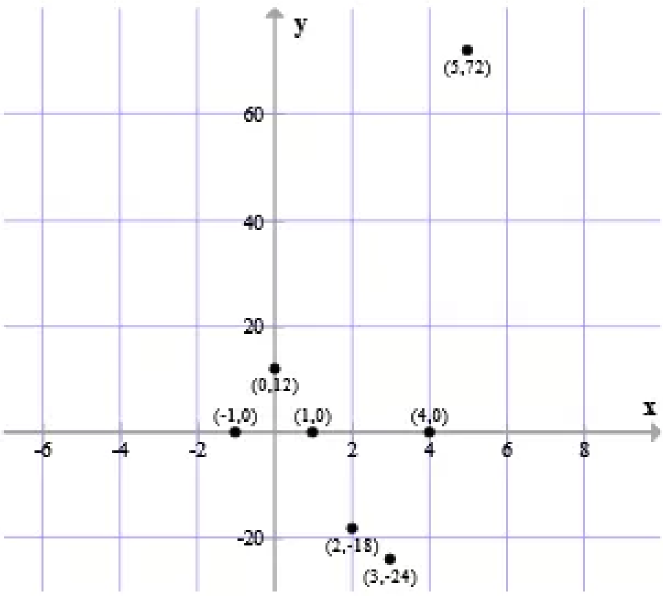
Since $1, -1$ and 4 are the zeros of f, we plot $(1,0), (-1,0)$ and $(4,0)$ as follows:



We plot points between and beyond the x-intercepts.

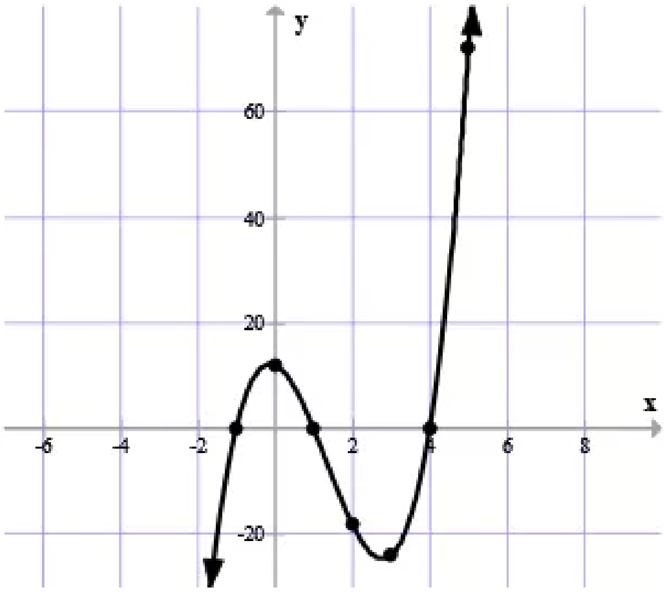
x	0	2	3	5
$f(x)$	12	-18	-24	72

The graph with these points is as follows:



We determine the end behavior. Because f has three factors of the form $(x-k)$ and a constant factor of 3, it is a cubic function with a positive leading coefficient. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Now we draw the graph so that it passes through the plotted points and has the appropriate end behavior. The graph is as follows:



Answer 9e.

STEP 1 **Use** the three x-intercepts to write the function in factored form.
From the given set of points, it can be identified that the graph has three x-intercepts at the points -5, 0 and 6, two of which in turn are among the real zeros of the function. This follows that $x+5$, $x-0$ and $x-6$ are factors of the function.

We can now apply the factor theorem to write the corresponding function.
$f(x) = a(x+5)(x-0)(x-6)$

Simplify.
$f(x) = a(x+5)(x-6)x$

STEP 2 **Find** the value of a by substituting the coordinates of the fourth point.

We are also given another point $(1, -12)$, which means that the value of the function is -12 when x is 1. Using this result, we can substitute 1 for x and -12 for $f(x)$ in the function.
$-12 = a(1+5)(1-6)(1)$

Simplify and solve for a.

$$-12 = a(6)(-5)$$

$$-12 = -30a$$

$$\frac{2}{5} = a$$

Now, replace a with $\frac{2}{5}$ in the function.

$$f(x) = \frac{2}{5}(x+5)(x-6)x$$

Remove the parentheses using the distributive property and simplify.

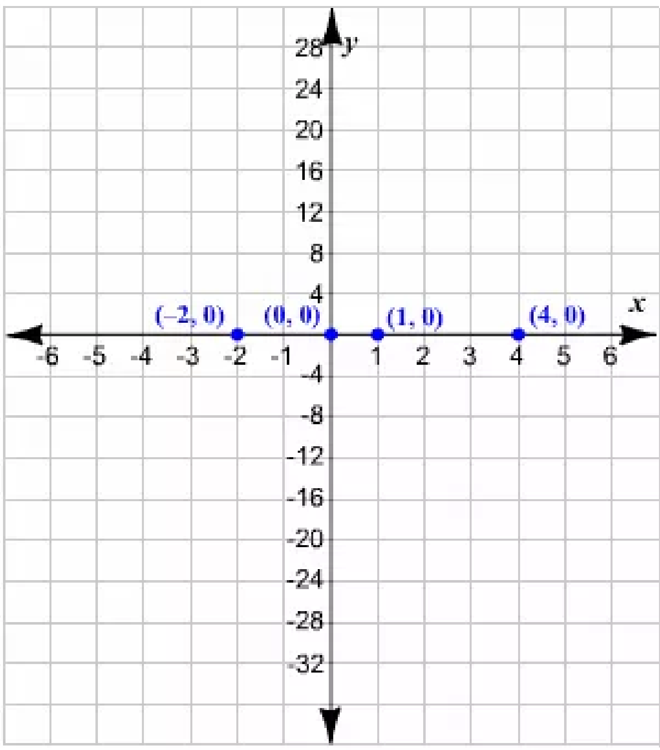
$$\begin{aligned} f(x) &= \frac{2}{5}\left(x^2 - x - 30\right)x \\ &= \frac{2}{5}\left(x^3 - x^2 - 30x\right) \\ &= \frac{2}{5}x^3 - \frac{2}{5}x^2 - 12x \end{aligned}$$

Therefore, the cubic function is $f(x) = \frac{2}{5}x^3 - \frac{2}{5}x^2 - 12x$.

Answer 9q.

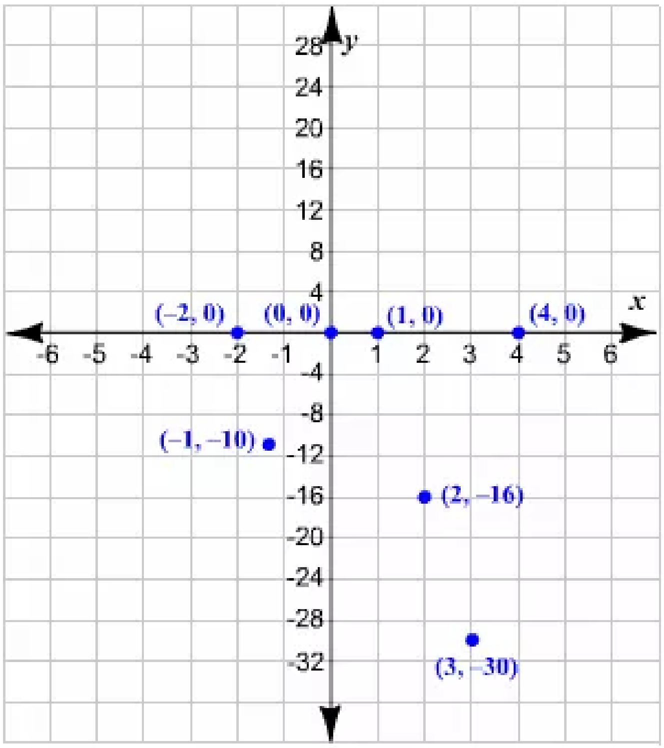
STEP 1 **Plot** the intercepts.

From the given function, we get the zeros, which are the x-intercepts of the graph of the function, as 0, 4, 1, and -2. Plot the points (0, 0), (4, 0), (1, 0) and (-2, 0).



STEP 2 **Plot** points between and beyond the x-intercepts.

x	-1	2	3
y	-10	-16	-30

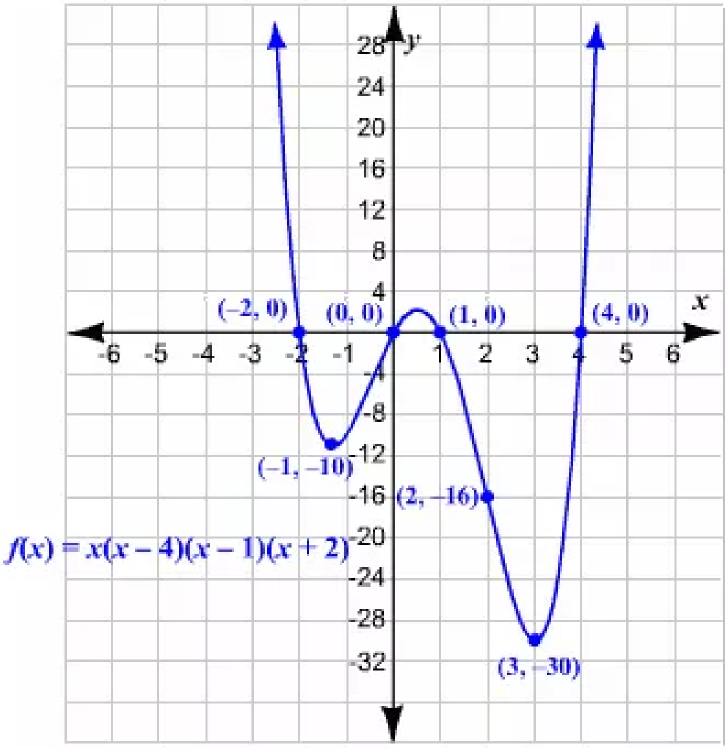


STEP 3 **Determine** end behavior.

Since $h(x)$ has four factors of the form $x - k$, the function is of the fourth power. The leading coefficient is 5, which is positive.

For a function with positive degree and positive leading coefficient, the end behavior is $h(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $h(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.

STEP 4 **Draw** the graph so that it passes through the plotted points and has the appropriate end behavior.



Answer 10e.

We need to find the cubic function whose graph passes through the points $(-3,0),(-1,0),(3,0),(0,3)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y = a(x+3)(x+1)(x-3) \quad \ldots\ldots (1)$$

To find the value of a by substituting the co-ordinate of the point $(0,3)$ in the equation (1), we have

$$y = a(x+3)(x+1)(x-3)$$

$$3 = a(0+3)(0+1)(0-3) \quad [\text{By putting } x=0 \text{ and } y=3]$$

$$3 = a(3)(1)(-3)$$

$$3 = -9a \quad [\text{By multiplying}]$$

$$\frac{3}{-9} = a \quad [\text{Dividing both sides by } -9]$$

$$a = -\frac{1}{3}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

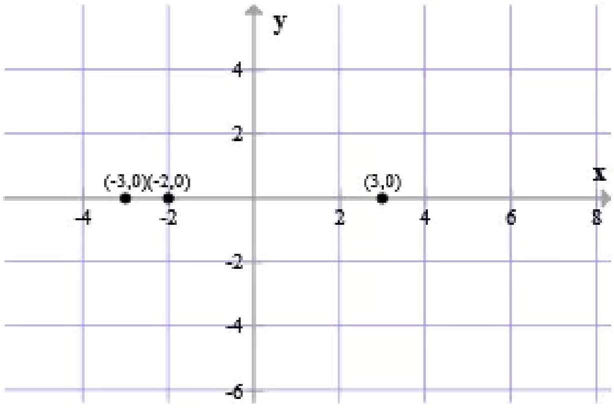
$$y=-\frac{1}{3}(x+3)(x+1)(x-3)$$

Therefore the answer is option $\boxed{B}$.

Answer 10q.

We need to graph the function $f(x)=(x-3)(x+2)^2(x+3)^2$.

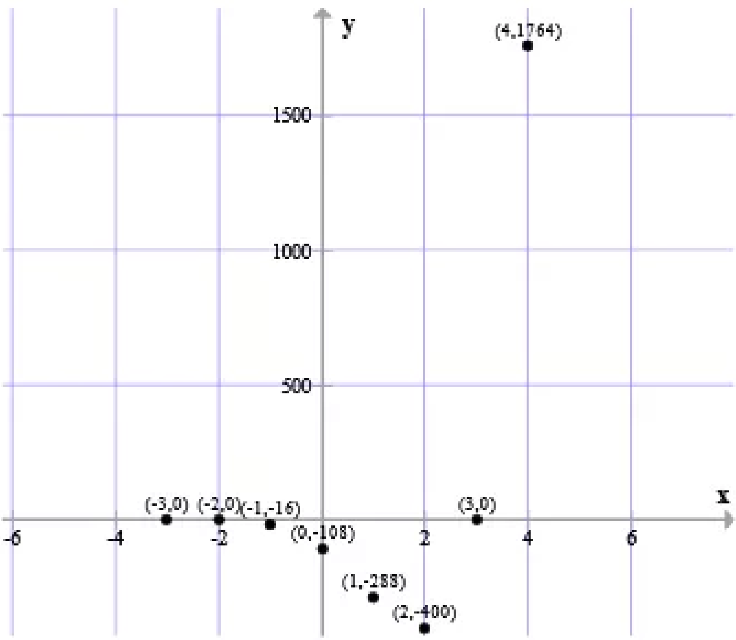
Since $3, -2$ and -3 are the zeros of f, we plot $(3,0), (-2,0)$ and $(-3,0)$ as follows:



We plot points between and beyond the x-intercepts.

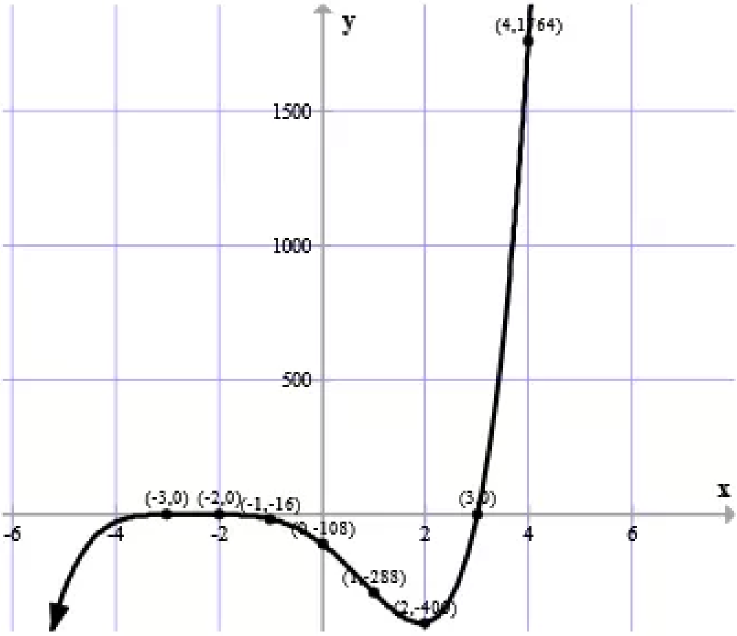
x	-1	0	1	2	4
$f(x)$	-16	-108	-288	-400	1764

The graph with these points is as follows:



We determine the end behavior. Because f has five factors of the form $(x-k)$ and a constant factor of 1, it is a pentagonal function with a positive leading coefficient. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Now we draw the graph so that it passes through the plotted points and has the appropriate end behavior. The graph is as follows:



Answer 11e.

It is given that the graph has three x-intercepts at the points -1, 2 and 5, which in turn are the three real zeros of the function. This follows that $x + 1$, $x - 2$ and $x - 5$ are factors of the function.

We can now apply the factor theorem to write the corresponding function.
$f(x) = a(x + 1)(x - 2)(x - 5)$

We are also given another point (1, 3), which means that the value of the function is 3 when x is 1. Using this result, we can substitute 1 for x and 3 for $f(x)$ in the function.
$3 = a(1 + 1)(1 - 2)(1 - 5)$

The student, on the other hand, has made mistake in this replacement. The value 1 has been substituted for y and 3 for x. The calculation, and hence, the value of the leading coefficient is not correct.

Now, let us proceed with our calculations and find a.
Simplify.

$$3 = a(2)(-1)(-4)$$

$$3 = 8a$$

$$\frac{3}{8} = a$$

Therefore, the correct value of the leading coefficient is $\frac{3}{8}$.

Answer 11q.

STEP 1 **Use** the three x-intercepts to write the function in factored form.

From the given set of points, it can be identified that the graph has three x-intercepts at the points -5, -2 and 2, two of which in turn are among the real zeros of the function. This follows that $x + 5$, $x + 2$ and $x - 2$ are factors of the function.

We can now apply the factor theorem to write the corresponding function.

$f(x) = a(x + 5)(x + 2)(x - 2)$

STEP 2 **Find** the value of a by substituting the coordinates of the fourth point.

We are also given another point (1, 9), which means that the value of the function is -12 when x is 1. Using this result, we can substitute 1 for x and 9 for $f(x)$ in the function.

$9 = a(1 + 5)(1 + 2)(1 - 2)$

Simplify and solve for a.

$$9 = a(6)(3)(-1)$$
$$9 = -18a$$
$$-\frac{1}{2} = a$$

Now, replace a with $-\frac{1}{2}$ in the function.

$$f(x) = -\frac{1}{2}(x+5)(x+2)(x-2)$$

Remove the parentheses using the distributive property and simplify.

$$f(x) = -\frac{1}{2}(x^3 + 5x^2 - 4x - 20)$$
$$= -\frac{1}{2}x^3 - \frac{5}{2}x^2 + 2x + 10$$

Therefore, the cubic function is $f(x) = -\frac{1}{2}x^3 - \frac{5}{2}x^2 + 2x + 10$.

Answer 12e.

We need to show that the nth order difference for the given function of degree n is nonzero and constant.

$$f(x) = 5x^3 - 10$$

The first six numbers using the given function are shown below:

$$f(1) = 5(1)^3 - 10 \qquad [\text{By putting } x = 1]$$
$$= -5$$
$$f(2) = 5(2)^3 - 10 \qquad [\text{By putting } x = 2]$$
$$= 30$$
$$f(3) = 5(3)^3 - 10 \qquad [\text{By putting } x = 3]$$
$$= 125$$
$$f(4) = 5(4)^3 - 10 \qquad [\text{By putting } x = 4]$$
$$= 310$$
$$f(5) = 5(5)^3 - 10 \qquad [\text{By putting } x = 5]$$
$$= 615$$
$$f(6) = 5(6)^3 - 10 \qquad [\text{By putting } x = 6]$$
$$= 1070$$

Now we find the first-order differences by subtracting consecutive numbers.

$$f(2) - f(1) = 30 - (-5)$$
$$= 35$$
$$f(3) - f(2) = 125 - 30$$
$$= 95$$
$$f(4) - f(3) = 310 - 125$$
$$= 185$$
$$f(5) - f(4) = 615 - 310$$
$$= 305$$
$$f(6) - f(5) = 1070 - 615$$
$$= 455$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$95 - 35 = 60$$
$$185 - 95 = 90$$
$$305 - 185 = 120$$
$$455 - 305 = 150$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$90-60=30$$
$$120-90=30$$
$$150-120=30$$

Thus we obtain that the function $f(x)=5x^3-10$ of degree 3 has the third order difference as nonzero and constant.

Answer 12q.

We need to find the cubic function whose graph passes through the points $(-1,0),(0,16),(2,0),(4,0)$.

By using the three given x-intercepts to write the function in factorized form, we have

$$y=a(x+1)(x-2)(x-4) \qquad \text{......} (1)$$

To find the value of a by substituting the co-ordinate of the point $(0,16)$ in the equation (1), we have

$$y=a(x+1)(x-2)(x-4)$$
$$16=a(0+1)(0-2)(0-4) \quad [\text{By putting } x=0 \text{ and } y=16]$$
$$16=a(1)(-2)(-4)$$
$$10=8a \quad [\text{By multiplying}]$$
$$\frac{10}{8}=a \quad [\text{Dividing both sides by } 8]$$
$$a=\frac{5}{4}$$

By putting the value of a in the equation (1), we will get the cubic function. Therefore the function is

$$\boxed{y=\frac{5}{4}(x+1)(x-2)(x-4)}$$

Answer 13e.

The given function is of the second degree.

By the properties of finite differences, if a polynomial function $f(x)$ has degree n, then the nth order differences of function values will be nonzero and constant for equally-spaced values of x.

This means that the second-order differences of the function will be constant.

Let us first find the values of $f(x)$ for some equally spaced x-values.

$$f(1) = -2(1)^2 + 5(1) = -2 + 5 = 3$$

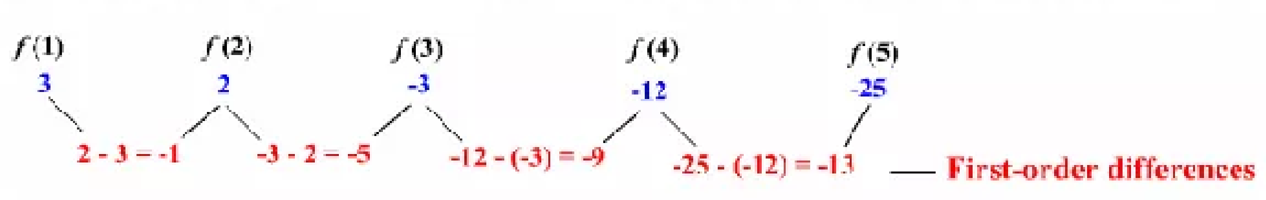
$$f(2) = -2(2)^2 + 5(2) = -8 + 10 = 2$$

$$f(3) = -2(3)^2 + 5(3) = -18 + 15 = -3$$

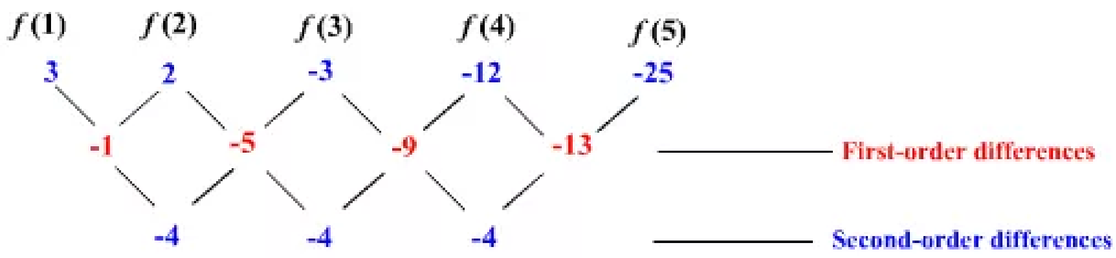
$$f(4) = -2(4)^2 + 5(4) = -32 + 20 = -12$$

$$f(5) = -2(5)^2 + 5(5) = -50 + 25 = -25$$

The difference between the consecutive y-values will give the first-order differences.



Now, find the second-order differences by subtracting consecutive first-order differences.



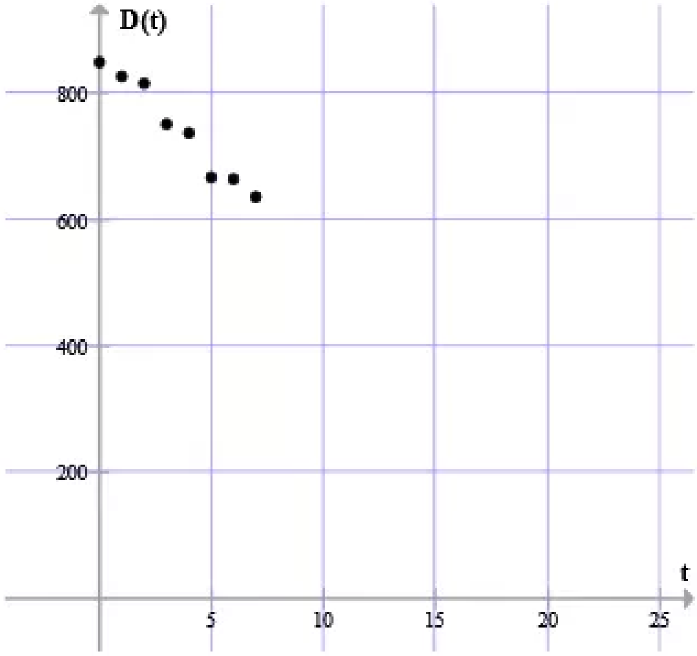
Each second-order difference of the function is –4.
Therefore, it is proved that the second-order differences for the given function are constant.

Answer 13q.

We need to find a polynomial function of the number of U.S. drive-in movie theaters for the year 1995 to 2002 for the following table of data.

Year since 1995 , t	0	1	2	3	4	5	6	7
Drive-in movie theaters, D	848	826	815	750	737	667	663	634

We enter the data into a graphing calculator and make a scatter plot.

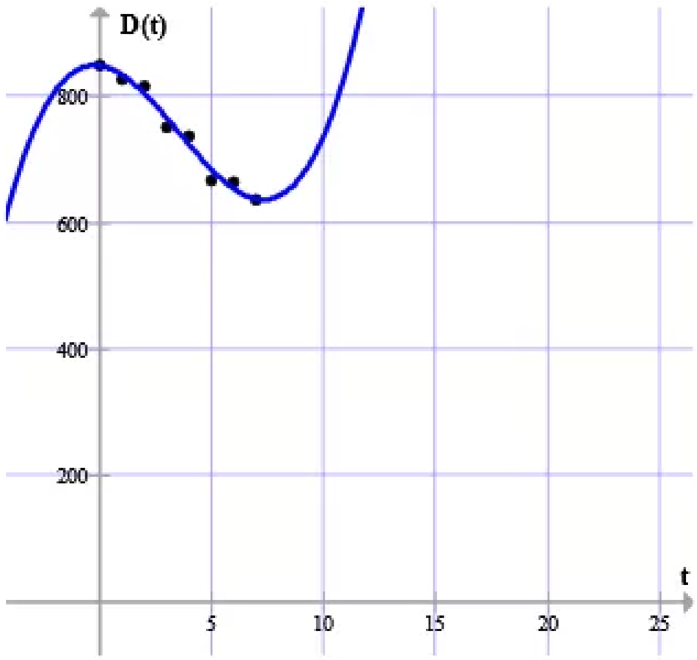


The points suggest a cubic model.

From the graph we obtain the polynomial model as:

$$D(t) = 0.989t^3 - 10.50t^2 - 5t + 846.77$$

Now we check the model by graphing it and the data in the same viewing window.



Thus the polynomial function for the number of U.S. drive-in movie theaters for the year 1995 to 2002 is $\boxed{D(t) = 0.989t^3 - 10.50t^2 - 5t + 846.77}$.

Answer 14e.

We need to show that the nth order difference for the given function of degree n is nonzero and constant.

$$f(x) = x^4 - 3x^2 + 2$$

The first six numbers using the given function are shown below:

$$f(1) = (1)^4 - 3(1)^2 + 2 \quad [\text{By putting } x = 1]$$
$$= 0$$
$$f(2) = (2)^4 - 3(2)^2 + 2 \quad [\text{By putting } x = 2]$$
$$= 6$$
$$f(3) = (3)^4 - 3(3)^2 + 2 \quad [\text{By putting } x = 3]$$
$$= 56$$
$$f(4) = (4)^4 - 3(4)^2 + 2 \quad [\text{By putting } x = 4]$$
$$= 210$$
$$f(5) = (5)^4 - 3(5)^2 + 2 \quad [\text{By putting } x = 5]$$
$$= 552$$
$$f(6) = (6)^4 - 3(6)^2 + 2 \quad [\text{By putting } x = 6]$$
$$= 1190$$
$$f(7) = (7)^4 - 3(7)^2 + 2 \quad [\text{By putting } x = 7]$$
$$= 2256$$

Now we find the first-order differences by subtracting consecutive numbers.

$$f(2) - f(1) = 6 - 0$$
$$= 6$$
$$f(3) - f(2) = 56 - 6$$
$$= 50$$
$$f(4) - f(3) = 210 - 56$$
$$= 154$$
$$f(5) - f(4) = 552 - 210$$
$$= 342$$
$$f(6) - f(5) = 1190 - 552$$
$$= 638$$
$$f(7) - f(6) = 2256 - 1190$$
$$= 1066$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$50-6=44$$
$$154-50=104$$
$$342-154=188$$
$$638-342=296$$
$$1066-638=428$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$104-44=60$$
$$188-104=84$$
$$296-188=108$$
$$428-296=132$$

Then we find the fourth order differences by subtracting consecutive third-order differences.

$$84-60=24$$
$$108-84=24$$
$$132-108=24$$

Thus we obtain that the function $f(x)=x^4-3x^2+2$ of degree 4 has the fourth order difference as nonzero and constant.

Answer 15e.

The given function is of the second degree.

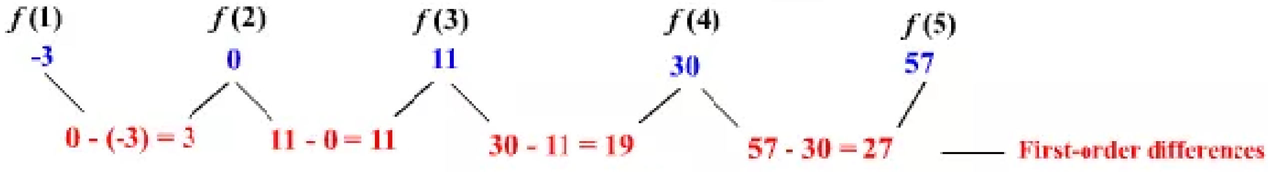
By the properties of finite differences, if a polynomial function $f(x)$ has degree n, then the nth order differences of function values will be nonzero and constant for equally-spaced values of x.

This means that the second-order differences of the function will be constant.

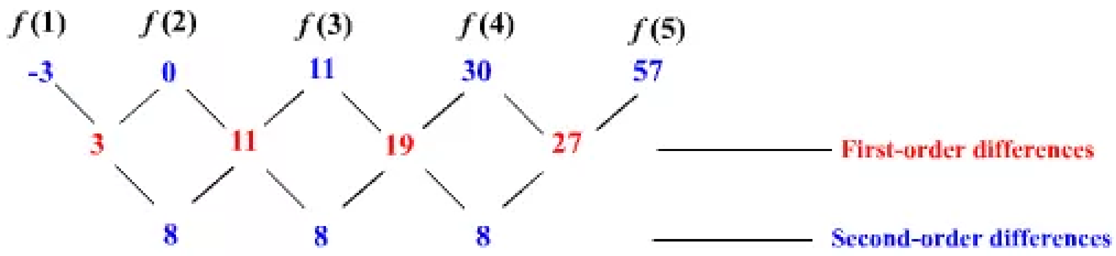
Let us first find the values of $f(x)$ for some equally spaced x-values.

$$f(1) = 4(1)^2 - 9(1) + 2 = 4 - 9 + 2 = -3$$
$$f(2) = 4(2)^2 - 9(2) + 2 = 16 - 18 + 2 = 0$$
$$f(3) = 4(3)^2 - 9(3) + 2 = 36 - 27 + 2 = 11$$
$$f(4) = 4(4)^2 - 9(4) + 2 = 64 - 36 + 2 = 30$$
$$f(5) = 4(5)^2 - 9(5) + 2 = 100 - 45 + 2 = 57$$

The difference between the consecutive y-values will give the first-order differences.



Now, find the second-order differences by subtracting consecutive first-order differences.



Each second-order difference of the function is 8.
Therefore, it is proved that the second-order differences for the given function are nonzero and constant.

Answer 16e.

We need to show that the nth order difference for the given function of degree n is nonzero and constant.

$$f(x)=x^3-4x^2-x+1$$

The first six numbers using the given function are shown below:

$$f(1)=(1)^3-4(1)^2-(1)+1 \quad [\text{By putting } x=1]$$
$$=-3$$

$$f(2)=(2)^3-4(2)^2-(2)+1 \quad [\text{By putting } x=2]$$
$$=-9$$

$$f(3)=(3)^3-4(3)^2-(3)+1 \quad [\text{By putting } x=3]$$
$$=-11$$

$$f(4)=(4)^3-4(4)^2-(4)+1 \quad [\text{By putting } x=4]$$
$$=-3$$

$$f(5)=(5)^3-4(5)^2-(5)+1 \quad [\text{By putting } x=5]$$
$$=21$$

$$f(6)=(6)^3-4(6)^2-(6)+1 \quad [\text{By putting } x=6]$$
$$=67$$

Now we find the first-order differences by subtracting consecutive numbers.

$$\begin{aligned} f(2)-f(1) &= -9-(-3) \\ &= -6 \\ f(3)-f(2) &= -11-(-9) \\ &= -2 \\ f(4)-f(3) &= -3-(-11) \\ &= 8 \\ f(5)-f(4) &= 21-(-3) \\ &= 24 \\ f(6)-f(5) &= 67-21 \\ &= 46 \end{aligned}$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$\begin{aligned} -2-(-6) &= 4 \\ 8-(-2) &= 10 \\ 24-8 &= 16 \\ 46-24 &= 22 \end{aligned}$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$\begin{aligned} 10-4 &= 6 \\ 16-10 &= 6 \\ 22-16 &= 6 \end{aligned}$$

Thus we obtain that the function $f(x) = x^3 - 4x^2 - x + 1$ of degree 3 has the third order difference as nonzero and constant.

Answer 17e.

The given function is of degree 5.

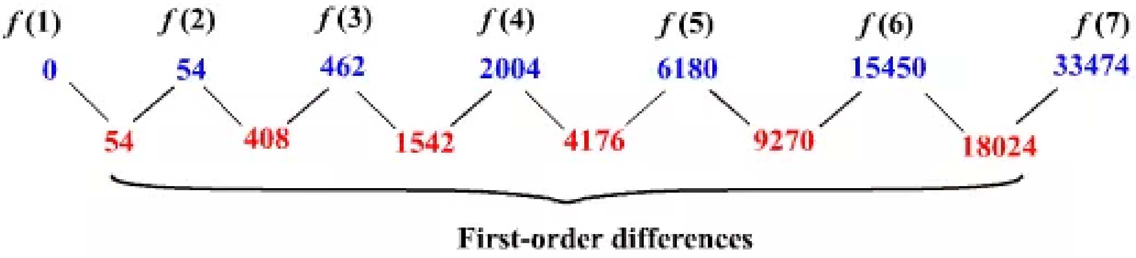
By the properties of finite differences, if a polynomial function $f(x)$ has degree n, then the nth order differences of function values will be nonzero and constant for equally-spaced values of x.

This means that the fifth-order differences of the function will be constant.

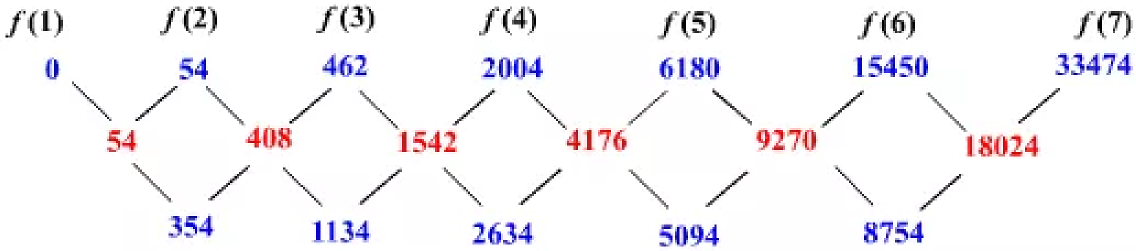
Let us first find the values of $f(x)$ for some equally spaced x-values.
For example, substitute 1 for x in $f(x)$ to find $f(1)$.
$f(1) = 2(1)^5 - 3(1)^2 + 1 = 2 - 2 + 1 = 0$

In a similar way, calculate $f(x)$ for values of x, say, up to 7.
$f(2) = 54, f(3) = 462, f(4) = 2004, f(5) = 6180, f(6) = 15450, f(7) = 33474$

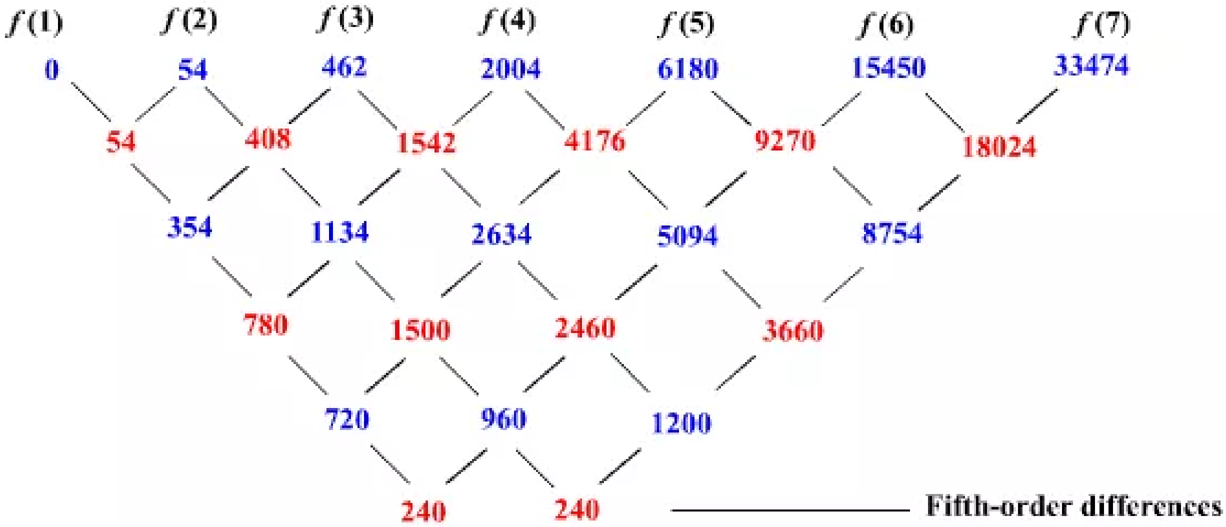
The difference between the consecutive y-values will give the first-order differences.



Now, find the second-order differences by subtracting the consecutive first-order differences.



Similarly, find the fifth-order differences.



Each fifth-order difference of the function is 240.
Therefore, it is proved that the fifth-order differences for the given function are nonzero and constant.

Answer 18e.

The given table of data is

x	1	2	3	4	5	6
$f(x)$	0	-3	-8	-15	-24	-35

We need to find the polynomial function by using the finite differences and system of equations.

Now we begin by finding the finite differences.

Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2)-f(1)=-3-0$$
$$=-3$$
$$f(3)-f(2)=-8-(-3)$$
$$=-5$$
$$f(4)-f(3)=-15-(-8)$$
$$=-7$$
$$f(5)-f(4)=-24-(-15)$$
$$=-9$$
$$f(6)-f(5)=-35-(-24)$$
$$=-11$$

Then we find the second order differences by subtracting consecutive first-order differences.

$$-5-(-3)=-2$$
$$-7-(-5)=-2$$
$$-9-(-7)=-2$$
$$-11-(-9)=-2$$

Since the second order difference is constant, the number can be represented by a quadratic function of the form $f(n)=an^2+bn+c$ ...... (1)

By substituting the first three numbers in the function (1) to obtain a system of three linear equations of three variables, we have

$$a(1)^2+b(1)+c=0$$
$$a(2)^2+b(2)+c=-3$$
$$a(3)^2+b(3)+c=-8$$

Or,

$$a+b+c=0 \quad \text{...... (2)}$$
$$4a+2b+c=-3 \quad \text{...... (3)}$$
$$9a+3b+c=-8 \quad \text{...... (4)}$$

By performing (3)-(2), we have

$$3a-b=-3$$

By performing (4)-(3), we have

$$5a-b=-5$$

By solving these two equations, we have

$$a=-1$$

By putting the value of a in $3a-b=-3$, we have

$$b=0$$

By putting the value of a and b in the equation (2), we have

$$c=1$$

Therefore from the equation (1) the polynomial function is

$$f(n)=-1n^2+0n+1$$

$$\boxed{f(n)=-n^2+1}$$

Answer 19e.

The given table of data is

x	1	2	3	4	5	6
$f(x)$	11	14	9	–4	–25	–54

We need to find the polynomial function by using the finite differences and system of equations.

Now we begin by finding the finite differences.
Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2)-f(1)=14-11$$
$$=3$$
$$f(3)-f(2)=9-14$$
$$=-5$$
$$f(4)-f(3)=-4-9$$
$$=-13$$
$$f(5)-f(4)=-25-(-4)$$
$$=-21$$
$$f(6)-f(5)=-54-(-25)$$
$$=-29$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$-5-3=-8$$
$$-13-(-5)=-8$$
$$-21-(-13)=-8$$
$$-29-(-21)=-8$$

Since the second order difference is constant, the number can be represented by a quadratic function of the form $f(n)=an^2+bn+c$ ...... (1)

By substituting the first three numbers in the function (1) to obtain a system of three linear equations of three variables, we have

$$a(1)^2+b(1)+c=11$$

$$a(2)^2+b(2)+c=14$$

$$a(3)^2+b(3)+c=9$$

Or,

$$a+b+c=11 \quad ......\ (2)$$
$$4a+2b+c=14 \quad ......\ (3)$$
$$9a+3b+c=9 \quad ......\ (4)$$

By performing (3)-(2), we have

$$3a-b=3$$

By performing (4)-(3), we have

$$5a-b=-5$$

By solving these two equations, we have

$$a=-4$$

By putting the value of a in $3a-b=3$, we have

$$b=-15$$

By putting the value of a and b in the equation (2), we have

$$c=30$$

Therefore from the equation (1) the polynomial function is

$$\boxed{f(n)=-4n^2-15n+30}$$

Answer 20e.

The given table of data is

x	1	2	3	4	5	6
$f(x)$	−12	−14	−10	6	40	98

We need to find the polynomial function by using the finite differences and system of equations.

Now we begin by finding the finite differences.
Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2)-f(1)=-14-(-12)$$
$$=-2$$

$$f(3)-f(2)=-10-(-14)$$
$$=4$$

$$f(4)-f(3)=6-(-10)$$
$$=16$$

$$f(5)-f(4)=40-6$$
$$=34$$

$$f(6)-f(5)=98-40$$
$$=58$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$4-(-2)=6$$
$$16-4=12$$
$$34-16=18$$
$$58-34=24$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$12-6=6$$
$$18-12=6$$
$$24-18=6$$

Since the third order difference is constant, the number can be represented by a cubic function of the form $f(n)=an^3+bn^2+cn+d$ ...... (1)

By substituting the first four numbers in the function (1) to obtain a system of four linear equations of four variables, we have

$$a(1)^3+b(1)^2+c(1)+d=-12$$
$$a(2)^3+b(2)^2+c(2)+d=-14$$
$$a(3)^3+b(3)^2+c(3)+d=-10$$
$$a(4)^3+b(4)^2+c(4)+d=6$$

Or,

$$a+b+c+d=-12 \quad ...... (2)$$
$$8a+4b+2c+d=-14 \quad ...... (3)$$
$$27a+9b+3c+d=-10 \quad ...... (4)$$
$$64a+16b+4c+d=6 \quad ...... (5)$$

By performing (3)-(2), we have

$$7a+3b+c=-2 \quad ...... (6)$$

By performing (4)-(3), we have

$$19a+5b+c=4 \quad ...... (7)$$

By performing (5)-(4), we have

$$37a+7b+c=16 \quad ...... (8)$$

By performing (7)-(6), we have

$$12a+2b=6$$

By performing (8)-(7), we have

$$18a+2b=12$$

By solving these two equations, we have

$$a=1, b=-3$$

By putting the value of a and b in the above equations, we have

$$c=0, d=-10$$

Therefore from the equation (1) the polynomial function is

$$f(n)=1n^3-3n^2+0n-10$$

$$\boxed{f(n)=n^3-3n^2-10}$$

Answer 21e.

The given table of data is

x	1	2	3	4	5	6
$f(x)$	5	14	27	41	53	60

We need to find the polynomial function by using the finite differences and system of equations.

Now we begin by finding the finite differences.
Now we find the first-order differences by subtracting consecutive pentagonal numbers.

$$f(2)-f(1)=14-5$$
$$=9$$
$$f(3)-f(2)=27-14$$
$$=13$$
$$f(4)-f(3)=41-27$$
$$=14$$
$$f(5)-f(4)=53-41$$
$$=12$$
$$f(6)-f(5)=60-53$$
$$=7$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$13-9=4$$
$$14-13=1$$
$$12-14=-2$$
$$7-12=-5$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$1-4=-3$$
$$-2-1=-3$$
$$-5-(-2)=-3$$

Since the third order difference is constant, the number can be represented by a cubic function of the form $f(n)=an^3+bn^2+cn+d$ ...... (1)

By substituting the first four numbers in the function (1) to obtain a system of four linear equations of four variables, we have

$$a(1)^3+b(1)^2+c(1)+d=5$$

$$a(2)^3+b(2)^2+c(2)+d=14$$

$$a(3)^3+b(3)^2+c(3)+d=27$$

$$a(4)^3+b(4)^2+c(4)+d=41$$

Or,

$$a+b+c+d=5 \quad \text{...... (2)}$$
$$8a+4b+2c+d=14 \quad \text{...... (3)}$$
$$27a+9b+3c+d=27 \quad \text{...... (4)}$$
$$64a+16b+4c+d=41 \quad \text{...... (5)}$$

By performing (3)-(2), we have

$$7a+3b+c=9 \quad \text{...... (6)}$$

By performing (4)-(3), we have

$$19a+5b+c=13 \quad \text{...... (7)}$$

By performing (5)-(4), we have

$$37a+7b+c=14 \quad \text{...... (8)}$$

By performing (7)-(6), we have

$$12a+2b=4$$

By performing (8)-(7), we have

$$18a+2b=1$$

By solving these two equations, we have

$$a=-\frac{1}{2}, b=5$$

By putting the value of a and b in the above equations, we have

$$c=-\frac{5}{2}, d=3$$

Therefore from the equation (1) the polynomial function is

$$\boxed{f(n)=-\frac{1}{2}n^3+5n^2-\frac{5}{2}n+3}$$

Answer 22e.

We need to find two different cubic functions whose graph pass through the points $(-3,0),(-1,0)$ and $(2,6)$.

Since two x-intercepts are given we suppose the third x-intercept as $(b,0)$ for the cubic functions.

By using the three given x-intercepts to write the function in factorized form, we have

$$y=a(x+3)(x+1)(x-b) \quad \text{...... (1)}$$

By substituting the co-ordinate of the point $(2,6)$ in the equation (1), we have

$$y = a(x+3)(x+1)(x-b)$$
$$6 = a(2+3)(2+1)(2-b) \quad [\text{By putting } x=2 \text{ and } y=6]$$
$$6 = a(5)(3)(2-b)$$
$$6 = 15a(2-b) \quad [\text{By multiplying}]$$
$$a(2-b) = \frac{6}{15} \quad [\text{Dividing both sides by } 15]$$
$$(2-b) = \frac{2}{5a}$$

By putting $b=1$ in the equation $(2-b)=\frac{2}{5a}$ we obtain $a=\frac{2}{5}$.

By putting $b=3$ in the equation $(2-b)=\frac{2}{5a}$ we obtain $a=-\frac{2}{5}$.

Therefore the two cubic functions are

$$\boxed{y=\frac{2}{5}(x+3)(x+1)(x-1)} \text{ and } \boxed{y=-\frac{2}{5}(x+3)(x+1)(x-3)}.$$

Answer 23e.

A function having fourth degree is termed as a quartic function. Likewise, a quintic function has degree 5.

We know that functions can be written from a set of data values. It is necessary that there should be more data point than the degree of the equation. For example, three data points would be required to determine a function which is of the second degree.

In a similar way, 5 points are required for determining a quartic function, and 6 for determining quintic functions.

Answer 24e.

The given function is

$$f(x) = ax^3 + bx^2 + cx + d \quad \ldots\ldots (1)$$

By substituting the expressions $k, k+1, k+2, \cdots, k+5$ for x in the given function we need to show that the third order differences are constant.

Therefore by putting $x = k, k+1, k+2, \cdots, k+5$ in the equation (1), we have

$$f(k) = a(k)^3 + b(k)^2 + c(k) + d$$
$$f(k+1) = a(k+1)^3 + b(k+1)^2 + c(k+1) + d$$
$$f(k+2) = a(k+2)^3 + b(k+2)^2 + c(k+2) + d$$
$$f(k+3) = a(k+3)^3 + b(k+3)^2 + c(k+3) + d$$
$$f(k+4) = a(k+4)^3 + b(k+4)^2 + c(k+4) + d$$
$$f(k+5) = a(k+5)^3 + b(k+5)^2 + c(k+5) + d$$

Now we find the first-order differences by subtracting consecutive expressions.

$$f(k+1)-f(k)=3ak^2+3ak+2bk+a+b+c$$
$$f(k+2)-f(k+1)=3ak^2+9ak+2bk+7a+3b+c$$
$$f(k+3)-f(k+2)=3ak^2+15ak+2bk+19a+5b+c$$
$$f(k+4)-f(k+3)=3ak^2+21ak+2bk+37a+7b+c$$
$$f(k+5)-f(k+4)=3ak^2+27ak+2bk+61a+9b+c$$

Now we find the second-order differences by subtracting consecutive first-order differences.

$$3ak^2+9ak+2bk+7a+3b+c-(3ak^2+3ak+2bk+a+b+c)=6ak+6a+2b$$
$$3ak^2+15ak+2bk+19a+5b+c-(3ak^2+9ak+2bk+7a+3b+c)=6ak+12a+2b$$
$$3ak^2+21ak+2bk+37a+7b+c-(3ak^2+15ak+2bk+19a+5b+c)=6ak+18a+2b$$
$$3ak^2+27ak+2bk+61a+9b+c-(3ak^2+21ak+2bk+37a+7b+c)=6ak+24a+2b$$

Now we find the third-order differences by subtracting consecutive first-order differences.

$$6ak+12a+2b-(6ak+6a+2b)=6a$$
$$6ak+18a+2b-(6ak+12a+2b)=6a$$
$$6ak+24a+2b-(6ak+18a+2b)=6a$$

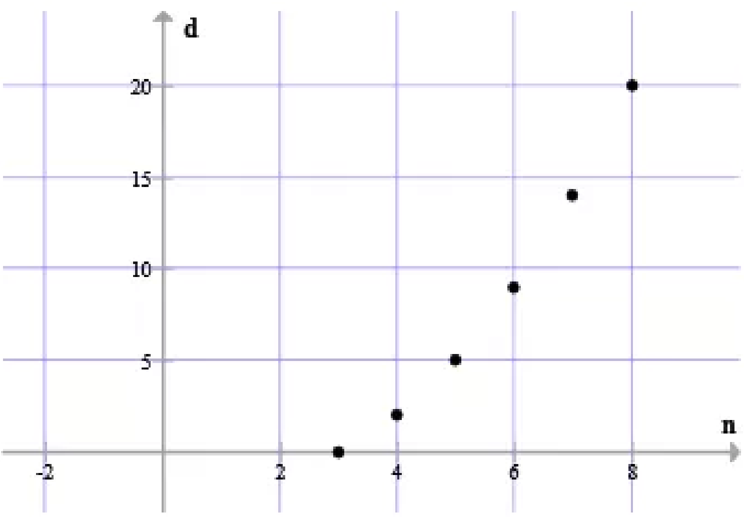
Therefore we obtain that the third order differences are constant.

Answer 25e.

We need to find a polynomial function that gives the number of diagonals d of a polygon with n sides for the given table of data.

Number of sides, n	3	4	5	6	7	8
Number of diagonals, d	0	2	5	9	14	20

We enter the data into a graphing calculator and make a scatter plot.

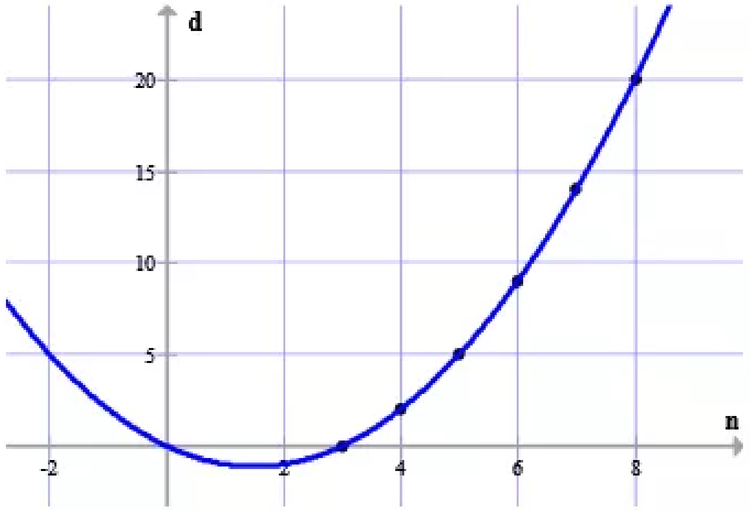


The points suggest a parabolic model.

From the graph we obtain the polynomial model as:

$$d(n) = 0.5n^2 - 1.5n$$

Now we check the model by graphing it and the data in the same viewing window.



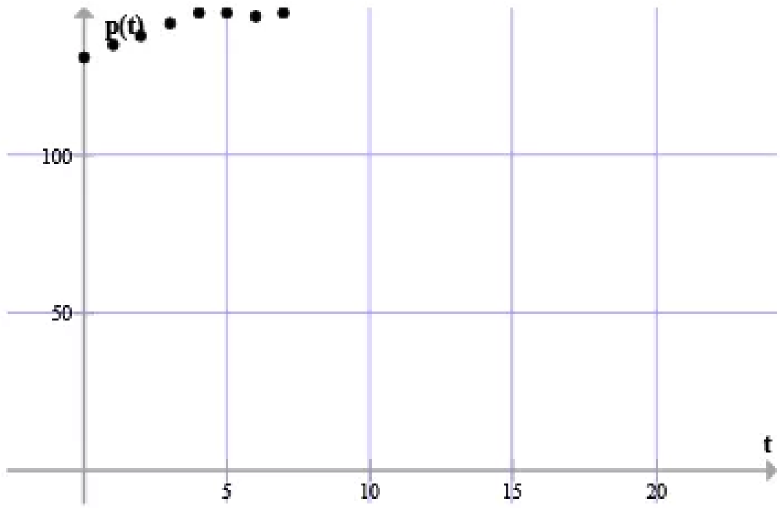
Thus the polynomial function that gives the number of diagonals d of a polygon with n sides is $\boxed{d(n) = 0.5n^2 - 1.5n}$.

Answer 26e.

We need to find a polynomial function for active pilots with airline transport licenses for the year 1997 to 2004 for the following table of data.

Year since 1997, t	0	1	2	3	4	5	6	7
Transport pilots, p	131	135	138	142	145	145	144	145

We enter the data into a graphing calculator and make a scatter plot.

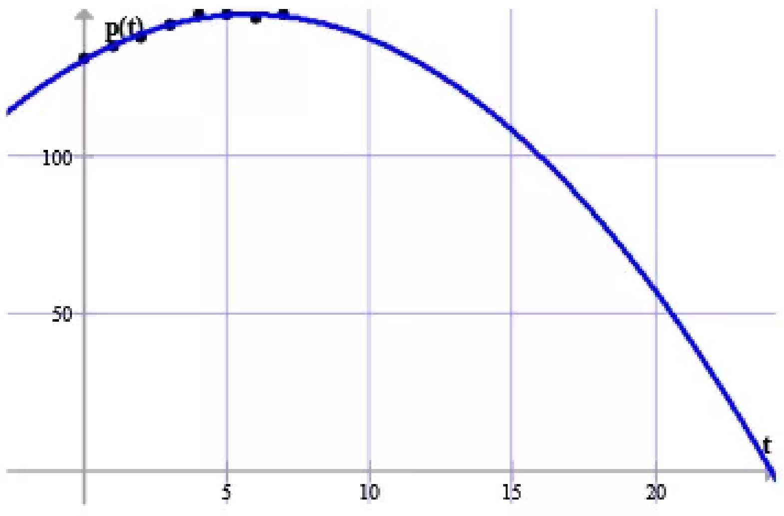


The points suggest a parabolic model.

From the graph we obtain the polynomial model as:

$$p(t) = -0.4345t^2 + 5.029t + 130.625$$

Now we check the model by graphing it and the data in the same viewing window.



Thus the polynomial function for active pilots with airline transport licenses for the year 1997 to 2004 is $\boxed{p(t) = -0.4345t^2 + 5.029t + 130.625}$.

Thus the polynomial function for active pilots with airline transport licenses for the year 1997 to 2004 is $\boxed{p(t) = -0.4345t^2 + 5.029t + 130.625}$.

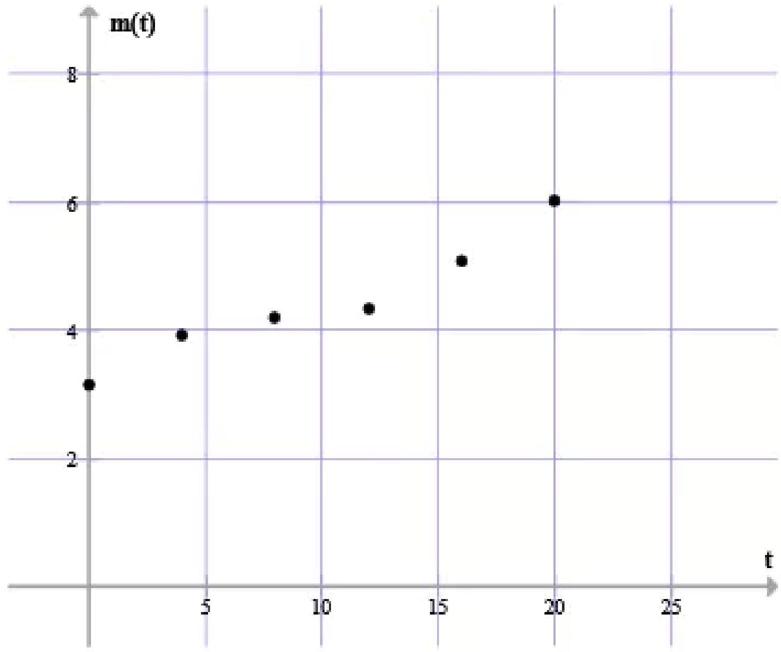
Answer 27e.

The following table shows the average U.S. movie ticket price (in dollars) for the years from 1983 to 2003.

Year since 1983, t	0	4	8	12	16	20
Movie ticket price, m	3.15	3.91	4.21	4.35	5.08	6.03

(a)

We need to find the polynomial model for these data by using the graphing calculator. We enter the data into a graphing calculator and make a scatter plot.

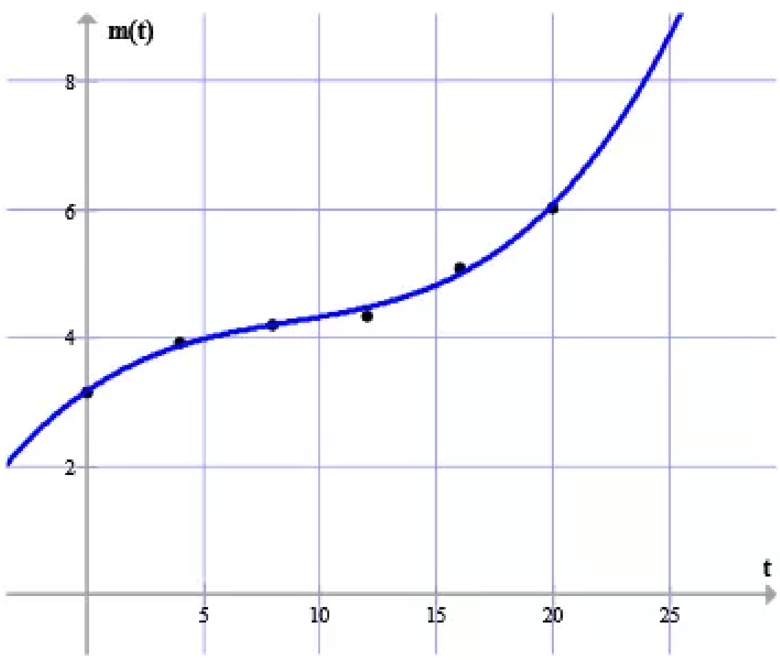


The points suggest a cubic model.

From the graph we obtain the polynomial model as:

$$m(t) = 0.000817t^3 - 0.0215t^2 + 0.249t + 3.168$$

Now we check the model by graphing it and the data in the same viewing window.



Thus the polynomial function representing the given data is

$$\boxed{m(t)=0.000817t^3-0.0215t^2+0.249t+3.168}$$

(b)

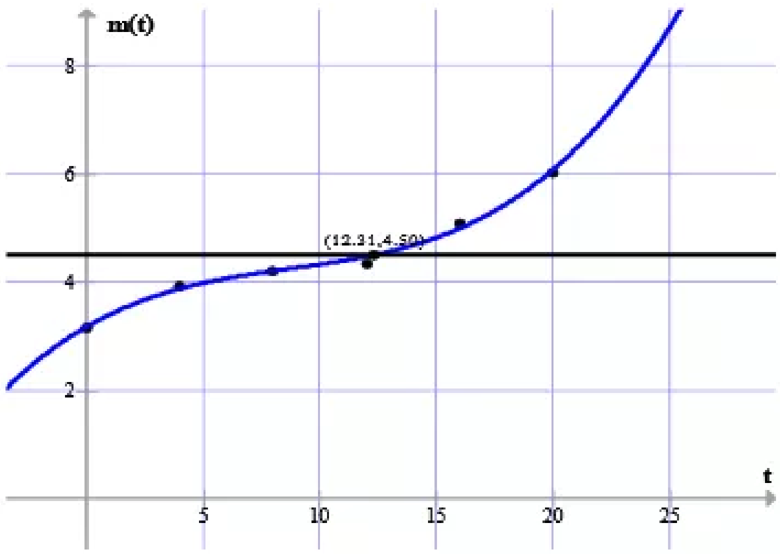
We need to estimate the average U.S. movie ticket price in 2010.
We take $t=27$ since year is taken from 1983. Therefore

$$\begin{aligned} m(27)&=0.000817(27)^3-0.0215(27)^2+0.249(27)+3.168\\ &=0.000817\times19683-0.0215\times729+0.249\times27+3.168\\ &=16.081011-15.6735+6.723+3.168\\ &=10.299 \end{aligned}$$

Therefore the average U.S. movie ticket price in 2010 is $\boxed{\$10.299}$.

(c)

We need to find the year in which the average U.S. movie ticket price is about \$4.50.
We graph the model and the line $y=4.50$ in the same window.



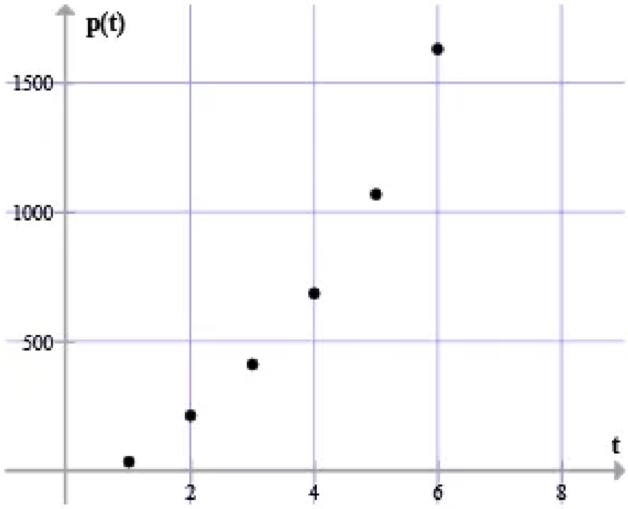
From the graph we use the intersecting point $(12.31, 4.50)$. Hence the movie ticket price is about \$4.50 in the year $(1983+12=)$ $\boxed{1995}$.

Answer 28e.

The following table shows the cumulative profit (in dollars) after each of six months for Yard Work business.

Month, t	1	2	3	4	5	6
Profit, p	30	210	410	680	1070	1630

We need to find the polynomial model for these data by using the graphing calculator. We enter the data into a graphing calculator and make a scatter plot.

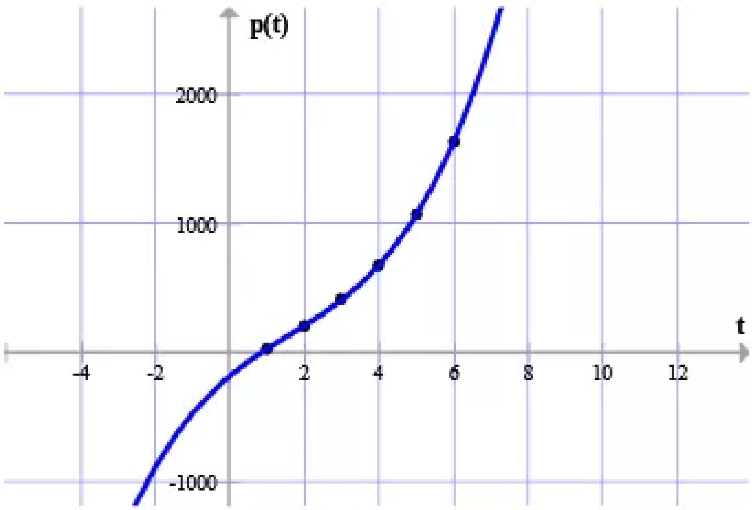


The points suggest a cubic model.

From the graph we obtain the polynomial model as:

$$p(t) = 8.33t^3 - 40t^2 + 241.67t - 180$$

Now we check the model by graphing it and the data in the same viewing window.



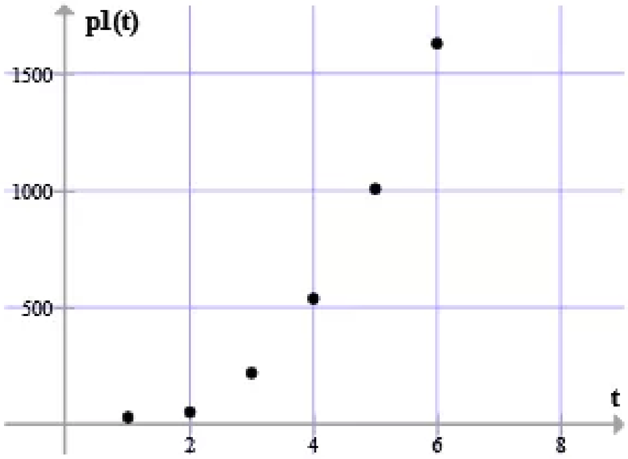
Thus the polynomial function representing the given data is

$$\boxed{p(t) = 8.33t^3 - 40t^2 + 241.67t - 180}$$

Again, the following table shows the cumulative profit (in dollars) after each of six months for Pet Care business.

Month, t	1	2	3	4	5	6
Profit, p_1	30	50	220	540	1010	1630

We need to find the polynomial model for these data by using the graphing calculator.
We enter the data into a graphing calculator and make a scatter plot.

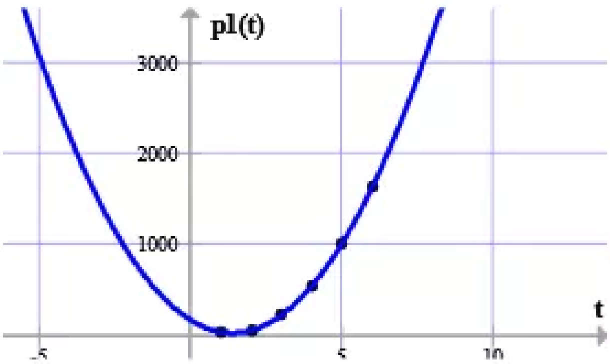


The points suggest a parabolic model.

From the graph we obtain the polynomial model as:

$$p_1(t) = 75t^2 - 205t + 160$$

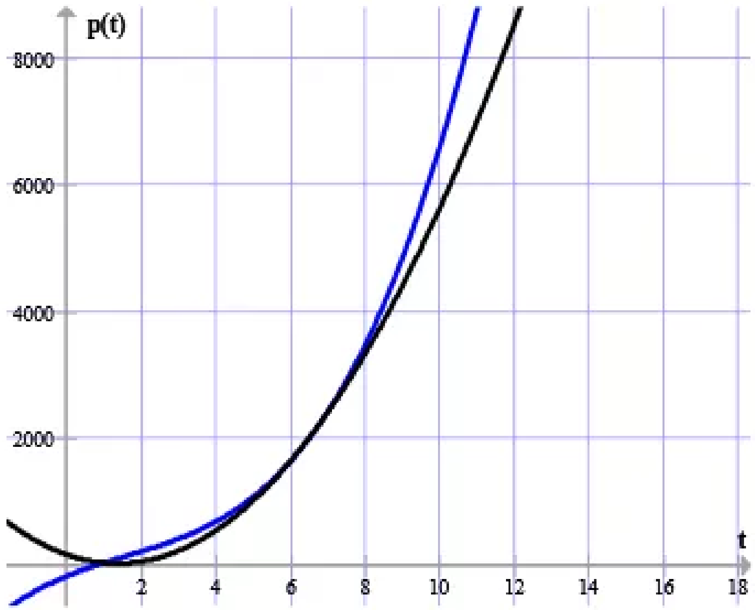
Now we check the model by graphing it and the data in the same viewing window.



Thus the polynomial function representing the given data is

$$\boxed{p_1(t) = 75t^2 - 205t + 160}$$

Now we combine the two graphs to check which model gives greatest long term profit.



From the graph we observe that the Yard Work model gives greatest long term profit since it is above the Pet Care model for the range 5 to ∞ .

Answer 29e.

The maximum number of regions R into which space can be divided by n intersecting spheres is given by

$$R(n)=\frac{1}{3}n^3-n^2+\frac{8}{3}n$$

We need to show that this function has constant third-order differences.

The first six numbers using the given function are shown below:

$$R(1)=\frac{1}{3}(1)^3-(1)^2+\frac{8}{3}(1) \quad \text{[By putting } n=1\text{]}$$
$$=2$$

$$R(2)=\frac{1}{3}(2)^3-(2)^2+\frac{8}{3}(2) \quad \text{[By putting } n=2\text{]}$$
$$=4$$

$$R(3)=\frac{1}{3}(3)^3-(3)^2+\frac{8}{3}(3) \quad \text{[By putting } n=3\text{]}$$
$$=8$$

$$R(4)=\frac{1}{3}(4)^3-(4)^2+\frac{8}{3}(4) \quad \text{[By putting } n=4\text{]}$$
$$=16$$

$$R(5)=\frac{1}{3}(5)^3-(5)^2+\frac{8}{3}(5) \quad \text{[By putting } n=5\text{]}$$
$$=30$$

$$R(6)=\frac{1}{3}(6)^3-(6)^2+\frac{8}{3}(6) \quad \text{[By putting } n=6\text{]}$$
$$=52$$

Now we find the first-order differences by subtracting consecutive numbers.

$$R(2)-R(1)=4-2$$
$$=2$$
$$R(3)-R(2)=8-4$$
$$=4$$
$$R(4)-R(3)=16-8$$
$$=8$$
$$R(5)-R(4)=30-16$$
$$=14$$
$$R(6)-R(5)=52-30$$
$$=22$$

Now we find the second order differences by subtracting consecutive first-order differences.

$$4-2=2$$
$$8-4=4$$
$$14-8=6$$
$$22-14=8$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$4-2=2$$
$$6-4=2$$
$$8-6=2$$

Thus we obtain that the function $R(n)=\frac{1}{3}n^3-n^2+\frac{8}{3}n$ has the third order difference $\boxed{\text{constant}}$.

Answer 30e.

The given table of data with values of c and $p(c)$ is

c	1	2	3	4	5	6
$p(c)$	2	4	8	15	26	42

where $p(c)$ represents maximum number of pieces of cakes by c planes. We need to find the maximum number of pieces when the cake is divided by 8 planes.

Now we begin by finding the finite differences.
Now we find the first-order differences by subtracting consecutive numbers.

$$p(2)-p(1)=4-2$$
$$=2$$
$$p(3)-p(2)=8-4$$
$$=4$$
$$p(4)-p(3)=15-8$$
$$=7$$
$$p(5)-p(4)=26-15$$
$$=11$$
$$p(6)-p(5)=42-26$$
$$=16$$

Then we find the second order differences by subtracting consecutive first-order differences.

$$4-2=2$$
$$7-4=3$$
$$11-7=4$$
$$16-11=5$$

Then we find the third order differences by subtracting consecutive second-order differences.

$$3-2=1$$
$$4-3=1$$
$$5-4=1$$

Since the third order difference is constant, the number can be represented by a cubic function of the form $p(c)=xc^3+yc^2+zc+w$ ...... (1)

By substituting the first four numbers in the function (1) to obtain a system of four linear equations of four variables, we have

$$p(1)=x(1)^3+y(1)^2+z(1)+w$$
$$p(2)=x(2)^3+y(2)^2+z(2)+w$$
$$p(3)=x(3)^3+y(3)^2+z(3)+w$$
$$p(4)=x(4)^3+y(4)^2+z(4)+w$$

Or,

$$x+y+z+w=2 \quad \text{...... (2)}$$
$$8x+4y+2z+w=4 \quad \text{...... (3)}$$
$$27x+9y+3z+w=8 \quad \text{...... (4)}$$
$$64x+16y+4z+w=15 \quad \text{...... (5)}$$

By performing (3)-(2), we have

$$7x+3y+z=2 \quad \text{...... (6)}$$

By performing (4)-(3), we have

$$19x+5y+z=4 \quad \text{...... (7)}$$

By performing (5)-(4), we have

$$37x+7y+z=7 \quad \text{...... (8)}$$

By performing (7)-(6), we have

$$12x+2y=2$$

By performing (8)-(7), we have

$$18x+2y=3$$

By solving these two equations, we have

$$x=\frac{1}{6}, y=0$$

By putting the value of x and y in the above equations, we have

$$z=\frac{5}{6}, w=1$$

Therefore from the equation (1) the polynomial function is

$$p(c)=\frac{1}{6}c^3+0c^2+\frac{5}{6}c+1$$

$$p(c)=\frac{c^3}{6}+\frac{5c}{6}+1$$

The maximum number of pieces when the cake is divided by 8 planes is

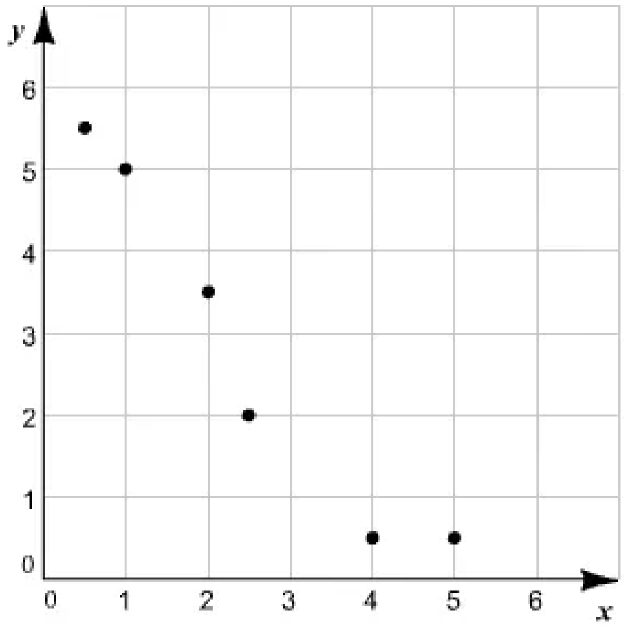
$$p(8) = \frac{8^3}{6} + \frac{5 \times 8}{6} + 1$$
$$= \frac{512}{6} + \frac{40}{6} + 1$$

$$= \frac{512 + 40 + 6}{6}$$
$$= \frac{558}{6}$$
$$= \boxed{93}$$

Answer 31e.

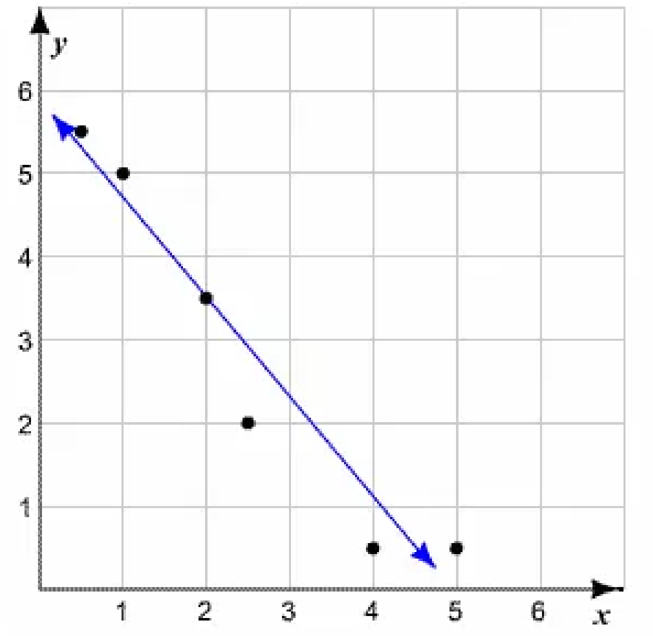
(a) The given data can be represented in the ordered pair form as (0.5, 5.5), (1, 5), (2, 3.5), (2.5, 2), (4, 0.5) and (5, 0.5).

Plot the above points on a graph to get the scatter plot.



(b) The line that closely follows the trends shown by the data points is called a best-fitting line.

First, sketch the line that best fits the data.



Now, choose two data points that appear to lie on the line. Let the points be (2, 3.5) and (4, 1.15).

Find the slope, m, using these points.

$$m = \frac{1.15 - 3.5}{4 - 2}$$

$$= \frac{-2.35}{2}$$

$$\approx 1.18$$

The point slope form of an equation is $y - y_1 = m(x - x_1)$. Choose (2, 3.5) as the point (x_1, y_1).

Substitute 1.18 for m, 2 for x_1, and 3.5 for y_1 in the above equation.

$y - 3.5 = 1.18(x - 2)$

Simplify.

$y - 3.5 = 1.18x - 2.36$

Add 3.5 to both sides of the equation.

$$y - 3.5 + 3.5 = 1.18x - 2.36 + 3.5$$

$$y = 1.18x - 1.14$$

Thus, an approximation of the best-fitting line is $y = 1.1x - 1.14$.

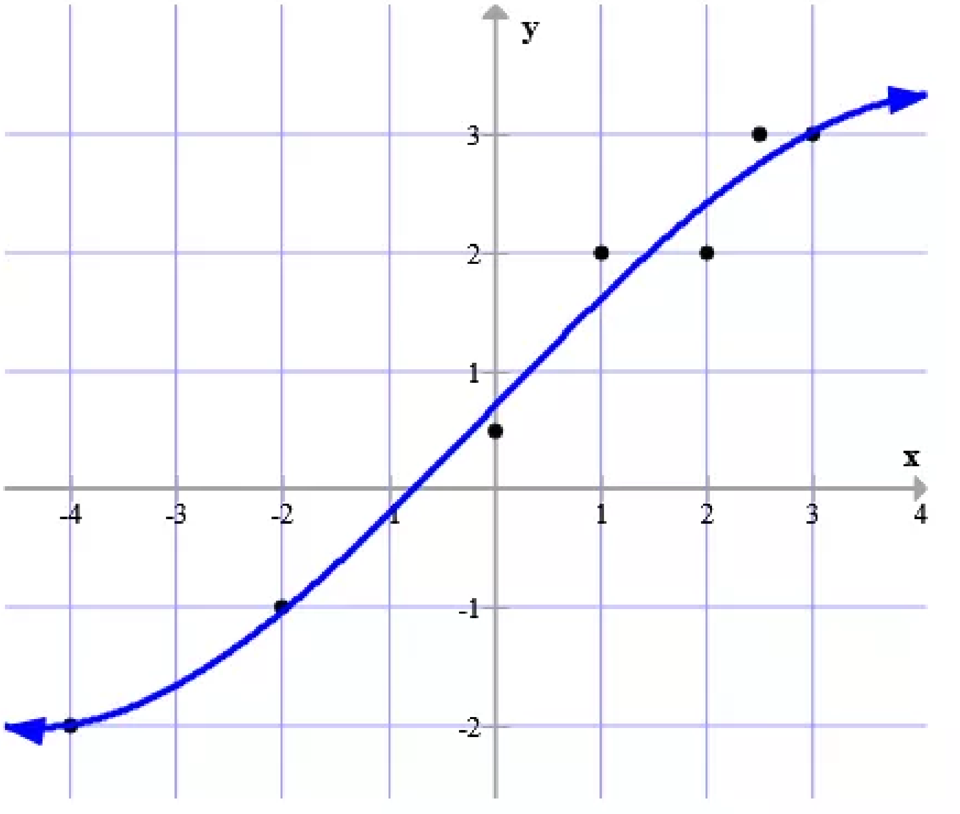
Answer 32e.

The given table of data is

x	–4	–2	0	1	2	2.5	3
y	–2	–1	0.5	2	2	3	3

We need to draw the scatter plot for the data and approximate the best-fitting line.

The scatter plot for the given data is drawn below:



Now we choose four points that appear to lie on the line. The chosen points are, $(2,2.4)$, $(-3,-1.6)$ which are not original data points and $(-2,-1),(3,3)$ which are the original data points.

Since the line looks like a cubic function we suppose the function as:

$$y = ax^3 + bx^2 + cx + d \qquad \text{...... (1)}$$

By putting the four points in the equation (1), we have

$$2.4 = a(2)^3 + b(2)^2 + c(2) + d \qquad [\text{By putting } x = 2, y = 2.4]$$

$$-1.6 = a(-3)^3 + b(-3)^2 + c(-3) + d \qquad [\text{By putting } x = -3, y = -1.6]$$

$$-1 = a(-2)^3 + b(-2)^2 + c(-2) + d \qquad [\text{By putting } x = -2, y = -1]$$

$$3 = a(3)^3 + b(3)^2 + c(3) + d \qquad [\text{By putting } x = 3, y = 3]$$

Or,

$$2.4 = 8a + 4b + 2c + d \qquad \text{...... (2)}$$

$$-1.6 = -27a + 9b - 3c + d \qquad \text{...... (3)}$$

$$-1 = -8a + 4b - 2c + d \qquad \text{...... (4)}$$

$$3 = 27a + 9b + 3c + d \qquad \text{...... (5)}$$

By performing (4) + (2), we have

$$4b + d = 0.7$$

By performing (5) + (3), we have

$$9b + d = 0.7$$

From these two equations, we have

$$b = 0, d = 0.7$$

By putting the values of b and d in the equations (2) and (3), we have

$$4a + c = \frac{1.7}{2}$$

$$9a + c = \frac{2.3}{3}$$

By solving these two equations, we have

$$a = -0.0167, c = 0.9168$$

Therefore from the equation (1), the best fitting line is

$$\boxed{y = -0.0167x^3 + 0.9168x + 0.7}$$

Answer 33e.

The given trinomial is of the form $x^2 + bx + c$, which when factored will be $(x + m)(x + n)$, where the product of m and n gives c, and their sum gives b.

Compare the given equation with $x^2 + bx + c = 0$. The value of b is -19 and of c is 48. We need to find m and n such that their product gives 48 and sum gives -19.

List the factors of 48 and find their sums.

Factors of 48: m, n	1, 48	2, 24	3, 16	4, 12	6, 8	−1, −48	−2, −24	−3, −16	−4, −12	−6, −8
Sum of factors: $m+n$	49	26	19	16	14	−49	−26	−19	−16	−14

From the table, it is clear that $m=-3$ and $n=-16$ gives the product 48 and sum −19.

Therefore, the given trinomial can be factored as $(x-3)(x-16)$.

Answer 34e.

We need to factor the polynomial $18x^2+30x-12$.

For factoring the polynomial ax^2+bx+c, we multiply a and c together and find factors of ac that add up to b.

Now, multiplication of 18 and −12 is −216 that is $18\times(-12)=-216$. By factoring

$$1\times(-216)=-216 \qquad (-1)\times 216=-216$$
$$2\times(-108)=-216 \qquad (-2)\times 108=-216$$
$$3\times(-72)=-216 \qquad (-3)\times 72=-216$$
$$4\times(-54)=-216 \qquad (-4)\times 54=-216$$
$$6\times(-36)=-216 \qquad (-6)\times 36=-216$$
$$8\times(-27)=-216 \qquad (-8)\times 27=-216$$
$$9\times(-24)=-216 \qquad (-9)\times 24=-216$$
$$12\times(-18)=-216 \qquad (-12)\times 18=-216$$

We take the factors −6 and 36 since these add up to 30.

Therefore, we have

$$\begin{aligned}18x^2+30x-12 &= 18x^2+36x-6x-12\\ &= 18x(x+2)-6(x+2) \quad \text{[By taking common]}\\ &= (x+2)(18x-6)\\ &= (x+2)6(3x-1) \quad \text{[By taking common]}\\ &= 6(x+2)(3x-1)\end{aligned}$$

Therefore the factors of the polynomial $18x^2+30x-12$ are $\boxed{6, (x+2) \text{ and } (3x-1)}$.

Answer 35e.

Check whether the given expression is a perfect square trinomial.

The first condition for an expression to be a perfect square trinomial is that the first and last terms of the trinomial must be perfect squares.

The first term of the given expression is $64x^2$, which is the perfect square of $8x$. Similarly, the last term 81 is the perfect square of 9. Thus, the first condition is satisfied.

The second condition is that the middle term must be twice the product of the square roots of the first and last terms of the trinomial.

The square root of $64x^2$ is $8x$ and of 81 is 9. Twice the product of $8x$ and 9 will be $2(8x)(9)$ or $144x$. Since the middle term of the given expression is also $144x$ (except for the negative sign), the second condition has also been satisfied.

Factor using the special factoring pattern $a^2 - 2ab + b^2 = (a - b)^2$.

$$\begin{aligned} 64x^2 - 144x + 81 &= (8x)^2 - 2(8x)(9) + 9^2 \\ &= (8x - 9)^2 \end{aligned}$$

Therefore, the given expression can be factored as $(8x - 9)^2$.

Answer 36e.

We need to factor the polynomial $18x^3 + 33x^2 - 30x$.

Now,

$$18x^3 + 33x^2 - 30x = x(18x^2 + 33x - 30)$$

For factoring the polynomial $ax^2 + bx + c$, we multiply a and c together and find factors of ac that add up to b.

Now, multiplication of 18 and -30 is -540 that is $18 \times (-30) = -540$. By factoring

$1 \times (-540) = -540$	$(-1) \times 540 = -540$
$2 \times (-270) = -540$	$(-2) \times 270 = -540$
$3 \times (-180) = -540$	$(-3) \times 180 = -540$
$4 \times (-135) = -540$	$(-4) \times 135 = -540$
$5 \times (-108) = -540$	$(-5) \times 108 = -540$
$6 \times (-90) = -540$	$(-6) \times 90 = -540$
$9 \times (-60) = -540$	$(-9) \times 60 = -540$
$10 \times (-54) = -540$	$(-10) \times 54 = -540$
$12 \times (-45) = -540$	$(-12) \times 45 = -540$
$15 \times (-36) = -540$	$(-15) \times 36 = -540$
$18 \times (-30) = -540$	$(-18) \times 30 = -540$
$20 \times (-27) = -540$	$(-20) \times 27 = -540$

We take the factors -12 and 45 since these add up to 33.

Therefore, we have

$$\begin{aligned}18x^3+33x^2-30x &= x\left(18x^2+33x-30\right)\\ &= x\left(18x^2+45x-12x-30\right)\\ &= x\{9x(2x+5)-6(2x+5)\} \quad \text{[By taking common]}\\ &= x(2x+5)(9x-6)\\ &= 3x(2x+5)(3x-2) \quad \text{[By taking common]}\end{aligned}$$

Therefore the factors of the polynomial $18x^3+33x^2-30x$ are $\boxed{3x,(2x+5)\text{ and }(3x-2)}$.

Answer 37e.

The given expression can be identified as the sum of two cubes.

Factor using the pattern $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$\begin{aligned}64x^3+27 &= (4x)^3+3^3\\ &= (4x+3)\left[(4x)^2-(4x)(3)+3^2\right]\\ &= (4x+3)\left(16x^2-12x+9\right)\end{aligned}$$

Therefore, the given polynomial in completely factored form is $(4x + 3)(16x^2 - 12x + 9)$.

Answer 38e.

We need to factor the polynomial $3x^5-66x^3-225x$.

Therefore, we have

$$\begin{aligned}3x^5-66x^3-225x &= 3x\left(x^4-22x^2-75\right)\\ &= 3x\left(x^4+3x^2-25x^2-75\right)\\ &= 3x\left\{x^2\left(x^2+3\right)-25\left(x^2+3\right)\right\} \text{[By taking common]}\\ &= 3x\left(x^2+3\right)\left(x^2-25\right)\\ &= 3x\left(x^2+3\right)\left(x^2-5^2\right) \quad \left[\text{Since } 5^2=25\right]\\ &= 3x\left(x^2+3\right)(x-5)(x+5) \quad \left[\text{Since } a^2-b^2=(a-b)(a+b)\right]\end{aligned}$$

Therefore the factors of the polynomial $3x^5-66x^3-225x$ are

$\boxed{3x,\left(x^2+3\right),(x-5)\text{ and }(x+5)}$.

Answer 39e.

First, we have to isolate x^2. For this, divide each side of the equation by 5.

$$\frac{5x^2}{5} = \frac{10}{5}$$
$$x^2 = 2$$

Take the square root on each side.

$$\sqrt{x^2} = \sqrt{2}$$
$$x = \pm\sqrt{2}$$

CHECK

Substitute the solutions for z in the original equation and evaluate.

Let $x = \sqrt{2}$	Let $x = -\sqrt{2}$
$5x^2 = 10$	$5x^2 = 10$
$5\left(\sqrt{2}\right)^2 \stackrel{?}{=} 10$	$5\left(-\sqrt{2}\right)^2 \stackrel{?}{=} 10$
$5(2) \stackrel{?}{=} 10$	$5(2) \stackrel{?}{=} 10$
$10 = 10$ ✓	$10 = 10$ ✓

Therefore, the solutions are $\sqrt{2}$ and $-\sqrt{2}$.

Answer 40e.

We need to solve the equation $24x^2 = 6$.

Now, we have

$24x^2 = 6$ [Original equation]

$\frac{\cancel{24}x^2}{\cancel{24}} = \frac{6}{24}$ [Dividing both sides by 24 to isolate x^2]

$x^2 = \frac{\cancel{2\cdot 3}}{2\cdot 2\cdot \cancel{2\cdot 3}}$ [To simplify the left side,we remove the common factor of 24 To simplify the right side of the equation,we factor 6 and 24 and remove the common terms]

$x^2 = \frac{1}{4}$ [Multiplying the remaining factors in the numerator and denominator]

$x^2 = \left(\pm\frac{1}{2}\right)^2$ [Since $\frac{1}{4} = \left(\pm\frac{1}{2}\right)^2$]

$x = \pm\frac{1}{2}$

Therefore solution for the equation $24x^2 = 6$ is $x = \boxed{\left(\frac{1}{2}, -\frac{1}{2}\right)}$.

Answer 41e.

We have to isolate x^2. For this, first subtract 2 from each side of the equation.

$$9x^2 + 2 - 2 = 6 - 2$$

$$9x^2 = 4$$

Now, divide each side by 9.

$$\frac{9x^2}{9} = \frac{4}{9}$$

$$x^2 = \frac{4}{9}$$

Take the square root on each side.

$$\sqrt{x^2} = \sqrt{\frac{4}{9}}$$

$$x = \pm\frac{2}{3}$$

CHECK

Substitute the solutions for x in the original equation and evaluate.

Let $x = \frac{2}{3}$

$$9x^2 + 2 = 6$$

$$9\left(\frac{2}{3}\right)^2 + 2 \stackrel{?}{=} 6$$

$$4 + 2 \stackrel{?}{=} 6$$

$$6 = 6 \checkmark$$

Let $x = -\frac{2}{3}$

$$9x^2 + 2 = 6$$

$$9\left(-\frac{2}{3}\right)^2 + 2 \stackrel{?}{=} 6$$

$$4 + 2 \stackrel{?}{=} 6$$

$$6 = 6 \checkmark$$

Therefore, the solutions are $\frac{2}{3}$ and $-\frac{2}{3}$.

Answer 42e.

We need to solve the equation $7x^2 - 4 = 8$.

Now, we have

$$7x^2 - 4 = 8 \quad \text{[Original equation]}$$

$$7x^2 = 8 + 4 \quad \text{[Taking } -4 \text{ to the right side]}$$

$$7x^2 = 12$$

$$\frac{\cancel{7}x^2}{\cancel{7}} = \frac{12}{7} \quad \left[\text{Dividing both sides by 7 to isolate } x^2\right]$$

$$x^2 = \frac{12}{7} \quad \text{[To simplify the left side, we remove the common factor of 7]}$$

$$x^2 = \left(\pm\sqrt{\frac{12}{7}}\right)^2 \quad \left[\text{Since } \frac{12}{7} = \left(\pm\sqrt{\frac{12}{7}}\right)^2\right]$$

$$x^2 = \left(\pm\frac{2\sqrt{3}}{\sqrt{7}}\right)^2$$

$$x = \pm\frac{2\sqrt{3}}{\sqrt{7}}$$

Therefore solution for the equation $7x^2 - 4 = 8$ is $x = \boxed{\left(\frac{2\sqrt{3}}{\sqrt{7}}, -\frac{2\sqrt{3}}{\sqrt{7}}\right)}$.

Answer 43e.

Begin by subtracting 16 from each side of the given equation to isolate x^2.

$$-x^2 + 16 - 16 = 5x^2 - 12 - 16$$

$$-x^2 = 5x^2 - 28$$

Now, subtract $5x^2$ from both the sides.

$$-x^2 - 5x^2 = 5x^2 - 28 - 5x^2$$

$$-6x^2 = -28$$

Divide each side by -6.

$$\frac{-6x^2}{-6} = \frac{-28}{-6}$$

$$x^2 = \frac{14}{3}$$

Take the square root on each side.

$$\sqrt{x^2} = \sqrt{\frac{14}{3}}$$

$$x = \pm\sqrt{\frac{14}{3}}$$

Apply the quotient property.

For any numbers $a, b > 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

$$x = \pm\frac{\sqrt{14}}{\sqrt{3}}$$

Now, we have to rationalize the denominator. For this, multiply the numerator and the denominator by $\sqrt{3}$.

$$x = \pm\frac{\sqrt{14}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

Apply the product property.

For any numbers $a, b > 0$, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

$$x = \pm\frac{\sqrt{14 \cdot 3}}{\sqrt{3 \cdot 3}}$$

$$= \pm\frac{\sqrt{42}}{3}$$

CHECK

Substitute the solutions for x in the original equation and evaluate.

Let $x = \frac{\sqrt{42}}{3}$

$$-x^2 + 16 = 5x^2 - 12$$

$$-\left(\frac{\sqrt{42}}{3}\right)^2 + 16 \stackrel{?}{=} 5\left(\frac{\sqrt{42}}{3}\right)^2 - 12$$

$$-\frac{42}{16} + 16 \stackrel{?}{=} \frac{210}{9} - 12$$

$$\frac{102}{9} = \frac{102}{9} \quad \checkmark$$

Let $x = -\frac{\sqrt{42}}{3}$

$$-x^2 + 16 = 5x^2 - 12$$

$$-\left(-\frac{\sqrt{42}}{3}\right)^2 + 16 \stackrel{?}{=} 5\left(-\frac{\sqrt{42}}{3}\right)^2 - 12$$

$$-\frac{42}{16} + 16 \stackrel{?}{=} \frac{210}{9} - 12$$

$$\frac{102}{9} = \frac{102}{9} \quad \checkmark$$

Therefore, the solutions are $\frac{\sqrt{42}}{3}$ and $-\frac{\sqrt{42}}{3}$.

Answer 44e.

We need to solve the equation $4x^2 + 3 = -4x^2 + 15$.

Now, we have

$$4x^2+3=-4x^2+15 \left[\text{Original equation}\right]$$

$$4x^2+4x^2=15-3 \quad \left[\text{Taking } 4x^2 \text{to the left side and 3 to the right side}\right]$$

$$8x^2=12$$

$$\frac{8x^2}{8}=\frac{12}{8} \qquad \left[\text{Dividing both sides by 8 to isolate } x^2\right]$$

$$x^2=\frac{\cancel{2}\cdot\cancel{2}\cdot 3}{\cancel{2}\cdot\cancel{2}\cdot 2} \qquad \left[\begin{array}{l}\text{To simplify the left side,we remove the common factor}\\ \text{of 8 and factor 12 and 8 and remove the common terms}\\ \text{from numerator and denominator}\end{array}\right]$$

$$x^2=\frac{3}{2}$$

$$x^2=\left(\pm\sqrt{\frac{3}{2}}\right)^2 \quad \left[\text{Since } \frac{3}{2}=\left(\pm\sqrt{\frac{3}{2}}\right)^2\right]$$

$$x=\pm\sqrt{\frac{3}{2}}$$

Therefore solution for the equation $4x^2+3=-4x^2+15$ is $x=\boxed{\left(\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}\right)}$.