Ch - :- Fourier Transform (F.T.)

=> Transformation is the process in which one domain is converted to another domain such that signal analysis becomes easy.

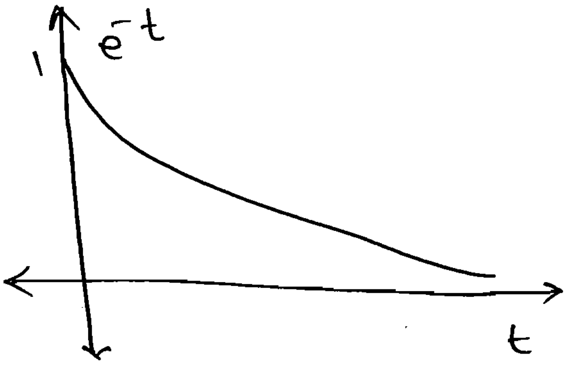
=> For any Non Periodic signal as $T \to \infty$ implies $\omega_0 \to 0$.

=> The discrete spectrum of Fourier series is converted to continuous spectrum in Fourier Transform.

=> Extension of F.S. is F.T.

=> F.T. is extension of F.S. to non-periodic signal.

=> $x(t) = e^{-t}.u(t)$, $u(t)$



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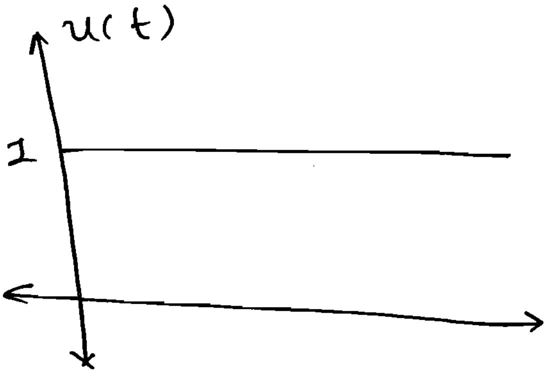
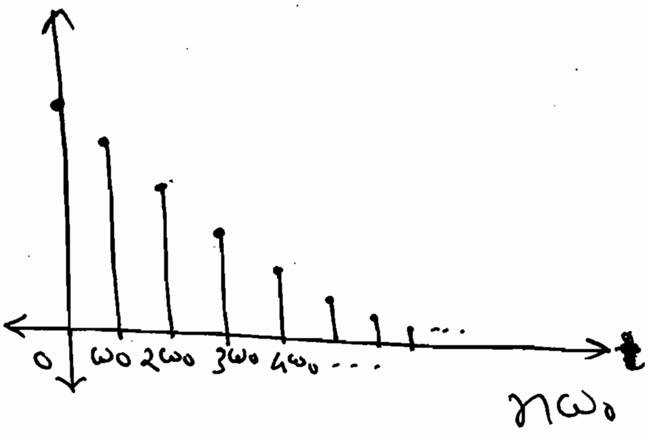


$T \to \infty$, Non-periodic.

⇒

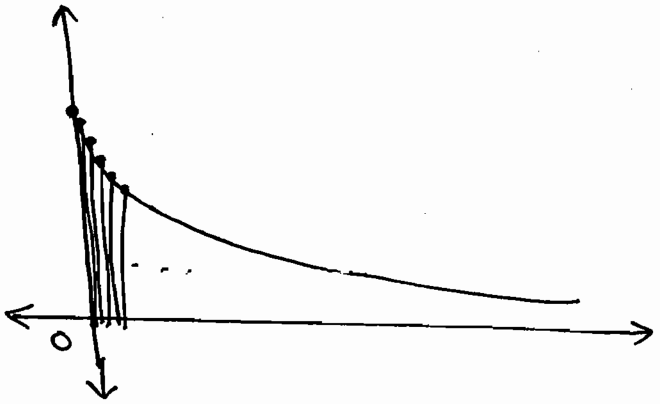


$T \to \infty$

$\omega \to 0$

i.e. $n\omega_0 \to \omega$.

⇓



$\omega = n\omega_0$

⇒ From F.S.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{j\omega_0 n t}.$$

&

$$C_n = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \cdot e^{-j\underset{\downarrow}{\underline{\omega_0 n}} t} \cdot dt.$$

$\boxed{T \to \infty}$

$$\lim_{T \to \infty} T\, C_n = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\underline{\omega} t} \cdot dt.$$

$$\Rightarrow \boxed{X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot \underbrace{e^{-j\omega t}}_{\to \text{kernel.}} \cdot dt.}$$

⇓

F.T. of $x(t)$.

⇒ I.F.T.

$$\therefore \boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).\, e^{j\omega t}\, .\, d\omega.}$$

$$\omega = 2\pi f$$

$$\therefore \; d\omega = 2\pi\, df.$$

I.F.T.

$$\therefore \boxed{x(t) = \int_{-\infty}^{\infty} X(f).\, e^{j2\pi f t}\, .\, df.}$$

⇒ F.T.

$$\boxed{X(f) = \int_{-\infty}^{+\infty} x(t).\, e^{-j2\pi f t}\, .\, dt}$$

$$\Rightarrow \boxed{2\pi\delta(\omega) = \delta(f).}$$

Proof:

$$2\pi\,\delta(\omega) = 2\pi\delta(2\pi f).$$

$$= \frac{\cancel{2\pi}}{|\cancel{2\pi}|}.\, \delta(f)$$

$$\boxed{2\pi\delta(\omega) = \delta(f).}$$

* Convergence of F.T. :

$$\Rightarrow \quad X(\omega) < \infty.$$

⇒ Fourier transform is defined for stable and energy signal. ✓

⇒ Fourier transform of a power signals is defined as approximation to energy signals (or) impulse functions. are Permitted.

⇒ Fourier transform is not defined for neither absolutely integrable nor square integrable signals.

P 4.1.1 If x(t) is a voltage waveform, then what are the units of X(f)?

Soln:

$$X(f) = \int_{-\infty}^{\infty} x(t) . e^{-j2\pi f t} . dt$$

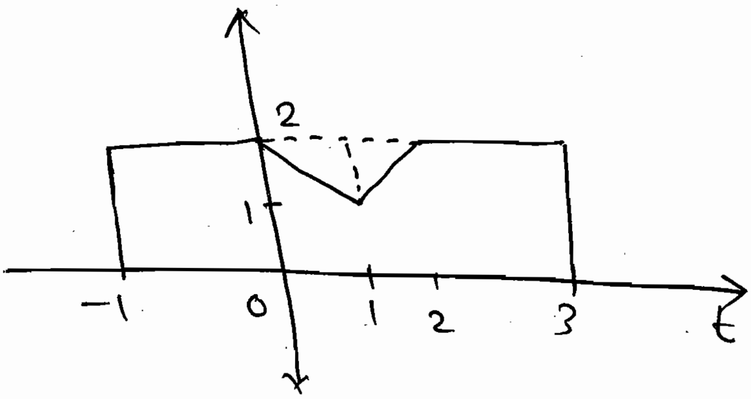
= Volts . sec.

So, X(f) unit is Volts. sec (or) $\boxed{\text{Volts/Hz}}$

P 4.1.2 For the signal x(t) shown in figure, find

(a) X(0).

(b) $\int_{-\infty}^{+\infty} X(\omega) \, d\omega$



Soln:

ⓐ $$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt.$$

make $\omega = 0$.

$$\therefore \quad X(0) = \int_{-\infty}^{+\infty} x(t) \cdot dt$$ = Area under the curve.

$$\therefore \quad X(0) = (4 \times 2) - \left(\tfrac{1}{2} \times 1 \times 2\right)$$

$$\boxed{X(0) = 7}$$

(b) $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega.$$

make $t = 0$.

$$\therefore \quad x(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{0} \cdot d\omega.$$

$$\Rightarrow \int_{-\infty}^{+\infty} X(\omega) \cdot d\omega = 2\pi x(0).$$

$$= 2\pi \times 2$$

$$= 4\pi.$$

Note:

→ Area under one domain corresponds to observing the other domain at origin.

P 4.1.3. Consider the signal $x(t) = \begin{cases} e^{-t} ; t \geq 0 \\ 0 ; t < 0 \end{cases}$

and $X(\omega)$ is the F.T. of this signal.

Then the Value of $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, d\omega$ is ____.

Soln.:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega).\, e^{j\omega t}.\, d\omega.$$

So,
$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega. = e^{-0} = 1.$$

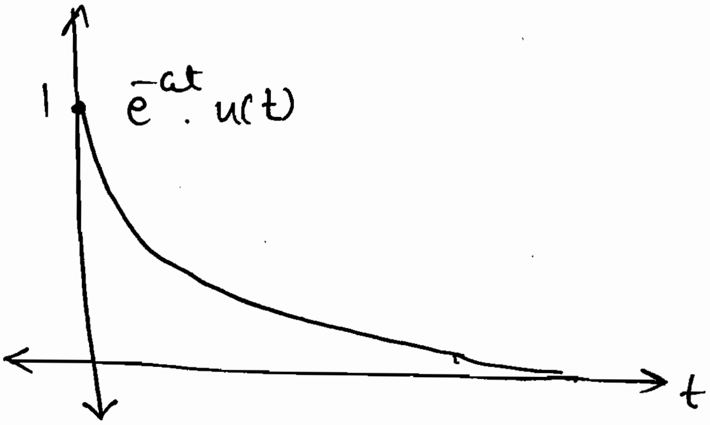
So,
$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega).d\omega = 1.}$$

* F.T. of Standard signals:

1) Decaying exponential

$x_1(t) = e^{-at} \cdot u(t)$, $a > 0$.

$\Rightarrow$



2. $$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} . dt.$$

$$= \int_{-\infty}^{+\infty} e^{-at} . e^{-j\omega t} . dt.$$

$$\therefore X(\omega) = \int_{+0}^{\infty} e^{-(a+j\omega)t} . dt.$$

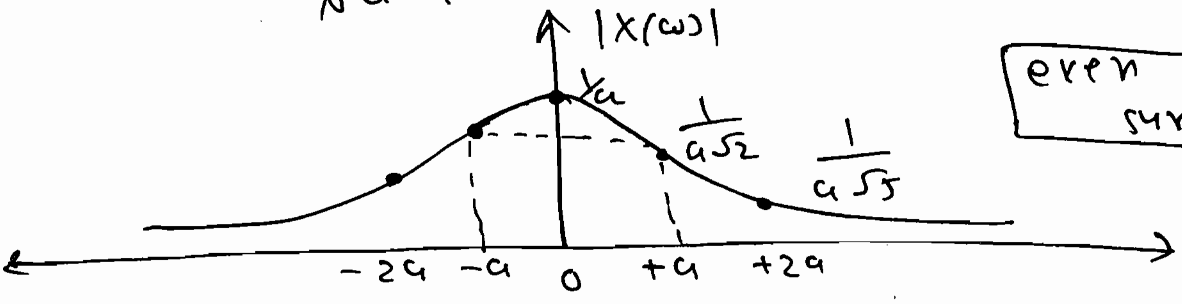
$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty}$$

$$X(\omega) = 0 + \frac{1}{a+j\omega}.$$

$$\therefore \boxed{e^{-at} . u(t) \xleftrightarrow{F.T.} \frac{1}{a+j\omega}.}$$

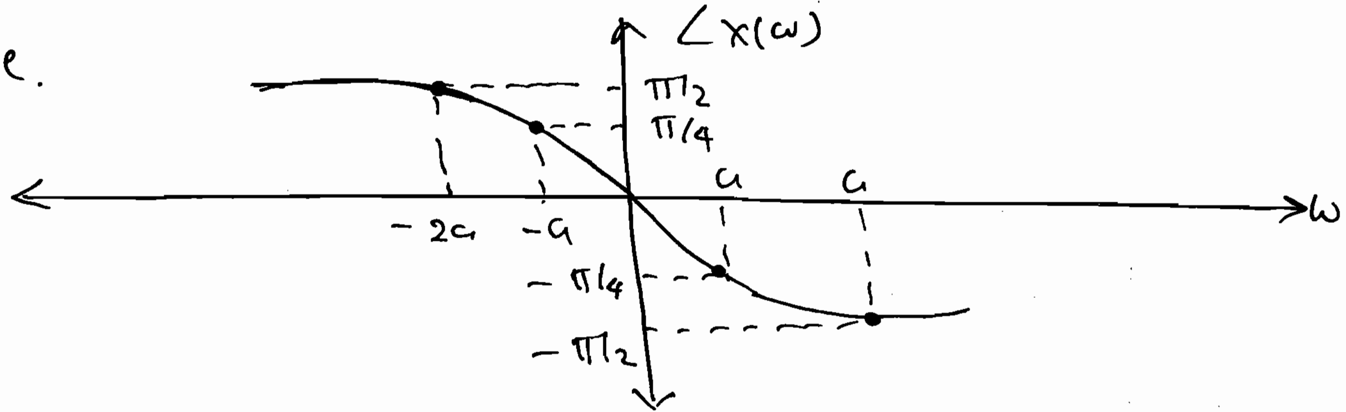
$$\Rightarrow |X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Mag.



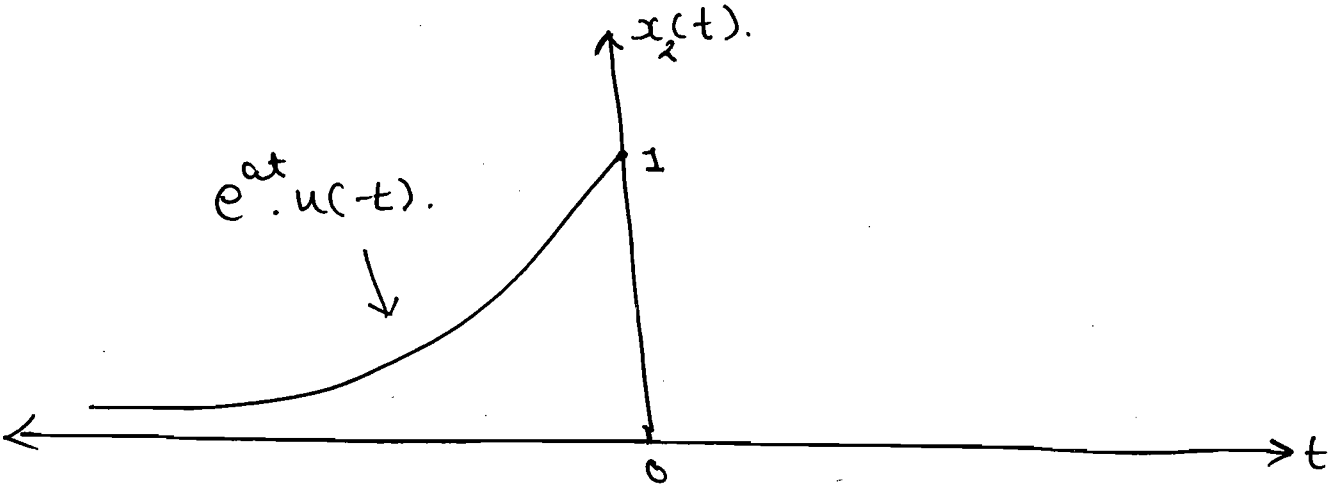
even sym.

Phase.



2) $e^{at}\cdot u(-t)$, $a>0$.

⇒



∴ $x_2(t) = x_1(-t)$

⇒ using time reversal property,

$$x(t) \xleftrightarrow{F.T.} X(\omega).$$

$$x(-t) \xleftrightarrow{F.T.} X(-\omega).$$

$$\therefore \quad X(-\omega) = \frac{1}{a-j\omega}.$$

$$\therefore \quad \boxed{e^{at}u(-t) \xleftrightarrow{F.T.} \frac{1}{a-j\omega}.}$$

③ $x(t) = \delta(t)$.

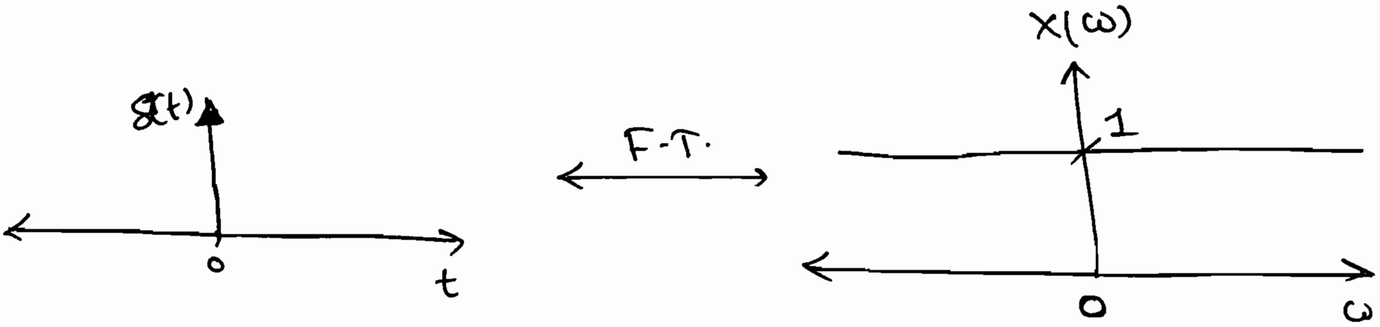
⇒

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\cdot e^{-j\omega t}\cdot dt$$

$$X(\omega) = e^{-j\omega(0)} \cdot dt \cdot$$ ($\because$ $t_0=0$ shifting property of impulse).

$$\boxed{X(\omega) = 1.}$$

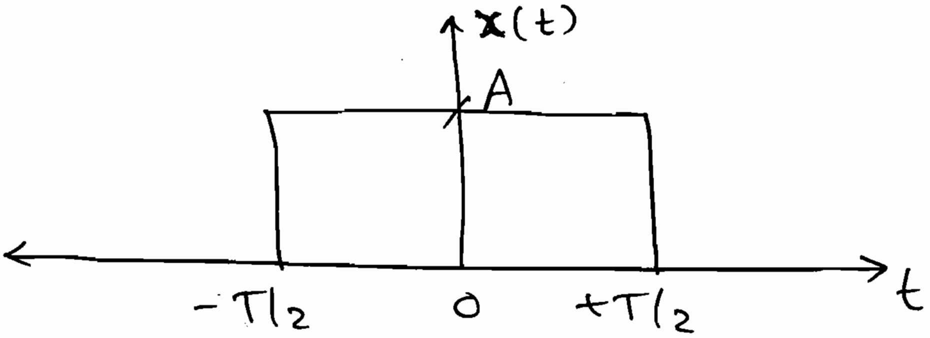
$$\therefore \quad \delta(t) \xleftrightarrow{F.T.} 1.$$



⇒ Spectrum of impulse is constant for all the frequency.

④ $x(t) = A\,\mathrm{rect}(t/T)$ (or) $A\,\Pi(t/T)$.

⇒



⇒ $$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t}\cdot dt.$$

$$\therefore \quad X(\omega) = \int_{-T/2}^{+T/2} A\cdot e^{-j\omega t}\cdot dt.$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{+T/2}.$$

$$\therefore \quad X(\omega) = \frac{A}{j\omega}\left[e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}}\right].$$

$$= \frac{A}{j\omega} \times 2j \left[\frac{e^{\frac{j\omega T}{2}} - e^{-j\frac{\omega T}{2}}}{2j} \right].$$

$$= \frac{2A}{\omega} \times \sin\frac{\omega T}{2}.$$

$$= \frac{2A}{\omega} \times \frac{1}{\frac{\omega T}{2}} \times \frac{\omega T}{2} \quad \sin\frac{\omega T}{2}.$$

$$= \frac{2A}{\cancel{\omega}} \times \frac{\cancel{\omega}T}{\cancel{2}} \times \frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}.$$

$$\therefore \boxed{X(\omega) = AT\, Sa\left(\frac{\omega T}{2}\right).}$$

$\Rightarrow$)

$$Sa(x) = \frac{\sin(x)}{x}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{x}. = \frac{\sin \pi x}{\pi x}.$$

$$\Rightarrow \text{sinc}\left(\frac{x}{\pi}\right) = \frac{\sin\left(\frac{\cancel{\pi} x}{\cancel{\pi}}\right)}{\frac{\cancel{\pi} x}{\cancel{\pi}}}$$

$$= \frac{\sin x}{x} = Sa(x).$$

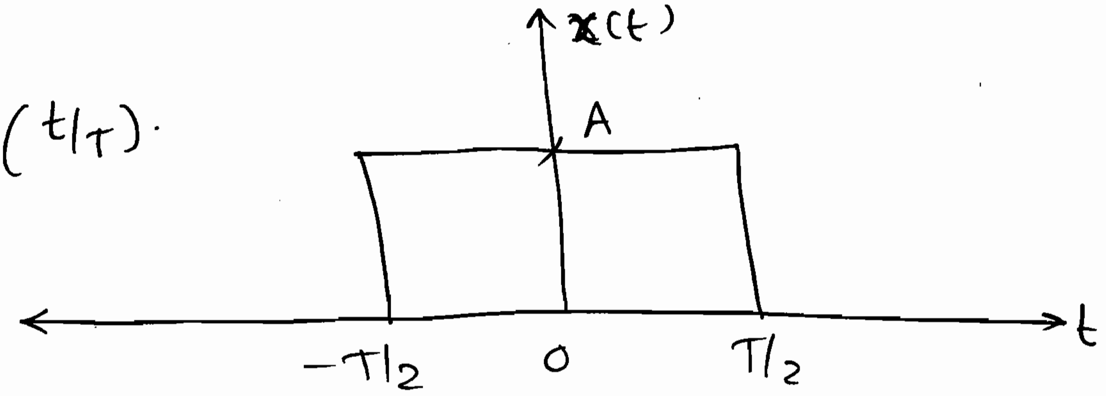
$$\Rightarrow \boxed{\text{sinc}\left(\frac{x}{\pi}\right) = Sa(x).}$$

$$\therefore \boxed{X(\omega) = AT\, \text{sinc}\left(\frac{\omega T}{2\pi}\right).}$$

$$= AT\, \text{sinc}(fT). \qquad \left(\because \omega = \frac{2\pi}{f}\right).$$

⇒

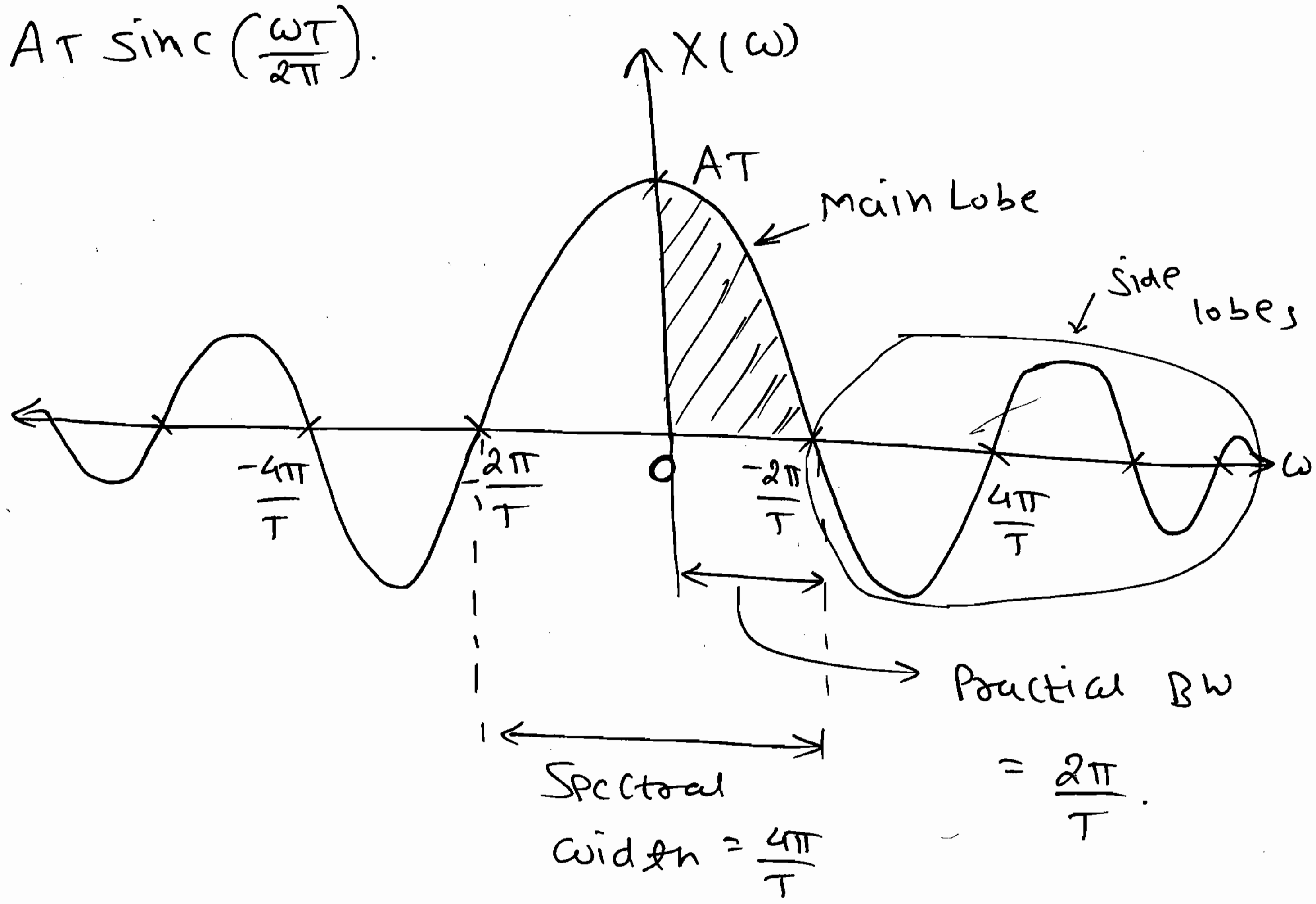
$A\,\text{rect}(t/T)$.



↓ F.T.

⇓ F.T.

$AT\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$.



⇒ Area under spectrum is

⇒ Value (Amp) of signal at $t = 0$.

⇒ Area under signal ⇒ Value (Amp) of spectrum at ~~t=0~~ $\omega = 0$.

⇒ Practical $BW = \frac{2\pi}{T}$.

Spectral width $= \frac{4\pi}{T}$.

⇒ Signal is zero at multiple of $\frac{2\pi}{T}$ of ω.

⇒ $\boxed{\text{Null to Null BW} = \text{Zero crossing BW} \Rightarrow \frac{2\pi}{T}.}$

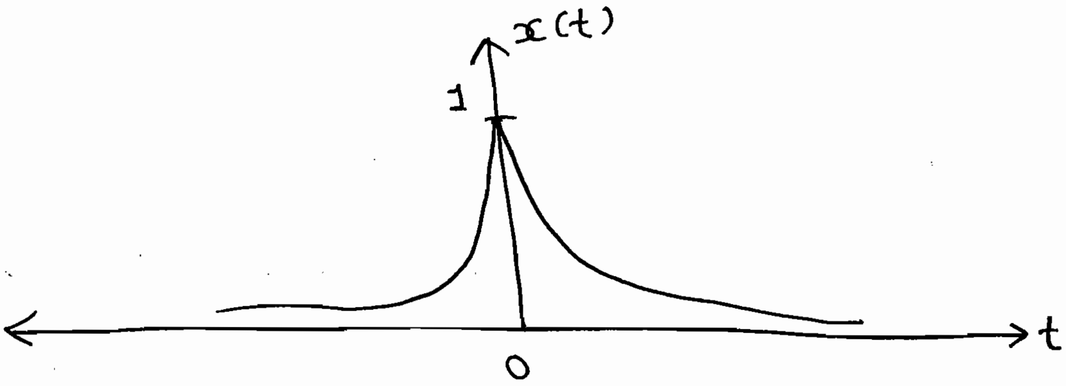
⇒ A signal can not be time limited and Band limited simultaneously.

⇒ Narrow Band in one domain becomes wideband in other domain.

⑤ $x(t) = e^{-\alpha|t|}$

⇒ $x(t) = e^{-\alpha t}$; $t>0$

$= e^{\alpha t}$; $t<0$.



$$\therefore \quad y(t) = e^{-\alpha t} \cdot u(t) + e^{\alpha t} \cdot u(-t).$$

$$= \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega}.$$

$$y(t) = \frac{2\alpha}{\alpha^2 + \omega^2}.$$

$$\therefore \quad \boxed{e^{-\alpha|t|} \xleftrightarrow{F.T.} \frac{2\alpha}{\alpha^2+\omega^2}.}$$

Method - II:

$\Rightarrow$ real $\longleftrightarrow$ even.

e.g. $x(t) = e^{-\alpha t} . u(t).$

$$X(\omega) = \frac{1}{\alpha + j\omega} \times \frac{\alpha - j\omega}{\alpha - j\omega}.$$

$$X(\omega) = \frac{\cancel{2}\alpha - j\omega}{\alpha^2 + \omega^2}.$$

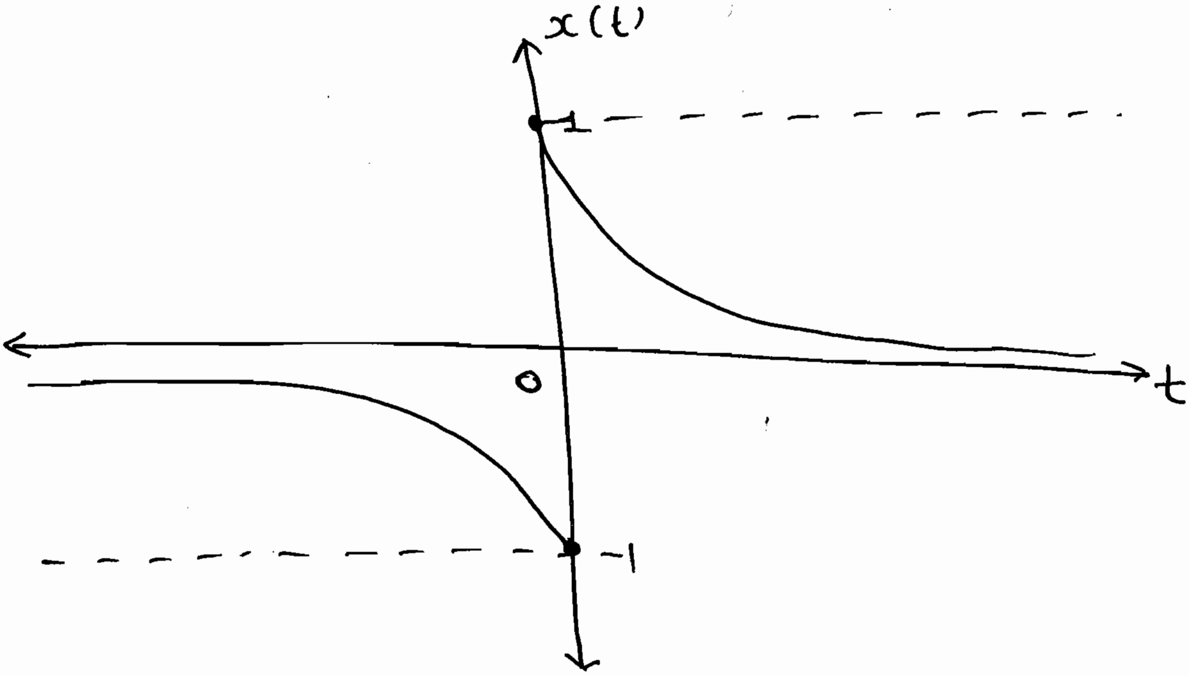
$$\Rightarrow \quad x_{even}(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{X(\omega) + X(-\omega)}{2}$$

$$\boxed{X_{even}(t). = \frac{\cancel{2}\alpha}{\alpha^2 + \omega^2}}$$

⑥ $x(t) = e^{-\alpha t} u(t) - e^{\alpha t} \cdot u(-t).$

$\Rightarrow$



$$\Rightarrow \quad X(\omega) = \frac{\alpha \mp j\omega}{\alpha^2+\omega^2} - \frac{\alpha + j\omega}{\alpha^2+\omega^2}.$$

$$\boxed{X(\omega) = \frac{-2j\omega}{\alpha^2+\omega^2}.} \quad \Rightarrow \quad \text{real} \longleftrightarrow \text{odd}.$$

Now, say $\alpha \to 0$

$$\lim_{\alpha \to 0} X(\omega) = \lim_{\alpha \to 0} \frac{-2j\omega}{\alpha^2+\omega^2} = \frac{-2j\cancel{\omega}}{\omega^{\cancel{2}}}$$

$$= \frac{2}{j\omega}.$$

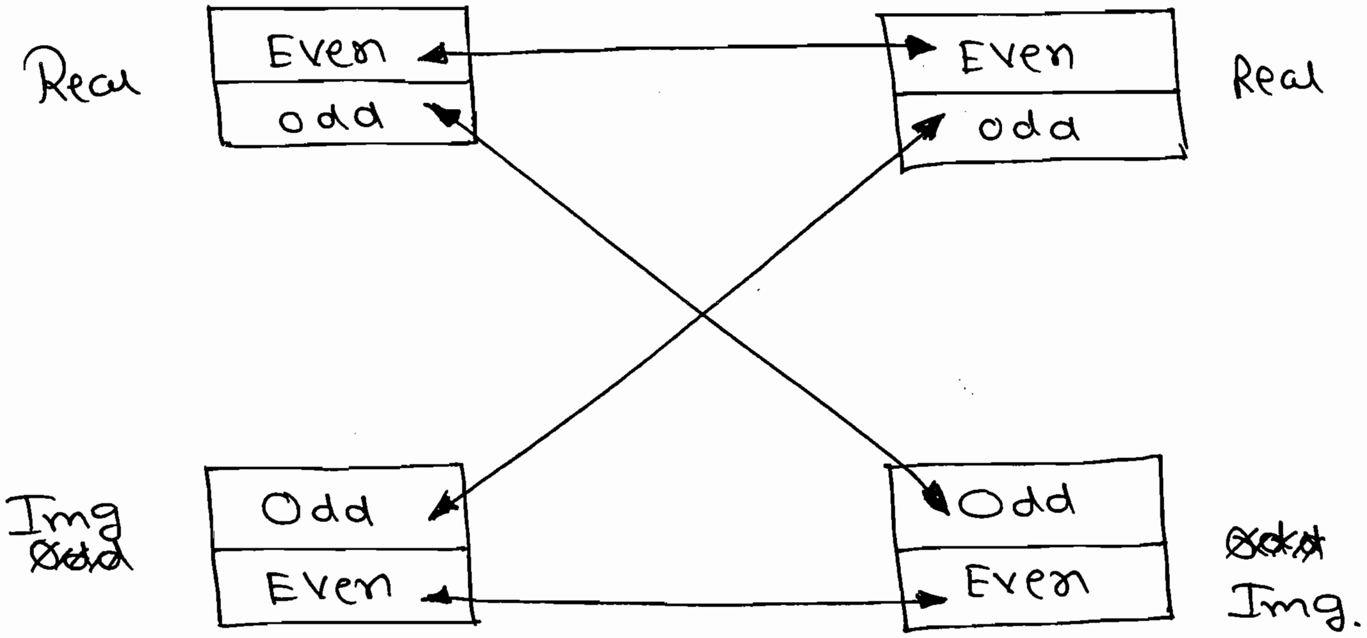
Now, $\lim_{\alpha \to 0} y(t) = \lim_{\alpha \to 0} e^{-\alpha t} \cdot u(t) - e^{\alpha t} \cdot u(-t).$

$$= u(t) - u(-t). = \text{sgn}(t).$$

$$\Rightarrow \boxed{Sgn(t) \xleftrightarrow{F.T.} \frac{2}{j\omega}}$$

real & odd Imag & odd.

*



* Properties of F.T. :

① Duality:

$$\Rightarrow \quad x(t) \longleftrightarrow X(\omega).$$

$$X(t) \longleftrightarrow 2\pi\, x(-\omega).$$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega.$$

replace 't' by '-t'.

$$\therefore \quad x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t} \cdot d\omega.$$

$$\therefore \quad 2\pi x(-t) = \int_{-\infty}^{+\infty} X(\omega) . e^{-j\omega t} . d\omega .$$

$$t \leftrightarrow \omega$$

$$\therefore \quad 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) . e^{-j\omega t} . dt .$$

$$\boxed{2\pi x(-\omega) = X(t).}$$

e.g.:

i)

$$e^{3t} u(-t) \longleftrightarrow \frac{1}{3-j\omega} .$$

$$\frac{1}{3-jt} \xleftrightarrow{t=-\omega} 2\pi e^{3(-\omega)} . u(\omega).$$

ii)

$$e^{-|t|} \longleftrightarrow \frac{2}{\omega^2+1} .$$

$$\frac{2}{t^2+1} \longleftrightarrow 2\pi e^{-|-\omega|}$$

$$\Rightarrow 2\pi e^{-|\omega|} \quad //$$

iii)

$$\delta(t) \longleftrightarrow 1$$

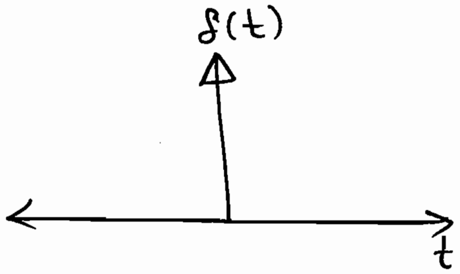
$$1 \longleftrightarrow 2\pi \delta(-\omega)$$

$$= \frac{2\pi}{|-1|} . \delta(\omega)$$

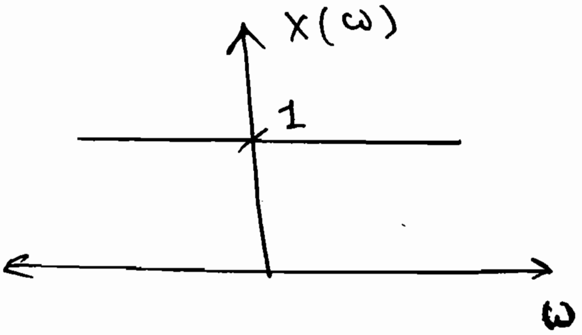
$$= 2\pi . \delta(\omega).$$

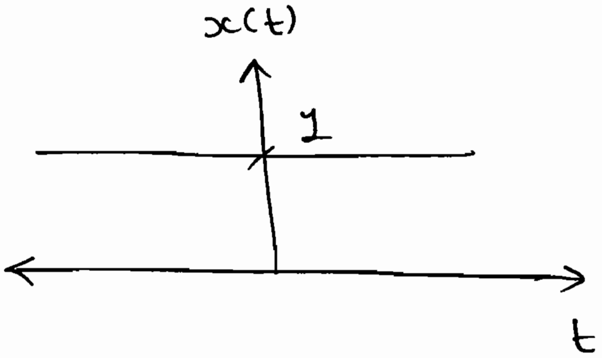
$$= \delta(f).$$

⇒

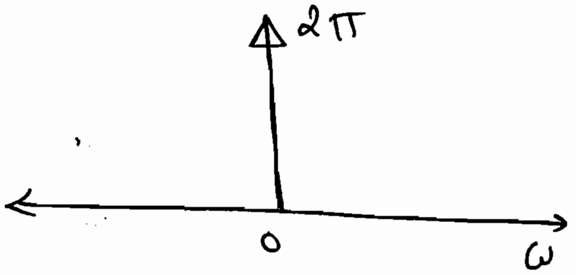


$\xleftrightarrow{F.T}$





$\xleftrightarrow{F.T}$



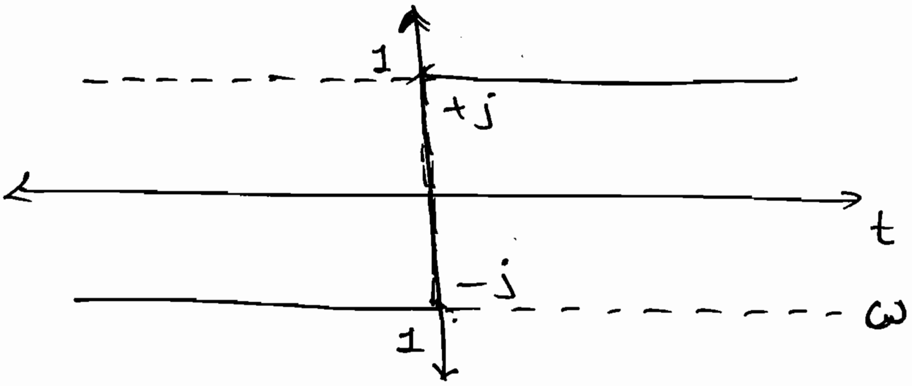
(iv)

$$\text{Sgn}(t) \xleftrightarrow{F.T} \frac{2}{j\omega}.$$

$$\frac{\cancel{2}}{jt} \xleftrightarrow{F.T} \cancel{2}\pi \,\text{Sgn}(-\omega)$$

$$\frac{1}{jt} \longleftrightarrow -\pi\,\text{Sgn}(\omega).$$

$$\therefore \boxed{\frac{1}{\pi t} \xleftrightarrow{F.T} -j\,\text{sgn}(\omega).}$$



⇒ Impulse response of Hibert Transform.

⇒ To set the additional π/2 phase angle we have to design $\frac{1}{\pi t}$ system in time domain.

$\Rightarrow$ $\mathrm{Sgn}(t) = 2u(t) - 1.$

$$\therefore \quad u(t) = \frac{1 + \mathrm{sgn}(t)}{2}.$$

$$\therefore \quad u(t) \xleftrightarrow{F.T.} \frac{2\pi\,\delta(\omega) + \frac{2}{j\omega}}{2}$$

$$\Rightarrow \quad \boxed{u(t) \xleftrightarrow{F.T.} \cancel{*}\ \frac{1}{j\omega} + \pi\,\delta(\omega).}$$ ✓

$\Rightarrow$ rect in time domain $\longleftrightarrow$ Sa in freq.

$$\mathrm{rect}\left(\frac{t}{2a}\right) \longleftrightarrow 1.\,2a\ \mathrm{Sa}\left(\frac{\omega \cancel{2}a}{\cancel{2}}\right).$$

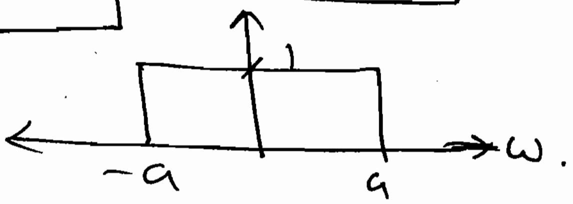
$$\mathrm{rect}\left(\frac{t}{2a}\right) \longleftrightarrow 2a.\ \mathrm{Sa}(\omega a).$$

Duality

$$2a.\ \mathrm{Sa}(at) \longleftrightarrow 2\pi.\mathrm{rect}\left(-\frac{\omega}{2a}\right).$$

$$a.\ \frac{\sin(at)}{at} \longleftrightarrow \pi.\ \mathrm{rect}\left(\omega/2a\right).$$ (even).

$$\therefore \quad \boxed{\frac{\sin(at)}{\pi t} \longleftrightarrow \mathrm{rect}\left(\omega/2a\right).}$$ ✓



② Time Scaling :-

$\Rightarrow$ If $x(t) \longleftrightarrow X(\omega)$.

$$\boxed{x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} \cdot X(\omega/\alpha).}$$

Compression ⟷ Expansion.

e.g. let, $x(t) = A\,\mathrm{rect}(t/T)$.

$\therefore \quad A\,\mathrm{rect}(t/T) \longleftrightarrow AT\,\mathrm{Sa}\left(\frac{\omega T}{2}\right)$.

Now, $y(t) = A\,\mathrm{rect}\left(\frac{2t}{T}\right)$.

$y(t) = x(2t)$.

$$\therefore \quad Y(\omega) = \frac{1}{2} X(\omega/2).$$

$$= \frac{AT}{2} \mathrm{Sa}\left(\frac{\frac{\omega}{2} \cdot T}{2}\right).$$

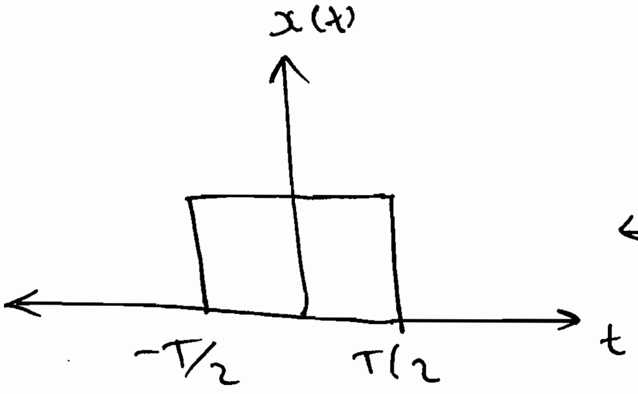
$$Y(\omega) = \frac{AT}{2} \mathrm{Sa}\left(\frac{\omega T}{4}\right).$$

$$\therefore \quad \frac{\omega T}{4} = \pm n\pi \Rightarrow \omega = \pm \frac{4n\pi}{T}.$$

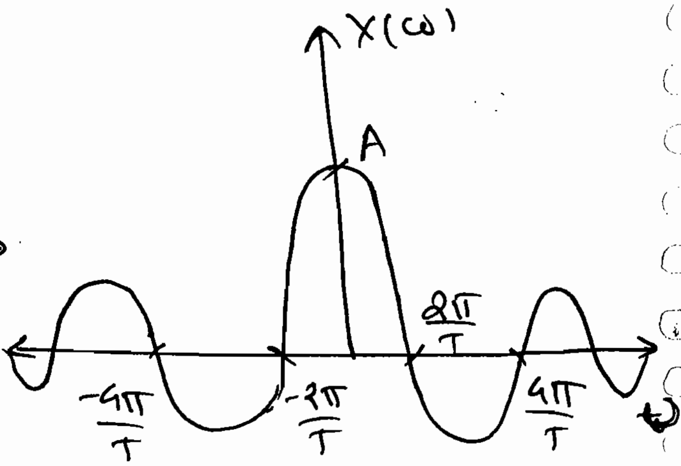
$$\therefore \quad Y(\omega) = \frac{AT}{2} \mathrm{sinc}\left(\frac{\omega T}{4\pi}\right).$$

$$\omega = \pm \frac{4n\pi}{T}.$$

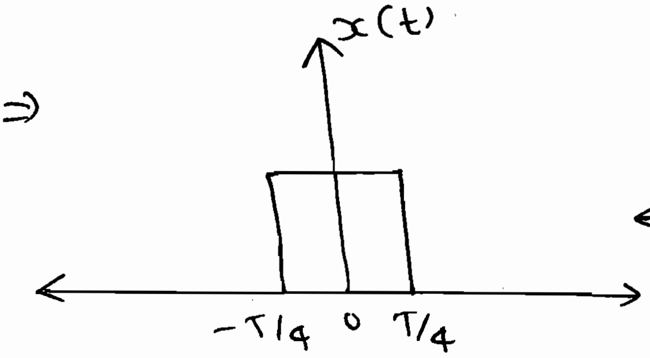
⇒



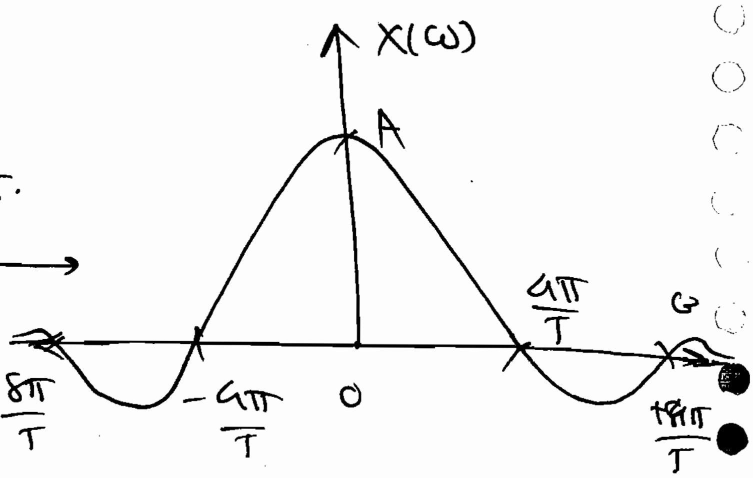
$\xleftrightarrow{F.T.}$



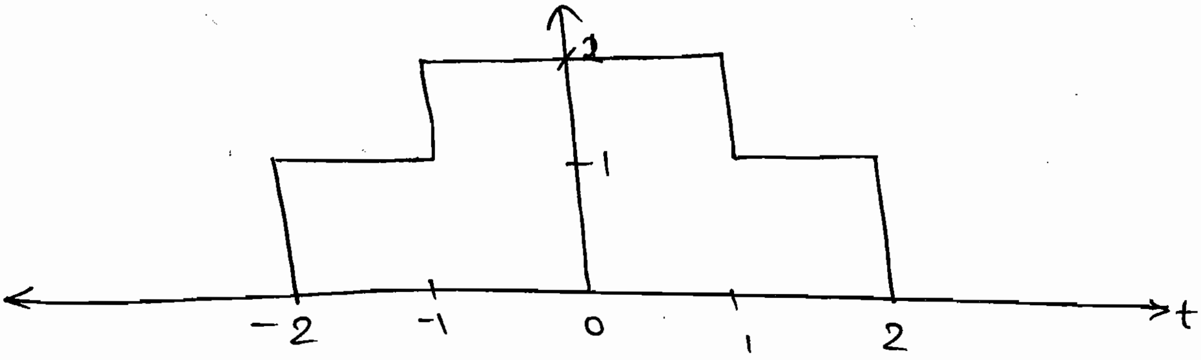
⇒



$\xleftrightarrow{F.T.}$

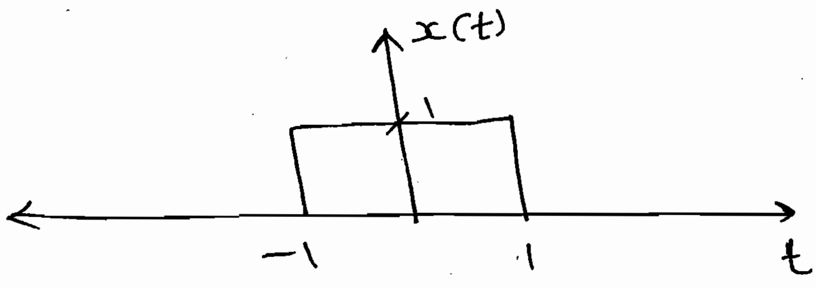


P 4.2.4. Find the F.T. of the signal shown in fig?



Sol^n:

let, $x(t) = 1 . rect(t/2)$.



$\therefore \; y(t) = x(t) + x(t/2)$.

$$\therefore \quad 1.\ \mathrm{rect}\,(t/2) \longleftrightarrow 2\,Sa\left(\frac{2\omega}{2}\right)$$

$$\Rightarrow 2\,Sa(\omega).$$

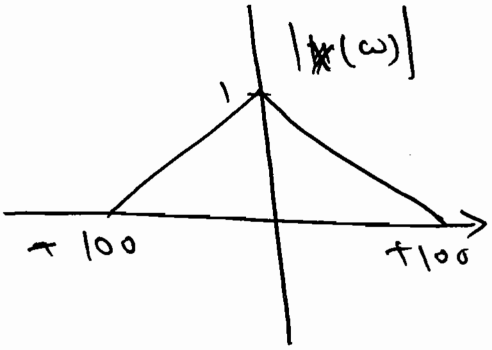
$$\therefore \quad y(t) = \cancel{2}\,x(t) + x(t/2).$$

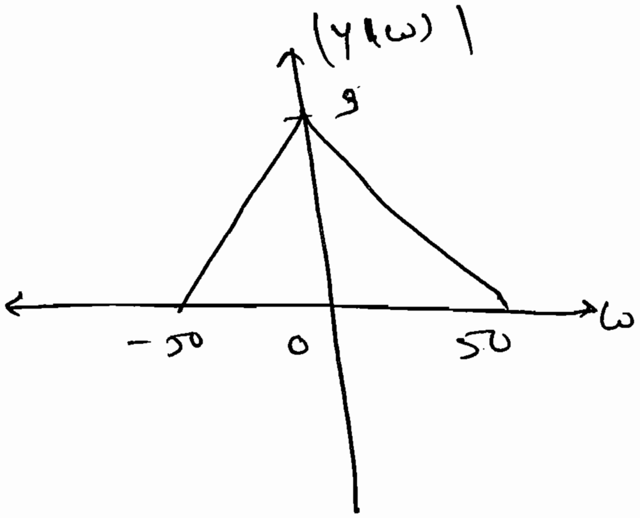
$$\therefore \quad y(\omega) = \overset{2}{\cancel{4}}\,Sa(\omega) + \frac{1}{|1/2|}\cdot X(2\omega/\cancel{2}).$$

$$\therefore \quad Y(\omega) = \overset{2}{\cancel{4}}\,Sa(\omega) + 2\cancel{2}\,Sa(2\omega/\cancel{2}).$$

$$\therefore \quad \boxed{Y(\omega) = 2Sa(\omega) + 4\,Sa(2\omega).}$$

P4.2.5 The magnitude of F.T. $X(\omega)$ of a function in fig (a) the magnitude of F.T. $Y(\omega)$ of other function $y(t)$ is shown below in fig (b). The phases $X(\omega)$ and $Y(\omega)$ are zero for all ω. The magnitude and frequency units are identical in both the figures. The f^n $y(t)$ can be expressed in terms of $x(t)$ as ______





Sol^n:
$$|Y(\omega)| = 3|X(2\omega)|$$
$$= 3 \cdot X(\omega/\tfrac{1}{2}).$$
$$= 3 \cdot \tfrac{1}{2} \times \frac{1}{1/2} X(\omega/\tfrac{1}{2}).$$
$$\therefore \boxed{y(t) = \frac{3}{2} x(t/2).}$$

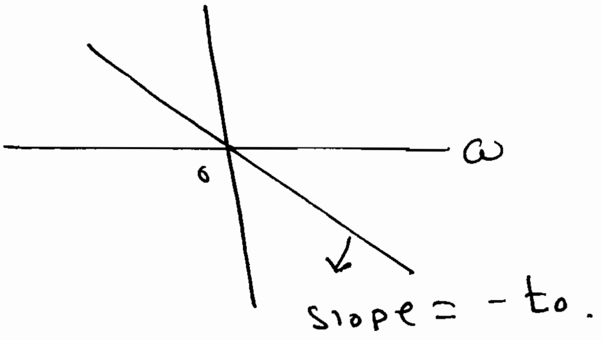
③ Time-Shifting:

⇒ If $x(t) \xleftrightarrow{F.T.} X(\omega)$.

then $\boxed{x(t-t_0) \xleftrightarrow{F.T.} e^{-j\omega t_0} X(\omega).}$

⇒ Time-delay in a signal causes a linear phase shift in its spectrum. shifting in time doesn't alter the amplitude spectrum of signal.

$$\therefore \theta(\omega) = -\omega t_0.$$



e.g. ① $y(t) = e^{2t} \cdot \cancel{x(t)} \cdot u(-t+3).$

Sol^n: $y(t) = e^{2(t-3)+6} \cdot u\,(-(t-3)].$

$$\therefore\ Y(\omega) = e^{6}\left[\frac{e^{-j\omega(3)}}{2-j\omega}\right].$$

② $y(t) = \text{rect}\left(\frac{t+1}{4}\right)$

Soln:

let, $x(t) = \text{rect}(t/4).$, $A=1,$ $T=4.$

$\therefore\ y(t) = x(t+1).$ ~~where~~,

$t_0 = -1.$

$\downarrow$

F.T.

$Y(\omega) = e^{j\omega}\cdot X(\omega).$

& $X(\omega) = 4.\ Sa\left(\frac{4\omega}{2}\right).$

$$\therefore\ \boxed{Y(\omega) = 4e^{j\omega}\cdot Sa(2\omega).}$$

✓ ✱

$\boxed{a}$ given $H(\omega) = \frac{4\sin 2\omega.\ \cos\omega}{\omega}$, find $h(0)$.

Soln:

$$H(\omega) = \frac{4\ \sin 2\omega.\ \cos\omega}{\omega}$$

$$= 2\left[\frac{2\sin 2\omega}{\omega}\right].\cos\omega \quad \times 2$$

$$= 2\left[\frac{2\ \sin\omega.\ \cos\omega.\ \cos\omega}{\omega}\right].\times 2$$

$$= 2\left[\frac{2\ \sin\omega}{\omega}\right]\cos^2\omega.\times 2$$

$$= \cancel{2}\left[\frac{2\ \sin\omega}{\omega}\right].\left[\frac{1+\cos 2\omega}{\cancel{2}}\right].\times 2$$

$$= \frac{2\sin\omega}{\omega}\times 2 + 2\left[\frac{2\sin\omega}{\omega}\right]\times\left[\frac{e^{-j2\omega}+e^{j2\omega}}{2}\right].$$

let, $x(t) = \text{rect}(t/2)$

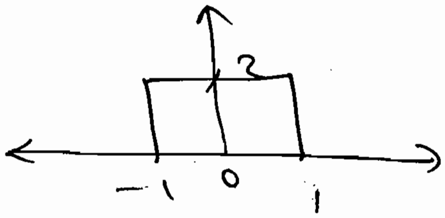
$$\Rightarrow X(\omega) = \frac{2\sin\omega}{\omega}.$$

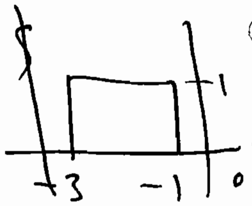
$$\therefore H(\omega) = 2X(\omega) + \frac{1}{2}\cancel{X(\omega)}\, e^{-j\omega 2} \cdot X(\omega)$$

$\downarrow$ IFT

$$+ e^{j2\omega} \cdot X(\omega).$$

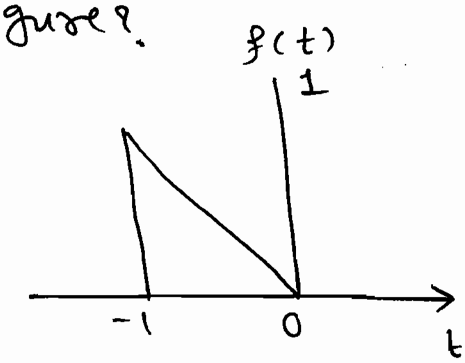
$$\therefore h(t) = 2x(t) + x(t-2) + x(t+2).$$



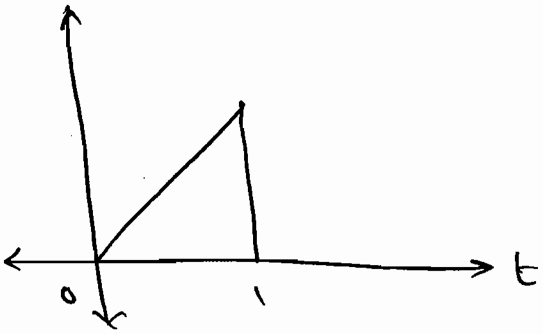


$$\therefore \boxed{h(0) = 2}$$

$\boxed{\text{P4.2.11}}$: The F.T. of a triangular pulse $f(t)$ shown in figure $F(\omega) = \frac{e^{j\omega} - j\omega e^{j\omega} - 1}{\omega^2}$ using this find the F.T. of the signals shown in figure?



Sol^n: ① $f_1(t)$

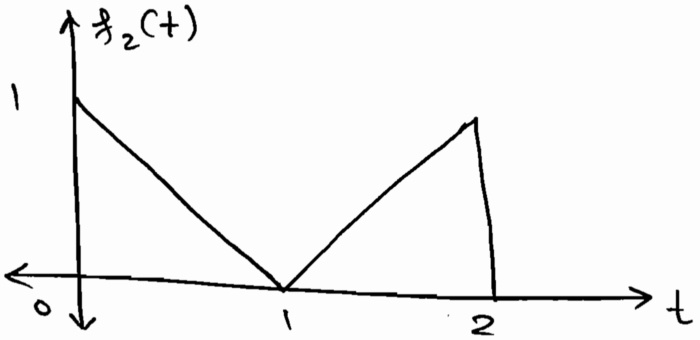


$$\Rightarrow \quad f_1(t) = f(-t).$$

$$\downarrow$$

$$F_1(\omega) = F(-\omega) = \frac{e^{-j\omega} + j\omega e^{-j\omega} - 1}{\omega^2}.$$

②



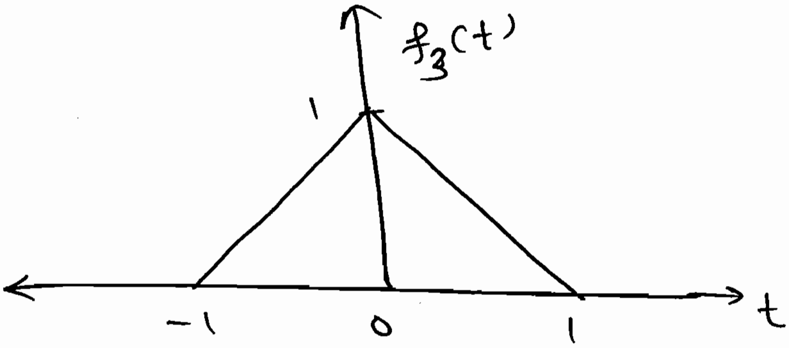
Sol^n:

$$f_2(t) = f_1(t-1) + f(t-1).$$

$\downarrow$ F.T.

$$\therefore \quad F_2(\omega) = e^{-j\omega} F_1(\omega) + e^{-j\omega} F_2(\omega).$$

③

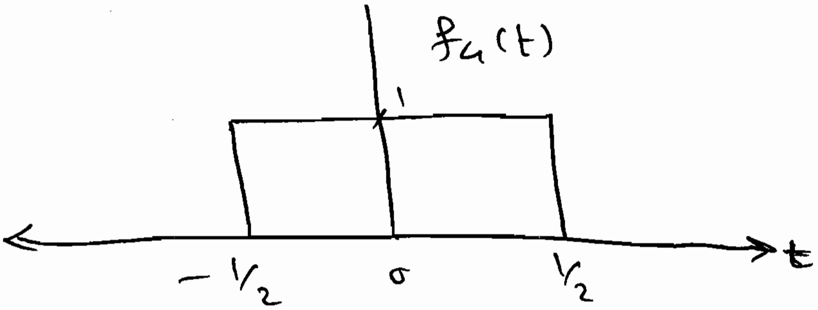


Sol^n:

$$f_3(t) = f_1(t+1) + f(t-1).$$

$$F_3(\omega) = e^{j\omega} F_1(\omega) + e^{-j\omega} F(\omega).$$

④

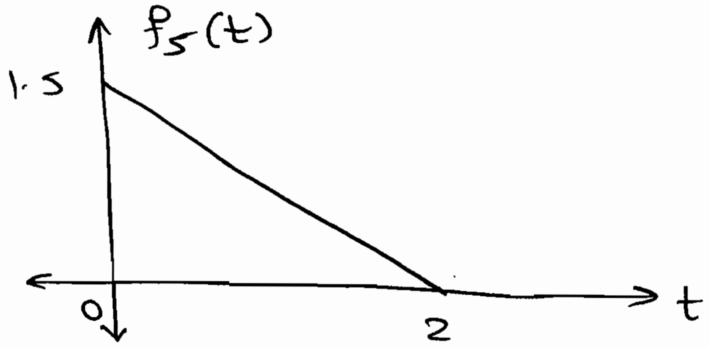


Sol^n:

$$f_4(t) = f\left(t - \tfrac{1}{2}\right) + f_1\left(t + \tfrac{1}{2}\right).$$

$$\therefore F_4(\omega) = e^{-j\frac{1}{2}\omega} \cdot F(\omega) + e^{+j\frac{\omega}{2}} \cdot F_1(\omega).$$

⑤



Sol^n:

$$f_5(t) = 1.5 f\left(\tfrac{t}{2} - 1\right).$$

$$= 1.5\, f\left(\frac{t-2}{2}\right). \quad t_0 = 2.$$

$$\alpha = \tfrac{1}{2}$$

$$\therefore F_5(\omega) = \frac{1.5}{\frac{1}{2}} \cdot e^{-j2\omega} \cdot F\left(\omega/\tfrac{1}{2}\right).$$

$$\therefore \boxed{F_5(\omega) = 3 \cdot e^{-j2\omega} \cdot F(2\omega).}$$

④ Frequency Shifting:

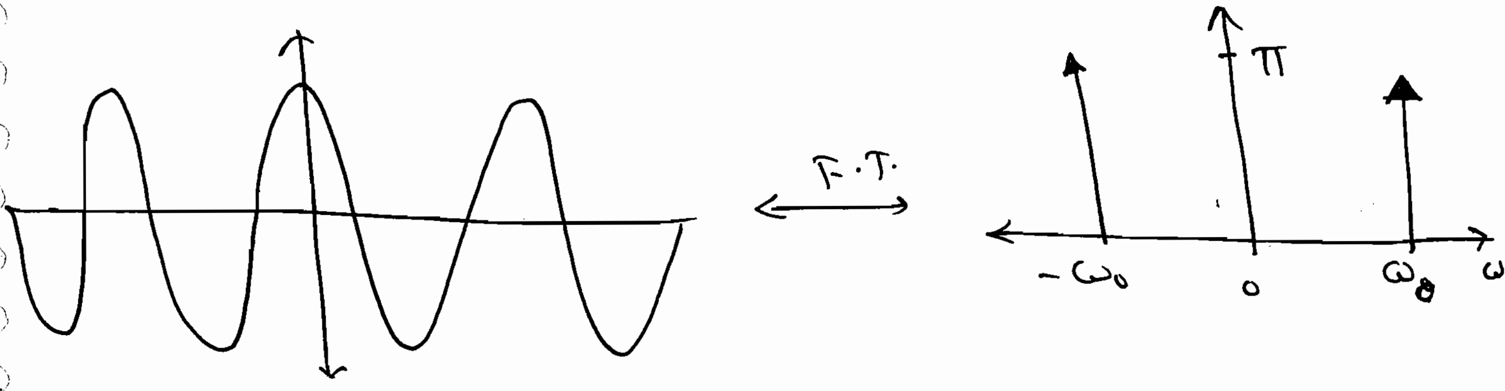
⇒ If $x(t) \longleftrightarrow X(\omega)$.

then $x(t) \cdot e^{j\omega_c t} \xleftrightarrow{F.T.} X(\omega - \omega_c)$.

e.g. $\cos\omega_0 t = \dfrac{1.e^{+j\omega_0 t} + 1.e^{-j\omega_0 t}}{2}$.

$$\xleftrightarrow{F.T} \; = \frac{2\pi\delta(\omega \mp \omega_0) + 2\pi\,\delta(\omega + \omega_0)}{2}.$$

∴ $\cos\omega_0 t \xleftrightarrow{F.T} \dfrac{\delta(f-f_c)+\delta(f+f_c)}{2}$.



$\Rightarrow$ $e^{-3t}\cdot\underbrace{\sin 6t}_{\text{reference signal.}}\cdot u(t)$ $\Rightarrow$ damped oscilliation.

$\Rightarrow$ $\text{rect}(t/4)\cdot\cos 6t$ ← reference pulse.

$\boxed{Q}$ F.T. of $y(t) = \text{sinc}(t).\cos 10\pi t$.

Sol:

$$\underset{\downarrow F.T.}{y(t)} = \underset{\substack{\downarrow\\ x(t)}}{\text{sinc}(t)}.\left[\frac{e^{j10\pi t}+e^{-j10\pi t}}{2}\right].$$

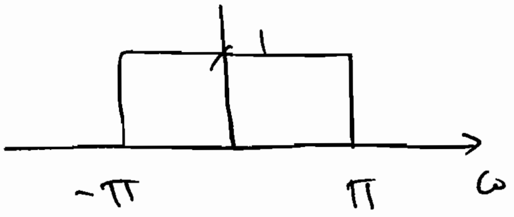
$$Y(\omega) = \frac{X(\omega-10\pi)+X(\omega+10\pi)}{2}.$$

$$x(t) = \text{sinc}(t). = \frac{\sin\pi t}{\pi t} \quad (a=\pi)$$

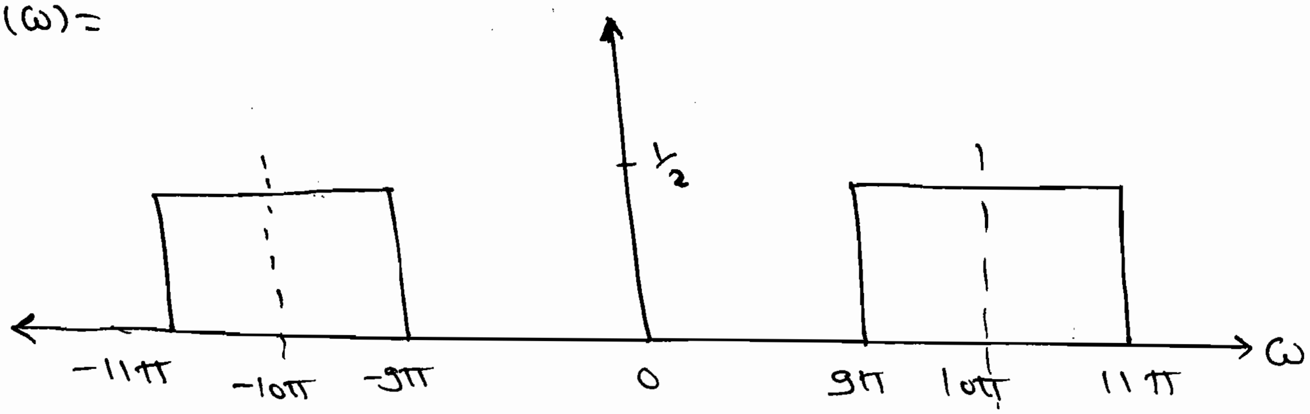
$\Rightarrow$ $X(\omega) = \text{rect}(\omega/2a)$.

$X(\omega) = \text{rect}(\omega/2\pi)$.

∴ $X(\omega) \Rightarrow$



∴ $Y(\omega)=$



$\boxed{\text{Q}}$ I.F.T. of $X(4\omega+3)$.

Soln:

$$Y(\omega) = X(4(\omega+3/4)).$$

$$= \frac{1}{4}\cdot\frac{1}{|1/4|}\cdot X(\omega/\tfrac{1}{4}).$$

$$\boxed{y(t) = \frac{1}{4}\cdot x(t/4).\ e^{-j(3/4)t}}$$

Note:

⇒ When we perform time differentiation $\delta(\omega)$ Component is loss since $j\omega\,\delta(\omega)=0$ and the original speltoum magnitudes are increased by the factor $|\omega|$ i.e. high freq. components are more ampiilied.

⑤ Differentiah in time:

⇒ If $x(t) \xleftrightarrow{F.T.} X(\omega)$

then $\boxed{\frac{d}{dt}x(t) \xleftrightarrow{F.T.} (j\omega)X(\omega).}$

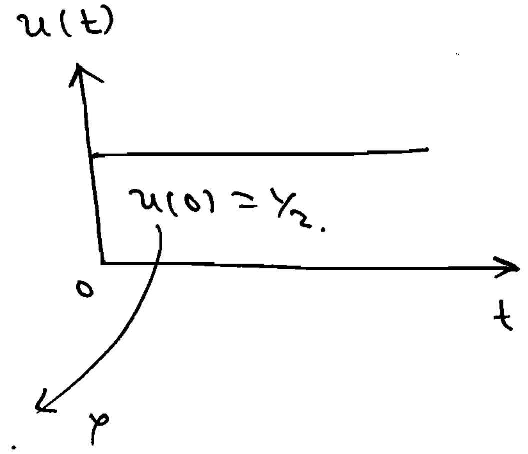
but, $X(\omega) \neq \dfrac{F.T\left\{\frac{d}{dt}x(t)\right\}}{j\omega}$.

e.g. ① $x(t) = u(t)$.

$$\frac{d}{dt}x(t) = \delta(t).$$

$$\therefore \; j\omega X(\omega) = 1.$$

$$X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$



(missed by differentiation).

$$\Rightarrow \quad j\omega\,\delta(\omega) = j(0).\,\delta(\omega) = 0.$$

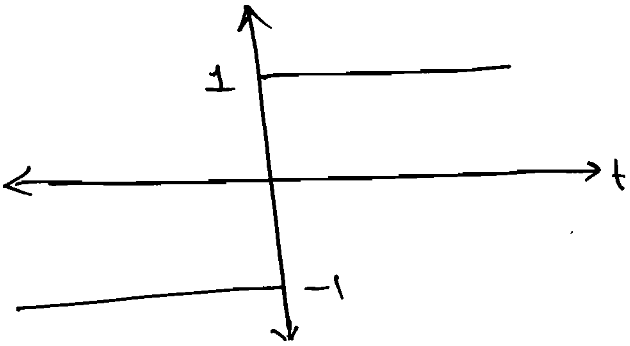
② $x(t) = \text{Sgn}(t)$

$$x(t) = 2u(t) - 1$$

$$\therefore \; \frac{d}{dt}x(t) = 2\delta(t)$$

$$\therefore \; j\omega X(\omega) = 2.$$

$$\boxed{X(\omega) = 2/j\omega.}$$



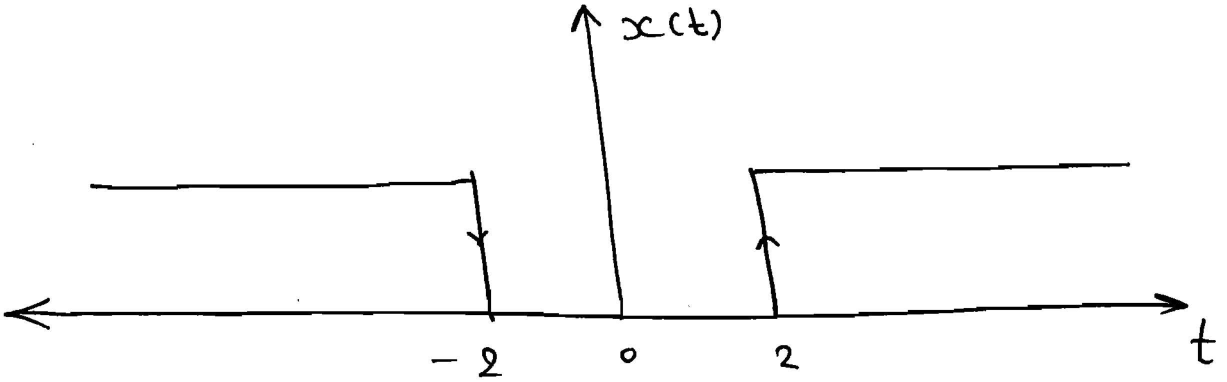
P. 4.2.16 Find the F.T. of the signal $y(t) = \frac{d}{dt}\left[u(t-2) + u(-t-2)\right]$.

Soln:

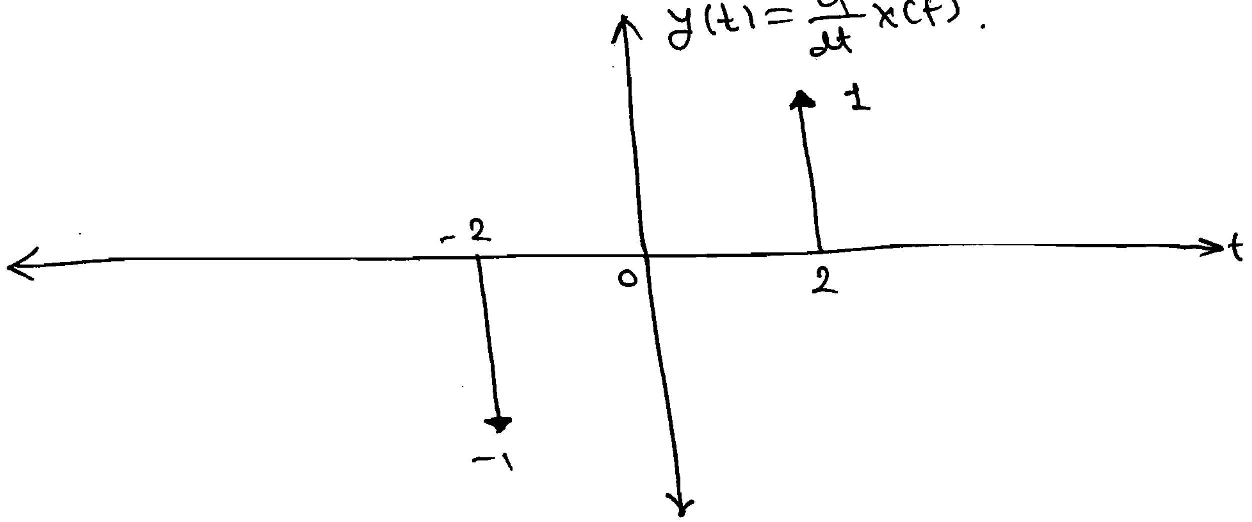
$$y(t) = \frac{d}{dt}\left[u(t-2) + u(-t-2)\right].$$

$$y(t) = \delta(t-2) \mp \delta(+t+2).$$

$\Rightarrow$



$\Downarrow$



$\therefore \quad y(t) = -1 \cdot \delta(t+2) + \delta(t-2).$

$$Y(\omega) = -e^{j2\omega} \cdot (1) + e^{-j2\omega} \cdot (1).$$

$$= -\frac{1}{2j}\left[\frac{e^{+j2\omega} - e^{-j2\omega}}{2j}\right].$$

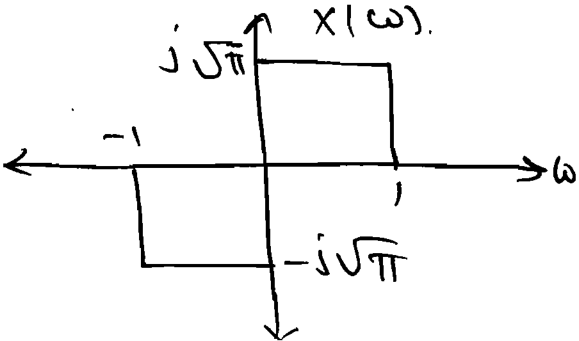
$$= -\frac{1}{2j} \times \sin(2\omega).$$

$$\boxed{Y(\omega) = \frac{j\sin(2\omega)}{2}}$$

☆

P 4.2.17 For the Speltrum X(ω) shown in figure,

find $\frac{d}{dt}x(t)$ at $t=0$?



Sol^n:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) . e^{j\omega t} . d\omega .$$

$$\therefore \frac{d x(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(\omega) . e^{j\omega t} . d\omega .$$

$$\left. \frac{d}{dt} x(t) \right|_{t=0} = \frac{j\omega}{2\pi} \int_{-\infty}^{\infty} X(\omega) . d\omega .$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (j\omega) X(\omega) . d\omega .$$

$$= \frac{1}{2\pi} \left[\int_{-1}^{0} \cancel{j(\omega)} (-j\sqrt{\pi}) (j\omega) d\omega \right.$$

$$\left. + \int_{0}^{1} (j\sqrt{\pi}) (j\omega) d\omega \right] .$$

$$= \frac{1}{2\pi} \left[\sqrt{\pi} \left(\frac{\omega^2}{2} \right)_{-1}^{0} - \sqrt{\pi} \left(\frac{\omega^2}{2} \right)_{0}^{1} \right] .$$

$$= \frac{1}{2\pi} \left[- \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \right] .$$

$$= \frac{-\sqrt{\pi}}{2\pi}$$

$$\therefore \boxed{\left. \frac{d}{dt} x(t) \right|_{t=0} = \frac{-1}{2\sqrt{\pi}}}$$

⑥ Frequency Differentiation:

⇒ Diff^n w.r.t. one variable corresponds to multiplication by other variable.

$$\boxed{-jt\,x(t) \xleftrightarrow{F.T.} \frac{d}{d\omega}X(\omega).}$$

e.g. ① $y(t) = t\cdot e^{-at}\cdot u(t).$

$$\therefore\ y(t) = \frac{-j}{-j}\cdot t\cdot e^{-at}\cdot u(t).$$

$$= \frac{-1}{j}\cdot\left(-jt\cdot e^{-at}\cdot u(t)\right).$$

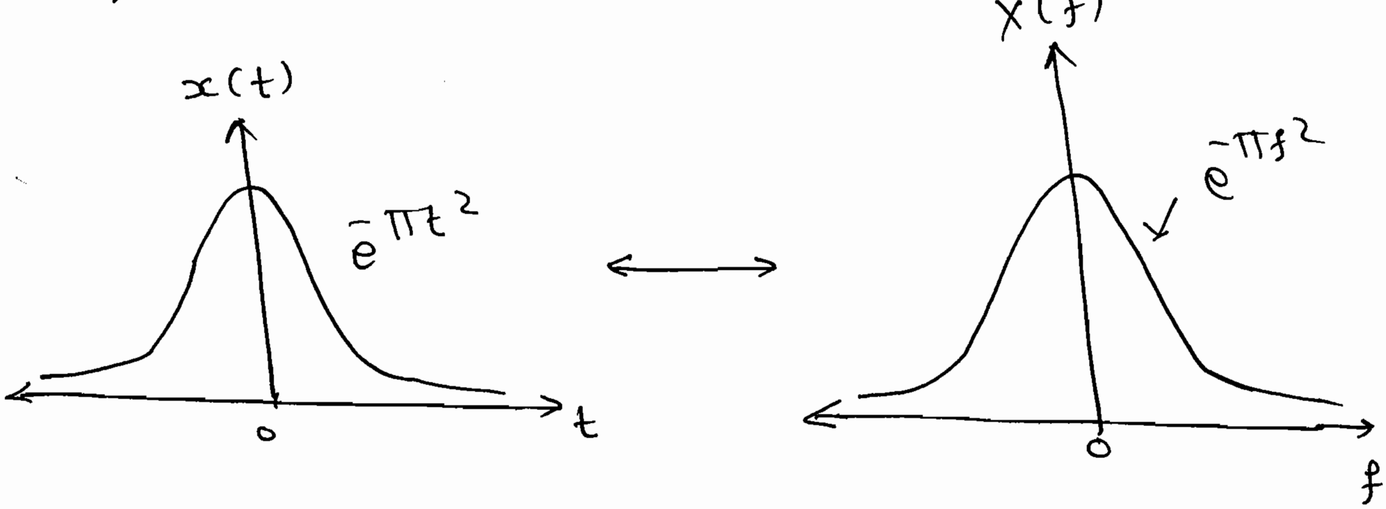
$$= +j\left[\frac{d}{d\omega} \times \frac{1}{a+j\omega}\right].$$

$$= +j\ \frac{-1}{(a+j\omega)^2} \times (j)$$

$$\therefore\ \boxed{Y(\omega) = \frac{1}{(a+j\omega)^2}}$$

* Gaussian function:

⇒



let, $x(t) = e^{-\pi t^2}.$

$$\therefore\ \frac{d}{dt}x(t) = -e^{-\pi t^2}(2\pi t). = -2\pi t\,x(t)$$

$$\therefore \boxed{\frac{d}{dt} x(t) = -2\pi t \, x(t)} \quad — \text{(I)}.$$

$$\therefore \; j2\pi f x(t) = -2\pi t \, . \, e^{-\pi t^2}.$$

$$j\,(j2\pi f x(t)) = -j2\pi t \cdot e^{-\pi t^2}.$$

$$\therefore \; -2\pi f \cdot x(t) = \frac{d}{df} X(f).$$

$$\therefore \boxed{\frac{d}{df} X(f) = -2\pi f \, X(f)} \quad — ②.$$

⇒ From eq I & II looking in similar manner $x(f)$ is same as $x(t)$.

⇒ In general,

$$\boxed{e^{-at^2} \; (a>0) \xleftrightarrow{F.T.} \sqrt{\frac{\pi}{a}} \cdot e^{-\omega^2/4a} \quad \text{(or)} = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{\pi^2 f^2}{a}}.}$$

P 4.2.20 Given $x(t) \longleftrightarrow X(\omega)$, express the F.T. of the following signals in terms of $X(\omega)$?

(i) $x_1(t) = x(2-t) + x(-t-2)$.

Soln: $x_1(t) = x(-(t-2)) + x(-(t+2))$.

$$X_1(\omega) = e^{-j2\omega} \cdot X(-\omega) + e^{j2\omega} X(-\omega).$$

$$= \frac{X(-\omega)}{2} \left[\frac{e^{j2\omega} + e^{-j2\omega}}{2} \right].$$

∴ $\boxed{X_1(\omega) = 2X(-\omega).\ \cos 2\omega.}$

② $x_2(t) = x(3t+6).$

Sol^n:

$x_2(t) = x(3(t-2)).$

∴ $\boxed{X_2(\omega) = \frac{1}{3}.\ e^{-j2\omega}.\ X(\omega/3).}$

③ $x_3(t) = \frac{d^2}{dt^2}x(t-3).$

Sol^n:

$x_3(t) = \frac{d^2}{dt^2}\ x(t-3).$

$X_3(\omega) = (j\omega)^2.\ e^{-j3\omega}.\ X(\omega).$

$\boxed{X_3(\omega) = -\omega^2.\ e^{-j3\omega}.\ X(\omega).}$

**
*
④ $x_4(t) = t\frac{dx(t)}{dt}.$

Sol^n:

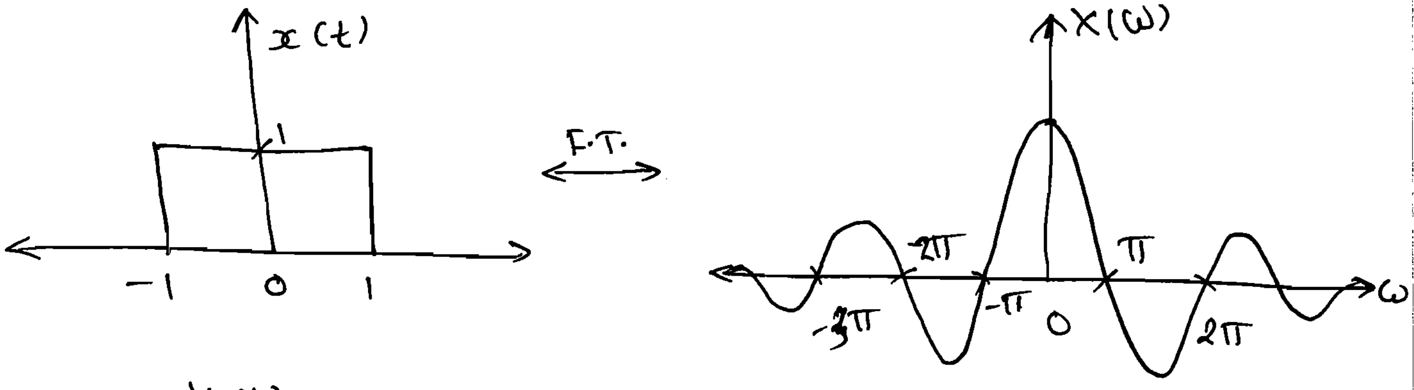
$x_4(t) = \underset{\downarrow\ j\frac{d}{d\omega}}{\textcircled{t}}.\ \textcircled{\frac{dx(t)}{dt}}$

$X_4(\omega) = j\frac{d}{d\omega}(j\omega.\ X(\omega)).$

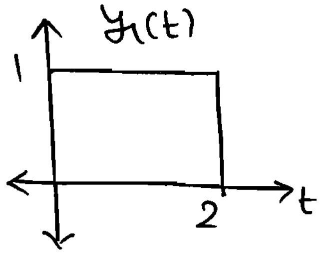
∴ $\boxed{X_4(\omega) = -\left(X(\omega) + \omega\frac{dX(\omega)}{d\omega}\right).}$

$\boxed{P4.2.21}$ Given $x(t) = \begin{cases} 1;\ |t|<1 \\ 0;\ \text{eleseawhere} \end{cases} \longleftrightarrow 2\frac{\sin\omega}{\omega}$ find the F.T. of the following signals?

⇒



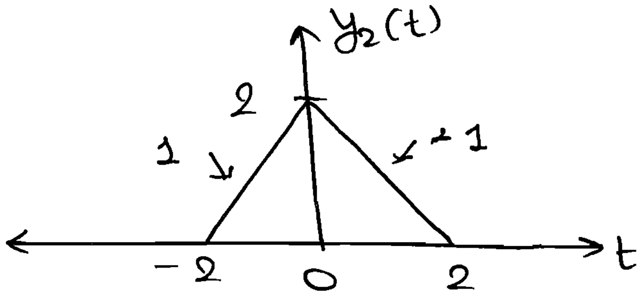
(a)



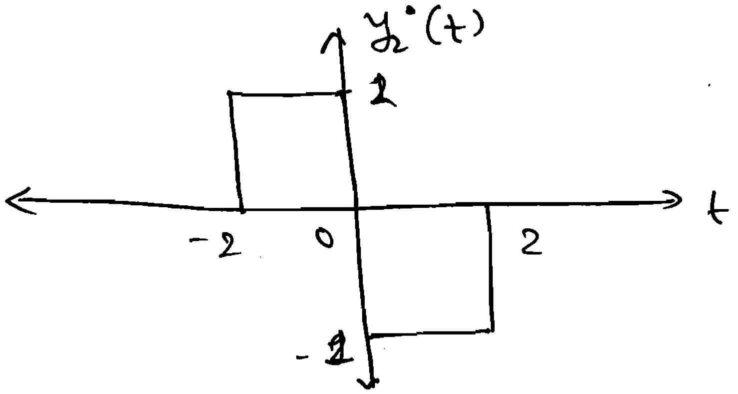
Soln:
$y_1(t) = x(t-1)$.

∴ $\boxed{Y_1(\omega) = e^{-j\omega} \cdot X(\omega).}$

ⓑ



Soln:

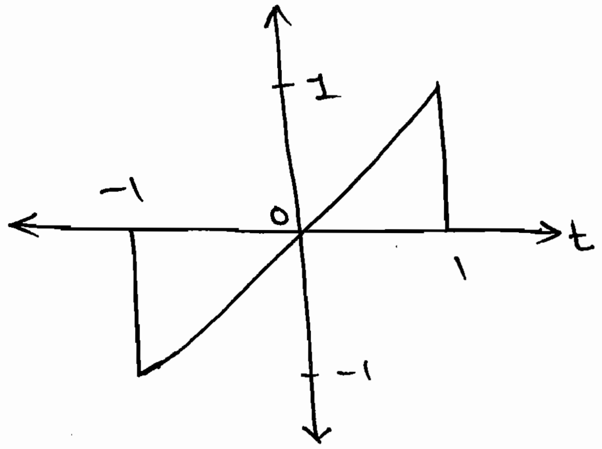


∴ $y_2'(t) = y_1(-t) \cancel{+} - y_1(t)$.

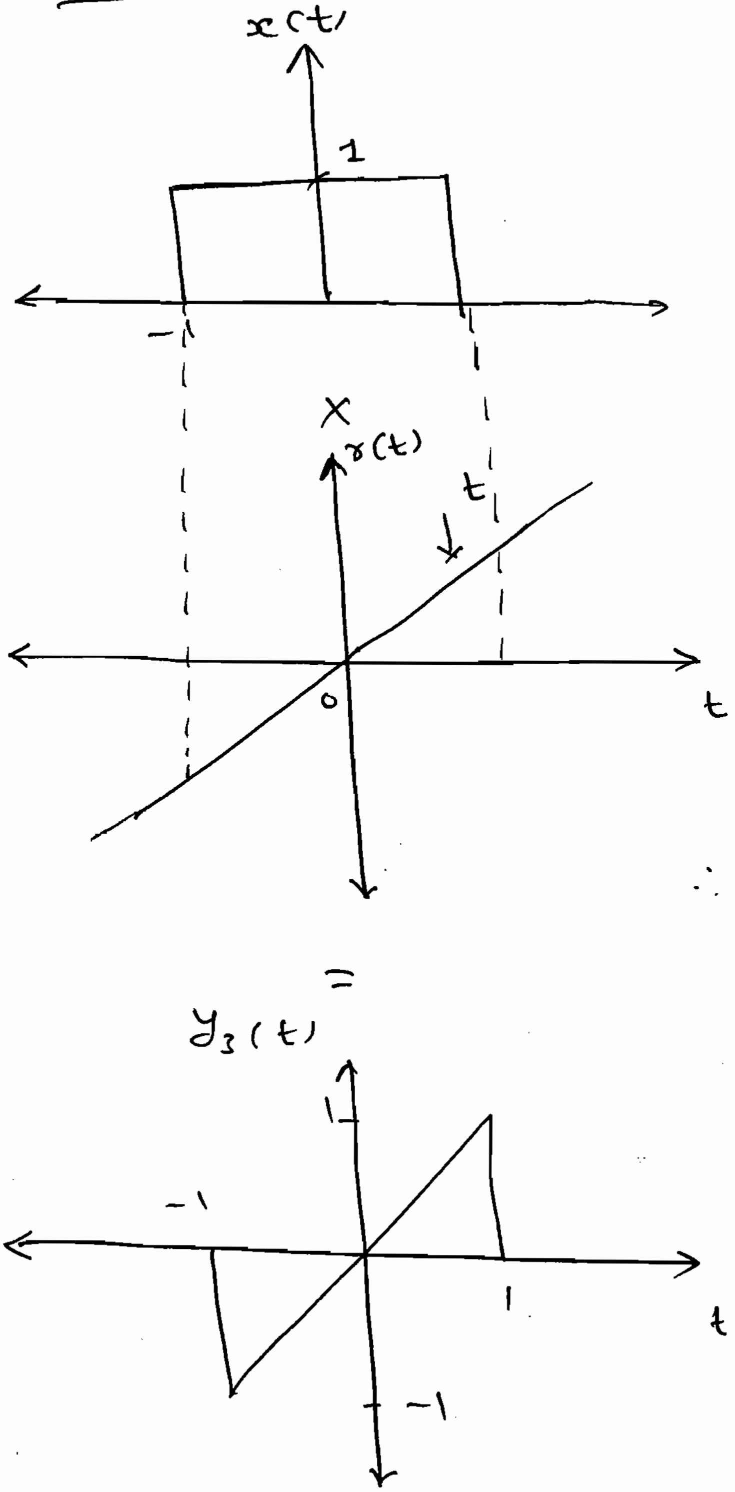
$j\omega Y_2(\omega) = Y_1(-\omega) - Y_1(\omega)$.

∴ $\boxed{Y_2(\omega) = \frac{1}{j\omega}\left[Y_1(-\omega) - Y_1(\omega)\right]}$.

(c)



Soln: <u>Method-I</u>



So, $y_3(t) = t \cdot x(t)$.

$$\therefore \; Y_3(\omega) = j \cdot \frac{dX(\omega)}{d\omega}.$$

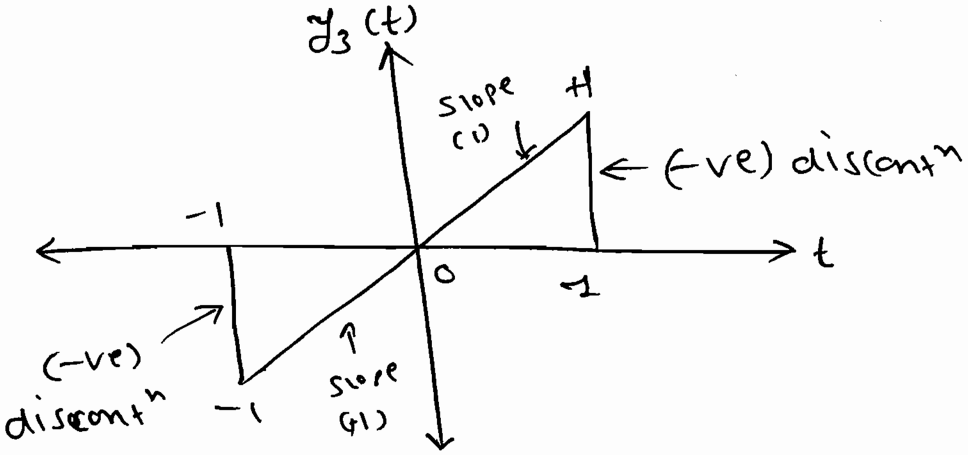
$$\therefore \; Y_3(\omega) = j \frac{d}{d\omega}\left(\frac{2\sin\omega}{\omega}\right)$$

$$= 2j\left[\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right]$$

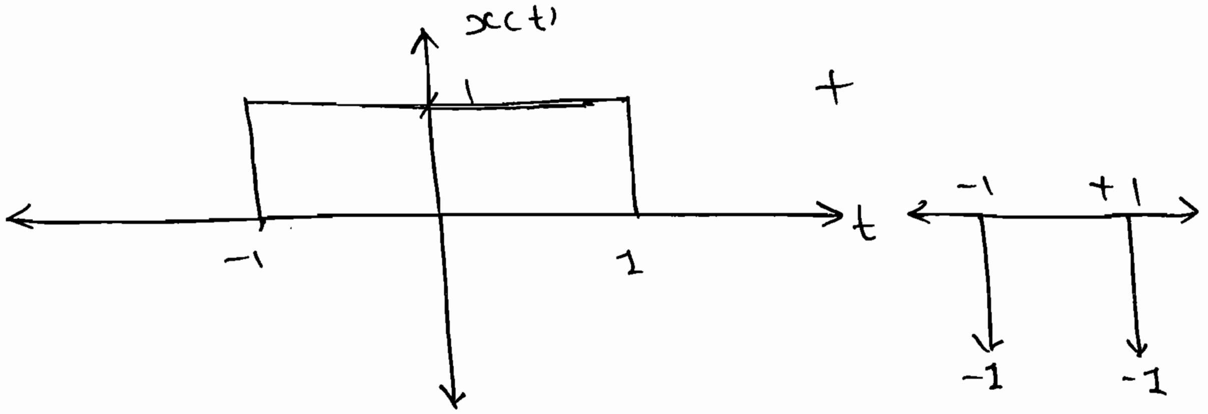
$$\therefore \; \boxed{y_3(\omega) = \frac{2j}{\omega^2}\left[\omega\cos\omega - \sin\omega\right]}$$

<u>Method-II</u> : By differentiation.

⇒



⇓



$$\therefore \quad \frac{dy_3(t)}{dt} = x(t) \mp \delta(t-1) \mp \delta(t+1).$$

$$\therefore \quad (j\omega)\, Y_3(\omega) = X(\omega) \mp \frac{e^{-j\omega} + e^{j\omega}}{1}.$$

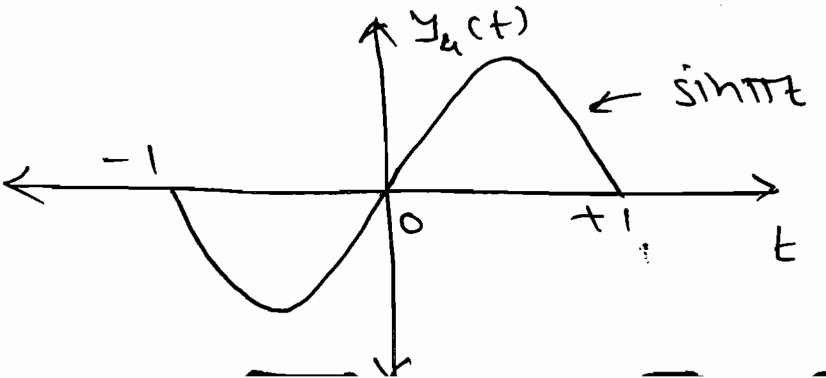
$$\therefore \quad Y_3(\omega) = \frac{X(\omega)}{j\omega} \mp \frac{2\cos\omega}{j\omega}.$$

$$= \frac{1}{j\omega}\left[\frac{2\sin\omega}{\omega} \mp 2\cos\omega\right]$$

$$= \frac{j}{j^2\omega^2}\left[2\sin\omega - 2\omega\cos\omega\right].$$

$$\boxed{Y_3(\omega) = \frac{2}{j\omega^2}\left[\cancel{2}\omega\cos\omega - \cancel{2}\sin\omega\right]}$$

(d)



Solⁿ:

$$Y_u(t) = \sin\pi t \cdot x(t).$$

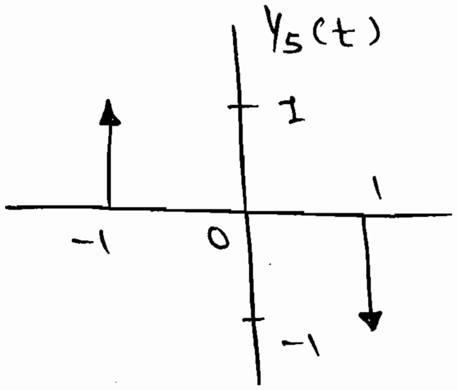
$$\therefore Y_u(\omega) \neq j \frac{d}{d\omega}(\cancel{x}(\omega)).$$

$$\therefore Y_u(t) = \frac{e^{j\pi t} - e^{-j\pi t}}{2j} \times x(t).$$

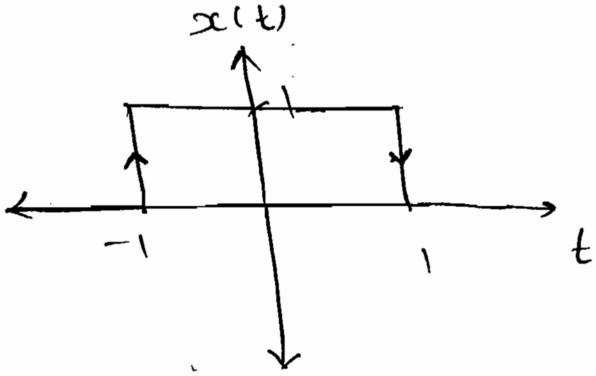
$$y_u(t) = \frac{1}{2j}\left[e^{+j\pi t} \cdot x(t) - e^{-j\pi t} \cdot x(t) \right].$$

$$\Rightarrow \boxed{Y(\omega) = \frac{1}{2j}\left[X(\omega - \pi) - X(\omega + \pi) \right].}$$

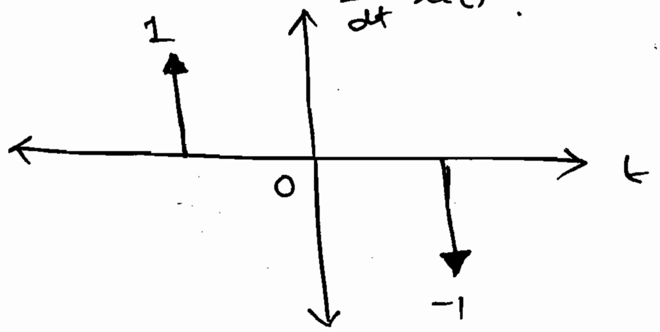
(e)



Solⁿ:



$$\Downarrow \frac{d}{dt}x(t).$$



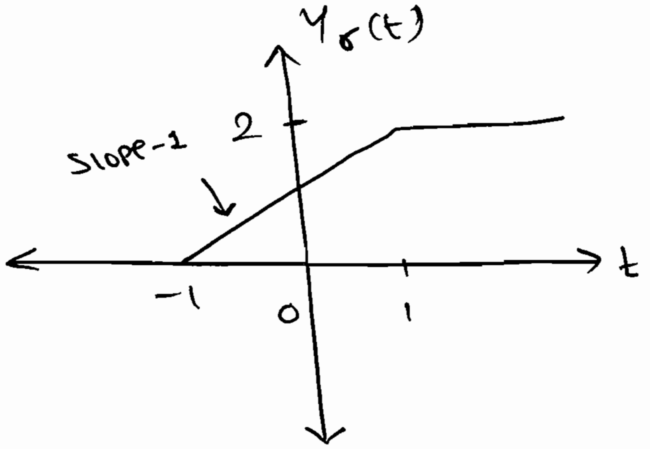
$$Y_s(t) = \frac{d}{dt} x(t).$$

$$\therefore Y_s(\omega) = j\omega \left[X(\omega) \right].$$

$$\therefore Y_s(\omega) = j\cancel{\omega}\left[\frac{2\sin\omega}{\cancel{\omega}} \right)$$

$$\boxed{Y_s(\omega) = 2j \sin\omega}$$

(f)



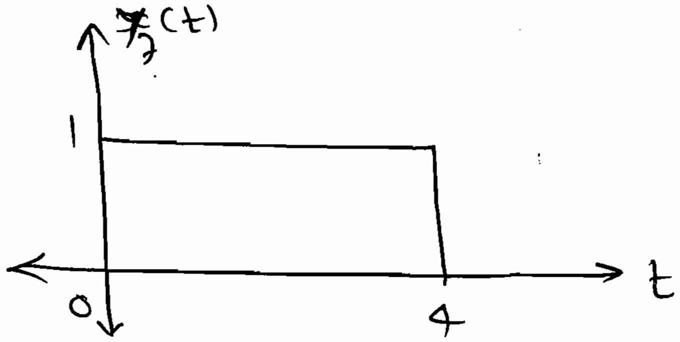
Sol^n: $y_6(t) = (t+1)\,x(t) + 2u(t-1)$.

$\therefore$ $y_6(t) = t\,x(t) + x(t) + 2u(t-1)$.

$\Big\downarrow$ F.T.

$$\therefore \boxed{Y_6(\omega) = j\frac{d}{d\omega}(X(\omega)) + X(\omega) + 2\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]e^{-j\omega}}$$

(g)



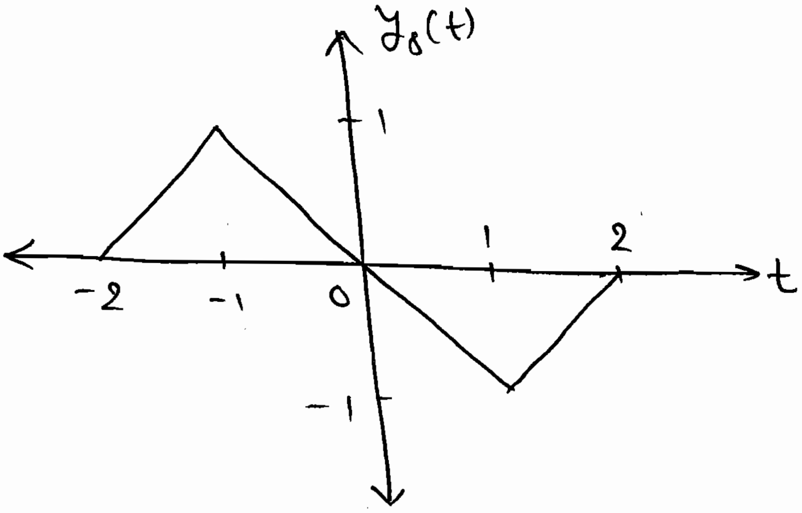
Sol^n: $y_7(t) = x\left(\frac{t}{4} - 1\right)$.

$y_7(t) = x\left(\frac{t-4}{4}\right)$.

$$\therefore Y_7(\omega) = \frac{1}{1/4} \cdot X\left(\omega/\tfrac{1}{4}\right) \cdot e^{-j4\omega}.$$

$$\therefore \boxed{Y_7(\omega) = 4 \cdot e^{-j4\omega} \cdot X(4\omega).}$$

(h)



Soln:

~~$y_8(t) =$ $y_2(t)$~~

$$y_8(t) = y_2\,(2(t+1)) - y_2\,(2(t-1)).$$

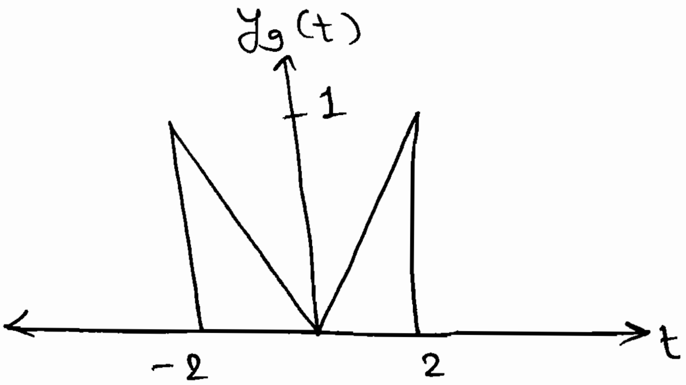
$$Y_8(\omega) = \frac{1}{(1/2)}\cdot e^{+j\omega}\,Y_2(\omega/{}_{1/2}) - \cancel{\frac{1}{(1/2)}}\cdot e^{-j\omega}\cdot Y_2\,(\omega/{}_{1/2}).$$

$$= 2e^{j\omega}\,Y_2\,(2\omega) - 2\,e^{-j\omega}\cdot Y_2\,(2\omega)$$

$$= 2\times 2j\times Y_2(2\omega)\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right].$$

$$\therefore \boxed{Y_8(\omega) = 4j\,Y_2(2\omega)\,.\,\sin\omega.}$$

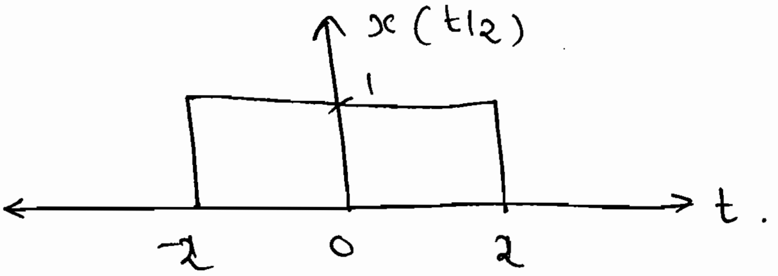
☆☆ (i)



Soln:

$$y_9(t) = x\,(t/2) - y_2(t).$$

$\Rightarrow$



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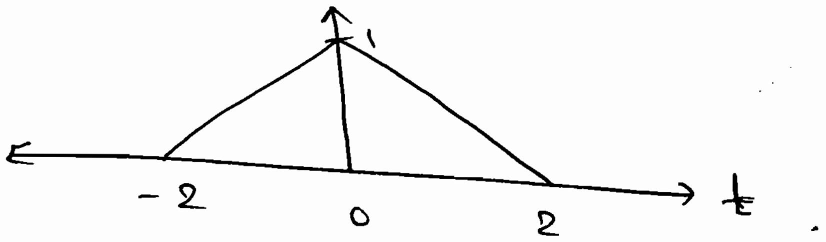

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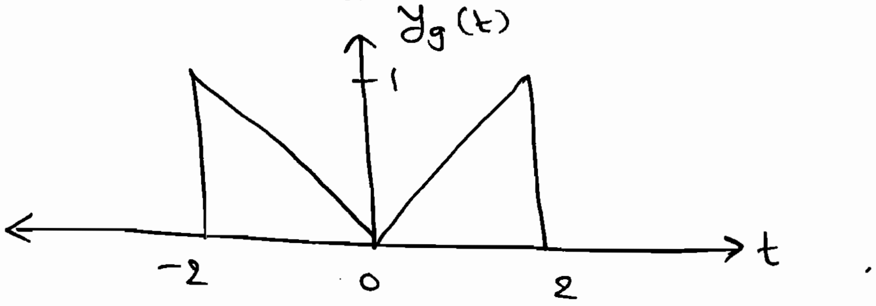
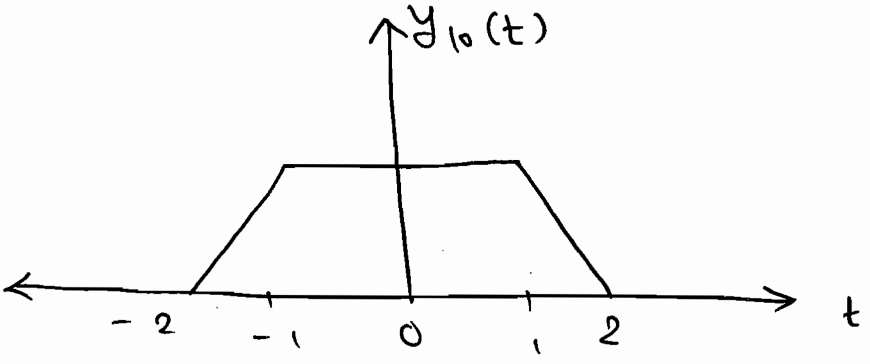


$$\therefore\ y_9(t) = x(t/2) - y_2(t).$$

$$\therefore\ \boxed{Y_9(\omega) = 2X(2\omega) - Y_2(\omega).}$$

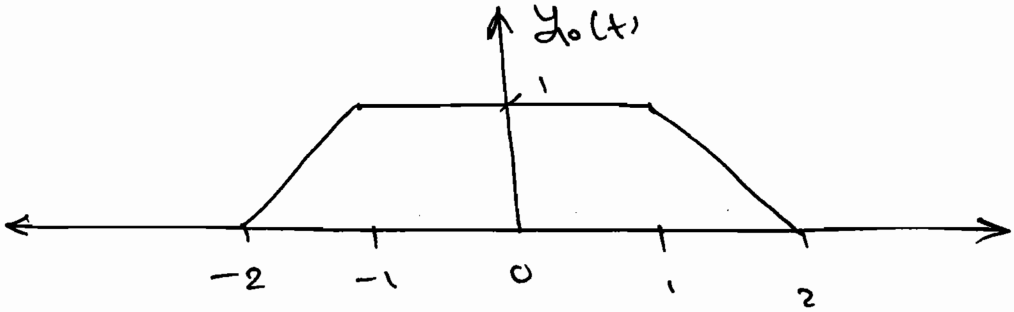
***
(j)



Sol^n: let, $z(t) = \frac{1}{2} y_2(2t).$

$$\therefore\ y_{10}(t) = z(t+1) + z(t) + z(t-1).$$

$$\therefore\ y_{10}(t) = \frac{1}{2}\left[y_2(2t+2)\right] + \frac{1}{2}\left[y_2(2t)\right]$$
$$+ \frac{1}{2}\left[y_2(2t-2)\right].$$



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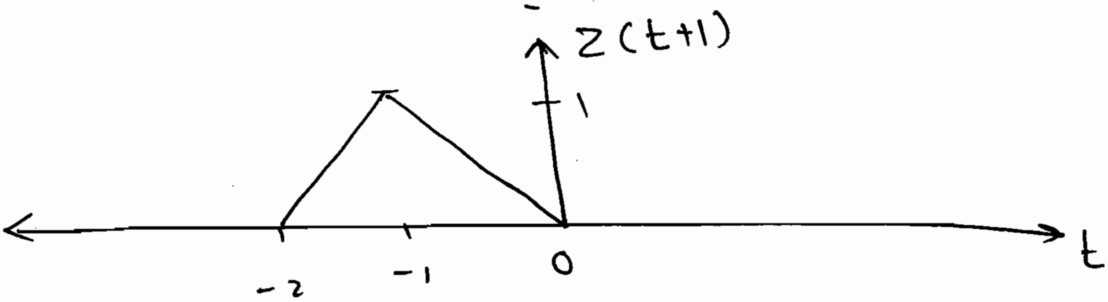
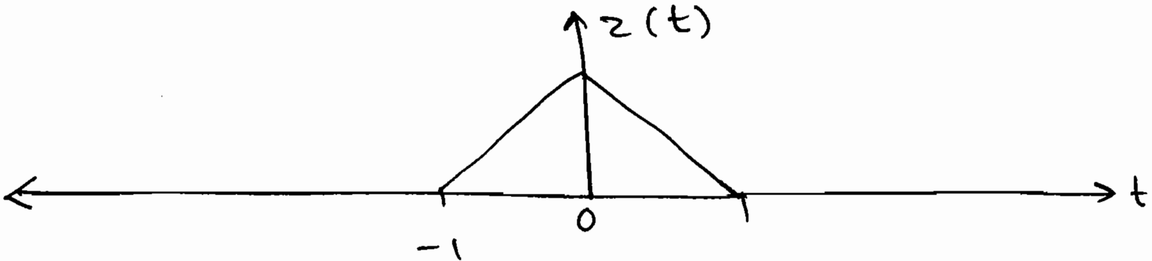
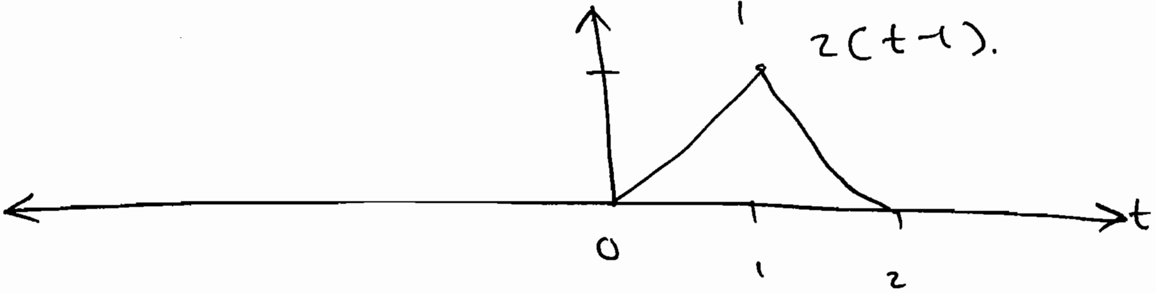






$$\therefore \quad Y_{10}(\omega) = \frac{1}{2}\Big[\frac{1}{2}e^{j\omega}\,Y_2(\omega/2) + \frac{1}{2}\cdot Y_2(\omega/2)$$

$$+ \frac{1}{2}\,Y_2(\omega/2)\cdot e^{-j\omega}\Big].$$

$$\boxed{Y_{10}(\omega) = \frac{Y_2(\omega/2)}{4}\Big[1 + 2\cos\omega\Big].}$$

⑦ Convolution in time :-

$\Rightarrow$ If $x(t) \longleftrightarrow X(\omega)$ & $h(t) \longleftrightarrow H(\omega)$.

then, $x(t) * h(t) \longleftrightarrow X(\omega)\cdot H(\omega)$.

⇒ Convolution in time corresponds to multiplication in frequency domain.

⇒ F.T. of of impulse response, $h(t)$ is known as frequency response, $H(\omega)$.

e.g. ① $e^{-at}\cdot u(t) * e^{-at}\cdot u(t) \xleftrightarrow{F.T.} \frac{1}{(a+j\omega)^2}$

$= t\cdot e^{-at}\cdot u(t)$

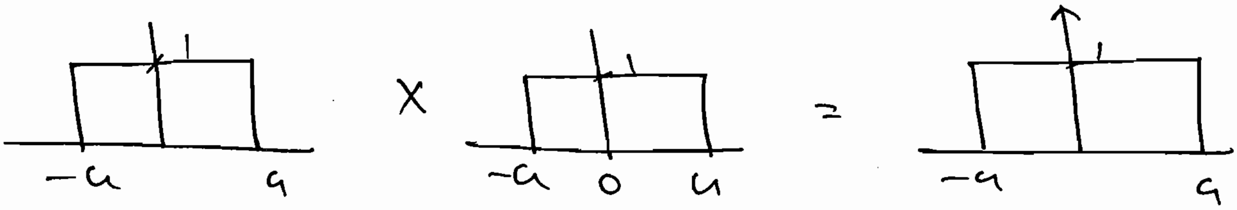
⇒ ②

∴ $rect(t/T) * rect(t/T) \xleftrightarrow{F.T.} T^2\, Sa^2\left(\frac{\omega T}{2}\right)$.

$T\, Sa\left(\frac{\omega T}{2}\right) \times T\, Sa\left(\frac{\omega T}{2}\right)$

③ $\boxed{\frac{\sin at}{\pi t} * \frac{\sin at}{\pi t} = \frac{\sin at}{\pi t}}$ ↖ I.F.T.

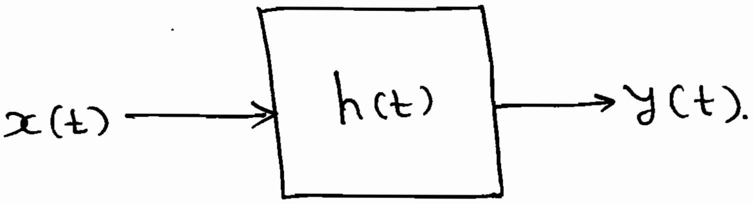
∴



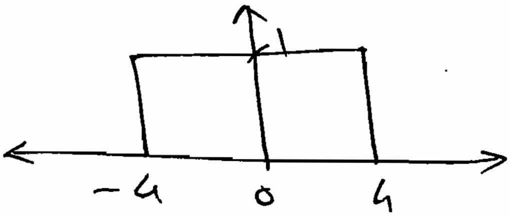
⇒ $\boxed{\text{Gaussian} * \text{Gaussian} = \text{Gaussian.}}$

$\boxed{P4.2.23}$ An L.T.I. system is having I.R. $h(t) = \frac{\sin 4t}{\pi t}$ for which the input applied is $x(t) = \cos 2t + \sin 6t$, find the o/p?

Soln:



∴ $H(\omega) =$ rect $(t/8)$.



← Ideal LPF.

So, o/p = $\cos 2t$.

*

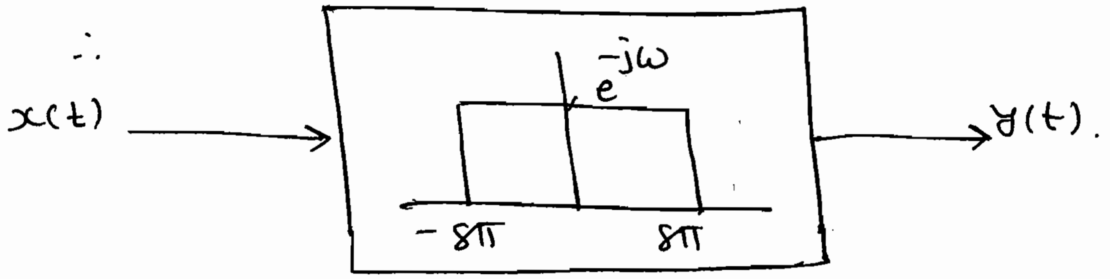
**

$\boxed{P4.2.26}$ (a) Find the o/p of a system having impulse response $h(t) = 8\,\mathrm{sinc}[8(t-1)]$ when the input applied is $x(t) = \cos \pi t$.

Soln: $h(t) = 8\,\mathrm{sinc}[8(t-1)]$.

$$h(t) = \frac{\cancel{8}\sin[8\pi(t-1)]}{\cancel{8}\pi(t-1)} \Rightarrow H(\omega) = e^{-j\omega}\cdot \mathrm{rect}(t/16\pi)$$

∴

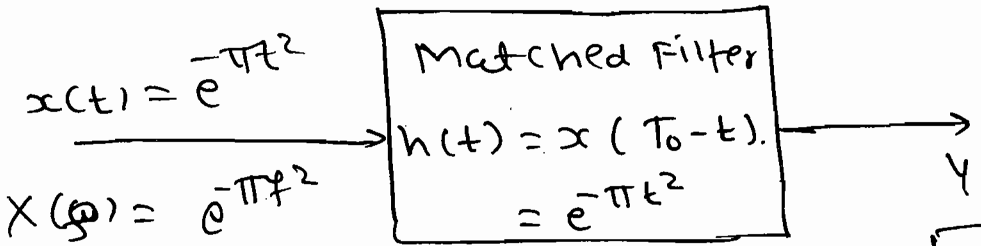


$\Rightarrow \boxed{y(t) = \cos \pi(t-1)}$ ← delay because of $e^{-j\omega}$.

b) Let $g(t) = e^{-\pi t^2}$, and $h(t)$ is a filter matched to $g(t)$. If $g(t)$ is applied as input to $h(t)$, then the Fourier transform of the output is,

(a) $e^{-\pi f^2}$ (b) $e^{-\pi f^2/2}$ (c) $e^{-\pi|f|}$ (d) $e^{-2\pi f^2}$.

Soln:



$x(t) = e^{-\pi t^2}$

$X(f) = e^{-\pi f^2}$

$h(t) = x(T_0 - t)$.

$= e^{-\pi t^2}$

$Y(F) = X(f) \cdot H(f)$

$$\boxed{Y(f) = e^{-2\pi f^2}.}$$

$H(f) = e^{-\pi f^2}$

(delay is negated)

P 4.2.25 Using Convolution Property of F.T. find the Convolution of following signals.

(a) $y(t) = \text{rect}(t) * \cos(\pi t)$.

Soln: $Y(\omega) = X(\omega) \cdot H(\omega)$.

$$\therefore Y(\omega) = Sa\left(\frac{\omega T}{2}\right) \cdot \left[\pi\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right]\right]$$

$$= \frac{\sin(\omega/2)}{(\omega/2)} \left[\pi\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right]\right].$$

$$\cancel{\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}}$$ [illegible]

$$= \cancel{\frac{2}{\omega}} \cdot \cancel{\pi} \times \cancel{\frac{\sin(\frac{\pi}{2})}{(\frac{\pi}{2})}} + \cancel{\frac{2}{\omega}} \cdot \cancel{\pi} \cdot \cancel{\frac{\sin(-\frac{\pi}{2})}{(-\frac{\pi}{2})}}$$

$$\therefore \quad Y(\omega) = \cancel{\pi} \times \frac{\cancelto{1}{\sin \frac{\pi}{2}}}{\cancel{\pi}/2} \delta(\omega - \pi)$$

$$+ \cancel{\pi} \times \frac{\cancelto{-1}{\sin\left(-\frac{\pi}{2}\right)}}{(-\cancel{\pi}/2)} . \delta(\omega + \pi).$$

$$\therefore \quad Y(\omega) = 2\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right].$$

$$\text{IFT}\downarrow \quad = \frac{2}{\pi}\left\{\pi\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right]\right.$$

$$\therefore \quad \boxed{Y(t) = \frac{2}{\pi} \cos \pi t.}$$

(b) $y_3(t) = \cancel{x(t) * \sin}\ \text{rect}(t) * \cos(2\pi t).$

$$\Rightarrow \quad y_3(t) = \text{rect}(t) * \cos(2\pi t).$$

$$= \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\omega/2\right)} \times \left\{\pi\left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right]\right\}$$

$$= \frac{\pi . \cancelto{0}{\sin\left(\frac{2\pi}{2}\right)}}{\left(\frac{2\pi}{2}\right)} . \delta(\omega - 2\pi)$$

$$+ \frac{\pi \cancelto{0}{\sin\left(-\frac{2\pi}{2}\right)}}{\left(-\frac{2\pi}{2}\right)} . \delta(\omega + 2\pi)$$

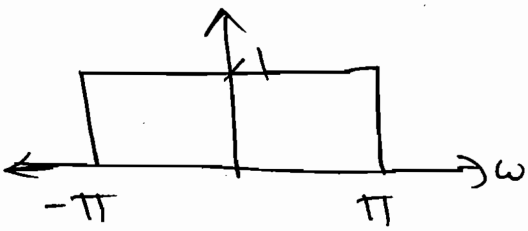
$$= 0 + 0$$

$$\boxed{Y_3(t) = 0}$$

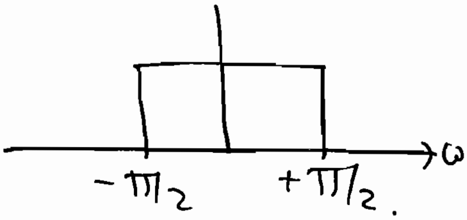
(c) $y_3(t) = \text{sinc}(t/2) * \sin(t/2).$

Soln: $y_3(t) = \text{sinc}(t) * \text{sinc}(t/2).$

$Y_3(\omega) = X_1(\omega) \cdot X_2(\omega).$ ~~$Sa((\omega/2)) \cdot Sa(\frac{2\omega}{2})$~~.



X



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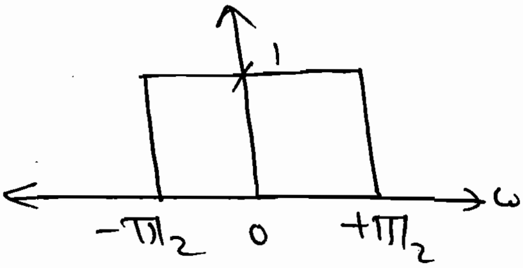
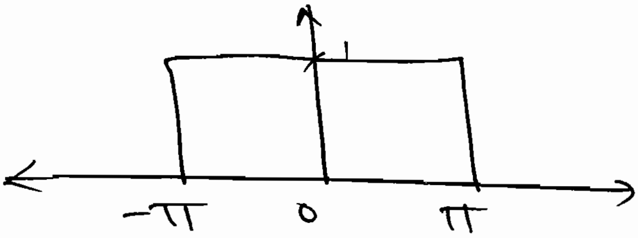


$$Y_3(\omega) = \left[\frac{\sin(\pi t)}{(\pi t)}\right] * \left[\frac{\sin\left(\frac{\pi t}{2}\right)}{\left(\frac{\pi t}{2}\right)}\right].$$

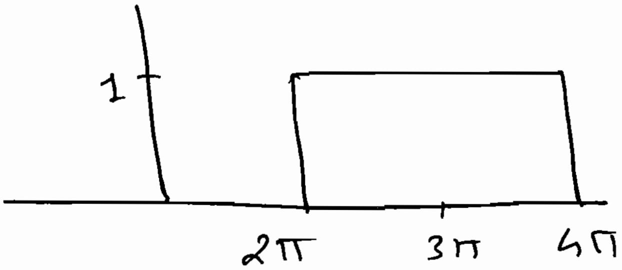
$$\boxed{y_3(t) = \text{sinc}(t/2).}$$

(d) $y_4(t) = \text{sinc}(t) * e^{j3\pi t} \cdot \text{sinc}(t).$

Soln: $y_4(t) = \left[\frac{\sin \pi t}{\pi t}\right] * \left[e^{j3\pi t} \cdot \frac{\sin \pi t}{\pi t}\right].$



X

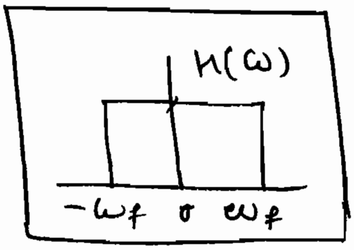


$= 0.$

So, $\boxed{y_4(t) = 0}$

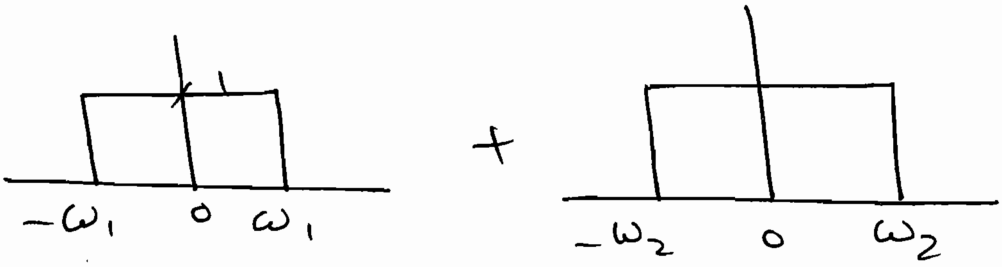
$\boxed{Q}$

I/P : $\frac{\sin\omega_1 t}{\pi t} + \frac{\sin\omega_2 t}{\pi t}$



$\longrightarrow$ O/P = ?

⇒ ① $\omega_f < \omega_1$



let, $\omega_f = 0.5$

$$O/P = \frac{\sin\omega_f t}{\pi t} + \frac{\sin\omega_f t}{\pi t}.$$

$$O/P = \frac{2\sin\omega_f t}{\pi t}$$

② $\omega_1 < \omega_f < \omega_2$.

let, $\omega_f = 1.5$

$1 < 1.5 < 2.$

$$O/P = \frac{\sin\omega_1 t}{\pi t} + \frac{\sin\omega_f t}{\pi t}.$$

③ $\omega_f > \omega_2$; $\omega_f = 2.5$.

∴ $\boxed{O/P = I/P}$

$\boxed{P\,4.2.22}$ Given $y(t) = x(t) * h(t)$ and $g(t) = x(3t) * h(3t)$ such that $g(t) = A\,y(Bt)$, find A & B?

Soln:

$$y(t) = x(t) * h(t).$$

$$\Rightarrow \quad Y(\omega) = X(\omega) \cdot H(\omega).$$

$$\longrightarrow \quad g(t) = x(3t) * h(3t).$$

$$G(\omega) = \frac{1}{3} X\left(\frac{\omega}{3}\right) \cdot \frac{1}{3} H\left(\frac{\omega}{3}\right).$$

$$G(\omega) = \frac{1}{9} H\left(\frac{\omega}{3}\right) \cdot X(\omega/3). \quad \text{— ①}$$

$$\rightarrow \quad g(t) = A\,y(Bt).$$

$$G(\omega) = \frac{A}{B} Y(\omega/B). \quad \text{— ②}$$

Compare eqn ① & ②.

$$\frac{A}{B} = \frac{1}{9}, \quad \& \quad B \neq \frac{1}{B}. \quad B \neq \frac{1}{3}$$

$$\boxed{B = 3}$$

$$\Rightarrow \quad A = 3/9$$

$$\boxed{A = 1/3}$$

$\boxed{P\,4.2.24}$ Let $x(t)$ be a signal whose F.T. is $X(\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$ & Let

$h(t) = u(t) - u(t-2).$

(a) is $x(t)$ periodic?

(b) is $x(t) * h(t)$ $h(t)$ is periodic?

Soln: ⓐ

$$X(\omega) = \delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5) .$$

$$x(t) = \frac{1}{2\pi} + \frac{e^{-j\pi t}}{2\pi}\cdot 1 + \frac{1}{2\pi} e^{-j5t} \cdot 1.$$

So, G.C.D (π , 5) ∄

So, Not periodic signal.

ⓑ $h(t) = u(t) - u(t-2)$

$$\xleftrightarrow{F.T.} e^{-j\omega} \cdot 2Sa\left(\frac{\omega \cdot 2}{2}\right)$$

$$= e^{-j\omega} \frac{2\sin\omega}{\omega} .$$

$$= \frac{2\sin\omega}{\omega} \cdot e^{-j\omega} .$$

$$\therefore \; Y(\omega) = X(\omega) \cdot H(\omega)$$

$$\therefore \; Y(\omega) = \left[\delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5)\right] \times \left[\frac{2\sin\omega}{\omega} \cdot e^{-j\omega}\right].$$

$$= \frac{e^{-j\omega}}{\omega} \cdot 2\sin\omega \cdot \delta(\omega) + \frac{e^{-j\omega}}{\omega} \cdot 2\sin\omega\, \delta(\omega-\pi).$$

$$+ \frac{e^{-j\omega}}{\omega} \cdot 2\sin\omega \cdot \delta(\omega-5).$$

$$Y(\omega) = 2\delta(\omega)\cancel{0} + 0 + \frac{1}{\omega}\, 2\sin 5 \cdot \delta(\omega-5).$$

$$\therefore \; \boxed{Y(\omega) = 2\delta(\omega) + \frac{1}{\omega} \cdot 2\sin 5\; \delta(\omega-5).}$$

So, o/p is periodic. ($\because$ GCD $(0,5) = 5$).

(8) Frequency Convolution:

$\Rightarrow$ If $x_1(t) \longleftrightarrow X_1(\omega)$ and

$x_2(t) \longleftrightarrow X_1(\omega)$ then.

$$\boxed{x_1(t)\cdot x_2(t) \xleftrightarrow{F.T.} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right].}$$

* Application:

① Modulator

② Samples in freq. domain.

e.g.: $x(t)\cdot\cos\omega_c t \xleftrightarrow{F.T.} \frac{1}{2\cancel{\pi}}\left[X(\omega) *^{\cancel{\pi}} [\delta(\omega-\omega_c) + \delta(\omega+\omega_c)]\right]$

$$= \frac{X(\omega-\omega_c) + X(\omega+\omega_c)}{2}.$$

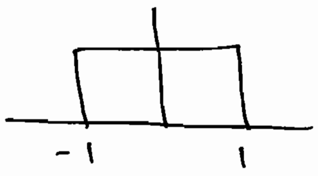
$\boxed{P\ 4.2.27}$ Find the F.T. of

(b) $x(t) = \frac{\sin t\cdot\sin(t/2)}{\pi t^2}.$

Sol^n: $x(t) = \frac{\sin t}{\pi t} \times \frac{\sin(t/2)}{\pi t} \times \pi$

$\therefore X(\omega) = \frac{1}{2\cancel{\pi}} \times \cancel{\pi}\left[\text{rect}(\omega/2) * \text{rect}(\omega).\right].$

$X(\omega) = \frac{1}{2}\Big[$



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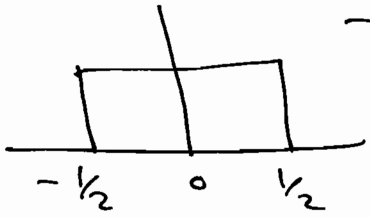
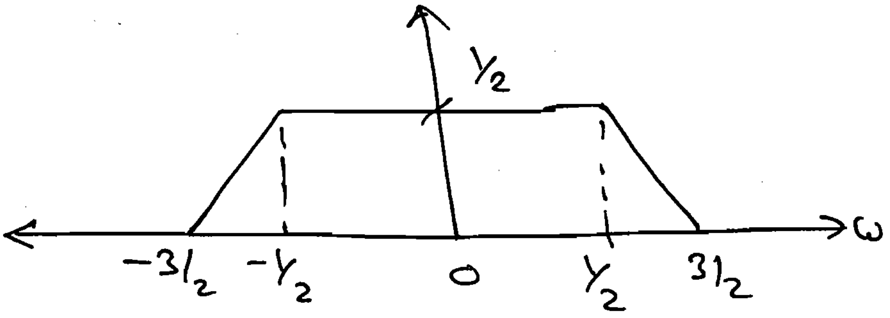


$\Big]$

$\therefore \quad X(\omega) =$



(g) Integration in Time:

$\Rightarrow$ If $x(t) \longleftrightarrow X(\omega)$

then $\boxed{\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega).}$

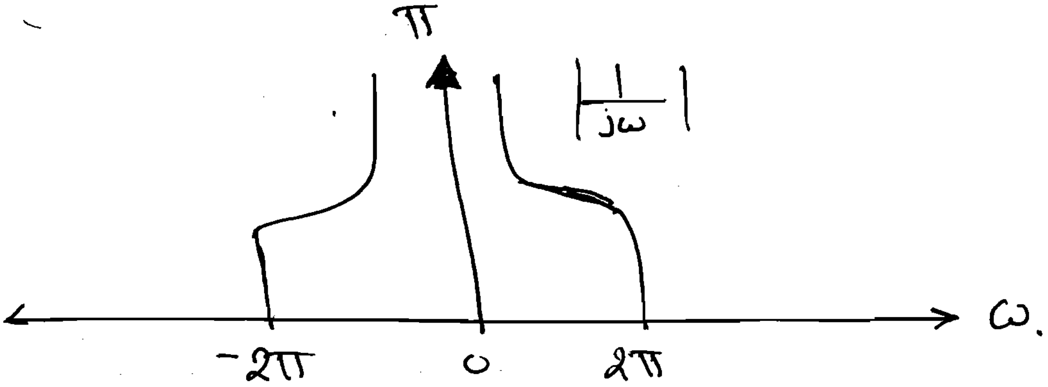
P. 4. 2. 28 Find the F.T. of $\int_{-\infty}^{t} \frac{\sin 2\pi t}{\pi t} dt$.

Sol^n:

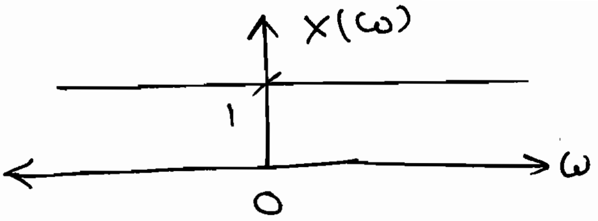
$x(t) = \int_{-\infty}^{t} \frac{\sin 2\pi t}{\pi t} \cdot dt$

$X(\omega) = \frac{X_1(\omega)}{j\omega} + \pi X(0)\delta(\omega).$

$X(\omega) = \frac{\text{rect}\,(\omega/4\pi)}{j\omega} + \pi (1).\delta(\omega).$



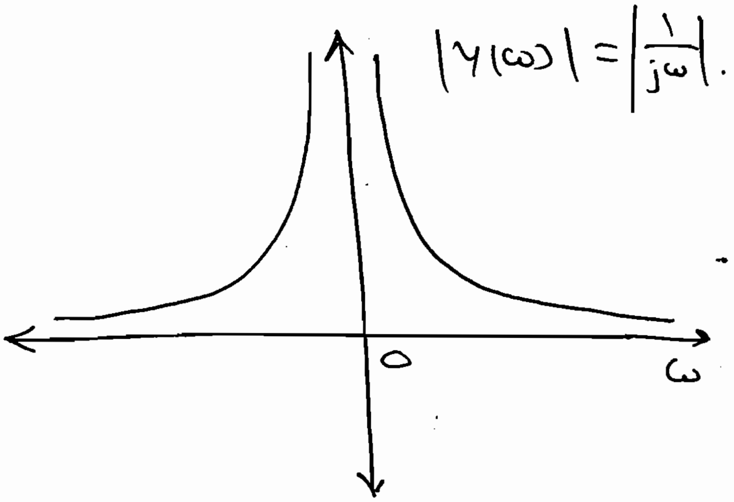
⇒ $x(t) = \delta(t) \longleftrightarrow$



$$\underbrace{\int_{-\infty}^{t} \delta(\tau)\,d\tau}_{=u(t).} \longleftrightarrow \underbrace{\frac{1}{j\omega} + \pi(1)\,\delta(\omega).}_{Y(\omega)}$$

i.e. $u(t) \xleftrightarrow{F.T.} \frac{1}{j\omega} + \pi\,\delta(\omega).$

$|Y(\omega)| = \left|\frac{1}{j\omega}\right|.$



⇒ Differentiation gives only $\frac{1}{j\omega}$ terms, $\pi\delta(\omega)$ [DC term] is missing there.

☆ Rayleigh's Energy theorem (or)
Parseval's Power theorem :-

⇒ Area under Spectral density represents energy (or) power in the signal.

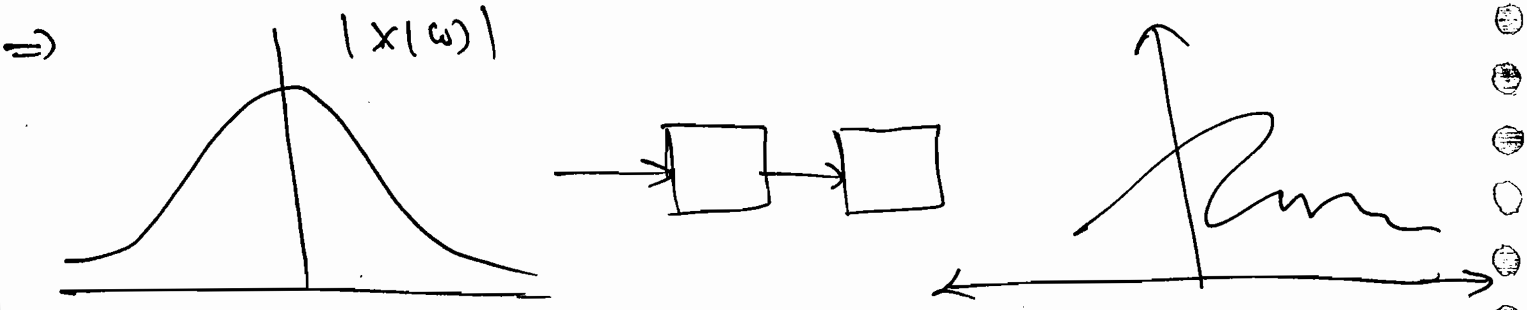
⇒ When a signal is added with noise by comparing the noise spectral density with signal spectral density we can suppress the noise component by designing an 'adaptive Filter'. [Adjustable coefficient].

⇒ IR → LTI filter.

Adaptive filter → LTI filter.

⇒ $$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 . d\omega.}$$

⇒



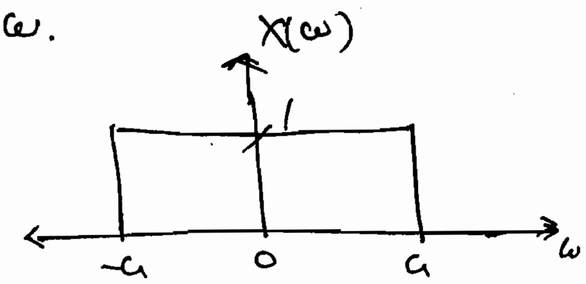
⇒

P 4.2.29 Find the energy in the signal

$x(t) = \frac{\sin at}{\pi t}$.

Soln: $E_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$

$x(t) = \frac{\sin at}{\pi t} \xleftrightarrow{F.T}$

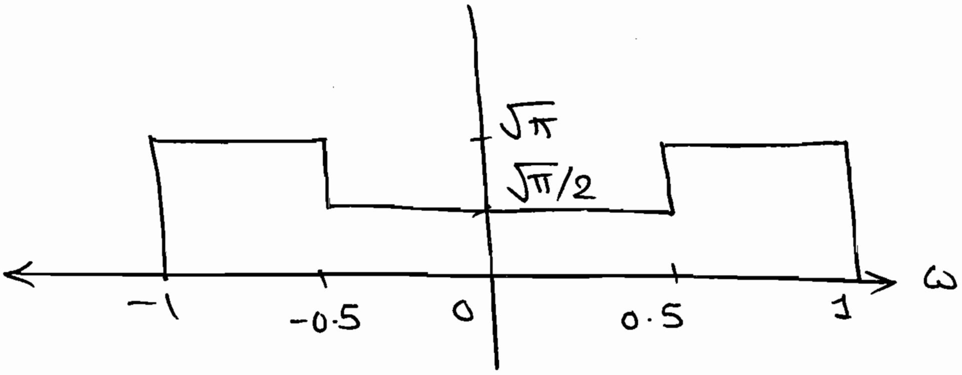


$$\therefore \quad E_{x(t)} = \frac{1}{2\pi} \int_{-a}^{a} (1)^2 \, d\omega.$$

$$= \frac{2a}{2\pi}$$

$$\therefore \boxed{E_{x(t)} = \frac{a}{\pi}}$$

P 4.2. 30 Find the energy in the spectrum shown in fig?



Sol^n:

$$E_{x(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

$$= \frac{1}{2\pi} \times 2 \times \int_{0}^{1} |X(\omega)|^2 d\omega$$

$$= \frac{1}{\pi} \times \left[\int_{0}^{0.5} \left(\frac{\sqrt{\pi}}{2}\right)^2 d\omega + \int_{0.5}^{1} (\sqrt{\pi})^2 d\omega \right].$$

$$= \frac{1}{\pi} \left[\frac{\pi}{4} \times \frac{1}{2} + \frac{\pi}{2} \right].$$

$$= \frac{1}{8} + 1/2$$

$$\boxed{E_{x(t)} = 5/8}$$

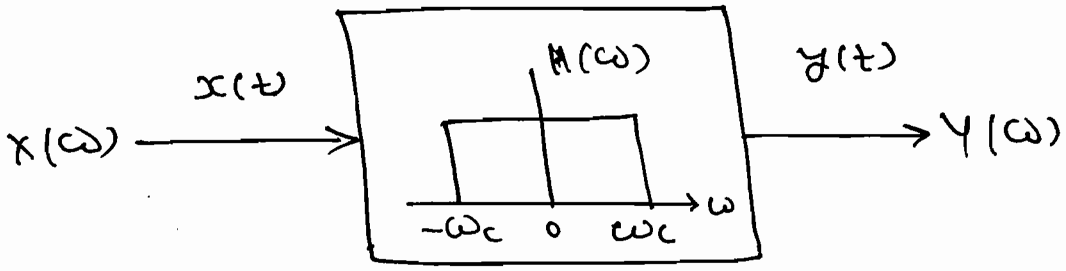
$\boxed{P4.3.21}$ An input signal $x(t) = e^{-2t} u(t)$ is applied to an ideal L.P.F. with freq. response char. $H(\omega) = 1 \; ; \; |\omega| < \omega_c$.

$= 0 \; ; \; |\omega| > \omega_c$.

√

Find ω_c , such that energy in the

Soln:



→ i/p : $x(t) = e^{-2t} . u(t)$.

$$X(\omega) = \frac{1}{2 + j\omega}.$$

$$|X(\omega)|^2 = \frac{1}{4+\omega^2}.$$

→ $$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-4t} . dt.$$

$$\boxed{E_{x(t)} = 1/4}$$

Now, given that

$$E_{y(t)} = \frac{1}{2} \times E_{x(t)}.$$

$$\therefore E_{y(t)} = 1/8.$$

Now, $Y(\omega) = X(\omega) \cdot H(\omega)$.

$$Y(\omega) = \frac{1}{\sqrt{\omega^2+4}} \cdot \quad (1).$$

$$\therefore \quad E_{y(t)} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{1}{\omega^2+4} \cdot d\omega.$$

$$= \frac{2}{2\pi} \times \int_{0}^{\omega_c} \frac{1}{\omega^2+4} \cdot d\omega.$$

$$\therefore \quad \frac{1}{8} = \frac{1}{\pi} \times \frac{1}{2} \times \left[\tan^{-1}(\omega/2)\right]_0^{\omega_c}$$

$$\therefore \quad \frac{\pi}{4} = \tan^{-1}(\omega_c/2).$$

$$\therefore \quad \omega_c = 2 \tan \frac{\pi}{4}.$$

$$\therefore \quad \boxed{\omega_c = 2 \text{ rad/sec.}}$$

P4.2.32 Consider $x(t) \longleftrightarrow X(\omega)$. Suppose we are given the following facts

(i) $x(t)$ is real and non negative.

(ii) $F^{-1}\{(1+j\omega)X(\omega)\} = A \cdot e^{-2t}u(t)$, where 'A' is independent of t.

(iii) $\int_{-\infty}^{+\infty} |X(\omega)|^2 \cdot d\omega = 2\pi$ find a closed-form expression for $x(t)$.

Soln:

$$F^{-1}\{(1+j\omega)\,X(\omega)\} = A\,e^{-2t}.u(t).$$

$$\therefore (1+j\omega)\;X(\omega) = F\{e^{-2t}.\;u(t)\}.$$

$$\therefore (1+j\omega)\,X(\omega) = \frac{1}{2+j\omega}.$$

$$\therefore X(\omega) = \frac{1}{(1+j\omega)\,(2+j\omega)}.$$

$$\therefore X(\omega) = A\left[\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right]$$

$$\therefore x(t) = A\left[e^{-t} - e^{-2t}\right]u(t).$$

Now, $$\int_{-\infty}^{+\infty} |X(\omega)|^2\,d\omega = 2\pi$$

$$\Rightarrow \boxed{E_{x(t)} = 1} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\,d\omega.$$

$$\therefore 1 = \int_{-\infty}^{\infty} |x(t)|^2\,dt.$$

$$\therefore 1 = A^2\left[\int_0^\infty \left[e^{-2t} - 2e^{-3t} + e^{-4t}\right]dt\right].$$

$$\therefore 1 = A^2\left[\left[\frac{e^{-2t}}{-2} - \frac{2}{-3}e^{-3t} + \frac{e^{-4t}}{-4}\right]_0^\infty\right].$$

$$\therefore 1 = A^2\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right].$$

$\therefore$ ~~$\frac{6}{8} = A^2$~~ $$1 = A^2\left[\frac{6-8+3}{12}\right].$$

$$\Rightarrow \quad \boxed{A = \sqrt{12}.}$$

$$\therefore \quad \boxed{x(t) = \sqrt{12}\left[e^{-t} - e^{-2t}\right]u(t).}$$

P 4.2.33 Find the value of the integral

$$\int_{-\infty}^{+\infty} \frac{8}{(\omega^2+4)^2}\, d\omega.$$

Soln:

$$\int_{-\infty}^{+\infty} \frac{8}{(\omega^2+4)^2} . d\omega$$

$$\rightarrow \quad e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\omega^2+\alpha^2}.$$

$$\Rightarrow \quad \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{2(2)}{\omega^2+\alpha^2}\right)^2 . d\omega.$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\pi \times \int_{-\infty}^{\infty} e^{-4|t|} . dt.$$

$$= \pi \times 2 \times \int_{0}^{\infty} e^{-4t} . dt.$$

$$= 2\pi \times \frac{1}{4}.$$

$$= \frac{\pi}{2}.$$

$$\therefore \text{So,} \int_{-\infty}^{+\infty} \frac{8}{(\omega^2+4)^2} d\omega = \pi/2.$$

$\boxed{a}$ Using Parseval's theorem find the signal energy in $x(t) = 4\,\text{sinc}(t/5)$.

Sol^n:

$$x(t) = 4\,\text{sinc}(t/5).$$

$$A = 4, \quad T = 5.$$

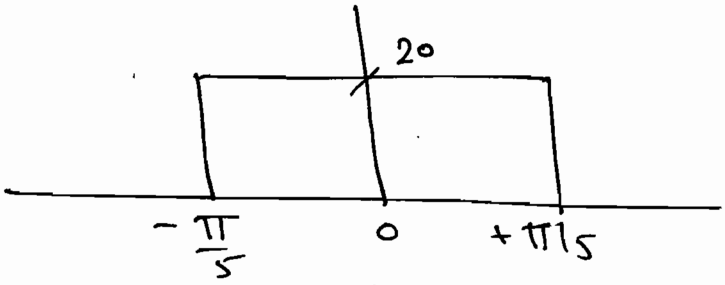
$$\therefore \; X(\omega) = AT\,\text{rect}(\omega/2a).$$

$$x(t) = \frac{4\sin(\pi t/5)}{\pi t/5}$$

$$x(t) = 20\,\frac{\sin\left(\frac{\pi t}{5}\right)}{\pi t}$$

$\downarrow$ F.T.

$$X(\omega) = 20.\,\text{rect}\left(\omega/2\left(\tfrac{\pi}{5}\right)\right).$$



$$\therefore \; E_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 .d\omega.$$

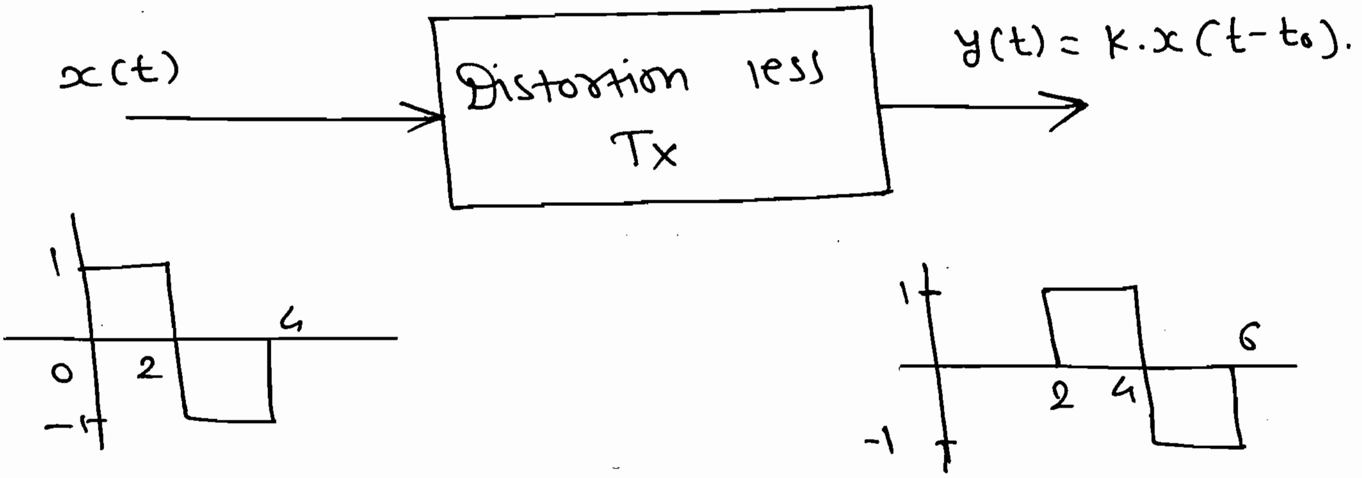
$$= \frac{1}{2\pi} \times 2 \times \int_{0}^{\pi/5} (400).d\omega.$$

$$= \frac{1}{\cancel{\pi}} \times \overset{}{\underset{80}{\cancel{400}}} \times \frac{\cancel{\pi}}{\cancel{5}}$$

$$\therefore \; \boxed{E_{x(t)} = 80}$$

* Applications:

① Distortionless Transmission:-



⇒ For distortionless transmission, output is replica of the input which scaling in its amplitude and possible delay.

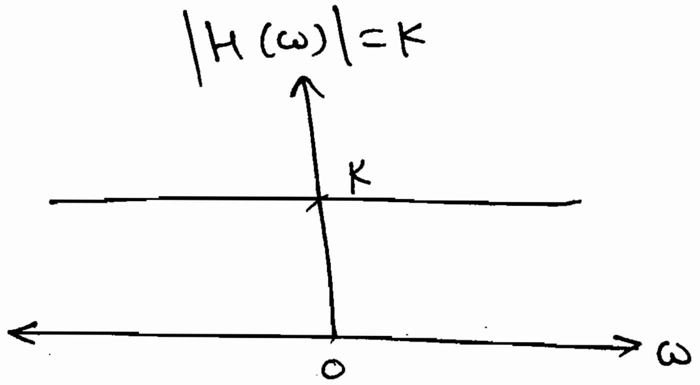
⇒ For Distortion less transmission, magnitude response must be a constant, phase response must be linear function of ω with slope $-t_0$, where t_0 is delay in output with respect to input.

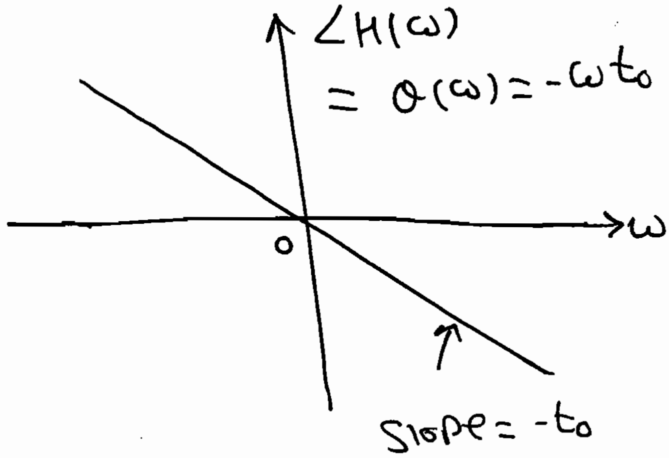
$$y(t) = K\, x(t - t_0).$$

$$\therefore Y(\omega) = K \cdot e^{-j\omega t_0} \cdot X(\omega).$$

$$\therefore Y(\omega) = |H(\omega)| \cdot e^{j\theta(\omega)}$$

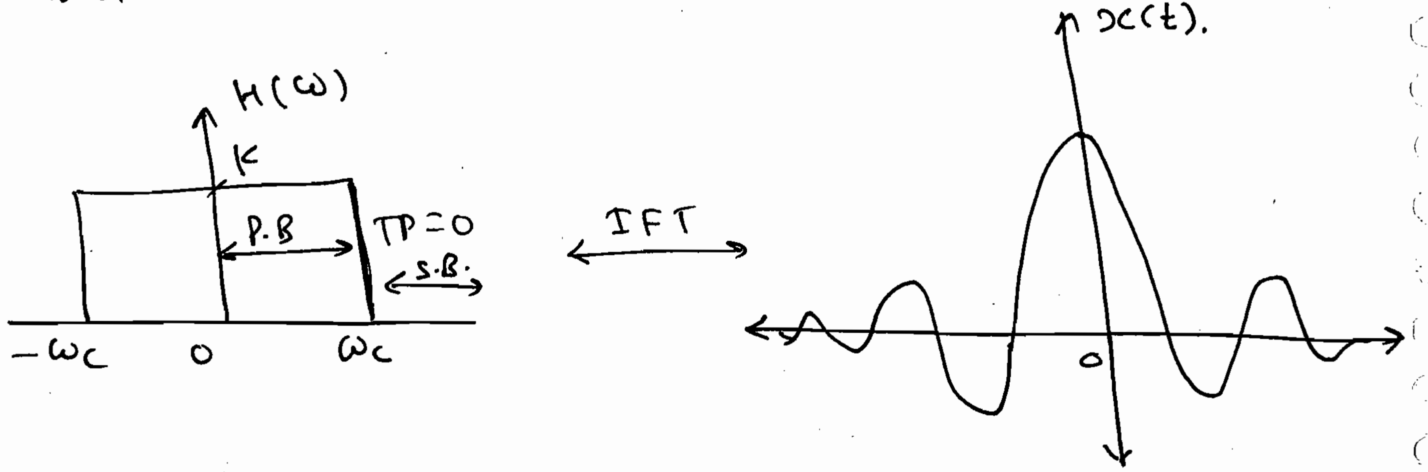
$$\Rightarrow |H(\omega)| = K, \quad \boxed{\theta(\omega) = -\omega t_0}$$





⇒ According to Rayleigh's theorem for a distortionless condition we require infinite energy which is impractical so we are limitting the range of freq. from 0 to ω_c i.e. Ideal filter (IFT of rectangular spectrum is a Sa f^n which extends for all time.).

⇒ All ideal filters are non-causal and unstable.



⇒ Transition width is deciding the order of the filter (no. of energy storing elements).

⇒ Most of the Practical Systems we are designing as non-linear phase response. To make it as linear we are defining two parameters.

① **Phase delay:**

⇒ It is the delay i.e. occuring at a single freq. which is due to carrier

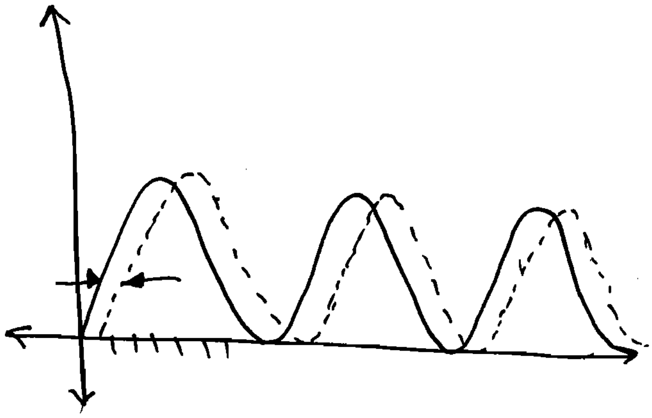
$$\boxed{t_p(\omega) = -\frac{\theta(\omega)}{\omega}.}$$

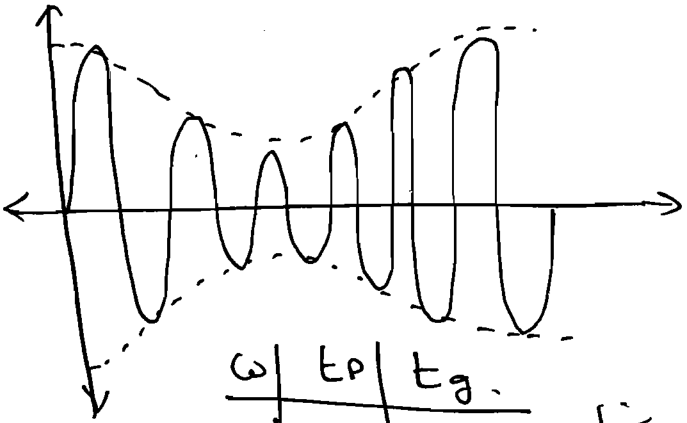
② **Group delay:**

⇒ It is the delay i.e. occuring at a group (or) narrow band of freq. which is due to envelope of the msg signal.

$$\boxed{t_g(\omega) = -\frac{d\theta(\omega)}{d\omega}.}$$

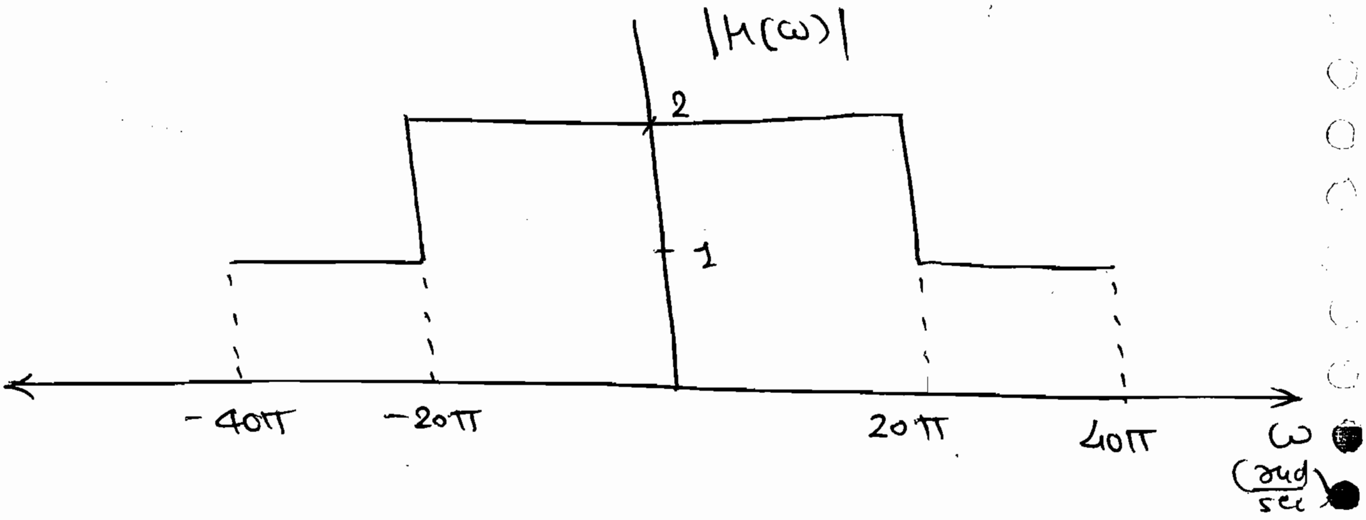
⇒ '−' indicates the phase lag present in the . system.

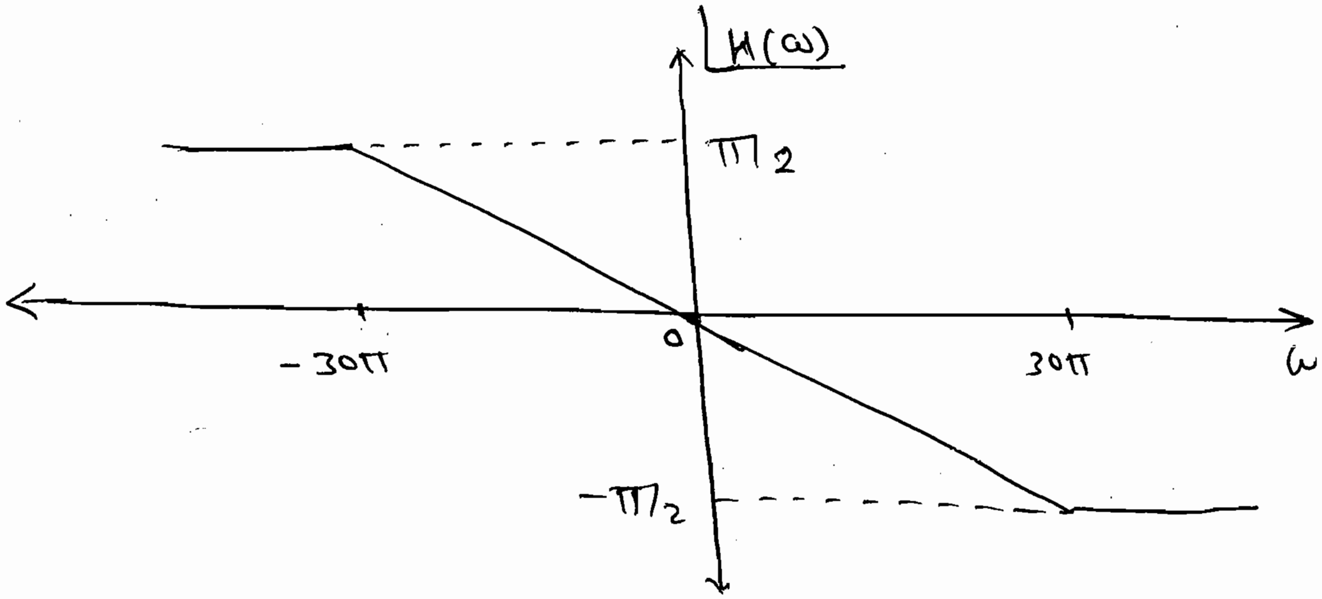




ω	tp	tg.

P4.3-1 Consider a transmission system H(ω) with magnitude and phase response as shown in figure. If an input $x(t) = 2\cos 10\pi t + \sin 26\pi t$ is given to the system the output will be ___.





Sol^n:

$x(t) = 2\cos 10\pi t + \sin 26\pi t.$

① $\cos 10\pi t$

at $\pm 30\pi$ $M = 1$

$\phi \Rightarrow \pm\pi/2$

So, 30π $-\pi/2$

10π $(?)$

$\Rightarrow \phi = -\pi/2 \times 1/3 = -\pi/6.$

So, at 10π, $M \Rightarrow 2$

$\phi \to -\pi/6.$

② $\sin 26\pi t$.

M at $26\pi \Rightarrow \cancel{2}\ 1.$

$\phi \Rightarrow \quad 30\pi \to \pi/2$

$26\pi \to (?)$

$$\Rightarrow \phi = -\frac{\overset{13}{\cancel{26\pi}}}{\cancel{30\pi}} \times \pi/\cancel{2}$$

$$\phi = -\frac{13\pi}{30}.$$

So, Ans: $(2)(2)\cos\left(10\pi t - \pi/6\right).$

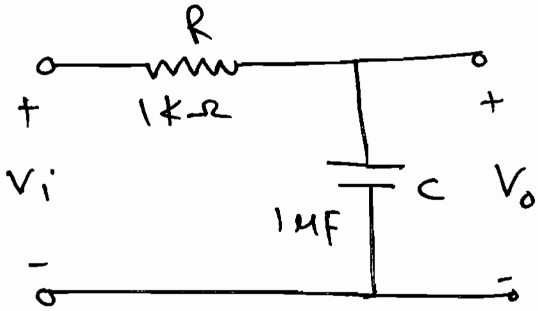
$+ \sin\left(30\pi t - \frac{13\pi}{30}\right).$

$$\boxed{y(t) = 4\cos(10\pi t - \pi/6) + \sin\left(26\pi t - \frac{13\pi}{30}\right).}$$

Q The system under consideration is an RC LPF with $R = 1\,k\Omega$ & $C = 1\mu F$

a) Let $H(f)$ denote the frequency response of an RC LPF. with Let f_1 be the highest freq. component such that $0 \le |f| \le f_1$, $\left|\frac{H(f_1)}{H(0)}\right| \geq 0.95$ then f_1 (in Hz) is ______.

Soln:



$$H(f) = \frac{1}{1 + j2\pi f RC}.$$

$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi f RC)^2}}$$

Now, given that

$$|H(f_1)| \geq 0.95 |H(0)|.$$

So, $$\frac{1}{\sqrt{1+(2\pi f_1 RC)^2}} \geq 0.95 \times \frac{1}{\sqrt{1+0}}.$$

$$\Rightarrow \frac{1}{1+(2\pi f_1 RC)^2} = 0.9025.$$

$$\Rightarrow 1+(2\pi f_1 RC)^2 = 1.108$$

$$\therefore (2\pi f_1 RC)^2 = 0.108$$

$$(f_1 \times 10^{-6} \times 10^3)^2 = \frac{0.108}{4 \times \pi^2}$$

$$\Rightarrow \boxed{f_1 = 52.2 \text{ Hz}}$$

b) Let $t_g(f)$ denote the group delay of RC LPF and $f_2 = 100\,\text{Hz}$, then $t_g(f_2)$ in msec, is ______.

Soln:

$$\angle H(f) = -\tan^{-1}(2\pi f RC). = \theta(f).$$

$$\therefore t_g = -\frac{d\theta(f)}{df}$$

$$= +\frac{1}{1+(2\pi f RC)^2}$$

$$= \frac{1}{1+(2\times\pi\times 100\times 10^{-6}\times 10^3)^2}$$

$$\boxed{t_g = 0.717 \text{ sec}}$$

$\boxed{P. 4.3.5}$. The input to a Channel is a band-pass signal. It is obtained by linearly modulating a sinusoidal Carrier with a single-tone signal. The output of the Channel due to this input is given by

$$y(t) = \frac{1}{100} \cos(100t - 10^{-6}) \cos(10^6 t - 1.56).$$

The group delay (t_g) & the phase delay (t_p) in seconds, of the Channel are, —?

Soln:

$$y(t) = \frac{1}{100} \cos\Big(\underbrace{100(t - 10^{-8})}_{\text{msg signal}}\Big) \cdot \cos\Big(\underbrace{10^6 (t - 1.56 \times 10^{-6})}_{\text{Carrier}}\Big).$$

So, $\boxed{t_g = 10^{-8}}$ $\boxed{t_p = 1.56 \times 10^{-6}.}$

⇒ Message signal gives t_g &
Carrier signal gives t_p.

$\boxed{Q}$ Which of the following below is distortion less?

Sol

$$\left.\begin{array}{l} (A)\ \theta(\omega) = -\omega^2 + \omega^3. \\ (B)\ \theta(\omega) = \ln\omega. \\ (C)\ \theta(\omega) = e^{\omega}. \end{array}\right\} \text{Non-linear}$$

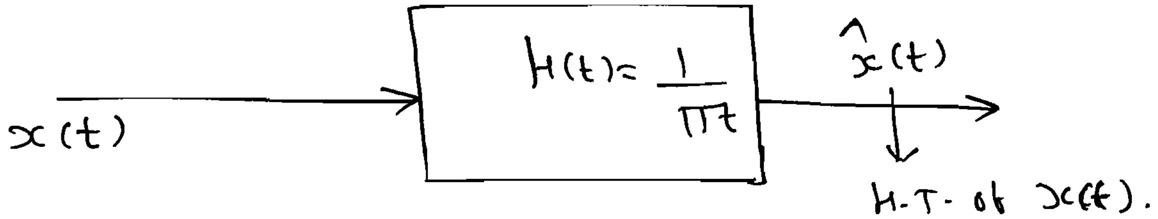
$$(D)\ \theta(\omega) = -3\omega \longrightarrow \text{Linear}$$

Soln: Ans – (D) → $\theta(\omega) = \underset{\text{Linear.}}{\underset{\checkmark}{-3\omega}}$

$\boxed{2}$ Hilbert Transform:-

⇒ The Hilbert transform is an operation that shifts the phase of $x(t)$ by $-\pi/2$, while the amplitude spectrum of the signal remains unaltered.

⇒



⇒ $$\hat{x}(t) = x(t) * \frac{1}{\pi t}.$$

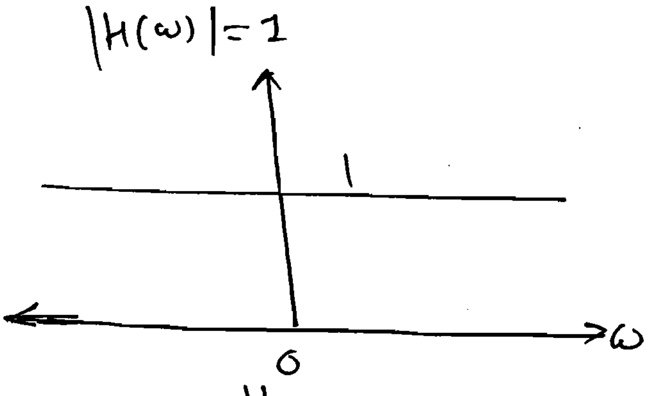
$$\Big\downarrow \text{F.T.}$$

$$\hat{X}(\omega) = X(\omega) \cdot \left[-j\,\text{sgn}(\omega)\right].$$

∴ freq. response of H.T. $= \boxed{H(\omega) = \frac{\hat{X}(\omega)}{X(\omega)} = -j\,\text{sgn}(\omega).}$

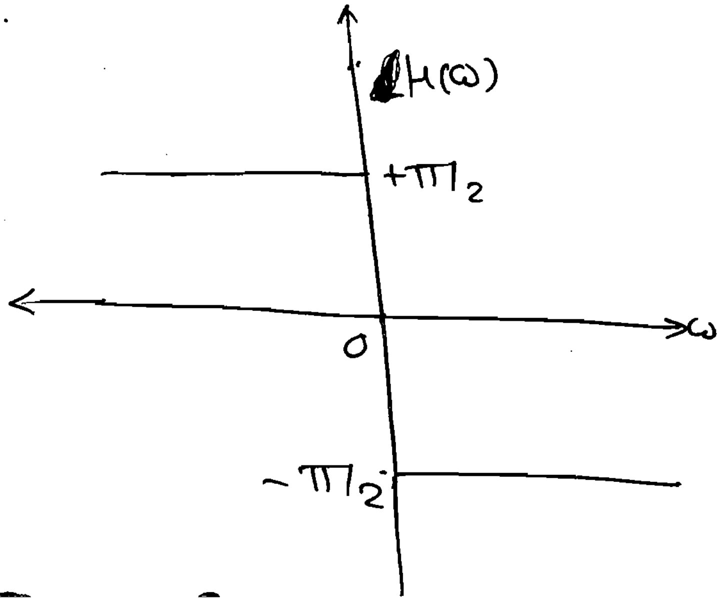
⇒ $H(\omega) = -j; \quad \omega > 0.$

$\quad\quad\;\; = +j; \quad \omega < 0.$



⇓

All pass filter



⇒ An ideal H.T. is an all pass 90° phase shifter.

⇒ It is obeying orthogonality,

Area under two signal must be zero.

i.e. $$\boxed{\int_{t_1}^{t_2} x(t).\,\hat{x}(t).\,dt = 0.}$$

* Properties of H.T.

1) H.T. doesn't change the domain of a signal.

2) H.T. doesn't alter the amplitude spectrum of a signal.

3) If $\hat{x}(t)$ is H.T. of $x(t)$, then H.T. of $\hat{\hat{x}}(t)$ is $-x(t)$.

4) $x(t)$ and $\hat{x}(t)$ are orthogonal to each other.

$\boxed{P4.4.1}$ Find the H.T. of

(1) $x(t) = \cos\omega_0 t$

(2) $x(t) = \sin\omega_0 t.$

Soln:

$$x(t) = \cos\omega_0 t \xleftrightarrow{H.T.} \cos(\omega_0 t - \pi/2)$$

$$= \cos(\pi/2 - \omega_0 t).$$

$$= \sin\omega_0 t.$$

$$x(t) = \sin\omega_0 t \xleftrightarrow{H.T.} \sin(\omega_0 t - \pi/2).$$

$$= -\sin(\pi/2 - \omega_0 t).$$

$$= -\cos\omega_0 t.$$

⇒ H.T. of $e^{j\omega_0 t}$ $(\omega_0 > 0)$ $= -je^{j\omega_0 t}$

↓

$$e^{j(\omega_0 t - \pi/2)} = e^{j\omega_0 t} \cdot e^{-j\pi/2}$$

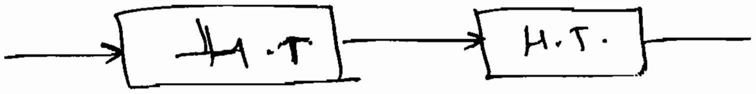
$$= e^{j\omega_0 t} \cdot (-j).$$

$$= (-j) \cdot e^{j\omega_0 t}.$$

⇒ H.T. of $\delta(t)$ is ______.

⇒ $\delta(t) * \frac{1}{\pi t} = \frac{1}{\pi t}$.

⇒ H.T. of $\frac{1}{\pi t}$ is $\underline{\ -1\ }$.



$(-j \,\text{sgn}\, \omega)^2 = j^2 (1)$

$= -1.$

(or)

$= \cos(180°)$

$+ \sin(180°)$

$= -1 + 0$

$= -1.$

* CORRELATION:- (correlogram).

- $x(t), x(t-\tau)$
  A.C.F
  Auto correlation f^n
  - Energy
  - Power
- $x(t), y(t-\tau)$.
  C.C.F
  Cross Correlation f^n.
  - Energy
  - Power.

⇒ It Provide a measure of the similarity between 2 waveforms as the function of search parameter (τ).

⇒ An application of coreletion to signal detection in a radar, when a signal pulse is transmitted in order to detect a suspect target. If a target is present, the pulse will be reflected by it. If the target is not present, there will be no reflection pulse, just noise. By detecting the presence (or) absense of the reflected pulse we confirm the presence (or) absence of target.

⇒

	A.C.F.
Energy	$R_x(\tau) = \int_{-\infty}^{+\infty} x(t).\, x(t-\tau)\, dt.$
Power	$R_x(\underset{\uparrow}{T}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t).\, x(t-T)\, dt.$

Lag (or) searching Parameter

* Properties of ACF :-

1) ACF is an even function of τ

i.e $R_x(\tau) = R_x(-\tau)$.

2) ACF at origin indicates either energy (or) Power in the signal.

3) Max. value of ACF is at origin,

i.e, $|R_x(\tau)| \leq |R_x(0)| \quad \forall \tau$.

4) $R_x(\tau) = x(\tau) * x(-\tau)$.

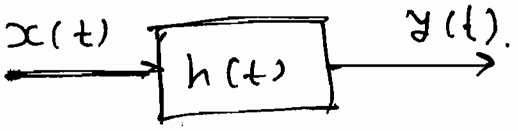
$$x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(-(\tau - t)) \cdot dt.$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot x(t-\tau) \cdot dt = R_x(\tau).$$

5) F.T. of ACF is known as ESD(or) PSD

$$R_x(\tau) \xleftrightarrow{F.T.} S_x(\omega) \quad ESD \,|\, PSD.$$

6) For an LTI system



$\therefore Y(\omega) = X(\omega) \cdot H(\omega)$.

$\therefore |Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2$.

$$\therefore \; |Y(\omega)|^2 = |X(\omega)|^2 . |H(\omega)|^2 .$$

$$\therefore \; S_Y(\omega) = S_X(\omega) . |H(\omega)|^2 .$$

Output Spectral density = [input spectral density] × $[|H(\omega)|^2]$.

$\boxed{P4.5.1}$ Find the Auto correlation and Power in the signal.

$$x(t) = 6\cos\left(6\pi t + \frac{\pi}{3}\right).$$

Soln:

$$P_{avg} = \frac{A^2}{2} = \frac{36}{2} = 18 \text{ W}.$$

$$\rightarrow R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 6\cos\left(6\pi t + \frac{\pi}{3}\right) . 6\cos\left(6\pi t - 6\pi\tau + \frac{\pi}{3}\right) . dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{36}{2} . \left[\cos\left(12\pi t - 6\pi\tau + \frac{\pi}{3}\right) \rightarrow 0 + \cos(6\pi\tau) . \right] dt .$$

$$= \lim_{T\to\infty} \frac{18}{2T} \int_{-T}^{T} \cos(6\pi\tau) . dt .$$

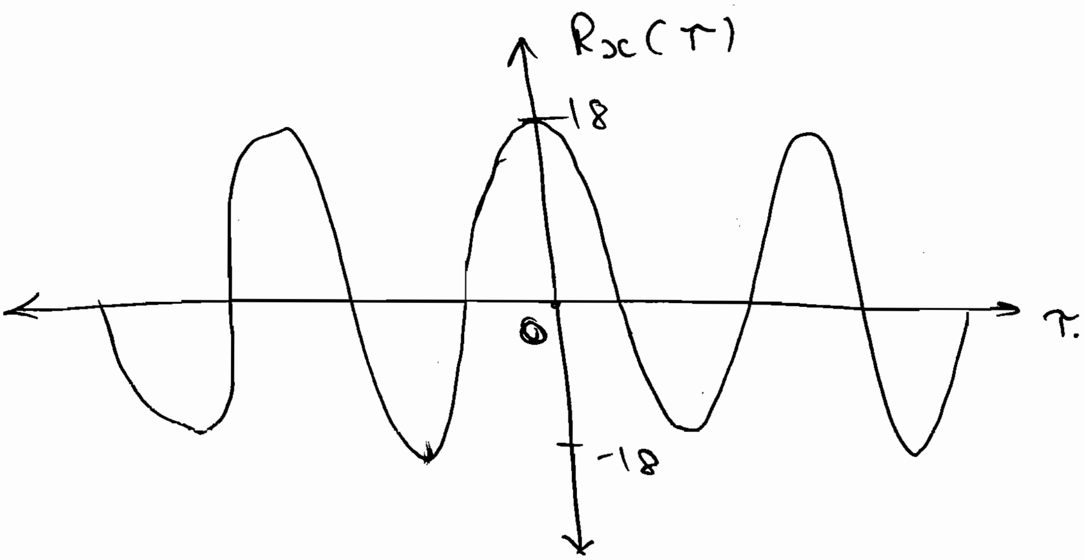
$$= \lim_{T\to\infty} \frac{18\cos(6\pi\tau)}{2T} \times \int_{-T}^{T} dt .$$

$$= \lim_{T\to\infty} \frac{18\cos(6\pi\tau)}{\cancel{2T}} \times \cancel{2T}$$

$\Rightarrow$ $\boxed{R_x(\tau) = 18\cos(6\pi\tau).}$

(or) Power $= R_x(0) = 18$ W.

$$\boxed{\left.\begin{array}{c} A\cos(\omega_0 t+\theta) \\ \text{(or)} \\ A\sin(\omega_0 t+\theta) \end{array}\right\} \longrightarrow \frac{A^2}{2}\cos(\omega_0\tau) \leftarrow \text{ACF is fixed.}}$$



$\boxed{\text{P 4.5.2.}}$ Find the ACF of $x(t) = e^{-3t}u(t)$.

$\downarrow$ energy signal.

Soln:

$$R_x(t) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\cdot dt.$$

$$= \int_{\cancel{-\infty}\,0}^{\infty} e^{-3t}\cdot e^{-3(t+\tau)} . dt$$

wrong

$$= e^{3\tau}\int_{0}^{\infty} e^{-6t}.dt.$$

$$= \frac{-e^{3\tau}}{6}[0-1] = \frac{e^{3\tau}}{6}.$$

$\Rightarrow \quad x(t) = e^{-3t} \cdot u(t)$ is energy signal.

$$\therefore R_x(\tau) = \int_{-\infty}^{\infty} x(t) . \; x(t-\tau)\, dt.$$

$$\therefore R_x(\tau) = \int_{-\infty}^{\infty} e^{-3t} \cdot u(t) \; . \; e^{-3(t-\tau)} \; . \; u(t-\tau) \, . dt$$

$u(t) . \; u(t-\tau)$

$t > \tau$

& $\quad t < \tau .$

(OR)

$$\Rightarrow R_x(\tau) = x(\tau) * x(-\tau).$$

$$= e^{-3\tau} . u(t) \quad * \quad e^{3\tau} . u(-\tau).$$

$$\text{F.T.} \downarrow \quad = \left[\frac{1}{3+j\omega}\right] \times \left[\frac{1}{3-j\omega}\right].$$

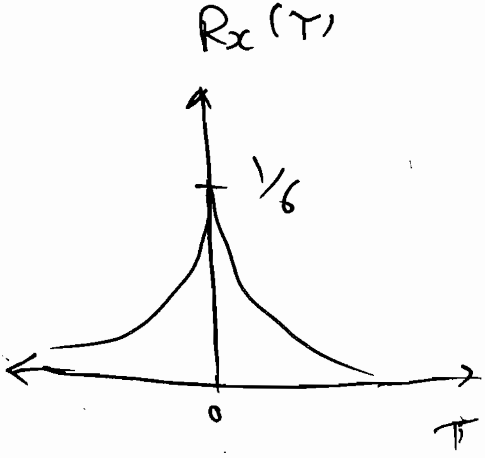
$$= \left(\frac{1}{\omega^2+9}\right)$$

$$= \frac{1}{6}\left(\frac{2(3)}{\omega^2+(3)^2}\right).$$

$$\downarrow \text{I.F.T}$$

$$\therefore \boxed{R_x(\tau) = \frac{1}{6} \; e^{-3|\tau|} .}$$

$$\therefore \boxed{R_x(0) = \text{Energy} = 1/6.}$$

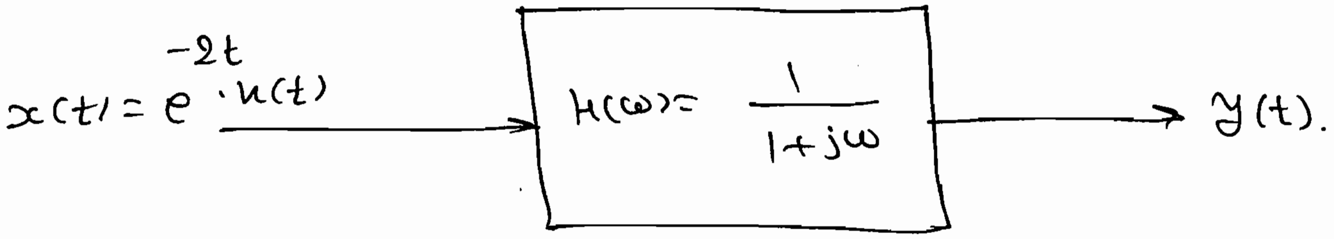


$\boxed{\text{Q P. 4.5.3}}$ Consider a filter with $H(\omega) = \frac{1}{1+j\omega}$ and input $x(t) = e^{-2t} \cdot u(t)$.

(a) Find the ESD of the output?

(b) Show that total energy in the o/p is one-third of the input energy.

Sol^n:



$\therefore\ x(t) = e^{-2t} \cdot u(t)$

$\downarrow$ F.T.

$\therefore\ X(\omega) = \frac{1}{2+j\omega}$.

$Y(\omega) = X(\omega) \cdot H(\omega)$.

$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2$.

$\therefore\ S_y(\omega) = |H(\omega)|^2 \cdot S_x(\omega)$.

$S_x(\omega) = |X(\omega)|^2 = \frac{1}{4+\omega^2}$.

$\therefore\ |H(\omega)|^2 = \frac{1}{1+\omega^2}$.

$\therefore\ \boxed{S_y(\omega) = \left(\frac{1}{1+\omega^2}\right) \times \left(\frac{1}{4+\omega^2}\right)}$. ESD at o/p.

(b) Energy = Area under ESD.

$\therefore\ E_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) \cdot d\omega$.

$$\therefore \quad E_y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\omega^2+1)} \cdot \frac{1}{(\omega^2+4)} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{3}}{\omega^2+1} - \frac{\frac{1}{3}}{\omega^2+4} \cdot d\omega.$$

$$= \frac{1}{2\pi} \times \frac{1}{3} \left[\tan^{-1}(\omega) - \frac{1}{2} \tan^{-1}(\omega/2) \right]_{-\infty}^{\infty}$$

$$= \frac{1}{6\pi} \left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{\cancel{\pi}}{2} - \frac{\pi}{4} \right].$$

$$= \frac{1}{6\cancel{\pi}} \left[-\frac{\cancel{\pi}}{2} \right].$$

$$\therefore \quad \boxed{E_y(\omega) = \frac{1}{12}.}$$

$\Rightarrow$ I/P $\quad x(t) = e^{-2t} u(t).$

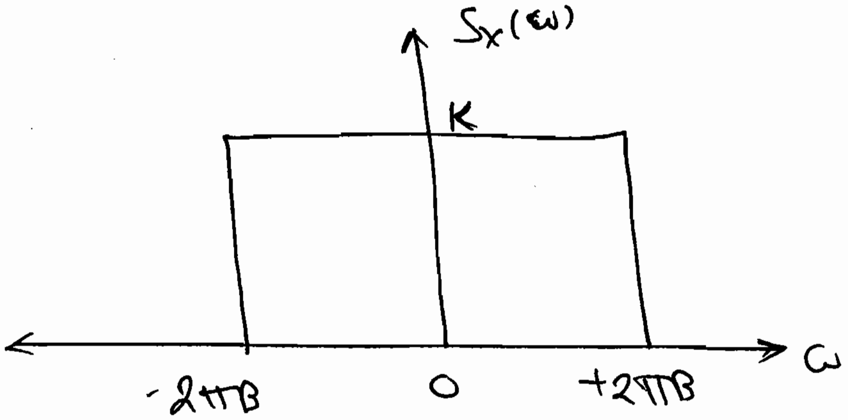
$$E_x = \int_0^{\infty} (e^{-2t})^2 \cdot dt$$

$$= \int_0^{\infty} e^{-4t} \cdot dt$$

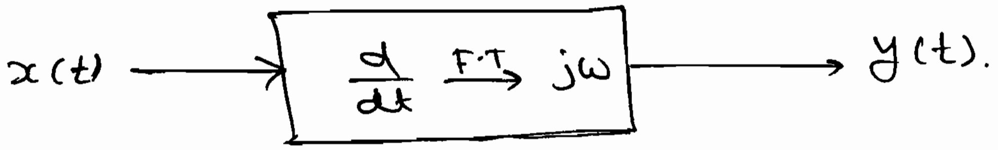
$$\boxed{E_x = 1/4}$$

So, $\boxed{E_{O/P} = \frac{1}{3} E_{I/P}}$

P 4.5.4 A power signal whose psd is shown in fig. is applied to an ideal differentiator, find the mean square value of the o/p of the differentiator.



Soln:



⇒ mean square value of o/p

$$= \text{Power} = \text{Area under o/p PSD}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega).\, d\omega.$$

Now, $S_y(\omega) = |H(\omega)|^2.\, S_x(\omega).$

so, $|H(\omega)|^2 = \omega^2.$

$$\therefore \text{M.S.V. of o/p} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2.\, S_x(\omega).\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2.\, S_x(\omega).d\omega.$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \omega^2.\, k\, d\omega.$$

$$= \frac{1}{2\pi} \times 2 \times K \times \int_0^{2\pi B} \omega^2 . d\omega$$

$$= \frac{K}{\pi} \times \left[\frac{\omega^3}{3}\right]_0^{2\pi B}$$

$$= \frac{K}{\pi} \times \frac{8 \times \pi^3 \times B^3}{3}$$

$$\boxed{\text{M.S.V. of o/p} = \frac{8\pi^2 B^3 K}{3}}$$

P4.5.5 Find the C.C.F of $x(t) = e^{-t} u(t)$ and $y(t) = e^{-3t} u(t)$?

Soln:

ACF $\Rightarrow R_x(\tau) = x(\tau) * x(-\tau)$.

CCF $\Rightarrow R_{xy}(\tau) = x(\tau) * y(-\tau)$.

& $R_{yx}(\tau) = Y(\tau) * x(-\tau)$.

$$\therefore \boxed{R_{yx} = R_{xy}}$$

So, $R_{xy}(\tau) = x(\tau) * y(-\tau)$.

$$\Big\downarrow = \left[e^{-t}. u(t)\right] * \left[e^{3t}. u(-t)\right].$$

F.T.

$$= \frac{1}{+1 + j\omega} \times \frac{1}{3 - j\omega}.$$

$$= \frac{1}{(1 + j\omega)(3 - j\omega)}.$$

$$= \frac{1}{4}\left[\frac{1}{1+j\omega} + \frac{1}{3-j\omega}\right].$$

$\downarrow$ I.F.T.

$$\boxed{R_x(t) = \frac{1}{4}\left[e^{-t}\cdot u(t) + e^{3t}\cdot u(-t).\right].}$$

* F.T. of Periodic Signals :-

$\Rightarrow$ Periodic and discrete in one domain is corresponds to discrete & periodic in other domain.

$$\Rightarrow \quad 1 \xrightarrow{F.T.} 2\pi\delta(\omega).$$

$$1.\, e^{j\omega_0 t} \xrightarrow{F.T.} 2\pi\,\delta(\omega-\omega_0)$$

$$\Rightarrow \quad \cos\omega_0 t = \frac{1.\, e^{j\omega_0 t} + 1.e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2}.\, e^{j(1)(\omega_0)t} + \frac{1}{2}.\, e^{j(-1)(\omega_0)t}.$$

$\downarrow$ F.T.

$$= \frac{2\pi}{2}\delta(\omega-\omega_0) + \frac{2\pi}{2}.\,\delta(\omega+\omega_0).$$

$$\cos\omega_0 t \xrightarrow{F.T.} = 2\pi\left[\frac{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)}{2}\right].$$

$$\boxed{\begin{array}{l} \text{Rect} \xleftrightarrow{F.T.} Sa \text{ (or) } \text{Sinc} \\ \Delta^{le} \xleftrightarrow{F.T.} Sa^2 \text{ (or) } \text{Sinc}^2. \end{array}}$$

Impulse $\xleftrightarrow{F.T.}$ Constant

Imupulse train $\xleftrightarrow{F.T.}$ Impulse train

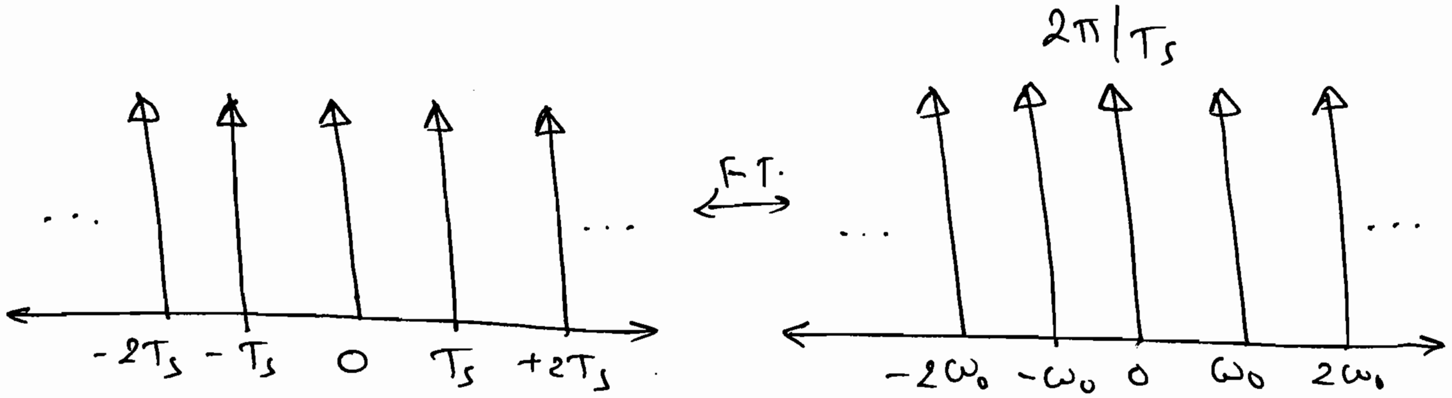
Gaussian $\xleftrightarrow{F.T.}$ Gaussian.

$$\Rightarrow \quad x_p(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}.$$

$$\Big\downarrow F.T.$$

$$X_p(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \cdot \delta(\omega - n\omega_0).$$

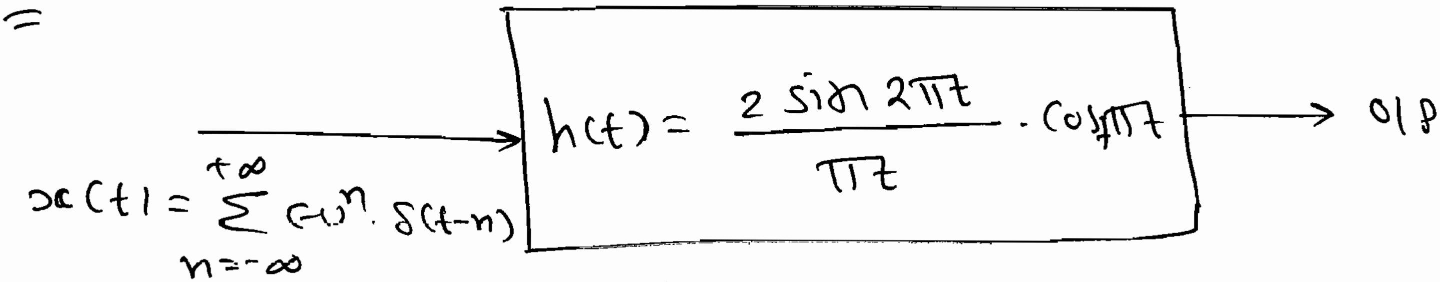
⇒ F.T. of a Periodic signal consist of a seauence of equidistant impulse located at harmonic frequencies of the signal.



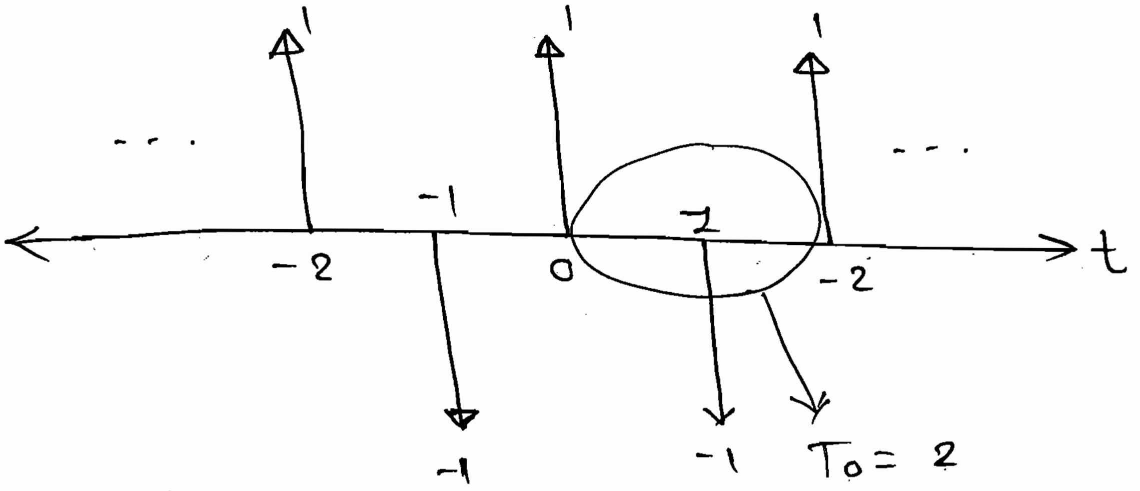
$$\Rightarrow \quad \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \longleftrightarrow \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0).$$

$\boxed{a}$ An L.T.I. sys. is having Impulse Response $h(t) = 2\frac{\sin 2\pi t}{\pi t} \cdot \cos 7\pi t$ for which the I/P applied is $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \cdot \delta(t-n)$, find the O/P.

Sol^n:



$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \, \delta(t-n).$$



$T_0 = 2$

$\omega_0 = \frac{2\pi}{T} \Rightarrow \boxed{\omega_0 = \pi}$

$\Rightarrow$ Now,
$$X_p(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} C_n \cdot \delta(\omega - n\omega_0).$$

So,
$$C_n = \frac{1}{T} \int_0^T x(t) \cdot e^{-j\omega_0 n t}.$$

$$\Rightarrow C_n = \frac{1}{2} \int_0^2 [\delta(t) - \delta(t-1)]\, e^{-jn\pi t}\, dt.$$

$$= \frac{1}{2}\left[\int_0^2 \underset{t_0=0}{\underset{\uparrow}{\delta(t)}}.\, e^{-jn\pi t}\, dt - \int_0^2 \underset{t_0=1}{\underset{\uparrow}{\delta(t-1)}} . e^{-jn\pi t} . dt \right]$$

shifting property of impulse.

$$\Rightarrow C_n = \frac{1}{2}\left[e^{-jn\pi(0)} - e^{-jn\pi(1)} \right].$$

$$= \frac{1}{2}\left[1 - e^{-jn\pi} \right].$$

$$C_n = \frac{1}{2}\left[1 - (-1)^n \right]$$

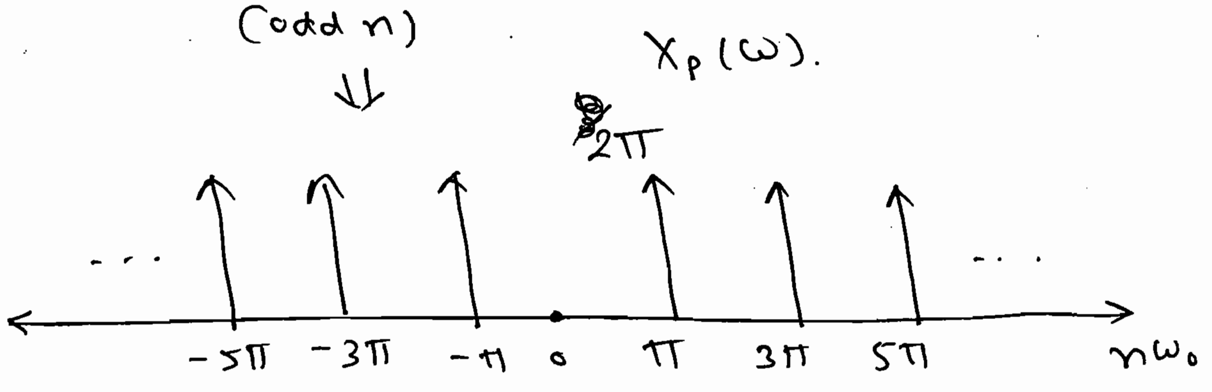
$$\Rightarrow C_n = 1 \quad ; \quad n = \text{odd}.$$

$$= 0 \quad ; \quad n = \text{even}.$$

$$\therefore X_p(\omega) = \frac{2\pi}{1} \sum_{n=-\infty}^{+\infty} (1).\ \delta(\omega - n\pi) \qquad (\because \omega_0 = \pi).$$

(odd n) & $T_0 = 1$

$\Downarrow$



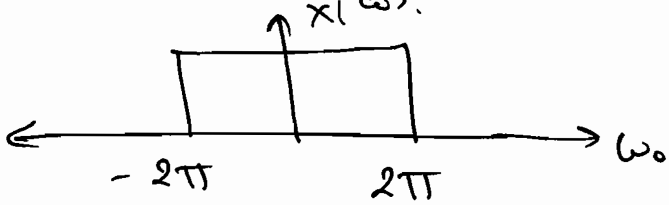
Now, $h(t) = 2\frac{\sin 2\pi t}{\pi t} . \cos 7\pi t.$

$$= \cancel{2}\frac{\sin 2\pi t}{\pi t} . \left[\frac{e^{-j7\pi t} + e^{j7\pi t}}{\cancel{2}} \right]$$

$$h(t) = \frac{\sin 2\pi t}{\pi 7} \cdot \left[\frac{e^{-j\pi 7} + e^{j\pi 7}}{1} \right]$$

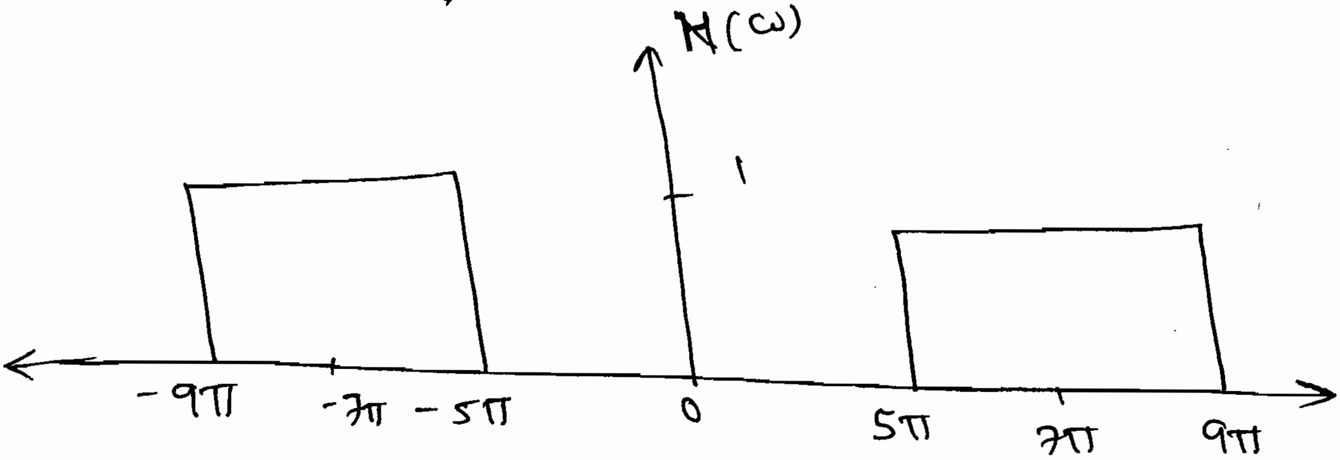
ie, $x(t) = \frac{\sin 2\pi t}{\pi t}$

$\downarrow$ F.T.

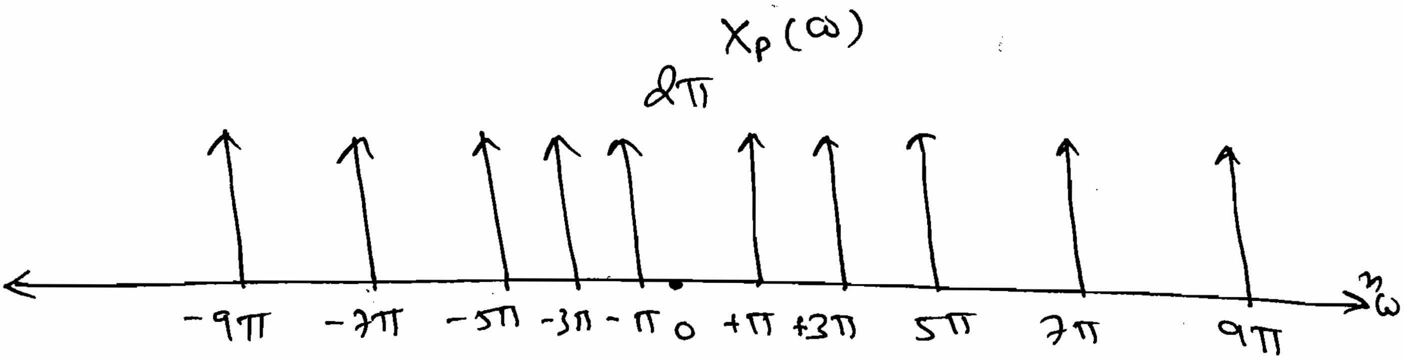


$$\therefore \quad H(\omega) = X(\omega - 7\pi) + X(\omega + 7\pi).$$

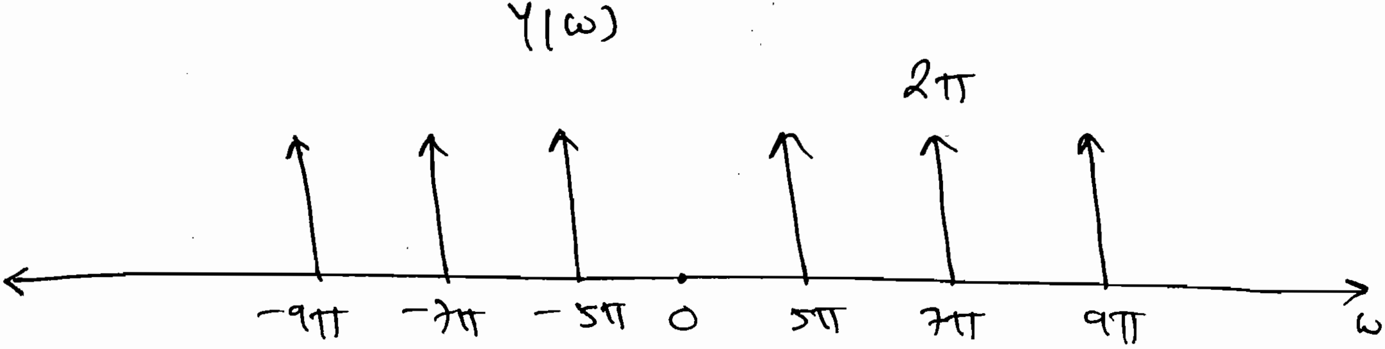
$\Downarrow$



$\times$



$\Downarrow$



$$\Rightarrow \quad Y(\omega) = 2\pi\Big[\delta(\omega-5\pi)+\delta(\omega+5\pi) + \delta(\omega-7\pi)+\delta(\omega+7\pi) + \delta(\omega-9\pi)+\delta(\omega+9\pi)\Big].$$

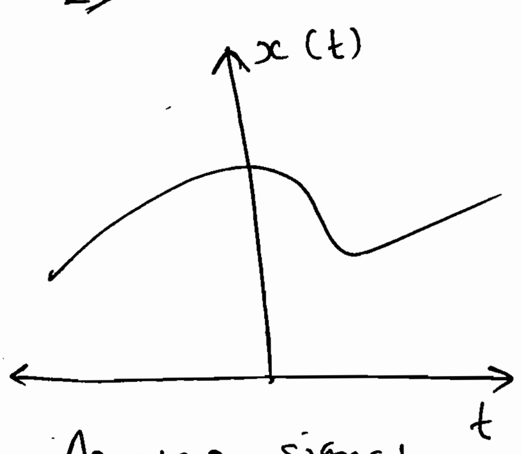
$$\therefore \boxed{Y(\omega) = 2\left[\cos 5\pi t + \cos 7\pi t + \cos 9\pi t\right].}$$

* Samping Theorem :-

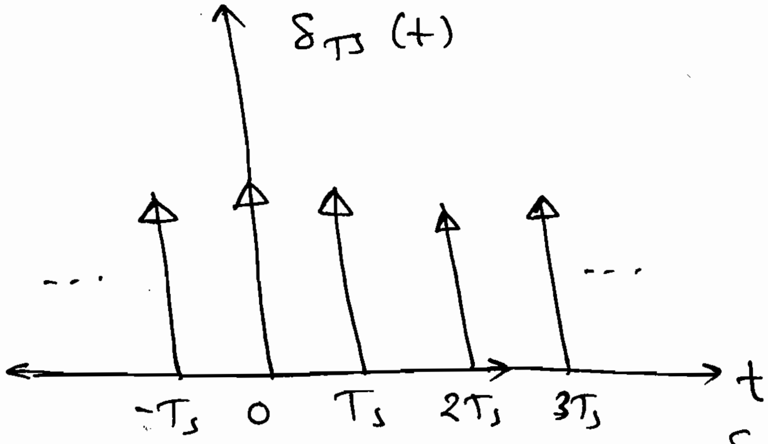
- Ideal
- Practical — $\omega_s > 2\omega_m$ (or) $f_s > 2f_m$
  - Natural
  - Flat-top.

① Ideal Samping:-

$\Rightarrow$

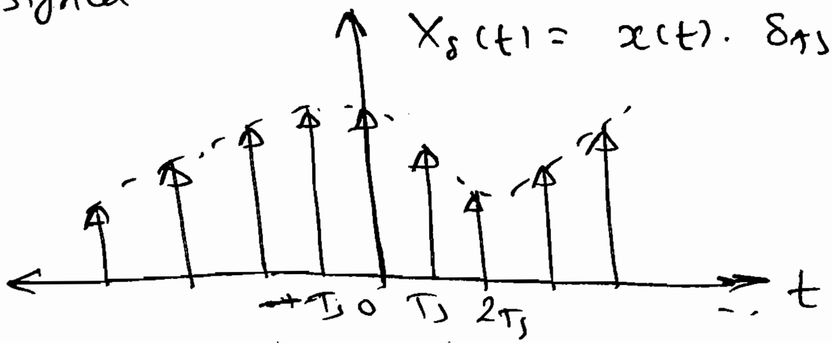


Analog signal



Samping fn.

$$X_s(t) = x(t).\ \delta_{Ts}(t).$$



Ideal Sampled signal

⇒ $x(t) \rightarrow$ Analog signal

⇒ $$\delta_{T_S}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

$\delta_{T_S}(t) =$ Samping function.

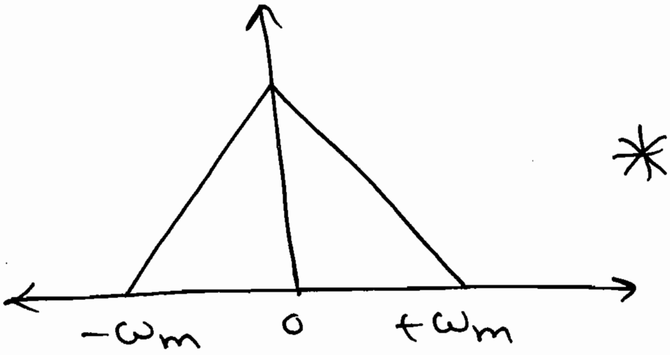
⇒ $x_\delta(t) \rightarrow$ Ideally Sampled signal.

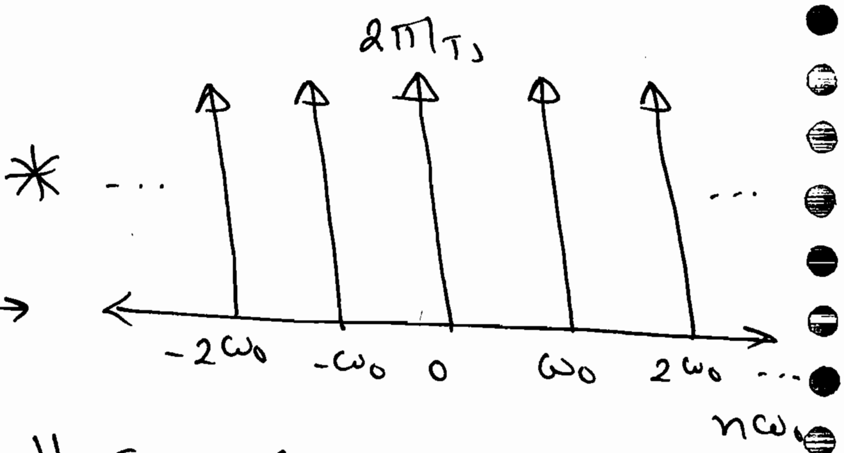
$$\boxed{x_\delta(t) = x(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t-nT_s).} \leftarrow \text{Time domain.}$$

⇒ Take F.T.

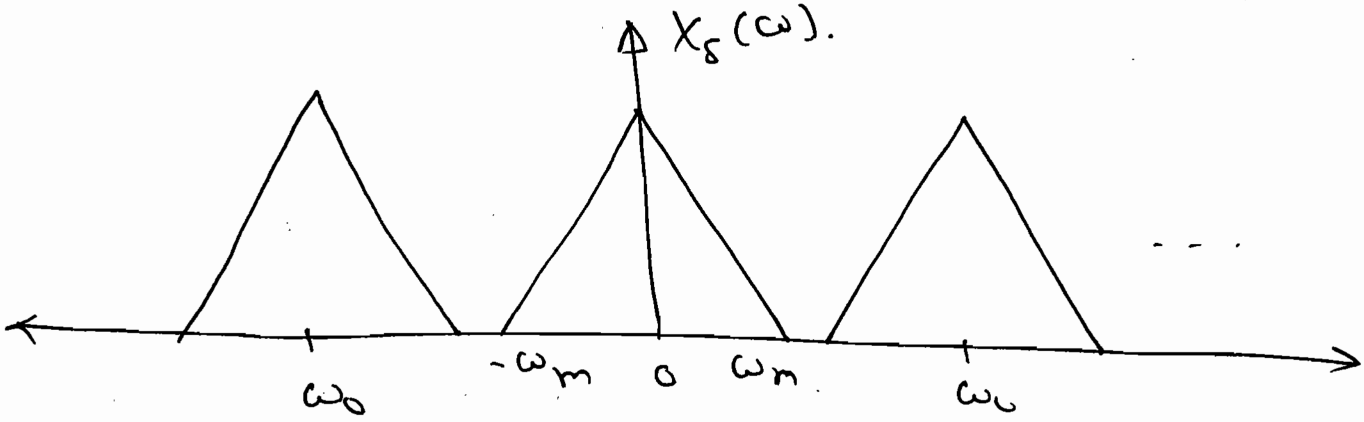
$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) \longleftrightarrow \frac{2\pi}{T_s}\sum_{n=\infty}^{\infty} \delta(\omega-\omega_0 n)$$

$x(t) \longleftrightarrow X(\omega)$





⇓ Convolve.



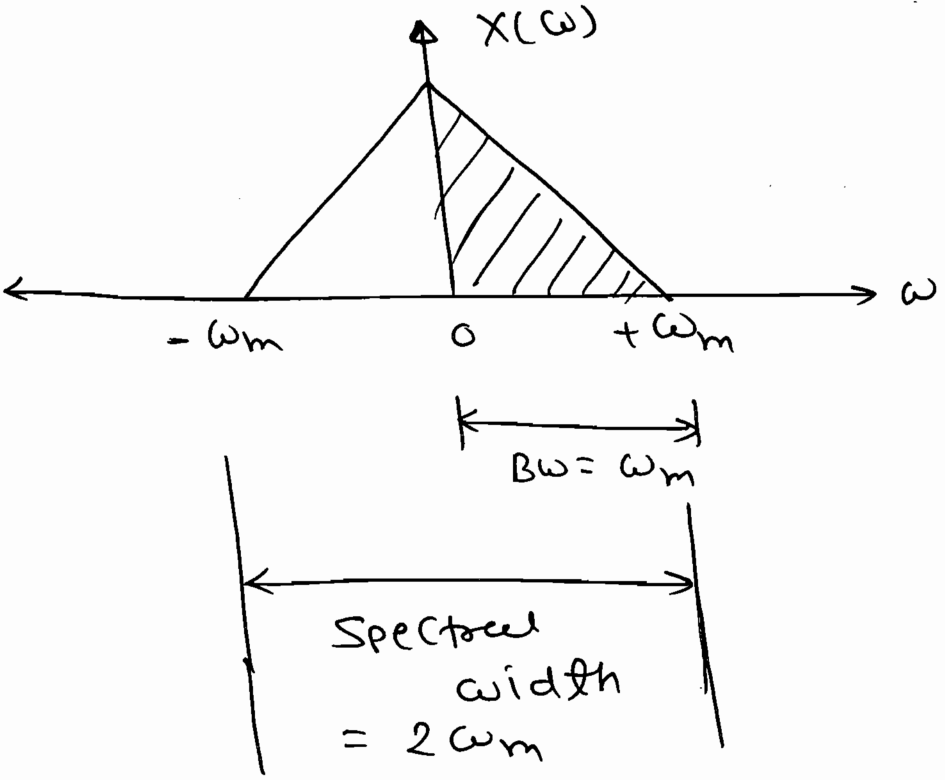
⇒ $$X_\delta(\omega) = \frac{1}{\cancel{2\pi}}\left[X(\omega) * \frac{\cancel{2\pi}}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)\right]$$

⇒ Spectrum of $x_\delta(t)$ ⇒

$$\boxed{X_\delta(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} x(\omega - n\omega_0).}$$

↑ Freq. domain.

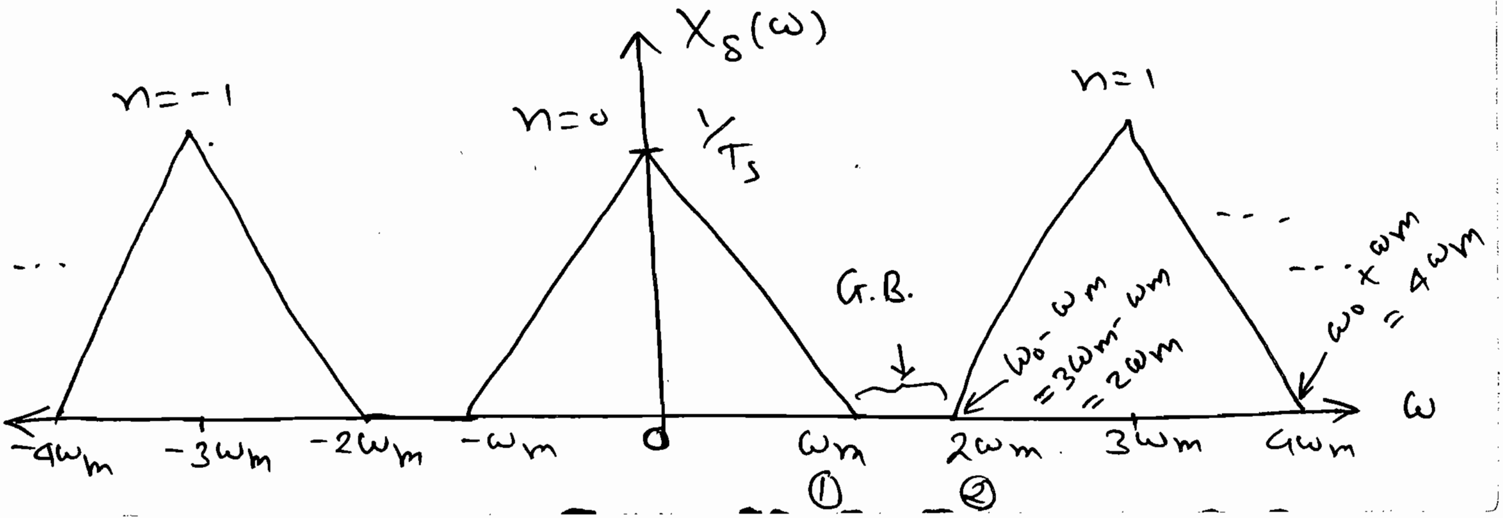
⇒ Assume Band limited spectrum,



<u>case- (i):</u> $\underline{\omega_0 > 2\omega_m}$ ⇒ <u>over samping</u>

⇒ Let, $\omega_0 = 3\omega_m$

$$X_\delta(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(\omega - 3n\omega_m).$$



⇒ Guard Band = ② − ①

$= (\omega_0 - \omega_m) - (\omega_m)$

$$\boxed{\text{Guard Band} = \omega_0 - 2\omega_m.}$$

⇒ Images don't overlap,

if ~~ω_0~~ ② > ①.

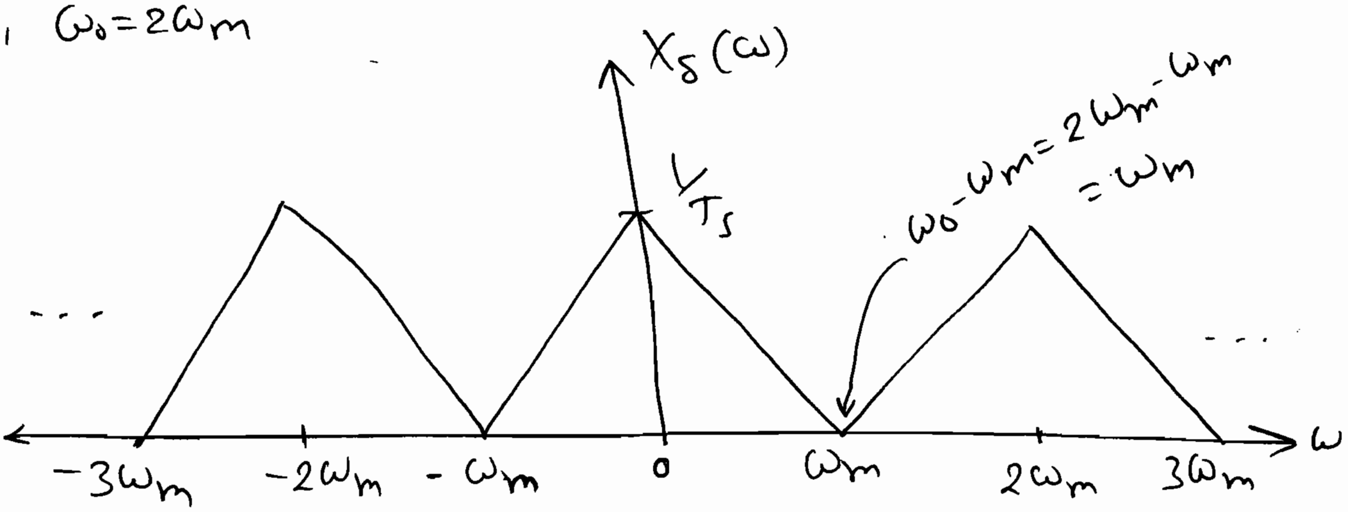
i.e. $\omega_0 - \omega_m > \omega_m$.

⇒ $\boxed{\omega_0 > 2\omega_m.}$

Case-(ii):- $\omega_0 = 2\omega_m$, Critical Sampling.

$$\Rightarrow X_\delta(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - 2\omega_{\overset{n}{m}}).$$

let, $\omega_0 = 2\omega_m$



⇒ G.B. = $\omega_0 - 2\omega_m = 2\omega_m = 2\omega_m = 0$.

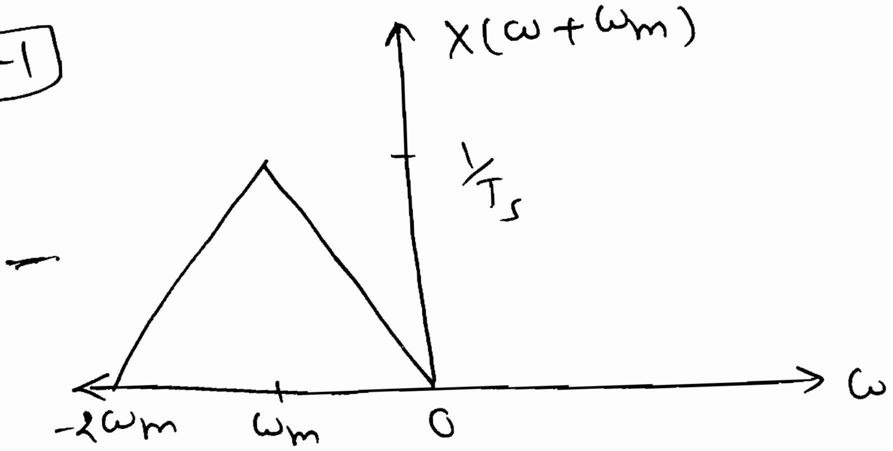
$\boxed{\text{G.B.} = 0}$.

⇒ Case-(iii):- $\omega_0 < 2\omega_m$ Under Sampling.

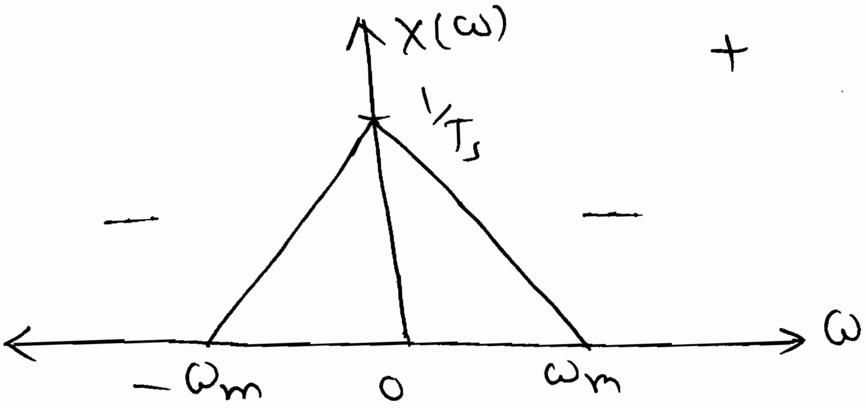
⇒ Let, $\omega_0 = \omega_m$.

$$\therefore \; X_\delta(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_m).$$

n=-1

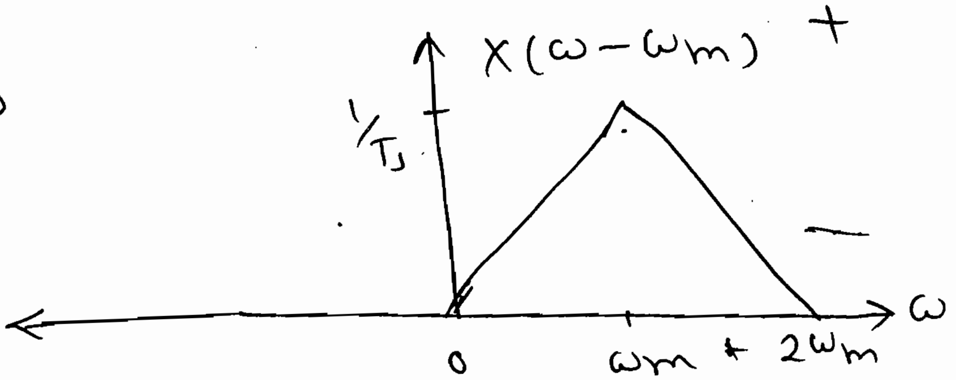


n=0



+

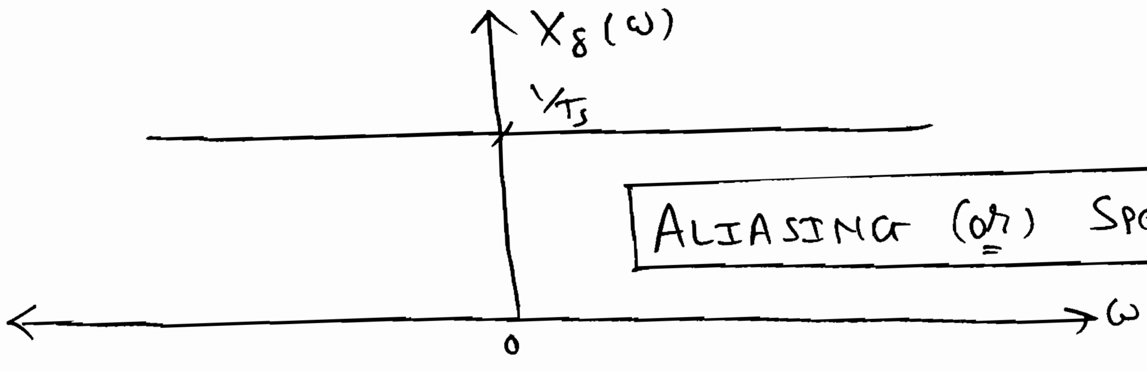
n=1



+

⇓

=

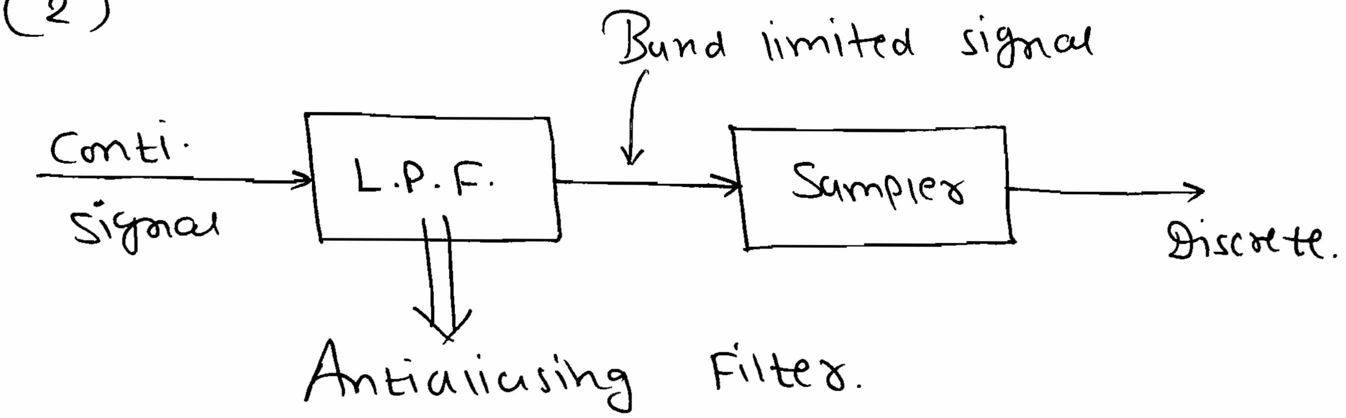


ALIASING (or) Spectral Folding.

⇒ To Avoid Aliasing

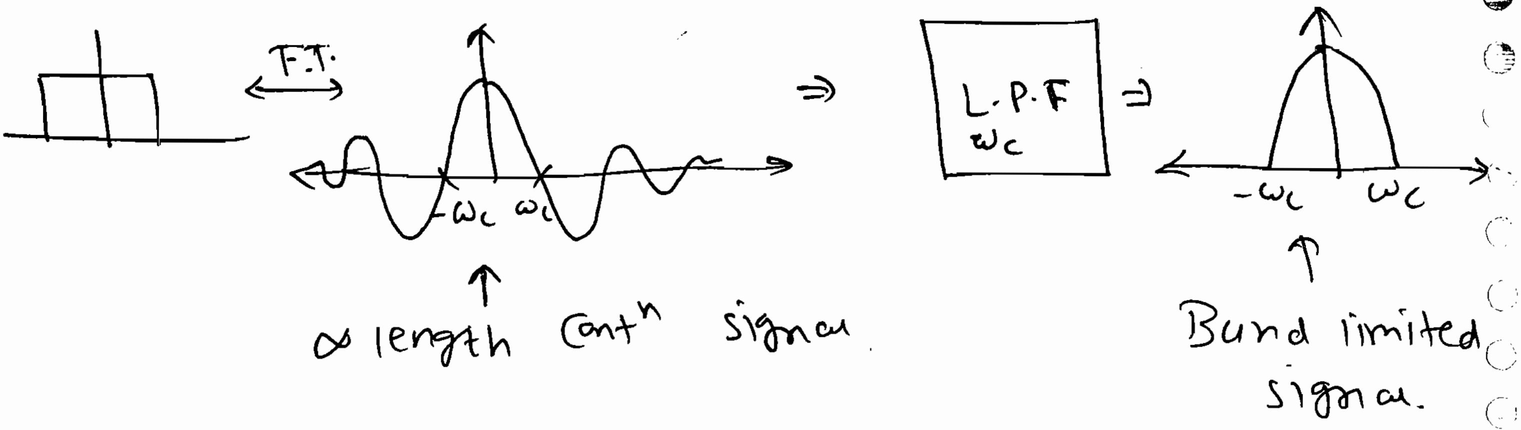
(1) $\boxed{\omega_0 > 2\omega_m.}$

(2)



⇒ No Spectrum in Real time is Band-limited. So, to make ∞ length Spectrum as Band limited, Continuous signal is applied to LPF before Sampiinging.

e.g.



⇒ Nyquist Rate = minimum Samping Rate

$= 2\omega_m$

= 2 (Height freq. of signal component).

$\boxed{P4.6\text{-}1}$ Find the Nyquist rate & Nyquist interval for each of the following signals:

(a) $x_1(t) = \left(\frac{\sin 200\pi t}{\pi t}\right)$.

Soln: Here, $\omega_m = 200\pi$.

$\therefore$ $N.R = 2\omega_m = 2(200\pi)$. rad/sec.

$\Rightarrow N.R = 200\ Hz. = f_s$

$T_s = \frac{1}{f_s} = \frac{1}{200}$ sec.

$\therefore$ $\boxed{T_s = 5ms}$

(b) b $x_2(t) = \left(\frac{\sin 200\pi t}{\pi t}\right)^2$.

Soln: $x_2(t) = \frac{1 + \cos 400\pi t}{4(\pi t)^2}$

So, $\omega_m = 400\pi$

N.R. $\Rightarrow$ $\omega_0 = 2\omega_m = 2(400\pi)$ rad/sec.

$f_s = 2f_m = 400\ Hz.$

$\Rightarrow$ $T_s = \frac{1}{f_s} = \frac{1}{400.}$ $\Rightarrow$ $\boxed{T_s = 2.5ms}$

(c) $x_3(t) = 5\cos 1000\pi t. \cos 4000\pi t.$

Soln: $x_3(t) = \frac{5}{2}. 2\cos 1000\pi t. 4000\pi t.$

$= \frac{5}{2}. (\cos 4000\pi t + \cos 5000\pi t)$

$\Rightarrow \quad \omega_m = 5000\pi$

$\Rightarrow \quad N.R. = \omega_0 = 2\omega_m = 2(5000\pi)$ rad/sec.

$f_s = 2f_m = 5\,KHz.$

$\Rightarrow \quad T_s = \frac{1}{5K} = 0.2\,msec.$

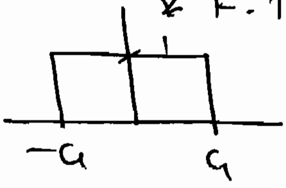
$$\boxed{T_s = 0.2\,ms}$$

(d) $x_4(t) = e^{-6t}u(t) * \frac{\sin at}{\pi t}.$

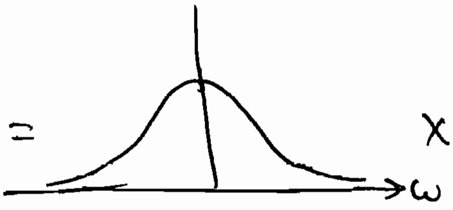
$\downarrow F.T \qquad \downarrow F.T$

Soln:

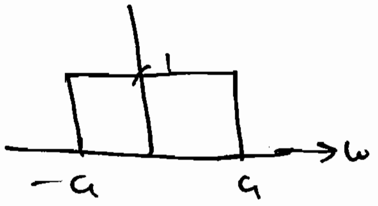
$X_4(\omega) = \frac{1}{6+j\omega} \times$



$=$



$\times$



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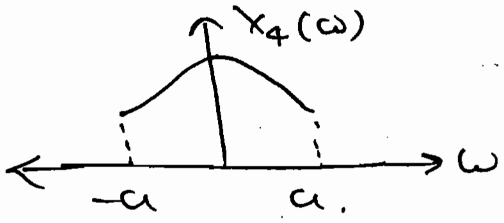


$\Rightarrow \quad \omega_m = a.$

$N.R. \Rightarrow \omega_0 = 2\omega_m = 2a.$ rad/sec.

$f_s = 2f_m = \cancel{2} \times \frac{\omega_m}{\cancel{2}\pi} = \frac{a}{\pi}$ Hz.

$\Rightarrow \quad \boxed{T_s = \frac{\pi}{a}\ sec}$

(e) $x_5(t) = \text{sinc}(100t) + 3\,\text{sinc}^2(60t).$

Sol^n: $x_5(t) = \frac{\sin(100\pi t)}{100\pi t} + 3\left(\frac{\sin 60\pi t}{60\pi t}\right)^2$.

$$x_5(t) = \frac{\sin(100\pi t)}{100\pi t} + 3\left[\frac{1 - \cos 120\pi t}{(2)^2 \times (60\pi t)^2}\right].$$

$\Rightarrow$ $\omega_m = 120\pi$

N.R. $\Rightarrow$ $\omega_0 = 2\omega_m = 2(120\pi)$ rad/sec.

$f_s = 2 f_m = 120\text{ Hz}.$

$\Rightarrow$ $\boxed{T_s = \frac{1}{120}\text{ sec.}}$

P 4.6.2. Let $x(t)$ be a signal with Nyquist rate ω_0. Determine the Nyquist rate for each of the following signals.

(a) $x(t) + x(t-1)$.

Sol^n: $x(t) \rightarrow$ B.W. $= \omega_m$.

& N.R $= \omega_0$.

i.e. $\boxed{\omega_0 = 2\omega_m}$

F.T.

$X(\omega) + e^{-j\omega(1)} \cdot X(\omega) \rightarrow$ N.R. $= \omega_0$

Bw not change hence N.R. not change. i.e. ω_0.

(b) $\frac{d}{dt} x(t)$.

$\Rightarrow$ Soln: $\downarrow$ F.T. $= j\omega X(\omega)$.

$\Rightarrow$ B.ω. not change hence N.R. Same.

i.e. ω_0.

(c) $x(3t)$.

Solⁿ: $x(3t) \xleftrightarrow{F.T.} \frac{1}{|3|} \cdot X(\omega/3)$.

New, $B\omega \Rightarrow \omega_m' = 3\omega_m$.

So New N.R.

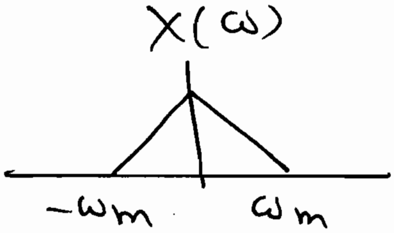
$\omega_0' = 2(3\omega_m)$

$= 3(2\omega_m)$

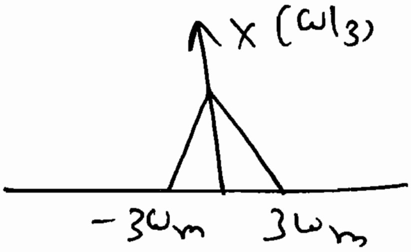
$\boxed{\omega_0' = 3\omega_0}$

~~$\omega_0' = 3(2(\omega_m)) = \omega_0/3$~~.

~~$\omega_0' = \omega_0/3$~~



,

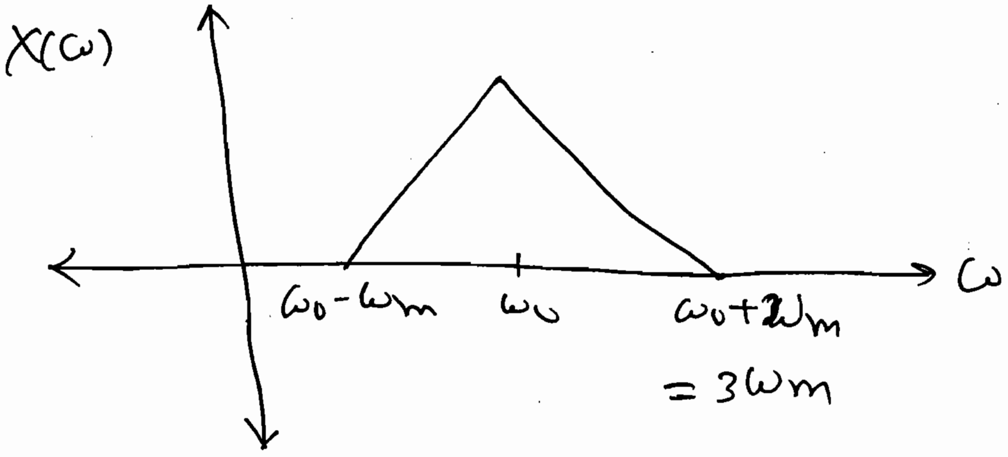


(d) $x(t) \cdot \cos(\omega_0 t)$.

Solⁿ: $x(t) \cdot \cos(\omega_0 t) \xleftrightarrow{F.T.} \pi\left[X(\omega_m - \cancelto{2\omega_m}{\omega_0}) + X(\omega_m + \cancelto{2\omega_m}{\omega_0})\right]$.

eg. $\omega_0 = 2\omega_m$

~~$= \pi\{X(-\omega_m) + X(3\omega_m)\}$~~.



So, New N.R. $\omega_0' = 2(3\omega_m)$

$= 3(2\omega_m)$

$$\boxed{\omega_0' = 3\omega_0}$$

P4.6.3 Two signals $x_1(t)$ & $x_2(t)$ are band limited to 2 KHz & 3 KHz respectively, find the Nyquist rate of the following signals.

Soln: (a) $x_1(2t)$.

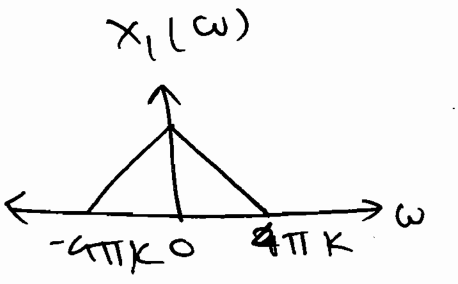
Soln: $x_1(2t) \xleftrightarrow{F.T.} \frac{1}{2} X_1(\omega/2)$.

$x_1(t) \rightarrow$ B.W. = 2 KHz

So, N.R. of $x_1(t) \Rightarrow$

$f_{s_1} = 2 f_{m_1} = 4$ KHz.

$\omega_{0_1} = 2\omega_{m_1} = 8\pi$ K rad/sec.



$X_1(\omega/2)$

So, N.R. of $x_1(2t)$

$\Rightarrow \omega_{0_1}' = 2\omega_{m_1}' = 2(8\pi)K$

$= 16K\pi$ rad/sec.

$f_{s_1}' = 2 f_{m_1}'$

$$\boxed{f_{s_1}' = 8 \text{ KHz}}$$

(b) $x_2(t-3)$.

Soln: B.w. of $x_2(t)$ is → 3 KHz.

$\Rightarrow f_{m_2} = 3$ KHz.

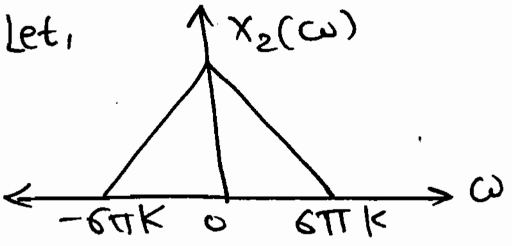
N.R. $f_{s_2} = 2 f_{m_2} = 6$ KHz

$\omega_{o2} = 2\omega_{m_2} = 6(2\pi) = 12\pi$ k rad/sec

Now, $x_2(t-3) \xleftrightarrow{F.T.} e^{-j3\omega} \cdot X_2(\omega)$.

B.w. is not change. Hence. N.R. is not change.

Let,



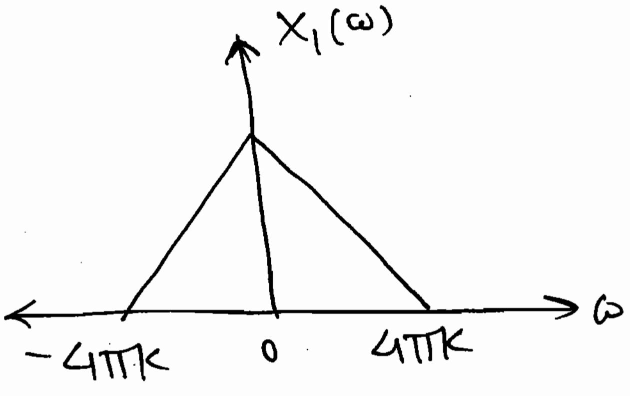
i.e. N.R. of $x_2(t-3)$

$\Rightarrow \omega'_{o2} = 2\omega_{m_2} = 12\pi$ k rad/sec.

& $\boxed{f'_{s_2} = 2 f_m = 6 \text{ KHz}}$

(c) $x_1(t) + x_2(t)$.

Soln: $x_1(t) + x_2(t) \xleftrightarrow{F.T.} X_1(\omega) + X_2(\omega)$.



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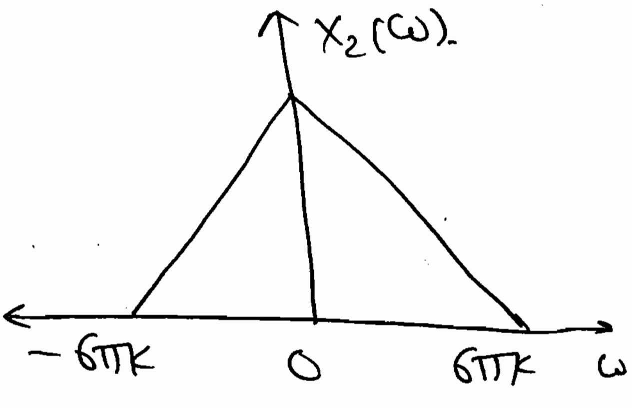


⇒ Max. freq. after addition of $X_1(\omega)$ & $X_2(\omega)$ is $6\pi k$ rad/sec.

$\Rightarrow \quad \omega_m' = 6\pi k$ rad/sec

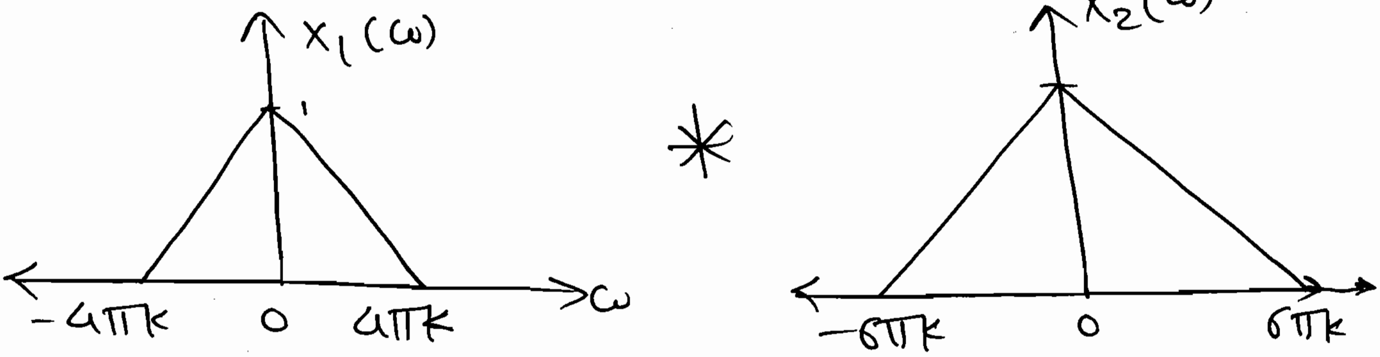
N.R. $\Rightarrow \quad \omega_o' = 2\omega_m' = 12\pi k$ rad/sec.

$\therefore$ ~~ω~~ $\boxed{f_s' = 2f_m' = 6k\ Hz.}$

(d) $x_1(t) \cdot x_2(t)$.

Soln: Mul. $\xleftrightarrow{F.T.}$ Conv.

$\therefore \; x_1(t) \cdot x_2(t) \xleftrightarrow{F.T.} X_1(\omega) * X_2(\omega)$.



$\Rightarrow$ After Convolution new upper & lower limit of the signal are.

Lower limit = { Sum of the lower limit of $x_1(\omega)$ & $x_2(\omega)$ }.

Upper limit = { Sum of the upper limit of $x_1(\omega)$ & $x_2(\omega)$ }.

$\Rightarrow$ So, Lower limit = {$-10\pi k$}.

upper limit = {$+10\pi k$}.

So, $\omega_m' = 10\pi k$ rad/sec.

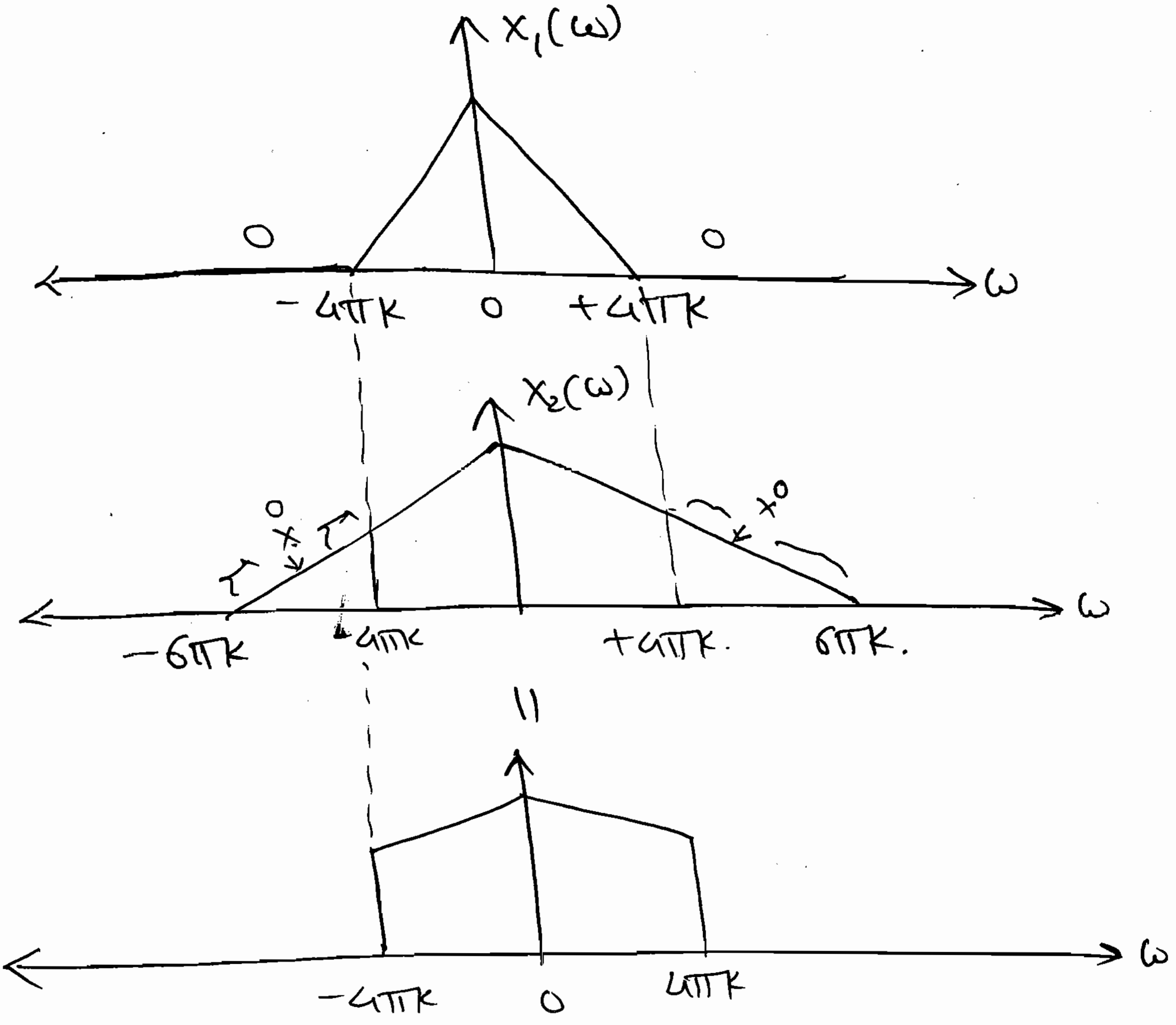
N.R. $\quad \omega_o' = 2\omega_m' = 2(10\pi k) = 20\pi k \frac{rad}{sec.}$

$\Rightarrow \boxed{f_s' = 2f_m = 10K\,Hz.}$

(e) $x_1(t) * x_2(t).$

Sol^n: Conv. $\xleftrightarrow{F.T.}$ mult.

$\Rightarrow x_1(t) * x_2(t) \xleftrightarrow{F.T.} X_1(\omega).\ X_2(\omega).$



$\Rightarrow$ After Multipiication of two signal $X_1(\omega)$ & $X_2(\omega)$ New highest freq. is $4\pi K$ rad/sec.

So, N.R. $\Rightarrow$ $\omega_0' = 2\omega_m' = 2(4\pi K).$
$= 8\pi K$ rad/sec.

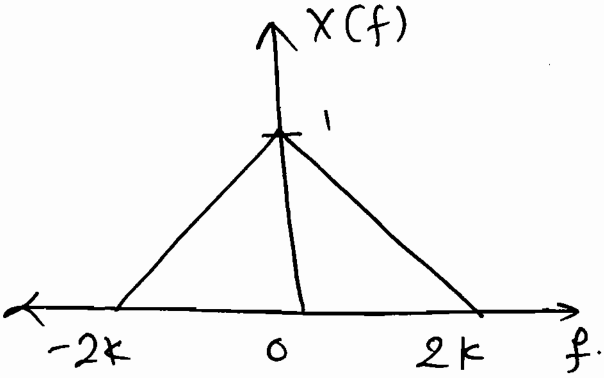
$\boxed{f_s' = 2f_m = 4KHz.}$

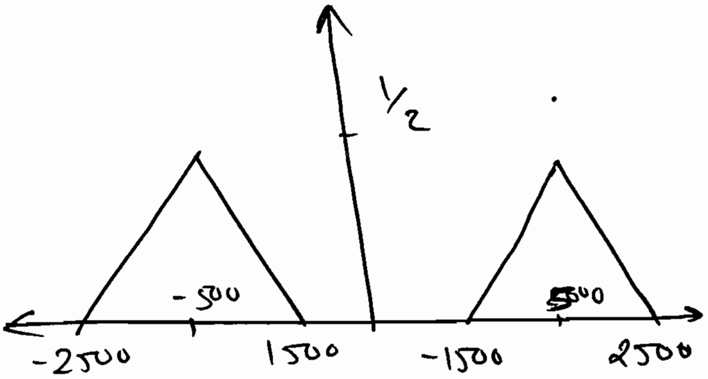
(f) $x_1(t) \cdot \cos(\underbrace{1000\pi t}) \rightarrow f_0 = 500\,Hz$.

Soln:

$= x_1(t) \cdot \cos(1000\pi t)$.

$$\xleftrightarrow{F.T.} \frac{X_1(f-f_0) + X_1(f+f_0)}{2}$$





$\Rightarrow$ $f_m' = 2500\,Hz$.

N.R. $2f_m = f_s = 5000\,H$.

$\therefore \boxed{f_s = 5\,KHz.}$

Note:-

$x_1(t) \longleftrightarrow X_1(\omega) \rightarrow$ B.W. ω_{m1}

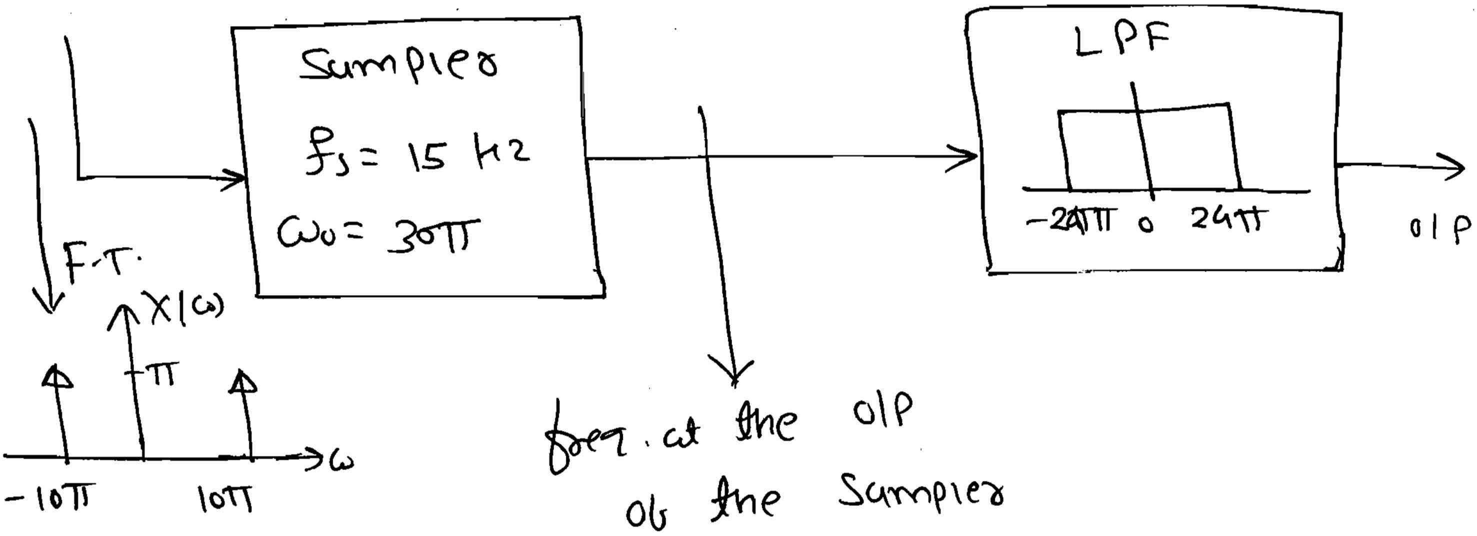
$x_2(t) \longleftrightarrow X_2(\omega) \rightarrow \omega_{m2}$.

	New BW.
① $x_1(t) \pm x_2(t) \xleftrightarrow{F.T.} X_1(\omega) \pm X_2(\omega)$	Max $(\omega_{m1}, \omega_{m2})$.
② $x_1(t) \cdot x_2(t) \longleftrightarrow X_1(\omega) * X_2(\omega)$	$(\omega_{m1} + \omega_{m2})$
③ $x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$.	Min $(\omega_{m1}, \omega_{m2})$.

Q A signal $x(t) = \cos(10\pi t)$ is a sampled at 15 Hz and is passed through an ideal L.P.F with cut of freq 12 Hz. What freq. will appear at the o/p of the filter?

Soln:

$x(t) = \cos(10\pi t)$



freq. at the o/p of the Sampler

$(\omega - n\omega_0)$ $n \rightarrow -\infty$ to $+\infty$.

$(\omega_m - n\omega_0) \Rightarrow (f_m - nf_s)$

$\Rightarrow$ here. $\omega_m = 10\pi$

$\omega_0 = 30\pi$.

let, $n=0 \Rightarrow \omega = \boxed{\pm 10\pi}$ o/p

$n=1 \Rightarrow \omega = \omega_m - \omega_0$

$\boxed{-20\pi}$ o/p -40π

$n=-1 \Rightarrow \omega = \omega_m + \omega_0$ $\boxed{20\pi}$ o/p, 40π

$n=2 \Rightarrow \omega = \omega_m - 2\omega_0$ -50π, -70π

So, freq. at ~~LPF~~ o/p of LPF i.e at o/p ⇒ ± 10π, ± 20π.

P4.6.4. A signal represented by $x(t) = 5\cos(400\pi t)$ is Sampled at a rate of 300 Hz. The resulting samples are Passed through an ideal LPF with Cut-off freq. of 150 Hz. Which of the following will be contained in the o/p of LPF?

(a) 100 Hz (b) 100 Hz, 150 Hz

(c) 50 Hz, 100 Hz (d) 20, 100, 150 Hz.

Soln:

$\omega_m = 400\pi \Rightarrow f_m = \pm 200$ Hz.

$f_s = 300$ Hz.

⇒ Freq. at o/p of the Samples is,

$f = f_m - n f_s$, $n \to -\infty$ to $+\infty$. ✓

$n=0$. $f = f_m = \pm 200$ Hz. ✓

$n=1$ $f = f_m - f_s = \boxed{100\text{ Hz}}, -500$ Hz.

$n=-1$ $f = f_m + f_s = 500$ Hz, $\boxed{100\text{ Hz}}$ ✓

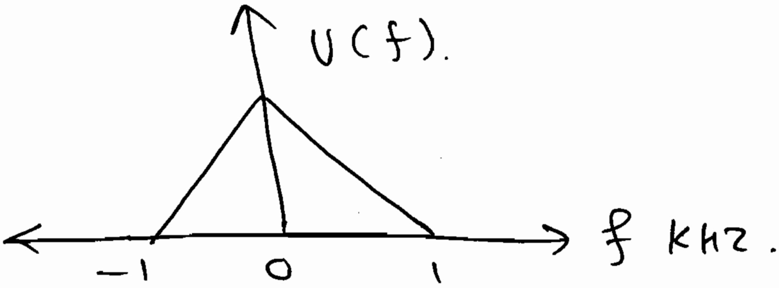
$n=2$ $f = f_m - 2f_s = 800$ Hz, -400 Hz.

⇒ Cut-off freq. of LPF is 150 Hz.

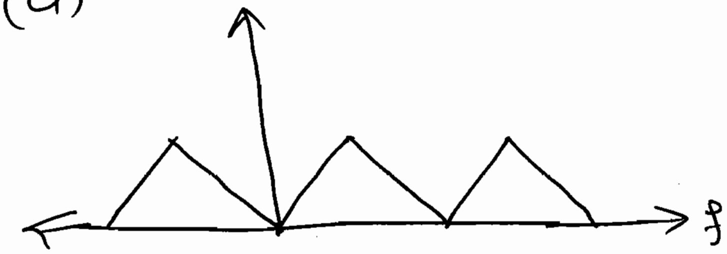
So, o/p freq. at LPF is in betn −150 to 150.

i.e ± 100 Hz So, ans - $\boxed{(a)\ 100\text{ Hz}}$

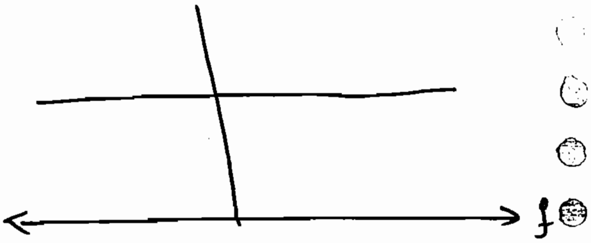
$\boxed{P4.6.5}$ The freq. Spectrum of a signal is shown in figure if this signal is ideally samped at intervals of 1 msec, then the freq. Spectrum of the sampled signal will, be



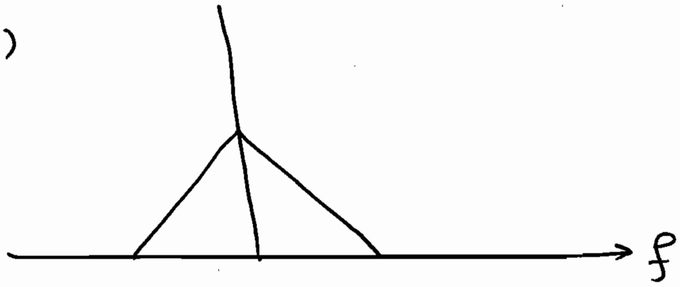
(a)



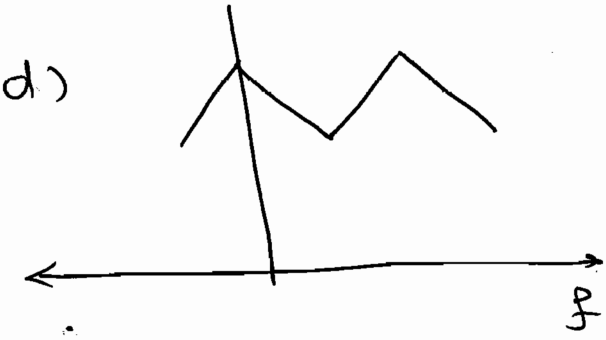
(b) ✓



(c)



(d)

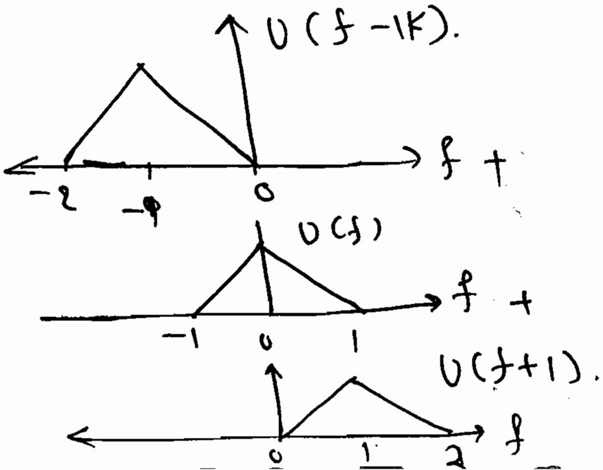


Soln:

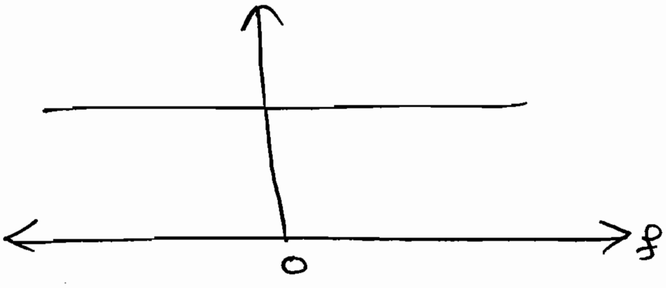
$f_m = 1$ KHz.

$f_s = 1$ msec.

$\therefore$ $f_s < 2 f_m$. So, under sampling.

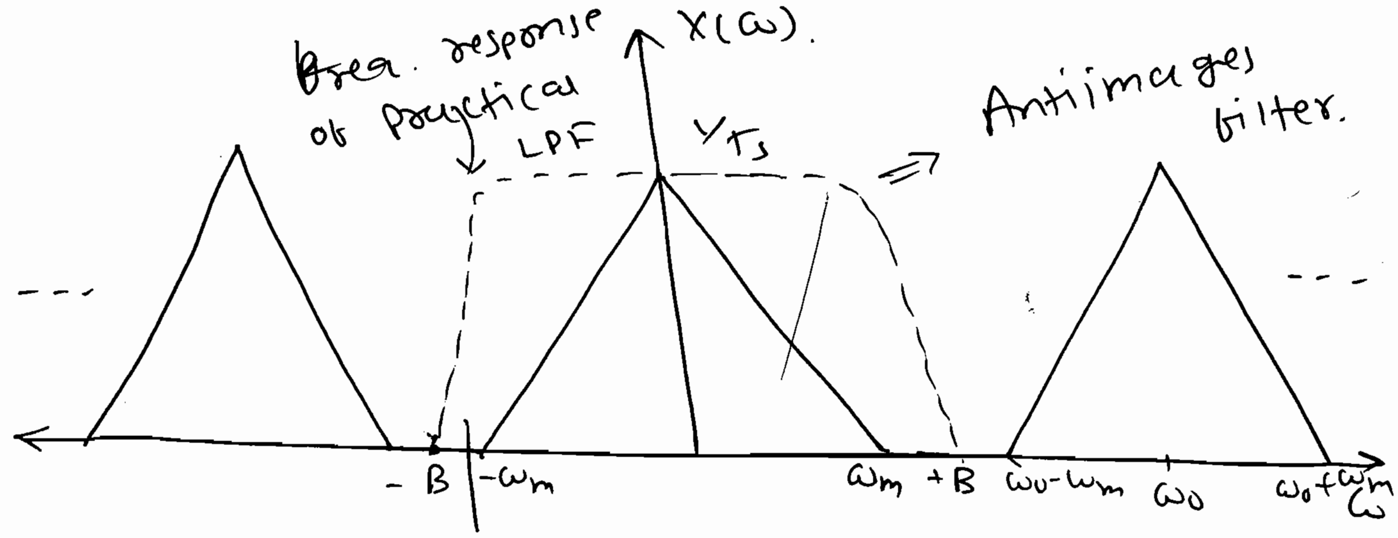


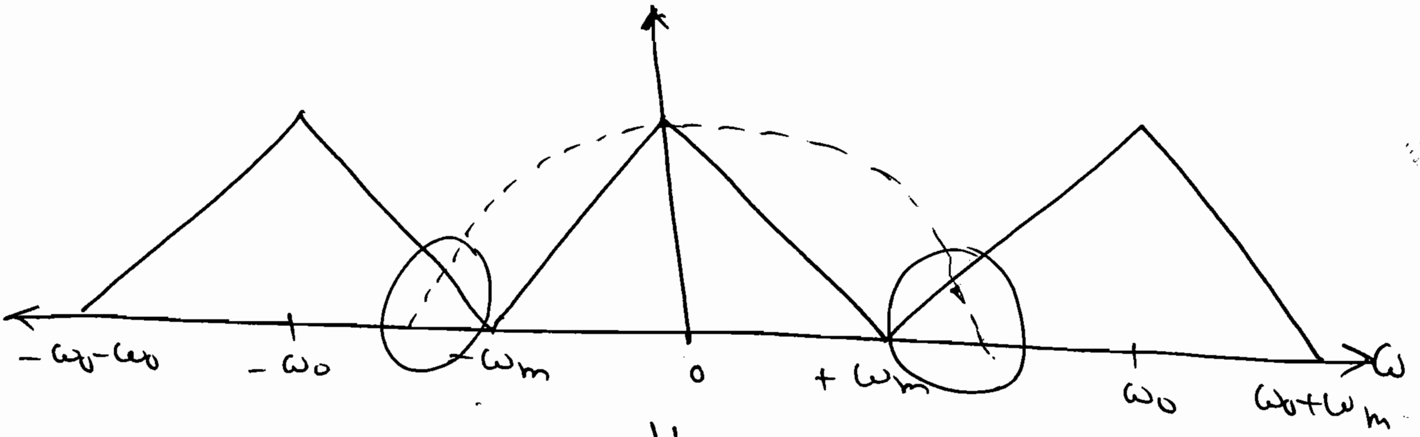
$\Rightarrow$



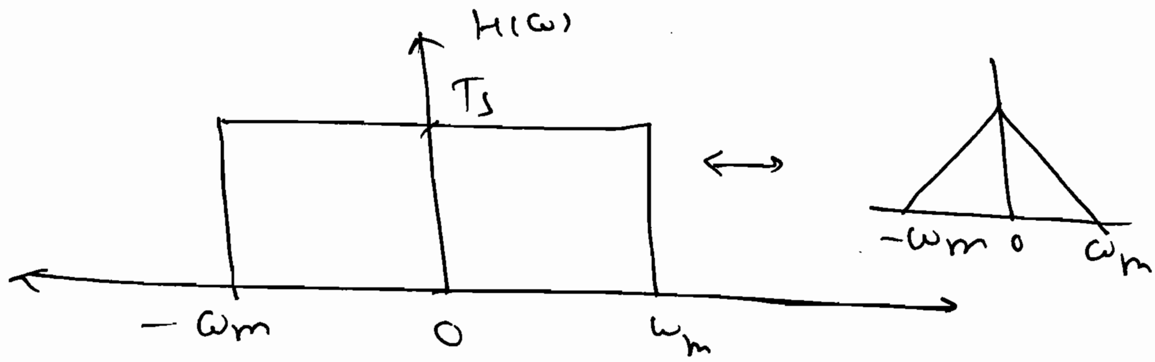
* Signal Reconstruction:-

⇒ The Process Of recovering original continous Spectrum from the Sampled Spectrum is signal reconstruction.





⇓



⇓ Ideal LPF

⇒ $\boxed{\omega_m < B < \omega_0 - \omega_m.}$ ✓

P 4.6.7 A signal $x(t) = 6\cos 10\pi t$ is sampled at a rate of 14 Hz to recover the original signal, cut-off freq. of the LPF should be ______.

(a) $5 < f_c < 9$ ✓ (b) 9 (c) 10 (d) 14.

Solⁿ: $\omega_m = 10\pi \Rightarrow f_m = 5$ Hz., $f_s = 14$ Hz

$\therefore$ $\boxed{f_m < B < f_s - f_m.}$

$\therefore$

$\therefore$ $5 < B < 14 - 5$

$\Rightarrow \boxed{5 < B < 9}$

P4.6.6 A signal with 2 freq. components at 6 kHz and 12 kHz is sampled at the rate of 16 kHz and then passed through a LPF having a cut-off freq. of 16 kHz. The output signal of the filter is ___.

(a) is an undistorted version of original signal.

(b) contains 6 kHz & spurious components of 4 kHz.

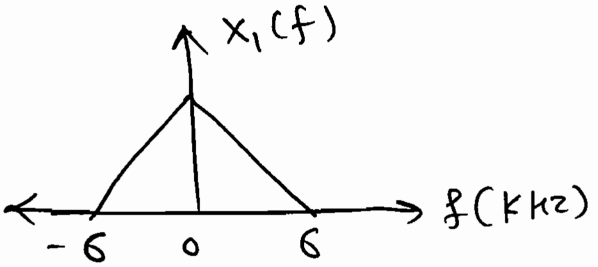
(c) contains only 6 kHz components.

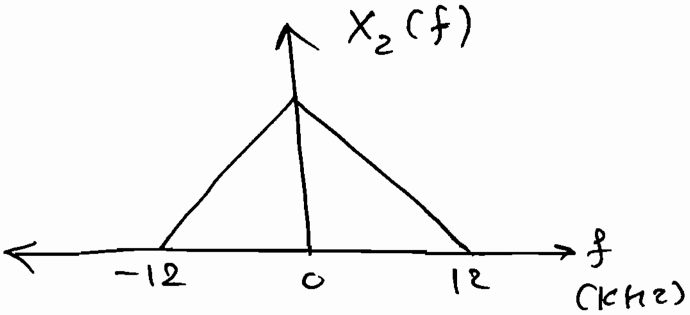
(d) containing both components of original signal and 2 spurious components of of 4 KHz & 10KHz.

Soln: f_{m1} = 6 KHz, f_{m2} = 12 KHz.

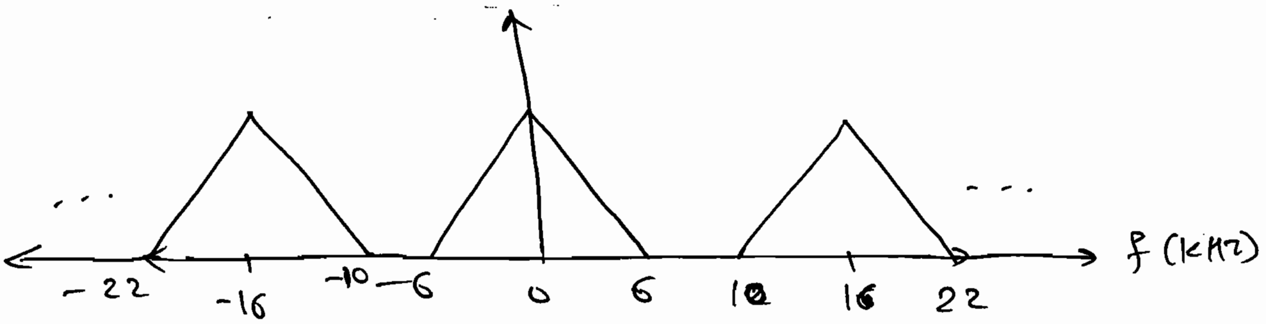
f_s = 16 KHz.

=>

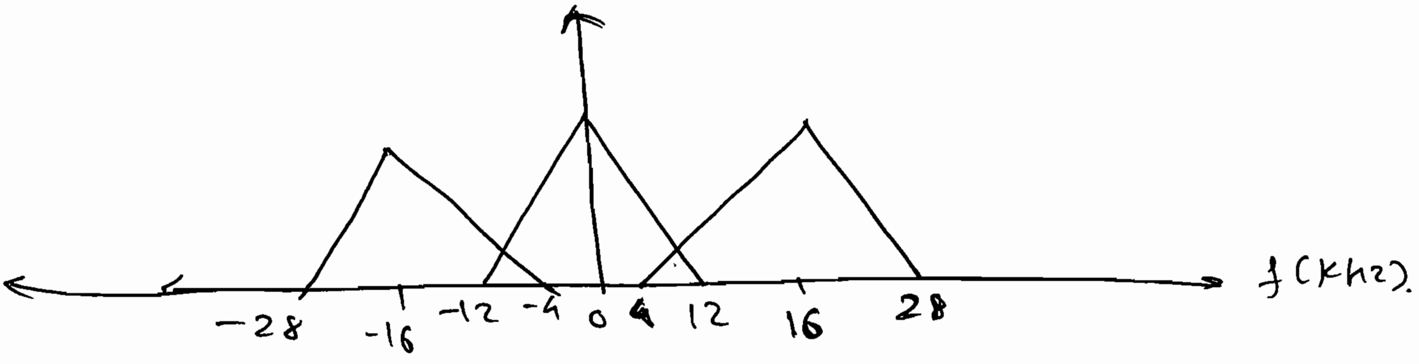


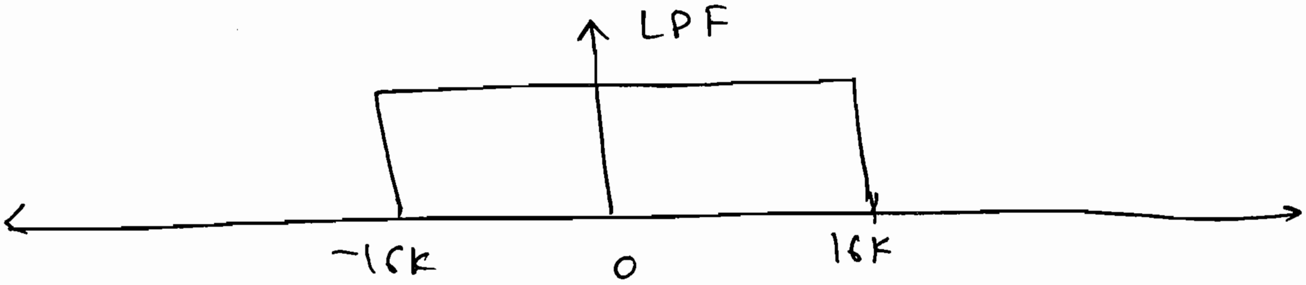


① (f_s = 16K) > (2 f_{m1} = 12KHz) => over sampling



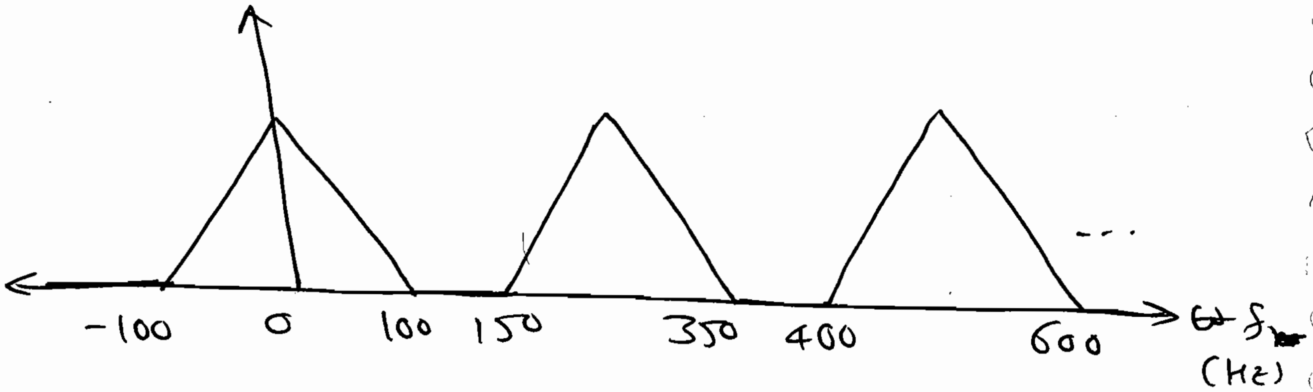
② (f_s = 16K) < (2 f_{m2} = 24 KHz). => under sampling





=> Ans-(b) Contains 6KHz & Spurious Components of 4 KHz.

P 4.6.8 The Spectrum of a bandlimited signal after Sampling is shown in figure. The value of Sampling interval is ______.



Sol^n:

$f_m = 100$ Hz.

$\therefore$ $f_s - f_m = 150$ Hz.

$\therefore$ $f_s = 150 + 100 = 250$ Hz.

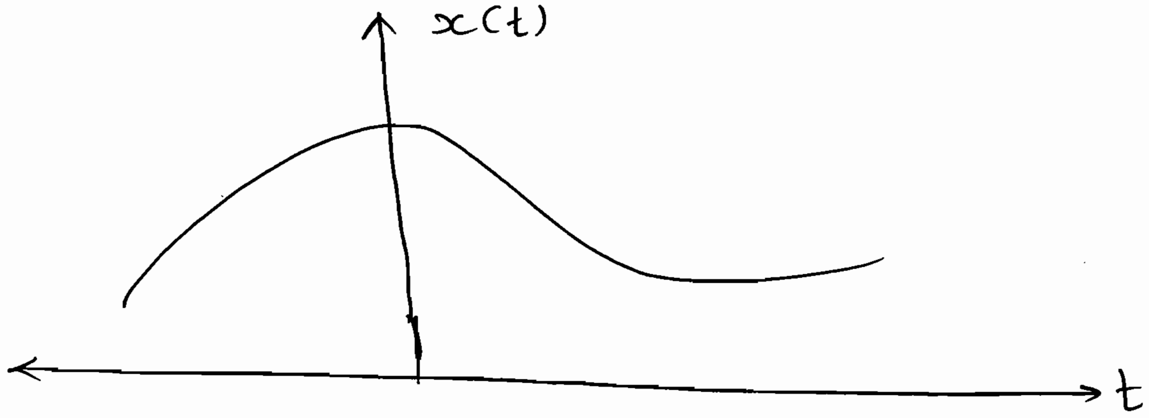
$\therefore$ $f_s = 250$ Hz.

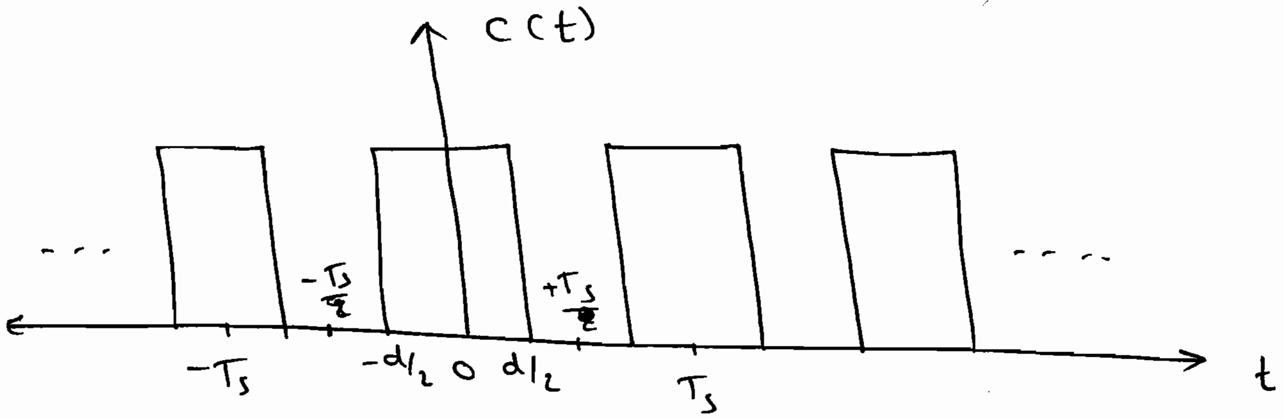
$\Rightarrow$ S.I. $T_s = \frac{1}{f_s} = \frac{1}{250} = 4$ m sec.

$$\boxed{T_s = 4\text{msec}}$$

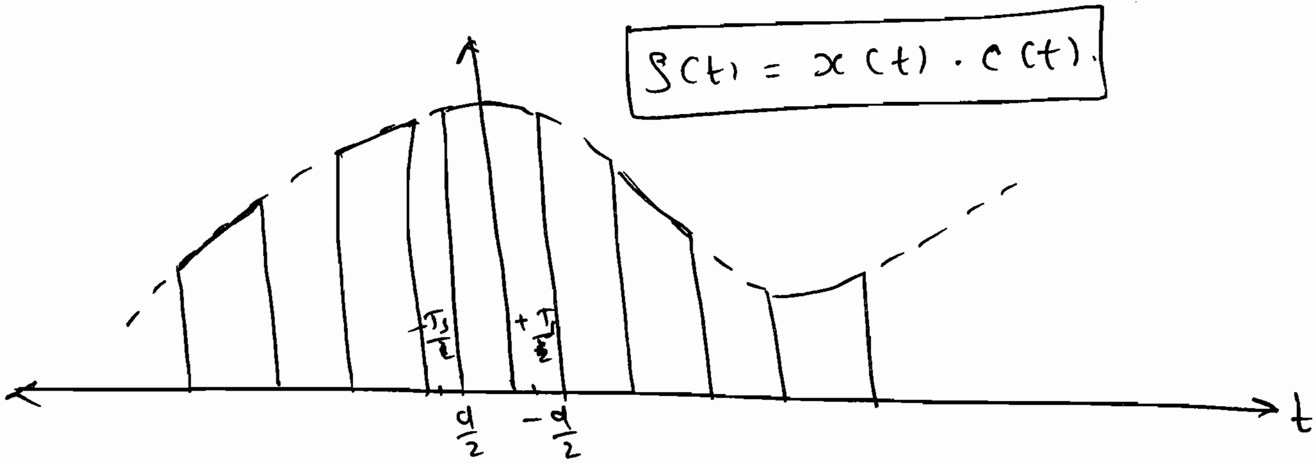
Note: The freq. at the o/p of Sampler is $\omega - n\omega_0$ in ideal Sampling. We can take the value of n from $-\infty$ to $+\infty$, but in natural Sampling n value is desided by C_n.

* ② Natural Sampiing:-





$$\boxed{s(t) = x(t) \cdot c(t).}$$



$$s(t) = x(t) \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t} .$$

$$\Big\downarrow \text{F.T.}$$

$$S(\omega) = \sum_{n=-\infty}^{\infty} C_n X(\omega - n\omega_0).$$

$$C_n = \frac{1}{T} \int_{-d/2}^{d/2} (1).\ e^{-jn\omega_0 t} \, dt$$

$\Rightarrow \quad T = T_s.$

$$\therefore C_n \; \cancel{S(\omega)} = \frac{1}{2T_s} \times \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right]_{-d/2}^{+d/2}$$

$$= \frac{1}{T_s} \times \left[\frac{e^{-jn\omega_0 \frac{d}{2}} - e^{jn\omega_0 \frac{d}{2}}}{-jn\omega_0} \right].$$

$$C_n = \frac{\sin\left(\frac{n\omega_0 d}{2}\right)}{n\omega_0 T_s}.$$

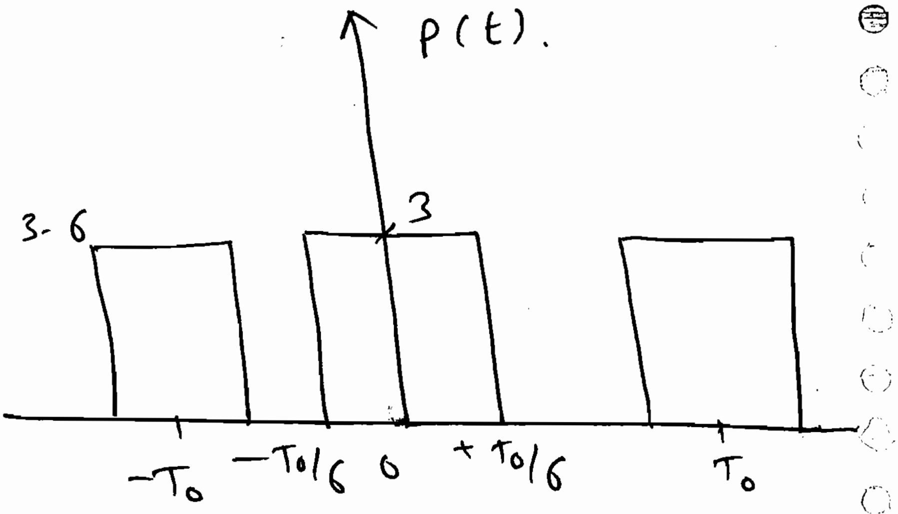
$\boxed{\text{P 4.6.9}}$ Let $x(t) = 2\cos(800\pi t) + \cos(1400\pi t)$ and $x(t)$ is sampled with the rectangular pulse train as shown in fig. The only spectral components (in kHz) in the sampled signal in the freq. range 2.5KHz to 3.5 KHz.

(a) 2.7, 3.4

(b) 3.3, 3.6

(c) 2.6, 2.7, 3.3, 3.4, 3.6

(d) 2.7, 3.3.



Soln:

$$C_n = \frac{1}{T_0} \int_{-T_0/6}^{T_0/6} (3).\, e^{-jn\omega_0 t}\, .dt$$

$$\omega_0 = \frac{2\pi}{2T_0}$$

$$\omega_0 = 2\pi \text{ K } \cancel{\text{Hz}} \text{ rad/sec}$$

$$f_s = \frac{1}{2} \text{ K Hz}.$$

$$C_n = \frac{\sin\left(\frac{n\omega_0 d}{2}\right)}{n\omega_0 T_s}.$$

$$= \frac{\sin\left(\frac{n}{2}\times\frac{2\pi}{2\times10^{-3}}\times\frac{10^{-3}}{6}\right)}{n\times\frac{\cancel{2}\pi}{\cancel{2}T_s}\times T_s}$$

$$C_n = \frac{\sin\left(\frac{n\pi}{3}\right)}{n\pi}$$

$C_n = 0$ for $n = 3, 6, 9, \ldots$

$C_n \neq 0$ for $n = 0, 1, 2, 4, 5, \ldots$

⇒ freq. at the o/p of natural sampler.

$$\left.\begin{array}{l} f_{m_1} \pm n f_s. \\ f_{m_2} \pm n f_s. \end{array}\right] \quad n = 0, 1, 2, 4, 5, 7, 8, \ldots$$

⇒ $n=0$ ⇒ $f_{m_1} = f = 0.4k$

$f_{m_2} = f = 0.7k.$

⇒ $n=1$ ⇒ $f_1 = f_{m_1} \pm f_s = 0.4+1k = 1.4$ kHz

$f_2 = f_{m_2} \pm f_s = 0.7+1k = 1.7$ kHz.

⇒ $n=2$ ⇒ $f_1 = f_{m_1} \pm 2f_s = 0.4+2k = 2.4$ kHz

$f_2 = f_{m_2} + 2f_s = 0.7+2k = 2.7$ kHz.

⇒ $n=4$ ⇒ $f_1 = f_{m_1} \pm 4f_s = 0.4 \mp 4k = 3.6$ k

$f_2 = f_{m_2} \mp 4f_s = 0.7 \mp 4k = 3.3$ k.

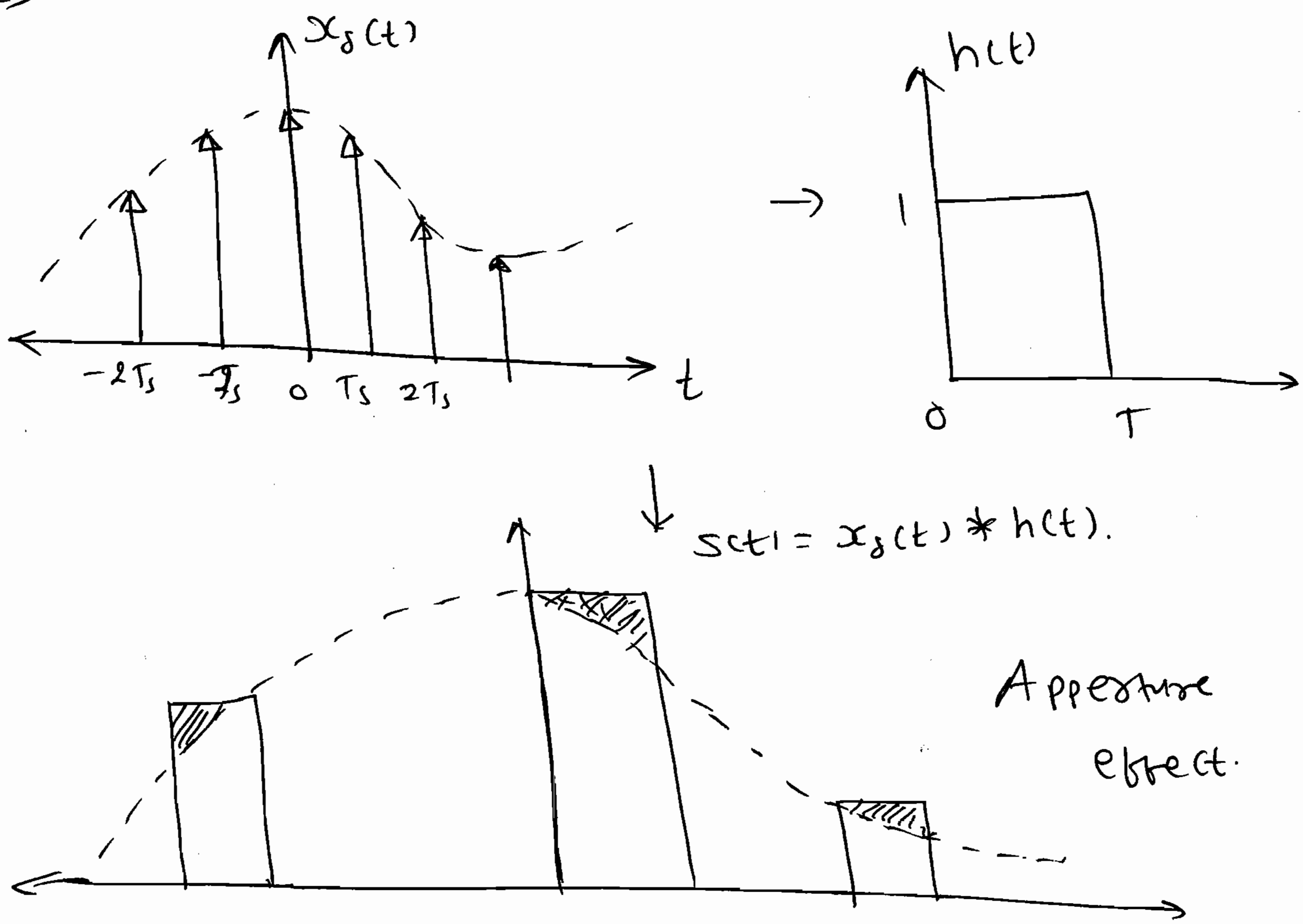
⇒ ~~$n=-1$ ⇒ $f_1 = f_{m_1} - f_s$~~

~~$f_2 = f_{m_2} - f_s$~~

So, Ans: ⓓ 2.7, 3.3.

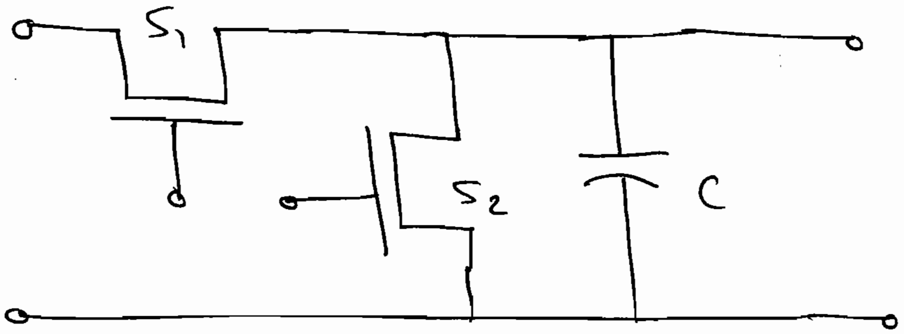
③ Flat - Top Sampling:

⇒



$s(t) = x_\delta(t) * h(t).$

⇒

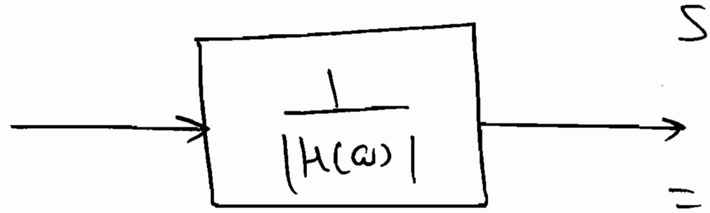


Sample and hold ckt.

⇒ because of maintaining Constant amplitude level we introduce amplitude distortion of $T\text{sinc}\left(\frac{\omega T}{2\pi}\right)$ & phase Delay of $-\frac{\omega T}{2}$, which is known as Apperture effect.

⇒ To Cancle this, Flat-Top Sampled sig. is applied to an eqnalizer $\frac{1}{|h(\omega)|}$.

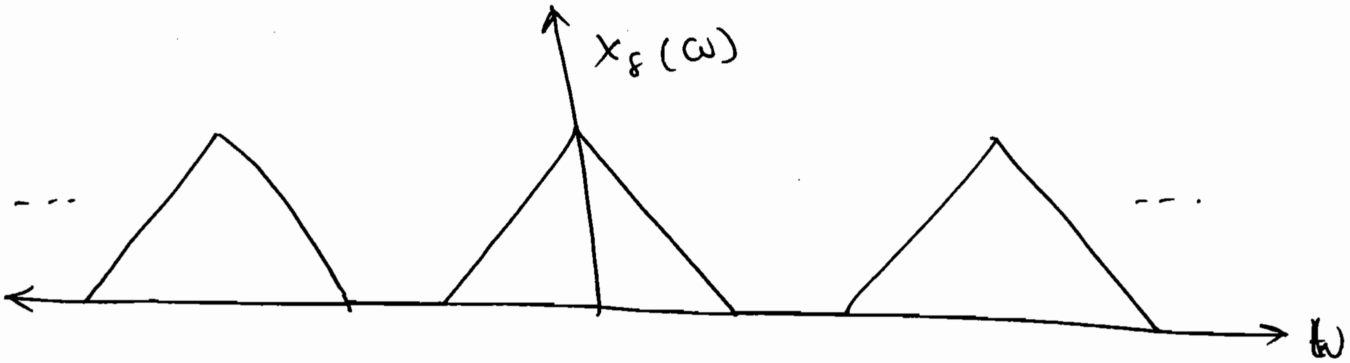
⇒



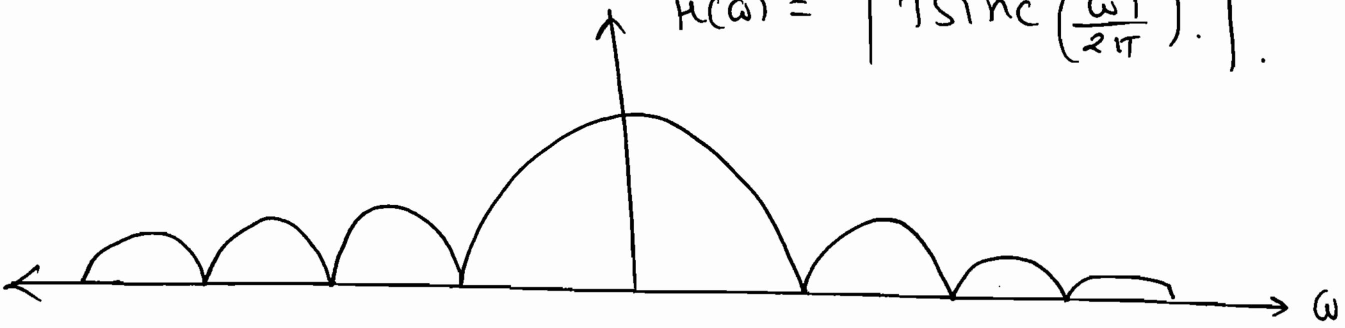
$$S(\omega) = X_\delta(\omega) \cdot H(\omega)$$

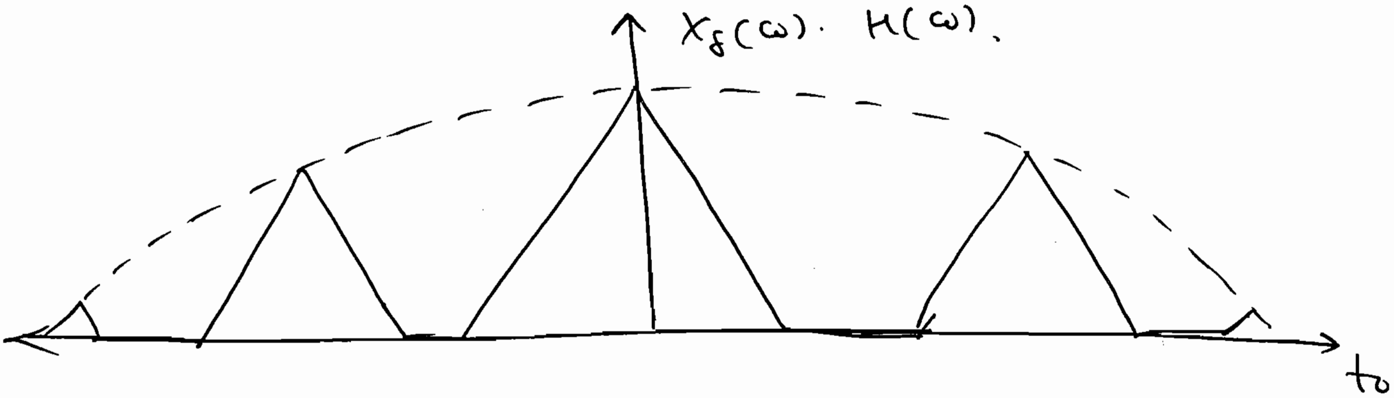
$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_0) \cdot H(\omega).$$

Eeqnalizer $H(\omega) = T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right) \cdot e^{-j\omega(T/2)}$.



$H(\omega) = \left| T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right). \right|$.





* Zero order Hold (ZOH) ckt.