INTEGRALS

CHAPTER – 7

INTEGRALS

INTRODUCTION

→ Integration is a method of adding or summing up the parts to find the whole. It is a reverse process of differentiation, where we reduce the functions into parts. This method is used to find the summation under a vast scale.

→ We know that differentiation is the process of finding the derivative of the functions and integration is the process of finding the antiderivative of a function. So, these processes are inverse of each other. So we can say that integration is the inverse process of differentiation or vice versa. The integration is also called the anti-differentiation.

→ To represent the antiderivative of "f", the integral symbol "∫" symbol is introduced. The antiderivative of the function is represented as ∫ f(x) dx. This can also be read as the indefinite integral of the function "f" with respect to x.

Therefore, the symbolic representation of the antiderivative of a function (Integration) is:

y = ∫ f(x) dx

∫ f(x) dx = F(x) + C.

→ **TWO TYPES OF INTEGRALS**

Indefinite Integral

Definite Integral

INDEFINITE INTEGRAL

Indefinite integrals are defined without upper and lower limits. It is represented as:

∫f(x)dx = F(x) + C

Where C is any constant and the function f(x) is called the integrand.

SUM AND DIFFERENCE RULE

- ∫ [f(x) + g(x)] dx = ∫ f(x) dx + ∫ g(x) dx
- ∫ [f(x) - g(x)] dx =∫ f(x) dx - ∫ g(x) dx

FORMULA

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, n $\neq -1$
- $\int \cos x \, dx$ =sin x +C
- $\int \sin x \, dx = -cosx + C$
- $\int sec^2 x \, dx = tanx + C$

- $\int \sec x \tan x \, dx = \sec x + C$
- $\int cosec^2 x \, dx = -\cot x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1} x + C$
- $\int \frac{dx}{1+x^2} = tan^{-1} x$ +C
- $\int \frac{dx}{\sqrt{1-x^2}} = -cos^{-1} x + C$
- $\int \frac{dx}{1+x^2} = -cot^{-1} x + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log a} + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = sec^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = -cosec^{-1} x + C$
- $\int \frac{1}{x} \, dx = \log|x| + C$
- ∫ ln(x) dx = x ln(x) -x + C

PROPERTIES OF INTEGRATION

A few properties of indefinite integrals are:

- ∫ [f(x) ± g(x)] dx =∫ f(x) dx ± ∫ g(x) dx
- ∫ k f(x) dx = k ∫ f(x) dx, where k is any real number.
- ∫f(x) dx = ∫ g(x) dx, if ∫ [f(x)-g(x)] dx = 0
- $\int (k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x))dx =$ $k_1 \int f_1(x)$dx+$k_2 \int f_2(x)$dx+.... +$k_n \int f_n(x)$dx+ c , where c is the constant of integration.

METHODS OF INTEGRATION

Sometimes, the inspection is not enough to find the integral of some functions. There are additional methods to reduce the function in the standard form to find its integral. Prominent methods are discussed below.

The methods of integration are:

- Decomposition method
- Integration by Substitution
- Integration using Partial Fractions
- Integration by Parts

DECOMPOSITION METHOD

The functions can be decomposed into a sum or difference of functions, whose individual integrals are known. The given integrand will be algebraic, trigonometric or exponential or a combination of these functions.

Example

Suppose we need to integrate $(x^2 -x +1)/x^3$ dx, we decompose the function as :

Solution: ∫ $(x^2 -x +1)/x^3$ dx = ∫ $(x^2 /x^3 - x /x^3 +1/x^3)$

= ∫ (1/x) dx - ∫ $(1/x^2)$ dx + ∫ $(1/x^3)$ dx

Applying the reciprocal rule and the power rule, we get

INTEGRATION BY SUBSTITUTION

The **integration by substitution** method lets us change the variable of integration so that the integrand is integrated in an easy manner.

Suppose, we have to find y =∫ f(x) dx.

Let x=g(t). Then, $\frac{dx}{dt}$ = g'(t)

So, y= ∫ f(x) dx can be written as y= ∫ f(g(t)) g'(t).

Example

let's find the integral of f (x) = sin(mx) using substitution.

Solution: Let mx = t. Then, m $\frac{dx}{dt}$=1

$y = \int \sin mx \, dx$

$= \frac{1}{m} \int \sin t \, dt$

$= -\frac{1}{m} \cos t + C$

$= -\frac{1}{m}$ cos(mx) + C

Note

The substitution for the variable of integration can also use trigonometric identities. A few important standard results are:

- ∫ tan x dx = log |sec x| +C
- ∫ cot x dx = log |sin x| +C
- ∫cosec x dx = log |cosec x -cot x| +C
- ∫ sec x dx = log |sec x + tan x| +C

PARTIAL FRACTION OF RATIONAL FUNCTION

Any number which can be easily represented in the form of p/q, such that p and q are integers and q≠0 is known as a rational number. Similarly, we can define a rational function as the ratio of two polynomial functions P(x) and Q(x), where P and Q are polynomials in x and Q(x)≠0.

A **rational function** is known as proper if the degree of P(x) is less than the degree of Q(x); otherwise, it is known as an improper rational function. With the help of the long division process, we can reduce improper rational functions to proper rational functions. Therefore, if P(x)/Q(x) is improper, then it can be expressed as:

$\frac{P(x)}{Q(x)}$ = A(x) +$\frac{R(x)}{Q(x)}$

Here, A(x) is a polynomial in x and R(x)/Q(x) is a proper rational function.

We know that the integration of a function f(x) is given by F(x) and it is represented by:

∫f(x)dx = F(x) + C

Here R.H.S. of the equation means integral of f(x) with respect to x and C is the constant of integration.

S.No	Rational Fraction	Partial Fraction Form
1	$\frac{p(x)+q}{(x-a)(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-b)}$
2	$\frac{p(x)+q}{(x-a)^2}$	$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}$
3	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
4	$\frac{px^2+q(x)+r}{(x-a)^2(x-b)}$	$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{B}{(x-b)}$
5	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Example

Integrate the function $\frac{1}{(x-3)(x+1)}$ with respect to x.

Solution: The given integrand can be expressed in the form of partial fraction as:

$\frac{1}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$ To determine the value of real coefficients A and B, the above equation is rewritten as:

1= A(x+1) + B(x-3)

⇒1=x(A+B) + A-3B

Equating the coefficients of x and the constant, we have

A + B = 0 A − 3B = 1

Solving these equations simultaneously, the value of A =1/4 and B = -1/4. Substituting these values in equation 1, we have $\frac{1}{(x-3)(x+1)} = \frac{1}{4(x-3)} - \frac{1}{4(x+1)}$

According to the properties of integration, the integral of the sum of two functions is equal to the sum of integrals of the given functions, i.e.,

∫[f(x) +g(x)] dx = ∫f(x)dx + ∫g(x)dx

Therefore,

$= \frac{1}{4}\int\frac{1}{x-3} - \frac{1}{4}\int\frac{1}{x+1}$

$= \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1|$ $= \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

(i) $\int\frac{dx}{a+b\sin^2 x}$ OR $\int\frac{dx}{a+b\cos^2 x}$ OR

$\int\frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put tan x = t.

(ii) $\int\frac{dx}{a+b\sin x}$ OR $\int\frac{dx}{a+b\cos x}$ OR $\int\frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2}$ = t

(iii) $\int\frac{a.\cos x+b.\sin x+c}{\ell.\cos x+m.\sin x+n}dx$

Express $Nr \equiv A(Dr) + B\,(Dr) + C$ & proceed.

INTEGRATION OF TYPE $\int \sin^m x.\cos^n x dx$

Case - I: If m and n are even natural number then converts higher power into higher angles.

Case - II: If at least one of m or n is odd natural number then if m is odd put $\cos x = t$ and vice-versa.

Case - III: When m + n is a negative even integer then put tan x = t.

(i) $\int \sqrt{\frac{x-\alpha}{\beta-x}} dx$ or $\int \sqrt{(x-\alpha)(\beta-x)} dx$; put $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

(ii) $\int \sqrt{\frac{x-\alpha}{x-\beta}} dx$ or $\int \sqrt{(x-\alpha)(x-\beta)} dx$ dx; put $x = \alpha \sec^2 \theta - \beta \tan^2 \theta$

(iii) $\int \frac{dx}{(x-\alpha)(x-\beta)}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

INTEGRATION BY PARTS

This Integration rule is used to find the integral of two functions.

By product rule of derivatives, we have $\frac{d}{dx}$(u. v) $= u\frac{dv}{dx} + v\frac{du}{dx} \cdots$ (1)

Integration on both sides of equation (1), we get

$\int u\frac{dv}{dx}dx = u.\ v - \int v\frac{du}{dx}dx \cdots$ (2)

Equation (2) can be written as u. v = $\int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$

Let u=f(x) and $\frac{dv}{dx}$ = g(x)

Then, we have $\frac{du}{dx}$=f′(x) and v = ∫ g(x)dx.

So, Equation (2) becomes

$\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]\ dx$.

Example

let's find the integral of xe^x using integration by parts.

Solution: $\int xe^x dx = x\int e^x dx - \int[\frac{d}{dx}x\int e^x dx]\ dx = xe^x - e^x + C$

A few important standard results (Bernoulli's formula):

- $\int e^{ax} \sin bx\ dx = \frac{e^x}{(a^2+b^2)}$ [a sin bx - b cos bx] + C

$\int e^{ax} \cos bx\ dx = \frac{e^{ax}}{(a^2+b^2)}$ [a cos bx + b sin bx] + C

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS

To integrate the rational algebraic functions whose numerator and denominator contain some positive integral powers of x with the constant coefficients, we use integration by partial fractions and arrive at a few standard results that could be directly applied as integration formulas.

- $\int 1/(a^2 - x^2)\ dx = (1/2a) \log|\ (a+x)/(a-x)\ | + C$
- $\int 1/(x^2 - a^2)\ dx = (1/2a) \log|(x-a)/(x+a)\ | + C$
- $\int 1/\sqrt{(x^2 - a^2)}\ dx = \log |x + \sqrt{(x^2 - a^2)}| + C$
- $\int 1/\sqrt{(x^2 + a^2)}\ dx = \log |x + \sqrt{(x^2 + a^2)}| + C$
- $\int 1/\sqrt{(a^2 - x^2)}\ dx = \sin^{-1}(x/a) + C$
- $\int 1/(a^2 + x^2)\ dx = (1/a) \tan^{-1}(x/a) + C$
- $\int \sqrt{x^2+a^2}dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$
- $\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Note

- Integration is an inverse process of differentiation.
- Always add the constant of integration after determining the integral of the function.
- If two functions, say f(x) and g(x) have same derivatives, then |f(x)-g(x)|= C, where C is some constant.

REDUCTION FORMULA

$\int \tan^n x\ dx$ $\int \cot^n x\ dx$, $\int \sec^n x\ dx$, $\int \text{cosec}^n x\ dx$

(i) $I_n = \int \tan^n x\ dx = \int \tan^2 x\ \tan^{n-2} x\ dx = \int (\sec^2 x - 1)\ \tan^{n-2} x\ dx$

$\Rightarrow I_n = \int \sec^2 x \tan^{n-2} x\ dx - I_{n-2}$

$\Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

(ii) $I_n = \int \cot^n x\ dx = \int \cot^2 x\ .\ \cot^{n-2} x\ dx = \int (\text{cosec}^2 x - 1)\cot^{n-2} x\ dx$

$\Rightarrow I_n = \int \text{cosec}^2 x \cot^{n-2} x\ dx - I_{n-2}$

$\Rightarrow I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

(iii) $I_n = \int \sec^n x\ dx = \int \sec^2 x\ \sec^{n-2} x\ dx$

$\Rightarrow I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2) \sec^{n-3} x. \sec x. \tan x\ dx.$

$\Rightarrow I_n = \tan x \sec^{n-2} x - (n-2)(\sec^2 x - 1) \sec^{n-2} x\ dx$

$\Rightarrow (n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}$

$\Rightarrow I_n = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

(iv) $I_n = \int \text{cosec}^n x \, dx = \int \text{cosec}^2 x \, \text{cosec}^{n-2} x \, dx$

$\Rightarrow I_n = -\cot x \, \text{cosec}^{n-2} x + \int (\cot x)(n-2)(-\text{cosec}^{n-3} x \, \text{cosec}\, x \cot x) \, dx$

$\Rightarrow -\cot x \, \text{cosec}^{n-2} x - (n-2) \int \cot^2 x \, \text{cosec}^{n-2} x \, dx$

$\Rightarrow I_n = -\cot x \, \text{cosec}^{n-2} x - (n-2) \int (\text{cosec}^2 x - 1) \, \text{cosec}^{n-2} x \, dx$

$\Rightarrow (n-1) I_n = -\cot x \, \text{cosec}^{n-2} x + (n-2) I_{n-2}$

$\Rightarrow I_n = \dfrac{\cot x \, \text{cosec}^{n-2} x}{-(n-1)} + \dfrac{n-2}{n-1} I_{n-2}$

DEFINITE INTEGRAL

A definite integral is denoted by $\int_a^b f(x)\,dx$, where ' a ' is called the lower limit of the integral and ' b ' is called the upper limit of the integral.

FUNDAMENTAL THEOREM OF CALCULUS

Fundamental theorem of calculus contains two important theorems namely:

First fundamental theorem of integral calculus

Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function. Then A′(x) = f (x), for all x ∈ [a, b].

Second fundamental theorem of integral calculus

Let f be a continuous function defined on the closed interval [a, b] and F be an antiderivative of f. Then $\int_a^b f(x)$= F(b) – F(a)

NEWTON – LEIBNITZ FORMULA

Let $\dfrac{d}{dx}(F(x)) = f(x) \ \forall\ x \in (a, (B)$. Then $\int_a^b f(x)\,dx = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)$.

Note

(i) If $a > b$, then $\int_a^b f(x)\,dx = \lim_{x \to b^+} F(x) - \lim_{x \to a^-} F(x)$.

(ii) If F(x) is continuous at a and b, then $\int_a^b f(x)\,dx = F(b) - F(a)$

PROPERTIES OF DEFINITE INTEGRATION

Property (1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ i.e. definite integral is independent of variable of integration

Example

Evaluate $\int_0^1 xe^{x^2} dx$

Solution: Let us assume that, $x^2 = t$
Then, differentiating w. r.t. x, we get,
2x dx = dt
So, substitute x = 0
t = 0
Again substitute x = 1
t = 1
Then, the given question becomes,
$\int_0^1 xe^{x^2} dx = \int_0^1 \frac{e^t}{2} dt$
$= \frac{1}{2}[e - 1]$

Property (2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Example

Prove that : $\int_0^1 x^2 dx = -\int_1^0 x^2 dx$

Solution: L.H.S $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

R.H.S $-\int_1^0 x^2 dx = -\left[\frac{x^3}{3}\right]_1^0 = \frac{1}{3}$

Thus , LHS = RHS

Property (3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b].

Example

Evaluate $\int_{-1}^2 |x^3 - x|\,dx$

Solution: We note $x^3 - x \geq 0$ on $[-1,0]$ $x^3 - x \leq 0$ on $[0,1]$ and that $x^3 - x \geq 0$ on $[1,2]$.
So by the above property we write
$\int_{-1}^2 |x^3 - x|\,dx = \int_{-1}^0 (x^3 - x)dx + \int_0^1 -(x^3 - x)dx + \int_1^2 (x^3 - x)dx$
$= \int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)dx + \int_1^2 (x^3 - x)dx$
$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{11}{4}$

Property (4) $\int_{-a}^a f(x)\,dx = \int_0^a (f(x) + f(-x))\,dx =$

$$\begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$$

Example

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx$

Solution: we observe that $\sin^2 x$ is an even function. Therefore, by above property, we get $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx = 2\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$

$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos(2x))}{2} dx \qquad = \int_0^{\frac{\pi}{4}} (1-\cos(2x))dx \qquad = \left[x - \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 \qquad = \frac{\pi}{4} - \frac{1}{2}$

Property (5) $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$. Further $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Example

Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$

Solution: Let I = $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$,Then by above property, we have

I = $\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$

I = $\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$ **=** $\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}$ – I

2I = $\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}$

I = $\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}$

Put cos x = t so that -sin x dx= dt. When x=0, t=1 and when x= π, t= -1. Therefore, by property 1 we get

I = $-\frac{\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$

I = $\pi\int_0^1 \frac{dt}{1+t^2}$ ($\frac{1}{1+t^2}$ *is even function*)

I = $\pi[\tan^{-1} 1 - \tan^{-1} 0] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$

Property (6) $\int_0^{2a} f(x)\, dx = \int_0^{a} (f(x)+f(2a-x))\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Example

Evaluate $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Solution: Let I = $\int_0^{\pi} \frac{x}{1+\sin x} dx$

I = $\int_0^{\pi} \frac{\pi-x}{1+\sin(\pi-x)} dx$

I = $\int_0^{\pi} \frac{\pi-x}{1+\sin(x)} dx$

Adding (i) and(ii) , we get

2 I = $\int_0^{\pi} \frac{x+\pi-x}{1+\sin(x)} dx$

2 I = $\pi\int_0^{\pi} \frac{1}{1+\sin x} dx$

2 I = $\pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} + \frac{1}{1+\sin(\pi-x)} dx$ $\qquad \left[using\ \int_0^{2a} f(x) = \int_0^a \{f(x) + f(2a-x)\}dx\right]$

2 I = $2\pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx$

2 I = $2\pi\int_0^{\frac{\pi}{2}} \frac{1-\sin x}{1-\sin^2 x} dx$

2 I = $2\pi\int_0^{\frac{\pi}{2}} (\sec^2 x - \tan x.\sec x)\, dx = 2\pi[\tan x - \sec x] = 2\pi\left[\frac{-\cos x}{1+\sin x}\right] = 2\pi$

I = π

Property - (7) Integration of Periodic functions:

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\, dx = n\int_0^{T} f(x)\, dx,\ n \in z$

(ii) $\int_a^{a+nT} f(x)\, dx = n\int_0^{T} f(x)\, dx,\ n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\, dx = (n-m)\int_0^{T} f(x)\, dx,\ m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_{0}^{a} f(x)\,dx$, $n \in z$, $a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_{a}^{b} f(x)\,dx$, $n \in z$, $a, b \in R$

ESTIMATION OF INTEGRALS:

Method (1) If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$$

Method (2) If $m \le f(x) \le M$ for $a \le x \le b$, then m

(b – ((A) ≤ dx ≤ $\int_a^b f(x) \le$ M (b – ((A)

Further if f(x) is monotonically decreasing in (a, b), then f(b) (b – a) < dx < $\int_a^b f(x)$ f(a) (b – a) and if f(x) is monotonically increasing in (a, b), then f(a) (b – a) < $\int_a^b f(x)$ dx < f(b) (b – a)

LEIBNITZ THEOREM:

If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\frac{dF(x)}{dx} = h'(x)\, f(h(x)) - g'(x)\, f(g(x))$

Proof : Let $P(t) = \int f(t)\,dt \Rightarrow F(x) = \int_{g(x)}^{h(x)} f(t)\,dt = P(h(x)) - P(g(x))$

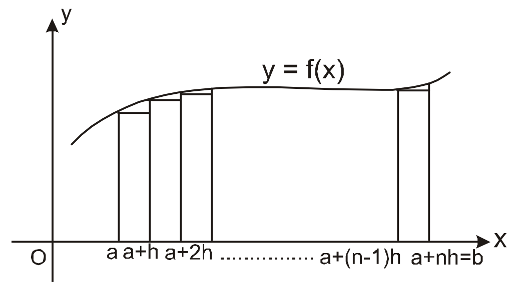
$\Rightarrow \frac{dF(x)}{dx} = P'(h(x))\, h'(x) - P'(g(x))\, g'(x) = f(h(x))\, h'(x) - f(g(x))\, g'(x)$

Example

$\frac{d}{dx}\int_{\sin x}^{x^3} \sqrt{t^2+1}\,dt = \sqrt{(x^3)^2+1}\,.3x^2 - \sqrt{\sin^2 x+1}\,.\cos x = 3x^2.\sqrt{x^6+1} - \cos x.\sqrt{\sin^2 x+1}$

DEFINITE INTEGRAL AS A LIMIT OF SUM:

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\frac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles.

Then, $S_n = hf(a) + hf(a+h) + hf(a+2h) + \ldots\ldots + hf(a+(n-1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}}\ S_n$

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\, f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)\,r}{n}\right)$$

Note

1. We can also write
   $S_n = hf(a+h) + hf(a+2h) + \ldots\ldots + hf(a+nh)$ and
   $\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right) r\right)$
2. If a = 0, b = 1, $\int_0^1 f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as definte integral:

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{\text{Lt}} \Sigma$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For

Example

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right)=\int_0^p f(x)\,dx \quad (\Theta\ \underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)\Big|_{r=1}=0,\ \underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)\Big|_{r=np}=p)$$

REDUCTION FORMULAE IN DEFINITE INTEGRALS:

(i) If $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$, then show that

$$I_n=\left(\frac{n-1}{n}\right)I_{n-2}$$

Proof: $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\ dx$

$$I_n=\left[-\sin^{n-1}x\ \cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\ \sin^{n-2}x\ .\ \cos^2 x\ dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\ .\ (1-\sin^2 x)\ dx=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\ dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\ dx$$

$$I_n+(n-1)\,I_n=(n-1)\,I_{n-2}$$

$$I_n=\left(\frac{n-1}{n}\right)I_{n-2}$$

(ii) If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\ dx$, then $I_n+I_{n-2}=\frac{1}{n-1}$

(ii) If $I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m x\ .\ \cos^n x\ dx$, then $I_{m,n}=\frac{m-1}{m+n}I_{m-2,n}$

WALLI'S FOMRULA

Let, $I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m x\ .\ \cos^n x\ dx$,

$$I_{m,n}=\begin{cases}\dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........}\ \dfrac{\pi}{2} & \text{when both m, n are even}\\[2ex] \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........} & \text{otherwise}\end{cases}$$

Example

$$\int_0^{\frac{\pi}{2}}\sin^7 x\,dx=\frac{6}{7}.\frac{4}{5}.\frac{2}{3}$$

$$=\frac{16}{35}$$

QUESTIONS

MCQ

Q1. $\int \frac{x^2+1}{x(x^2-1)}dx$ is equal to
(a) $-\ln|x|+\ln|x+1|-\ln|x-1|+C$
(b) $-\ln|x|-\ln|x+1|+\ln|x-1|+C$
(c) $-\ln|x|-\ln|x+1|-\ln|x-1|+C$
(d) $-\ln|x|+\ln|x+1|+\ln|x-1|+C$

Q2. Find $\int \frac{dx}{5-8x-x^2}$.
(a) $\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$
(b) $\frac{1}{\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$
(c) $\frac{-1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$
(d) None

Q3. Evaluate $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$
(a) $I=\frac{\pi(\pi+2)}{2}$ (b) $I=\frac{-\pi(\pi-2)}{2}$
(c) $I=\frac{-\pi(\pi+2)}{2}$ (d) $I=\frac{\pi(\pi-2)}{2}$

Q4. Evaluate : $\int_1^4\{|x-1|+|x-2|+|x-4|\}dx$
(a) $I=\frac{23}{2}$ (b) $I=\frac{-23}{2}$
(c) $I=\frac{27}{2}$ (d) $I=\frac{-27}{2}$

Q5. $\int_0^1\frac{dx}{(1+x^2)}=?$
(a) $\pi/4$ (b) $\pi/2$
(c) $\pi/3$ (d) None

Q6. $\int_0^{\pi/4}\cos^2 x dx=?$
(a) $\frac{\pi}{4}+\frac{1}{4}$ (b) $\frac{\pi}{8}+\frac{1}{4}$
(c) $\frac{\pi}{8}-\frac{1}{4}$ (d) None

Q7. $\int_0^{\pi/2}x^3\sin 3x dx=?$
(a) $\left(\frac{2}{27}-\frac{\pi^2}{40}\right)$ (b) $\left(\frac{2}{27}-\frac{\pi^2}{12}\right)$
(c) $\left(\frac{1}{27}-\frac{\pi^2}{12}\right)$ (d) None

Q8. $\int_1^e e^x\left(\frac{1+x\log x}{x}\right)dx=?$
(a) e^e (b) e^2
(c) e^3 (d) e

Q9. $\int_0^a\frac{x}{\sqrt{a^2+x^2}}dx=?$
(a) $a(\sqrt{2}+1)$ (b) $-a(\sqrt{2}-1)$
(c) $a(\sqrt{2}-1)$ (d) $a(\sqrt{2})$

Q10. Evaluate $\int\tan^2 x dx$
(a) $\tan x-x+C$ (b) $\tan x+x+C$
(c) $-\tan x-x+C$ (d) $\tan x-2x+C$

Q11. Evaluate $\int\frac{\cos x-\cos 2x}{1-\cos x}dx$
(a) $-2\sin x+x+C$ (b) $2\sin x+x+C$
(c) $2\sin x-x+C$ (d) $-2\sin x-x+C$

Q12. Evaluate $\int\frac{x^3}{x+2}dx$
(a) $\frac{x^3}{3}-x^2-4x-8\log|x+2|+C$
(b) $\frac{x^3}{3}-x^2+4x+8\log|x+2|+C$
(c) $\frac{x^3}{3}-x^2+4x-8\log|x+2|+C$
(d) $-\frac{x^3}{3}-x^2+4x-8\log|x+2|+C$

Q13. Evaluate
$\int 5^{\log_e x}dx$
(a) $-\frac{x^{\log_e 5+1}}{\log_e 5+1}+C$ (b) $\frac{x^{\log_e 5+1}}{\log_e 5-1}+C$
(c) $\frac{x^{\log_e 5-1}}{\log_e 5-1}+C$ (d) $\frac{x^{\log_e 5+1}}{\log_e 5+1}+C$

Q14. Evaluate $\int\frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}dx$
(a) $\frac{x^2}{2}-x+C$ (b) $\frac{x^2}{2}+x+C$
(c) $\frac{-x^2}{2}-x+C$ (d) $\frac{x^2}{2}-2+C$

Q15. Evaluate $\int\frac{1+2x^2}{x^2(1+x^2)}dx$
(a) $-\frac{1}{x}+\tan^{-1}x+C$ (b) $\frac{1}{x}+\tan^{-1}x+C$
(c) $x+\tan^{-1}x+C$ (d) $-x-\tan^{-1}x+C$

Q16. Evaluate $\int\left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx$
(a) $\frac{2}{3}x^{3/2}+\frac{4}{\sqrt{x}}+C$ (b) $-\frac{2}{3}x^{\frac{3}{2}}-\frac{4}{\sqrt{x}}+C$
(c) $\frac{8}{3}x^{3/2}+\frac{4}{\sqrt{x}}+C$ (d) $-\frac{2}{3}x^{3/2}+\frac{4}{\sqrt{x}}+C$

Q17. $\int\frac{1}{\sin(x-a)\cos(x-b)}dx=$
(a) $\frac{-1}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$
(b) $\frac{-2}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$
(c) $\frac{1}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$
(d) $\frac{2}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$

Q18. $\int\frac{\sin(x+a)}{\sin(x+b)}dx=$
(a) $(x+b)\cos(a-b)+\sin(a-b)\log|\sin(x+b)|+C$
(b) $(x+b)\cos(a-b)-\sin(a-b)\log|\sin(x+b)|+C$
(c) $-(x+b)\cos(a-b)+\sin(a-b)\log|\sin(x+b)|+C$
(d) $-(x+b)\cos(a-b)-\sin(a-b)\log|\sin(x+b)|+C$

Q19. $\int\frac{\sin(\log x)}{x}dx=$
(a) $\cos(\log x)+C$ (b) $-\cos(-\log x)+C$
(c) $-\sin(\log x)+C$ (d) $-\cos(\log x)+C$

Q20. $\int x\sin(4x^2+7)dx =$

(a) $\frac{1}{8}\cos(4x^2+7)+C$

(b) $-\frac{1}{8}\cos(4x^2+7)+C$

(c) $-\frac{1}{8}\cos(4x^2-7)+C$

(d) $\frac{1}{8}\cos(4x^2-7)+C$

Q21. $\int \cos 4x\cos 7x dx =$

(a) $\frac{\sin 3x}{6}-\frac{\sin 11x}{22}+C$ (b) $\frac{-\sin 3x}{6}+\frac{\sin 11x}{22}+C$

(c) $\frac{\sin 3x}{6}+\frac{\sin 11x}{22}+C$ (d) $-\frac{\sin 3x}{6}-\frac{\sin 11x}{22}+C$

Q22. $\int \sin x\cos x\cdot\cos 2x\cdot\cos 4x dx =$

(a) $\frac{\cos 8x}{64}+C$ (b) $\frac{\cos 8x}{32}+C$

(c) $\frac{-\cos 8x}{32}+C$ (d) $\frac{-\cos 8x}{64}+C$

Q23. $\int \frac{1+\cos^2 x}{1+\cos 2x}dx =$

(a) $\frac{1}{2}(\tan x+x)+C$

(b) $\frac{1}{2}(\tan x-x)+C$

(c) $\frac{-1}{2}(\tan x+x)+C$

(d) $\frac{1}{2}(-\tan x+x)+C$

Q24. $\int \frac{1-\tan^2 x}{1+\tan^2 x}dx =$

(a) $-\frac{\sin 2x}{2}+c$ (b) $\frac{\sin 2x}{4}+c$

(c) $\frac{\sin 2x}{2}+c$ (d) $\frac{\sin 2x}{-4}+c$

Q25. $\int \frac{1}{x^2-x+1}dx =$

(a) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

(b) $\frac{-2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

(c) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

(d) $\frac{-2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

Q26. $\int \frac{a^x}{\sqrt{1-a^{2x}}}dx =$

(a) $\frac{-1}{\log a}\sin^{-1}(a^x)+C$

(b) $\frac{1}{\log a}\sin^{-1}(-a^x)+C$

(c) $\frac{1}{\log a}\sin^{-1}(a^x)+C$

(d) $\frac{1}{\log a}\cos^{-1}(a^x)+C$

Q27. $\int \frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}dx =$

(a) $\log\left|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right|+C$

(b) $\log\left|(\sin x-1)+\sqrt{\sin^2 x+2\sin x-3}\right|+C$

(c) $-\log\left|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right|+C$

(d) $-\log\left|(\sin x-1)-\sqrt{\sin^2 x-2\sin x-3}\right|+C$

Q28. $\int \frac{1}{4\sin^2 x+9\cos^2 x}dx =$

(a) $\frac{-1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right)+C$

(b) $\tan^{-1}\left(\frac{2\tan x}{3}\right)+C$

(c) $\tan^{-1}\left(\frac{-2\tan x}{3}\right)+C$

(d) $\frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right)+C$

Q29. $\int \frac{dx}{2+\sin x+\cos x} =$

(a) $-\sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)+C$

(b) $\sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)+C$

(c) $\sqrt{2}\tan^{-1}\left(\frac{\cot\frac{x}{2}+1}{\sqrt{2}}\right)+C$

(d) $-\sqrt{2}\tan^{-1}\left(\frac{\cot\frac{x}{2}+1}{\sqrt{2}}\right)x+C$

Q30. $\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx =$

(a) $-e^x\tan x+C$ (b) $e^x\cot x+C$

(c) $e^x\tan x+C$ (d) $-e^x\cot x+C$

SUBJECTIVE QUESTIONS

Q1. Evaluate: $\int\left(e^{2\ell nx}+e^{a\ell nx}+e^{4\ell nx}\right)dx$, a > 0

Q2. Evaluate: $\int\frac{(1+x)^3}{\sqrt{x}}\,dx$

Q3. Evaluate $\int\frac{x^2}{(x^2+4)(x^2+1)}dx$

Q4. Evaluate $\int_0^{2\pi}\sin^{100}x\cos^{99}x\,dx$

Q5. Evaluate $\int_0^{\pi}\frac{dx}{1+2\sin^2 x}$ dx.

NUMERICAL TYPE QUESTIONS

Q1. Evaluate $\lim\limits_{x\to\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$ __________.

Q2. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos^2 x(\sin x+\cos x)\,dx$ __________.

Q3. The solution of $\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$ is $\lambda\tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\}$ + C ,then $\lambda=$ _____.

Q4. The solution of $\int \frac{dx}{\sqrt[3]{\sin^{11}x\cos x}}$ is A $(\tan x)^{\frac{-8}{3}}$ + B(tan x)$^{-2/3}$ + C ,then A + B__________.

Q5. The solution of $\int \frac{dx}{1+3\cos^2 x}$ is K. $\tan^{-1}\left(\frac{\tan x}{2}\right)$ + C ,then the value of 2k = ________.

TRUE AND FALSE

Q1. The solution of $\int \sec^2 x \operatorname{cosec}^2 x\, dx$ is tan x + cot x + C.

Q2. The solution of $\int \frac{1}{4+9x^2}$ dx is - $\frac{1}{6}$ $\tan^{-1}\left(\frac{3x}{2}\right)$ + C

Q3. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

Q4. The solution of $\int_1^2 \frac{dx}{(x+1)(x+2)}$ is $\ell n\left(\frac{9}{8}\right)$

Q5. If f(x) = $\begin{cases} x+5 & : x<2 \\ 2x^2+1 & : x\ge 2 \end{cases}$, then the value of $\int_0^4 f(x)\,dx$ is $\frac{154}{3}$

ASSERTION AND REASONING

Directions : (Q1 -5) In the following questions , A statement of Assertion (A) is followed by a statement of Reason (R).

(a) Both A and R are true but R is the correct explanation of A

(b) Both A and R are true but R is Not the correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

Q1. **Assertion (A) :** $\int_0^{\frac{\pi}{4}} cos^5\, x\, dx = 0$

Reason (R): If f(x) is an odd function , then $\int_a^b f(x)\,dx = 0$

Q2. **Assertion (A) :** $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx = 0$

Reason (R): $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Q3. **Assertion (A) :** $\int_1^e log^2 x\, dx = e - 2$

Reason (R) ; $I_n = \int_1^e log^n\, x\, dx = e - n.I_{n-1}$

Q4. **Assertion (A) :** 16 < $\int_4^6 2x\,dx < 24$

Reason (R): If m is the smallest and M is the greatest value of a function f(x) in an interval (a, b) , then the value of the integral $\int_a^b f(x)dx$ is such that for a < b ,we have m (a – b) ≤ $\int_a^b f(x)dx\ \le M\,(b-a)$.

Q5. **Assertion (A) :** $\int_0^{\pi} sin\,^7 x\,dx = 2\int_0^{\frac{\pi}{2}} sin^7 x\,dx$

Reason (R): $sin^7 x$ is an odd function.

HOMEWORK

MCQ

Q1. If f′(x) = x^2 + 5 and f(0) = –1, then f(x) =

(a) x^3 + 5x – 1 (b) x^3 + 5x + 1

(c) $\frac{1}{3}x^3$ + 5x – 1 (d) $\frac{1}{3}x^3$ + 5x + 1

Q2. $\int \frac{\cos 2x}{\cos x}$ dx is equal to

(a) 2 sin x – ln (sec x + tan x) + c

(b) 2 sin x – ln (sec x – tan x) + c

(c) 2 sin x + ln (sec x + tan x) + c

(d) None of these

Q3. $\int \frac{1+\cos^2 x}{\sin^2 x}dx$ =

(a) –cot x – 2x + c (b) –2cot x – 2x + c

(c) –2cot x – x + c (d) –2cot x + x + c

Q4. $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}$ dx is equal to

(a) $\frac{a^{\sqrt{x}}}{\sqrt{x}}$ + c (b) $\frac{2a^{\sqrt{x}}}{\ell na}$ + c

(c) $2a^{\sqrt{x}}$. λn a + c (d) none of these

Q5. $\int (x-1)e^{-x}\,dx$ is equal to

(a) $-x\,e^{x} + C$ (b) $x\,e^{x} + C$

(c) $-x\,e^{-x} + C$ (d) $x\,e^{-x} + C$

Q6. $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}\,dx =$

(a) $\frac{4}{3}\left(\sqrt{2}+1\right)$ (b) $\frac{4}{3}\left(\sqrt{2}-1\right)$

(c) $\frac{3}{4}\left(\sqrt{2}-1\right)$ (d) $\frac{3}{4}\left(\sqrt{2}-2\right)$

Q7. $\int_0^2 \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx$, equals-

(a) $\frac{2}{\ell n3}(3^{\sqrt{2}}-1)$ (b) 0

(c) $\frac{2\sqrt{2}}{\ell n3}$ (d) $\frac{3^{\sqrt{2}}}{\sqrt{2}}$

Q8. $\int_0^{\pi/2} \sqrt{1+\sin 2x}\,dx$ equals -

(a) 1/2 (b) 1

(c) 2 (d) $\frac{3}{2}$

Q9. $\int_1^2 e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$ equals

(a) $e\left(\frac{e}{2}-1\right)$ (b) 1

(c) e (e – 1) (d) $\frac{e}{2}$

Q10. If $f(x) = \begin{cases} x & x<1 \\ x-1 & x\geq 1\end{cases}$, then $\int_0^2 x^2 f(x)\,dx$ is equal to :

(a) 1 (b) $\frac{4}{3}$

(c) $\frac{5}{3}$ (d) $\frac{5}{2}$

SUBJECTIVE QUESTIONS

Q1. Evaluate $\int_2^8 |x-5|\,dx.$

Q2. Show that $\int_0^2 (2x+1)\,dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Q3. Evaluate $\int_0^1 \frac{e^{-x}dx}{1+e^x}\,dx$

Q4. Evaluate: $\frac{1-x^2}{1+x^2+x^4}\,dx$

Q5. Evaluate: $\int \frac{x+1}{(x-1)\sqrt{x+2}}\,dx$

NUMERICAL TYPE QUESTIONS

Q1. The solution of $\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$ is $A\sqrt{\frac{x-1}{x+1}} + C$ then $\frac{A}{2}$ ______.

Q2. The solution of $\int \frac{3\sin x + 2\cos x}{4\cos x + 5\sin x}\,dx$ is $\frac{23}{41}x$ - $A\ln|4\cos x + 5\sin x| + C$ then A = _____.

Q3. The solution of $\int \sin^5 x \cos^4 x\,dx$ is $A\frac{\cos^9 x}{9}$ + B $\frac{\cos^7 x}{7}$ + C $\frac{\cos^5 x}{5}$ + D ,then A + B + C ________.

Q4. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\;\;dx$ __________.

Q5. Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right)dx$ _________-.

TRUE AND FALSE

Q1. $\int_0^{\frac{\pi}{2}} \frac{g\;(\sin x)}{g\;(\sin x)+g\;(\cos x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{g\;(\cos x)}{g\;(\sin x)+g\;(\cos x)}\,dx = \frac{\pi}{4}$.

Q2. The solution of $\int \frac{1}{\sqrt{33+8x-x^2}}\,dx$ is $\sin^{-1}\left(\frac{x-4}{7}\right)$ + C

Q3. The solution of $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$ is $2\sqrt{x^2+4x+1}$ – ln $|x+2+\sqrt{x^2+4x+1}|$+C

Q4. The solution of $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ is $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)$ + C

Q5. The solution of $\int\sqrt{x^2+2x+5}\,dx$ is $\frac{1}{2}$ (x + 1) $\sqrt{x^2+2x+5}$ + 2 ln |(x + 1) $+\sqrt{x^2+2x+5}$ | + C

ASSERTION AND REASONING

Directions : (Q1 -5) In the following questions , A statement of Assertion (A) is followed by a statement of Reason (R).

(a) Both A and R are true but R is the correct explanation of A

(b) Both A and R are true but R is Not the correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

Q1. Assertion (A) : $\int \frac{xe^x}{(1+x)^2}\,dx = \frac{e^x}{x+1} + C$

Reason (R): $\int e^x(f(x) + f'(x))dx = f(x)e^x + C$

Q2. Assertion (A) : $\int e^{x\log a} . e^x\,dx = \frac{(ae)^x}{\log(ae)} + C$

Reason (R): $\int a^x dx = \frac{a^x}{\log_e a} + C$

Q3. Assertion (A) : $\int \frac{e^x}{x}(1 + x\log x\,)dx = e^x \log x + C$

Reason (R) ; $\int e^x[f(x) + f'(x)]dx = e^x f'(x) + C$

Q4. Assertion (A) : Let us define f'(x) = $\frac{1}{-x+\sqrt{x^2+1}}$ and f(0) = - $\left(\frac{1+\sqrt{2}}{2}\right)$, then f(1) is equal to log |1 + $\sqrt{2}$ |.

Reason (R): f(x) is not defined for every value of x.

Q5. Assertion (A) : $\int \frac{e^x}{x}(1 + xlog\,x)dx = e^x \log x + C$

Reason (R): $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$

SOLUTIONS

MCQ

S1. (d):

$\int \frac{x^2+1}{x(x^2-1)} dx$

$\int \frac{x^2+1}{x^3-x} dx$

$\int \frac{x^2+1}{x(x-1)(x+1)} dx$

$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$x^2 + 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

When x=1

$2 = A(0) + B(2) + 0(C)$

$B = 1$

When x = -1

$2 = A(0) + B(0) + C(-1)(-2)$

$2 = 2C$

$C = 1$

When x=0

$1 = A(-1) + B(0) + C(0)$

$A = -1$

Hence, $\int \frac{-1}{x} dx + \int \frac{dx}{x+1} + \int \frac{dx}{x-1} = -\ln|x| + \ln|x+1| + \ln|x-1| + C$

S2. (a) $\int \frac{dx}{5-8x-x^2} \& = \int \frac{dx}{5-8x-x^2+(4)^2-(4)^2}$

$= \int \frac{dx}{21-[(4)^2+8x+(x)^2]}$

$= \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$

$= \frac{1}{2\sqrt{21}} \log \left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$

S3. (d) $I = \int_0^\pi \frac{x\tan x}{\sec x+\tan x} dx$...(i)

Now,

$I = \int_0^\pi \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}$

$I = \int_0^\pi \frac{(\pi-x)\tan x}{\sec x+\tan x} dx$ ... (ii)

Adding (i) and (ii), we get

$2I\& = \int_0^\pi \frac{x\tan x}{\sec x+\tan x} dx + \int_0^\pi \frac{(\pi-x)\tan x}{\sec x+\tan x} dx$

$\Rightarrow 2I\& = \pi \int_0^\pi \frac{\tan x}{\sec x+\tan x} dx$

$\Rightarrow I\& = \frac{\pi}{2} \int_0^\pi \frac{\tan x}{\sec x+\tan x} \times \frac{\sec x-\tan x}{\sec x-\tan x} dx$

$\Rightarrow I\& = \frac{\pi}{2} \int_0^\pi \frac{(\sec x\tan x-\tan^2 x)}{\sec^2 x-\tan^2 x} dx$

$\Rightarrow I\& = \frac{\pi}{2} \int_0^\pi (\sec x\tan x - \sec^2 x + 1) dx$

$\Rightarrow I\& = \frac{\pi}{2} [\sec x - \tan x + x]_0^\pi$

$\Rightarrow I = \frac{\pi}{2}[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0]$

$\Rightarrow I = \frac{\pi}{2}[(-1+\pi) - (1)]$

$\therefore I = \frac{\pi(\pi-2)}{2}$

S4. (a) Let $I\& = \int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$

$\Rightarrow I\& = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-4| dx$

$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^2 |x-2| dx + \int_2^4 |x-2| dx + \int_1^4 |x-4| dx$

$\Rightarrow I = \int_1^4 (x-1) dx - \int_1^2 (x-2) dx + \int_2^4 (x-2) dx - \int_1^4 (x-4) dx$

$\Rightarrow I = \frac{1}{2}[(x-1)^2]_1^4 - \frac{1}{2}[(x-2)^2]_1^2 + \frac{1}{2}[(x-2)^2]_2^4 - \frac{1}{2}[(x-4)^2]_1^4$

$= \frac{1}{2}(9-0) - \frac{1}{2}(0-1) + \frac{1}{2}(4-0) - \frac{1}{2}(0-9)$

$\Rightarrow I = \frac{9}{2} + \frac{1}{2} + \frac{4}{2} + \frac{9}{2}$

$\therefore I = \frac{23}{2}$

S5. (a):

$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]$

$= [\tan^{-1} 1 - \tan^{-1} 0]$

$= \pi/4$

S6. (b):

$\int_0^{\frac{\pi}{4}} \cos^2 x dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 2x) dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\sin\left(\frac{\pi}{2}\right)}{2} - 0 - \frac{\sin 0}{2}\right]$

$= \frac{\pi}{8} + \frac{1}{4}$

S7. (b):

$\int_0^{\frac{\pi}{2}} x^3 \sin(3x) dx = -\frac{x^3\cos(3x)}{3} - \int -x^2\cos(3x) dx$

$= -\frac{x^3\cos(3x)}{3} + \frac{x^2\sin(3x)}{3} - \int \frac{2x\sin(3x)}{3} dx$

$= -\frac{x^3\cos(3x)}{3} + \frac{x^2\sin(3x)}{3} + \frac{2x\cos(3x)}{9} + \frac{2}{3}\int -\frac{\cos(3x)}{3} dx$

$= -\frac{x^3\cos(3x)}{3} + \frac{x^2\sin(3x)}{3} + \frac{2x\cos(3x)}{9} - \frac{2\sin(3x)}{27}$

$= -0 + \frac{\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{3\pi}{2}\right)}{3} + 0 - \frac{2\sin\left(\frac{3\pi}{2}\right)}{27} + 0 - 0 - 0 + 0$

$= \left(\frac{2}{27} - \frac{\pi^2}{12}\right)$

S8. (a): $\int_1^e e^x \left(\frac{(1+x\log(x))}{x}\right) dx = \int_1^e e^x \left(\frac{1}{x} + \log(x)\right) dx$

$= \log(x) e^x$

$= e^e . \log e$

$\& = e^e$

S9. (c)

$I = \int_0^a \frac{x}{\sqrt{a^2+x^2}} dx$

Let $a^2 + x^2 = t^2$

$\Rightarrow x dx = t dt$

Also, when $x = 0, t = a$
and when $x = a, t = \sqrt{2}a$
Hence,
$I = \int_a^{\sqrt{2}a} \frac{t}{\sqrt{t^2}} dt$
$= [t]$
$= a(\sqrt{2} - 1)$

S10. (a) $I = \int \tan^2 xdx \Rightarrow I = \int (\sec^2 x - 1)dx$
$\Rightarrow I = \int \sec^2 xdx - \int 1dx$ [using $\int \sec^2 xdx = \tan x + C$]
$\Rightarrow I = \tan x - x + C$

S11. (b) $I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$ [using $\cos 2x = 2\cos^2 x - 1$.]
$I\& = \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x} dx$
$= \int \frac{-2\cos^2 x + \cos x + 1}{1 - \cos x} dx$
$\Rightarrow I\& = \int \frac{-(2\cos x + 1)(\cos x - 1)}{-(\cos x - 1)} dx$
$\Rightarrow I\& = \int (2\cos x + 1)dx$
$I\& = 2\sin x + x + C$

S12. (c) $I = \int \frac{x^3}{x+2} dx = \int \frac{x^3 + 8 - 8}{x+2} dx$
$\Rightarrow I = \int \left(\frac{(x^3 + 2^3)}{x+2} - \frac{8}{x+2}\right) dx$
$I = \int \left(\frac{(x+2)(x^2 - 2x + 4)}{x+2} - \frac{8}{x+2}\right) dx$
$\Rightarrow I = \int \left(x^2 - 2x + 4 - \frac{8}{x+2}\right) dx$
$\therefore I = \frac{x^3}{3} - x^2 + 4x - 8\log |x + 2| + C$

S13. (d)
$I = \int 5^{\log_e x} dx = \int x^{\log_e 5} dx$ [Using $a^{\log_c b} = b^{\log_c a}$]
$= \frac{x^{(\log_e 5)+1}}{(\log_e 5+1)} + C$
$\therefore \int 5^{\log_e x} dx = \frac{x^{(\log_e 5)+1}}{\log_e 5+1} + C$

S14. (a)
Here,$I = \int \frac{(\sqrt{x}+1)\cdot\sqrt{x}(x^{3/2}-1)}{\sqrt{x}(x+\sqrt{x}+1)} dx$
$\therefore I = \int \frac{(\sqrt{x}+1)[(\sqrt{x})^3 - 1^3]}{(x+\sqrt{x}+1)} dx$
$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+\sqrt{x}+1)}{(x+\sqrt{x}+1)} dx$
[Using, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]
$= \int (x - 1)dx = \frac{x^2}{2} - x + C$

S15. (a)
Here, $I = \int \frac{1+2x^2}{x^2(1+x^2)} dx = \int \frac{1+x^2+x^2}{x^2(1+x^2)} dx$
$= \int \frac{1+x^2}{x^2(1+x^2)} dx + \int \frac{x^2}{x^2(1+x^2)} dx$
$= \int \frac{1}{x^2} dx + \int \frac{1}{1+x^2} dx = -\frac{1}{x} + \tan^{-1} x + C$

S16. (d)
Here, $I = \int \left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}} - \frac{2}{x^{3/2}} + \frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right) dx$
$= \int \left(\frac{(1-x^{-2})+(x^{-2}-x)}{x^{1/2}-x^{-1/2}} - \frac{2}{x^{3/2}}\right) dx$
$= \int \left(\frac{1-x}{\sqrt{x}-\frac{1}{\sqrt{x}}} - \frac{2}{x^{3/2}}\right) dx = \int \left(\frac{1-x}{\frac{x-1}{\sqrt{x}}} - \frac{2}{x^{3/2}}\right) dx$
$= \int (-\sqrt{x} - 2x^{-3/2})dx$
$= \left(-\frac{x^{3/2}}{3/2} - 2\cdot\frac{x^{-1/2}}{-1/2}\right) + C = -\frac{2}{3}x^{3/2} + \frac{4}{\sqrt{x}} + C$

S17. (c)

$I = \int \frac{1}{\sin (x-a)\cos (x-b)} dx$
$I = \frac{\cos (a-b)}{\cos (a-b)} \cdot \int \frac{dx}{\sin (x-a)\cos (x-b)}$
$= \frac{1}{\cos (a-b)} \cdot \int \frac{\cos ((x-b)-(x-a))}{\sin (x-a)\cos (x-b)} dx$
$= \frac{1}{\cos (a-b)} \cdot \int \left\{\frac{\cos(x-b).\cos(x-a)+\sin(x-b)\sin (x-a)}{\sin(x-a).\cos (x-b)}\right\}$
$= \frac{1}{\cos (a-b)} \int [\cot (x - a) + \tan (x - b)] dx$
$= \frac{1}{\cos (a-b)} \{\log |\sin (x - a)| - \log |\cos (x - b)|] + C$
$= \frac{1}{\cos(a-b)} \log_e \frac{\sin(x-a)}{\cos(x-b)} + C$

S18. (a) Let $I = \int \frac{\sin (x+a)}{\sin (x+b)} dx$. Put $x + b = t \Rightarrow dx = dt$
$\therefore I = \int \frac{\sin (t-b+a)}{\sin t} dt$
$= \int \left\{\frac{\sin t\cos (a-b)}{\sin t} + \frac{\cos t\sin (a-b)}{\sin t}\right] dt$
$= \cos (a - b)\int 1dt + \sin (a - b)\int \cot (t)dt$
$= t\cos (a - b) + \sin (a - b)\log |\sin t| + C$

$= (x + b)\cos (a - b) + \sin (a - b)\log |\sin (x + b)| + C$

S19. $I = \int \frac{\sin (\log x)}{x} dx$
We know that, $\frac{d}{dx}(\log x) = \frac{1}{x}$
Thus, let $\log x = t$
$\Rightarrow \frac{1}{x} dx = dt$
$\therefore I = \int \sin (t)dt = -\cos (t) + C$
$= -\cos (\log x) + C$

S20. (b)
$I = \int x\sin (4x^2 + 7)dx$
Let $4x^2 + 7 = t \Rightarrow 8xdx = dt \Rightarrow xdx = \frac{1}{8} dt$
$\therefore I = \int \sin (t) \frac{dt}{8} = -\frac{1}{8}\cos (t) + C$
$= -\frac{1}{8}\cos (4x^2 + 7) + C$

S21. (c) $\int \cos 4x\cos 7xdx$
Here; $\cos 4x\cos 7x = \frac{1}{2}(\cos 3x + \cos 11x)$
[using $\cos mx \cdot \cos nx = \frac{1}{2}\{\cos (m - n)x + \cos (m + n)x]$
$I = \int \cos 4x\cos 7xdx = \frac{1}{2}\int (\cos 3x + \cos 11x)dx$
$= \frac{1}{2}\int \cos 3xdx + \frac{1}{2}\cos\int 11xdx$
$= \frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$

S22. (d) $I = \int \sin x \cos x \cdot \cos 2x \cos 4x dx$

$= \frac{1}{2}\int 2\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x dx$

Here, $= \frac{1}{2\times 2}\int 2\sin 2x \cdot \cos 2x \cdot \cos 4x \cdot dx$

$= \frac{1}{4}\int \sin 4x \cdot \cos 4x dx = \frac{1}{2\times 4}\int 2\sin 4x \cos 4x dx$

$= \frac{1}{8}\int \sin 8x dx = \frac{-\cos 8x}{64} + C$

S23. (a)

Here, $I = \int \frac{1+\cos^2 x}{1+\cos 2x} dx$

$= \int \frac{1+\cos^2 x}{1+2\cos^2 x - 1} dx = \int \frac{1+\cos^2 x}{2\cos^2 x} dx$

[Using, $\cos 2x = 2\cos^2 x - 1$]

$= \frac{1}{2}\int (\sec^2 x + 1) dx = \frac{1}{2}(\tan x + x) + C$

S24. (c)

$I = \int \frac{1-\tan^2 x}{1+\tan^2 x} dx$

$\therefore I = \int \cos 2x dx$ Using, $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$.

$= \frac{\sin 2x}{2} + c$

S25. (a) $I = \int \frac{dx}{x^2-x+1}$

$= \int \frac{dx}{x^2-x+1/4-1/4+1} = \int \frac{dx}{(x-1/2)^2+3/4}$

$I = \int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$

$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C$

$I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$\left[\text{using } \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$

S26. (c) Here, $I = \int \frac{a^x}{\sqrt{1-a^{2x}}} dx$. Let, $a^x = t$

$\therefore a^x \log a dx = dt, a^x dx = \frac{dt}{\log a}$

$\therefore I = \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{dt}{\log a} = \frac{1}{\log a} \cdot \sin^{-1}(t) + C$

$I = \frac{1}{\log a}\sin^{-1}(a^x) + C$

S27. (a) $I = \int \frac{\cos x dx}{\sqrt{\sin^2 x - 2\sin x - 3}}$

Put $\sin x = t \therefore \cos x dx = dt$

$\Rightarrow I\& = \int \frac{dt}{\sqrt{t^2-2t-3}} = \int \frac{dt}{\sqrt{t^2-2t+1-1-3}}$

$I\& = \int \frac{dt}{\sqrt{(t-1)^2-(2)^2}}$

$= \log\left|(t-1) + \sqrt{(t-1)^2-(2)^2}\right| + C$

$\therefore I\& = \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}\right| + C$

S28. (d) $I = \int \frac{dx}{4\sin^2 x + 9\cos^2 x}$

Here, dividing numerator and denominator by $\cos^2 x$.

$I = \int \frac{\sec^2 x}{4\tan^2 x + 9} dx$

Put $\tan x\& = t$

$\Rightarrow \sec^2 x dx\& = dt$

$\therefore I = \int \frac{dt}{4t^2+9} = \frac{1}{4}\int \frac{dt}{t^2+(3/2)^2}$

$= \frac{1}{4} \cdot \frac{1}{3/2}\tan^{-1}\left(\frac{t}{3/2}\right) + C$

$I\& = \frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right) + C$

S29. (b) Let $I = \int \frac{dx}{2+\sin x+\cos x}$

$= \int \frac{dx}{2+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$= \int \frac{\sec^2 \frac{x}{2} dx}{2+2\tan^2 \frac{x}{2}+2\tan \frac{x}{2}+1-\tan^2 \frac{x}{2}}$

$I = \int \frac{\sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2}+2\tan \frac{x}{2}+3}$

Put $\tan \frac{x}{2} = t$

$\Rightarrow \frac{1}{2}\sec^2 \frac{x}{2} dx = dt = \int \frac{2dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$

$= 2\int \frac{dt}{(t+1)^2+(\sqrt{2})^2}$

$= 2 \cdot \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + C$

$I = \sqrt{2}\tan^{-1}\left(\frac{\tan \frac{x}{2}+1}{\sqrt{2}}\right) + C$

S30. (c)

$I = \int e^x \left(\frac{1+\sin x \cos x}{\cos^2 x}\right) dx$

$I = \int e^x \left\{\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}\right\} dx$

$I = \int e^x\{\tan x + \sec^2 x\} dx$

$I = \int e^x \cdot \tan x dx + \int e^x(\sec^2 x) dx$

$I = \tan x \cdot e^x - \int \sec^2 x \cdot e^x dx + \int e^x \cdot \sec^2 x dx + C$

$I = e^x \tan x + C$

SUBJECTIVE QUESTIONS

S1. $\int \left(e^{2\ell nx} + e^{a\ell nx} + e^{4\ell nx}\right) dx$

$= \int \left(e^{\ell nx^2} + e^{\ell nx^a} + e^{\ell nx^4}\right) dx$

$= \int (x^2 + x^a + x^4) dx = \frac{x^3}{3} + \frac{x^{a+1}}{a+1} + \frac{x^5}{5} + c$

S2. $\int \frac{(1+x)^3}{\sqrt{x}} dx = \int \frac{1+3x+3x^2+x^3}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} + 3$

$\int x^{\frac{1}{2}} dx + 3\int x^{\frac{3}{2}} dx + \int x^{\frac{5}{2}} dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} +$ C=

$2\sqrt{x} + 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + C$

S3. $\int \frac{x^2}{(x^2+4)(x^2+1)}dx = \frac{1}{3}\int\left[\frac{4}{x^2+4}-\frac{1}{x^2+1}\right]dx$

$=\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}x + C = \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)$

$-\frac{1}{3}\tan^{-1}x + C$

S4. $I = \int_0^{2\pi} \sin^{100}x\cos^{99}x\,dx$

here, $f(x) = \sin^{100}x\cos^{99}x$ for which $f(2\pi - x) = f(x)$

$I = 2\int_0^{\pi} \sin^{100}(\pi - x)\cos^{99}(\pi - x)\,dx$

$I = -2\int_0^{\pi} \sin^{100}x\cos^{99}x\,dx$

$-I = 2I\int_0^{\pi}\sin^{100}$

$\therefore 3I = 0$

$\therefore I = 0$

S5. Let $f(x) = \frac{1}{1+2\sin^2 x} \Rightarrow \quad f(\pi - x) = f(x)$

$\Rightarrow \int_0^{\pi}\frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}}\frac{dx}{1+2\sin^2 x} = 2$

$\int_0^{\frac{\pi}{2}}\frac{\sec^2 x\,dx}{1+\tan^2 x+2\tan^2 x} = 2\int_0^{\frac{\pi}{2}}\frac{\sec^2 x\,dx}{1+3\tan^2 x}$

$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\frac{\pi}{2}}$ Since, $\tan\frac{\pi}{2}$ is undefined, we take limit

$=\frac{2}{\sqrt{3}}\left[\lim_{x\to\frac{\pi}{2}}\tan^{-1}\left(\sqrt{3}\tan x\right)-\tan^{-1}\left(\sqrt{3}\tan 0\right)\right] = \frac{2}{\sqrt{3}}$

$\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$

NUMERICAL TYPE QUESTIONS

S1. (0)

$\lim_{x\to\infty}\frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$

Applying L′ Hospital rule

$= \lim_{x\to\infty}\frac{2\,.\int_0^x e^{t^2}dt\,.\,e^{x^2}}{1\,.\,e^{2x^2}} = \lim_{x\to\infty}\frac{2\,.\int_0^x e^{t^2}dt}{e^{x^2}} = \lim_{x\to\infty}$

$\frac{2\,.\,e^{x^2}}{2x\,.\,e^{x^2}} = 0$

2.$\left(\frac{4}{15}\right)$ Given integral $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3 x\cos^2 x\,dx +$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos^3 x\,dx \quad = 0 + 2\int_0^{\frac{\pi}{2}}\sin^2 x\cos^3 x\,dx$

(Since, $\sin^3x\cos^2x$ is odd and $\sin^2x\cos^3x$ is even)

$= 2.\frac{1.2}{5.3.1}$

$= \frac{4}{15}$

S3. $\left(\frac{2}{\sqrt{3}}\right)$ Let $I = \int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$

Putting $x + 1 = t^2$, and $dx = 2t\,dt$, we get $I =$

$\int\frac{(t^2+1)\,2t\,dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$

$\Rightarrow \quad 2\int\frac{(t^2+1)}{t^4+t^2+1}dt = 2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}dt$

$\left\{\text{put } t-\frac{1}{t} = u\right\}$

$= 2\int\frac{du}{u^2+\left(\sqrt{3}\right)^{2'}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}$

$\tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\} + C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\}$

$+ C$

Then $\lambda = \frac{2}{\sqrt{3}}$

S4. $\left(\frac{-15}{8}\right)$ $I = \int\frac{dx}{\sin^{\frac{11}{3}}x\cos^{\frac{1}{3}}x}$

Divide and multiply by $\cos^{11/3}x = \int\frac{dx}{\tan^{\frac{11}{3}}x\cos^4 x}$

$= \int\frac{(1+\tan^2 x)\sec^2 x}{\tan^{\frac{11}{3}}x\,dx}$

$= \int \frac{(1+t^2)}{t^{\frac{11}{3}}} dt$ [put tan x = t]

$= -\frac{3}{8} t^{-8/3} - \frac{3}{2} t^{-2/3} + C$ (where t = tan x)

$= \frac{-3}{8}(\tan x)^{\frac{-8}{3}} - \frac{3}{2}(\tan x)^{\frac{-2}{3}} + C$

On comparing with $A(\tan x)^{\frac{-8}{3}} + B(\tan x)^{-2/3} + C$

$A = \frac{-3}{8}$ and $B = -\frac{3}{2}$, then $A + B = \frac{-3}{8} - \frac{3}{2} = \frac{-3-12}{8} = \frac{-15}{8}$

S5. (1) Multiply Nr. & Dr. of given integral by $\sec^2 x$, we get

$I = \int \frac{\sec^2 x \, dx}{\tan^2 x + 4} = \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$

On comparing with $K.\tan^{-1}\left(\frac{\tan x}{2}\right) + C$,

then $k = \frac{1}{2}$

$\therefore 2k = 1$

TRUE AND FALSE

S1. (False) $I = \int \sec^2 x \operatorname{cosec}^2 x = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} = \int\left(\sec^2 x + \operatorname{cosec}^2 x\right) dx = \tan x - \cot x + C$

S2. (False) We have

$\int \frac{1}{4+9x^2} dx = \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2} dx = \frac{1}{9}\int \frac{1}{(2/3)^2 + x^2} dx$

$= \frac{1}{9}\cdot\frac{1}{(2/3)}\tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$

S3. (True) By the property of integration

$\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

S4. (True) Since $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$\int_1^2 \frac{dx}{(x+1)(x+2)} = [\ell n(x+1) - \ell n(x+2)]_1^2$

$= \ln 3 - \ln 4 - \ln 2 + \ln 3 = \ell n\left(\frac{9}{8}\right)$

S5. (True)

$\int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_0^2 (x+5)\,dx + \int_2^4 (2x^2+1)\,dx = \left[\frac{x^2}{2}+5x\right]_0^2 + \left[\frac{2x^3}{3}+x\right]_2^4$

$= (2+10) + \left(\frac{128}{3}+4\right) - \left(\frac{16}{3}+2\right) = 12 + \frac{112}{3} + 2 = 14 + \frac{112}{3} = \frac{154}{3}$

ASSERTION AND REASONING

S1. (d)

.

Let $f(x) = \cos^5 x$

$\therefore$ f(-x) $= \cos^5(-x) = \cos^5 x = f(x)$

$\therefore$ f(x) is an even function

$\therefore \int_0^{\frac{\pi}{4}} \cos^5 x\,dx \neq 0$

Thus ,A is false but R is true.

S2. (a)

.

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x} dx$ ....(i)

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx$

$I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$ ....(ii)

On adding equation (i) and (ii) ,we get

$2I = \int_0^{\frac{\pi}{2}} 0\,dx = 0$

$\Rightarrow I = 0$

Thus , both A and R are individually true and R is the correct explanation of A.

S3. (a) (A) $I = \int_1^e \log^2 x dx$

Integrating by parts , we get

$I = \int_1^e (\log x)^2 dx = [x(\log x)^2]_1^e - \int_1^e x \times \frac{2\log x}{x} dx$

$I = e(\log e)^2 - \log 1 - 2\int_1^e \log x\,dx$

Again , integrating by parts ,we get

$= e - 0 - 2\left\{[x\log x]_1^e - \int_1^e x \times \frac{1}{x} dx\right\}$

$= e - 2\left(e\log e - \log 1 - \int_1^e dx\right)$

$= e - 2e + 2(e-1) = e - 2$

(R) $I_n = \int_1^e \log^n x dx$

Integrating by parts , we get

$= [x(\log x)^n]_1^e - \int_1^e x \times n\frac{(\log x)^{n-1}}{x} dx$

$= e(\log e)^n - (\log 1) - n\int_1^e (\log x)^{n-1}dx$

$= e - 0 - nI_{n-1}$

$\therefore I_{n-1} = e - n\,I_{n-1}$

Hence ,A and R are both true and R is the correct explanation of A.

S4. (a) (A)Let $I = \int_2^6 2x dx = \left[2.\frac{x^2}{2}\right]_4^6 = 36 - 16 = 20$

$\Rightarrow 16 < I < 24$

(R) $f(x) = 2x, [4, 6]$

Smallest value = $2 \times 4 = 8 = m$

Greatest value = $2 \times 6 = 12 = M$

$\Rightarrow 8(6-4) < \int_4^6 2x\,dx < 12(6-4)$

$\Rightarrow 16 < \int_4^6 2x\,dx < 24$

Hence , A and R are both true and R is the correct explanation of A.

S5. (b) Since , $f(\pi - x) = \sin^7(\pi - x) = \sin^7 x = f(x)$

$\therefore \int_0^{\pi} \sin^7 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$

Also , $\sin^7 x$ is an odd function (since f(-x) = - f(x))

Hence , both A and R are individually true but R is not the correct explanation of A.

HOMEWORK

MCQ

S1. (c) If $f'(x) = x^2 + 5$ and $f(0) = -1$, then $f(x) =$

$\Rightarrow \int f'(x)dx = \int (x^2+5)dx$

$\Rightarrow f(x) = \frac{x^3}{3} + 5x + C$

$\Rightarrow f(0) = -1$ (given)

$\Rightarrow -1 = C$

$\Rightarrow f(x) = \frac{x^3}{3} + 5x - 1$

S2. (c) $\int \frac{\cos 2x}{\cos x} dx$ is equal to

$\Rightarrow \int \frac{2\cos^2 x - 1}{\cos x} dx = \int 2\cos x\,dx - \int \sec x\,dx$

$\Rightarrow 2\sin x - \log|\sec x + \tan x| + C$

S3. $\int \frac{1+\cos^2 x}{\sin^2 x} dx =$

$\Rightarrow \int \mathrm{cosec}^2 x\,dx + \int \cot^2 x\,dx$

$\Rightarrow -\cot x + \int \mathrm{cosec}^2 x\,dx - \int dx$

$\Rightarrow -\cot x - \cot x - x + C$

$\Rightarrow -2\cot x - x + C$

S4. (b) $\int \frac{a^{\sqrt{x}}}{\sqrt{x}} dx$ is equal to

Let $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\frac{dx}{\sqrt{x}} = 2\,dt$

$\Rightarrow \int 2a^t dt = 2\frac{a^t}{\ln a} + C$

$= 2\frac{a^{\sqrt{x}}}{\ln a} + C$

S5. (c) $\int (x-1)e^{-x} dx$ is equal to

$\Rightarrow \int x.e^{-x}dx - \int e^{-x}dx$

$\Rightarrow x\int e^{-x}dx - \int\left[\frac{d}{dx}x\int e^{-x}dx\right]dx + e^{-x} + C$

$\Rightarrow -xe^{-x} + \int e^{-x}dx + e^{-x} + C$

$\Rightarrow -x\,e^{-x} + C$

S6. (b) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}} dx =$

$\Rightarrow \int_0^1 \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}dx = \int_0^1 (\sqrt{x+1} - \sqrt{x})dx$

$\Rightarrow \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \left[\frac{2}{3} \times (2)^{\frac{3}{2}} - \frac{2}{3} - \frac{2}{3}\right] = \frac{4}{3}(\sqrt{2} - 1)$

S7. (a) $\int_0^2 \frac{3^{\sqrt{x}}}{\sqrt{x}}$ dx, equals-

Let $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} dx = 2dt$

$\Rightarrow 2\int 3^t dt$

$\Rightarrow 2 \times \frac{3^t}{\ln 3}$

$\Rightarrow 2\left[\frac{3^{\sqrt{x}}}{\ln 3}\right]_0^2 = 2\left(\frac{3^{\sqrt{2}}}{\ln 3} - \frac{1}{\ln 3}\right)$

S8. (c) $\int_0^{\pi/2} \sqrt{1+\sin 2x}$ dx equals –

$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{(\sin^2 x + \cos^2 x + 2\sin x\cos x)}\,dx$

$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{(\sin x + \cos x)^2}\,dx$

$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin x + \cos x)\,dx$

$\Rightarrow [-\cos x + \sin x]_0^{\frac{\pi}{2}} = \left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos 0 - \sin 0\right] = 2$

S9. (a) $\int_1^2 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$ dx equals2

$\Rightarrow \frac{1}{x}\int e^x - \int_1^2\left[\frac{d}{dx}\left(\frac{1}{x}\right)\int e^x dx\right]dx - \int_1^2 \frac{e^x}{x^2}$

$\Rightarrow \left[\frac{e^x}{x}\right]_1^2 = \left[\frac{e^2}{2} - e\right]$

S10. (c) If $f(x) = \begin{cases} x & x < 1 \\ x-1 & x \geq 1 \end{cases}$, then $\int_0^2 x^2 f(x)$ dx is equal to :

$\Rightarrow \int_0^1 x^3 dx + \int_1^2 x^2(x-1)dx$

$\Rightarrow \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = \left(\frac{1}{4}\right) + \left(\frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}\right) = \frac{5}{3}$

SUBJECTIVE QUESTIONS

S1. $\int_2^8 |x-5|\ dx = \int_2^5 (-x+5)\ dx + \int_5^8 (x-5)\ dx = 9$

S2. L.H.S. = $x^2 + x\ ]^2_0 = 4 + 2 = 6$

R.H.S. = 25 + 5 − 0 + (4 + 2) − (25 + 5) = 6

∴ L.H.S. = R.H.S

S3. $I = \int_0^1 \frac{e^{-x}dx}{1+e^x} dx = \int_0^1 \frac{dx}{e^x(1+e^x)}$ Put e^x = t ∴ $e^x dx = dt$

$\int_0^1 \frac{dy}{t^2(t+1)} = \int_1^e \left(\frac{1}{1+t} - \frac{t-1}{t^2}\right)dt$ = $\left|\log(1+t) - \log t - \frac{1}{t}\right|_1^e$

= (log(1 + e) − log e − $\frac{1}{e}$) −(log 2 − log 1 − 1)

log(1 + e) − $\frac{1}{2}$ − log 2

S4. Let $I = \frac{1-x^2}{1+x^2+x^4}\ dx = -\frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$

{put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$}

$\therefore I = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\left|\frac{t-1}{t+1}\right|\ln + C = -\frac{1}{2}\ln \left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

Let $I = \frac{1-x^2}{1+x^2+x^4}\ dx = -\frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$

{put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$}

$\therefore I = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = -\frac{1}{2}\ln \left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

S5. $I = \int \frac{x+1}{(x-1)\sqrt{x+2}} dx$

Let $x + 2 = t^2$ or $dx = 2t\ dt$

$I = \int \frac{t^2-1}{(t^2-3)t} \times 2t\ dt$

$2\int \frac{t^2-3+2}{(t^2-3)} = 2\int\left(1+\frac{2}{t^2-3}\right)dt$

$= 2t + \frac{2}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C$

$= 2\sqrt{x+2} + \frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + C$

NUMERICAL TYPE QUESTIONS

S1. $(\frac{1}{2})$ Let $x + 1 = \frac{1}{t}$ $dx = -\frac{1}{t^2}dt$

$I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}\left(-\frac{1}{t^2}\right)dt = \int \frac{dt}{\sqrt{1-2t}} = -$

$\int (1-2t)^{-\frac{1}{2}} dt = -\frac{(1-2t)^{\frac{1}{2}}}{(-2)\times\frac{1}{2}} + C = \sqrt{1-2t} + C$

$= \sqrt{1-\frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C$

On comparing with $A\sqrt{\frac{x-1}{x+1}} + C$, then A = 1

$\therefore \frac{A}{2} = \frac{1}{2}$

S2. $(\frac{2}{41})$ $I = \int \frac{3\sin x + 2\cos x}{4\cos x + 5\sin x} dx$

Let 3sinx + 2cosx = λ(4cosx + 5sinx) + $\mu\frac{d}{dx}$(4cosx + 5sinx)

⇒ 3sinx + 2cosx = λ(4cosx 5sinx) + μ(5cosx − 4sinx)

comparing coefficients of sin x and cos x

4λ + 5μ = 2

5λ − 4μ = 3

$\lambda = \frac{23}{41}$ and $\mu = -\frac{2}{41}$

$I = \frac{23}{41}\int 1.dx - \frac{2}{41}\int\frac{5\cos x - 4\sin x}{4\cos x + 5\sin x} dx$

$= \frac{23}{41}x - \frac{2}{41}\ln|4\cos x + 5\sin x| + C$

On comparing with $\frac{23}{41}x$ - Aln|4cosx + 5sinx| + C

Then $A = \frac{2}{41}$

S3. **(0)** Let $I = \int \sin^5 x \cos^4 x\, dx$

put cos x = t

$\Rightarrow \quad - \sin x\, dx = dt$

$\Rightarrow \quad I = -\int (1-t^2)^2 \, . \, t^4 \, .dt = -\int (t^4 - 2t^2 + 1)\, t^4 dt$

$= -\int (t^8 - 2t^6 + t^4)\, dt$

$= -\frac{t^9}{9} + \frac{2t^7}{7} - \frac{t^5}{5} + C$

$= -\frac{\cos^9 x}{9} + 2\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

On comparing with $A\frac{\cos^9 x}{9} + B\frac{\cos^7 x}{7} + C\frac{\cos^5 x}{5} + D$

A = -1 , B = 2 , C = -1 then , A + B + C = -1 + 2 -1 = 0

S4. **(2)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx = 2\int_0^{\frac{\pi}{2}} \cos x\, dx = 2$

(Since cos x is even function)

S5. **(0)** Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right) \Rightarrow f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function

$\therefore \quad \int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

TRUE AND FALSE

S1. **(True)** Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x) + g(\cos x)} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2} - x\right)\right)}{g\left(\sin\left(\frac{\pi}{2} - x\right)\right) + g\left(\cos\left(\frac{\pi}{2} - x\right)\right)} = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x) + g(\sin x)} dx$

on adding, we obtain

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x) + g(\cos x)} + \frac{g(\cos x)}{g(\cos x) + g(\sin x)}\right) dx = \int_0^{\frac{\pi}{2}} dx$

$\Rightarrow I = \frac{\pi}{4}$

S2. **(True)** $\int \frac{1}{\sqrt{33 + 8x - x^2}} dx = \int \frac{1}{\sqrt{-\{x^2 - 8x - 33\}}} dx = \int \frac{1}{\sqrt{-\{x^2 - 8x + 16 - 49\}}} dx$

$= \int \frac{1}{\sqrt{-\{(x-4)^2 - 7^2\}}} dx = \int \frac{1}{\sqrt{7^2 - (x-4)^2}} dx = \sin^{-1}\left(\frac{x-4}{7}\right) + C$

S3. **(True)** $\int \frac{2x+3}{\sqrt{x^2+4x+1}} dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}} dx = \int \frac{2x+4}{\sqrt{x^2+4x+1}} dx - \int \frac{1}{\sqrt{x^2+4x+1}} dx$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2 - (\sqrt{3})^2}} dx$, where $t = (x^2 + 4x + 1)$ for I^{st} integral

$= 2\sqrt{t} - \ln|(x+2) + | + \sqrt{x^2+4x+1}C$

$= 2\sqrt{x^2+4x+1} - \ln|x+2+\sqrt{x^2+4x+1}| + C$

S4. **(False)** Put $x = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2} dt$

$\Rightarrow \quad I = \int t \frac{dt}{(t^2+1)\sqrt{t^2-1}}$ {put $t^2 - 1 = y^2$

$\Rightarrow t\, dt = y\, dy$ }

$\Rightarrow \quad I = -\int \frac{y\, dy}{(y^2+2)\, y} = -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C = -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

S5. **(True)**

We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\, dx = \int \sqrt{(x+1)^2 + 2^2}$

$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2} . (2)^2 \ln|(x+1) + \sqrt{(x+1)^2+2^2}| + C$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\ln|(x+1) + \sqrt{x^2+2x+5}| + C$

ASSERTION AND REASONING

S1. (a) (A)$\int e^x\left(\frac{x}{(x+1)^2}\right)dx = \int e^x\left(\frac{x+1-1}{(x+1)^2}\right)dx = \int e^x\left(\frac{1}{1+x} - \frac{1}{(x+1)^2}\right)dx = \frac{e^x}{1+x} + C$

(A) is true.

Reason (R) is always true.

S2. (a) (A)Let $I = \int e^{x\log a}.e^x dx$

$= \int e^{\log a^x}.e^x dx$

$= \int (ae)^x dx = \frac{(ae)^x}{\log_e ae} + C$

(R) $\int a^x dx = \frac{a^x}{\log_e a} + C$

∴ Both A and R are individually true and R is the correct explanation of A.

S3. (c) (A)$\int \frac{e^x}{x}(1 + x\log x)dx = \int e^x\left(\frac{1}{x} + \log x\right)dx$

$= \int e^x.\frac{1}{x}dx + \int e^x \log x\, dx$

$= e^x \log x + C$

(R) $\int e^x[f(x) + f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx = e^x \log x + C$

Here , A is true but R is false.

S4. (c) (A)Given , $f'(x) = \frac{1}{-x+\sqrt{x^2+1}}$

∴ $f(x) = \int\left(x + \sqrt{x^2+1}\right)dx$

$= \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \log\left|x + \sqrt{x^2+1}\right| + C$

Put x = 0 , we get

$f(0) = C = - -\left(\frac{1+\sqrt{2}}{2}\right)$

∴ $f(1) = \frac{1}{2} + \frac{1}{2}\sqrt{2} + \log\left|1 + \sqrt{2}\right| - \left(\frac{1+\sqrt{2}}{2}\right) = \log\left|1 + \sqrt{2}\right|$

A is true.

Hence , we see that f(x) is defined for every value of x , so R is not correct.

S5. (a) (A)$\int \frac{e^x}{x}(1 + x\log x)dx$

$= \int \frac{e^x}{x}dx + \int e^x \log x dx$

$= e^x \log x - \int e^x \log x\, dx + \int e^x \log\ x dx$

$= e^x \log x + C$

(R) $\int e^x[f(x) + f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx$

$= e^x f(x) - \int e^x f'(x) + \int e^x f'(x)dx$

$= e^x f(x) + C$

Both A and R are individually true but R is the correct explanation of A