03

The word 'Trigonometry' is derived from the Greek words '*Trigon*' and '*Metron*' and it means **Measuring the sides of a triangle.**

Trigonometry is that branch of Mathematics which deals with the measurements of the sides, angles of triangles and the problems related to these angles.

TRIGONOMETRIC FUNCTIONS

|TOPIC 1|
Measure of an Angle

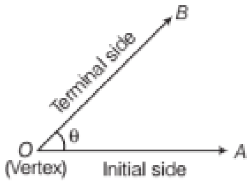
CHAPTER CHECKLIST

- Measure of an Angle
- Trigonometric Functions
- Trigonometric Functions of Compound Angles
- Transformation Formulae
- Trigonometric Functions of Multiple, Sub-multiple Angles and Trigonometric Equations
- Applications of Sine and Cosine Formulae

ANGLE

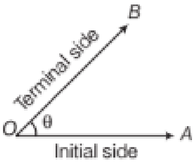
When a ray $\overrightarrow{OA}$ starting from its initial position OA rotates about its end point O and takes the final position OB, we say that $\angle AOB$ has been formed.

Here, initial position OA and final position OB are respectively known as the **initial side** and the **terminal side** of $\angle AOB$ and the point O, is called its **vertex.**

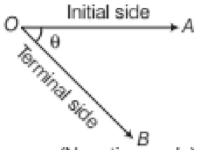


Positive and Negative Angles

If the direction of rotation is **anti-clockwise**, then the angle is said to be positive and if the direction of rotation is **clockwise**, then the angle is said to be negative.



(Positive angle)



(Negative angle)

MEASURE OF AN ANGLE

The amount of rotation from the initial side to the terminal side, is called the measure of an angle.

There are two systems for measuring an angle, *which are given below*

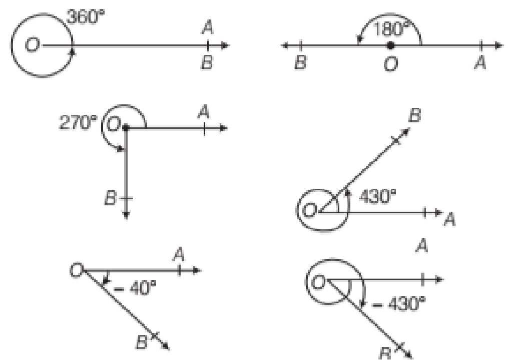
1. Sexagesimal System (Degree Measure)

If the rotation from initial side to terminal side is $\left(\frac{1}{360}\right)$th of a complete revolution, then the angle is said to have a measure of one degree (1°).

One degree is divided into 60 equal parts, called **minutes** and 1 minute is denoted by **1′**. One minute is divided into 60 equal parts, called **second** and 1 second is denoted by **1″**. Thus,

$$1° = 60' ; \quad 1' = 60''$$

Some of the angles, whose measures are 360°, 180°, 270°, 430°, − 40°, − 430°, are shown below



Note (i) The angle between two consecutive digits in a clock is $30° (= \pi/6 \text{ radian})$.

(ii) The hour hand rotates through an angle of 30° in one hour i.e. $\left(\frac{1}{2}\right)^{\circ}$ in one minute.

(iii) The minute hand rotates through an angle of 6° in one minute.

> **Knowledge Plus**
>
> Other system for measuring angles known as centasimal system (French system). In this system, a right angle is divided into 100 equal parts, called grade. It is denoted by 1^g.
>
> The symbols 1^g, $1'$ and $1''$ are used to denote a grade, a minute and a second respectively.
>
> Thus, 1 right angle = 100 grad = 100^g
>
> $1^g = 100'$ (minutes)
>
> $1' = 100''$ (seconds)

2. Circular System (Radian Measure)

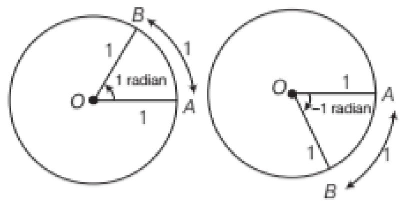
Angle subtended at the centre by an arc of length one unit in a circle of radius 1 unit is said to have a measure of 1 radian.

In general, a circle of radius r having an arc of length r will subtend an angle of 1 radian at the centre. 1 radian is written as 1^c.

Also, a circle of radius r having an arc of length l will subtend an angle θ radian at the centre, where

$$\theta = \frac{l}{r} = \frac{\text{Length of arc}}{\text{Radius}} \quad \text{or} \quad l = r\theta$$

The figures show the angles, whose measures are 1 radian (1^c) and −1 radian (-1^c).



Remembering Points

(i) In the sexagesimal system, we measure angles in degree, minutes and seconds.

1 right angle = 90°, 1° = 60′ and 1′ = 60″

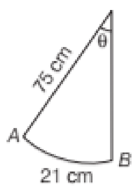
(ii) In the circular measure, we measure angles in radians.

EXAMPLE |1| Find the angle in radian through which a pendulum swings, if its length is 75 cm and tip describes an arc of length 21 cm. **[NCERT]**

Sol. Given, length of pendulum = 75 cm

Radius (r) = length of the pendulum = 75 cm

Length of arc (l) = 21 cm



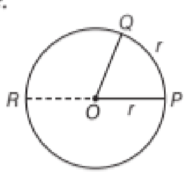
Now, $\theta = \frac{l}{r} = \frac{21}{75} = \frac{7}{25}$ rad

Theorem *A radian is a constant angle.*

Proof Let $C(O, r)$ be a circle.

Let $\widehat{PQ} = r$, join OP and OQ.

Now, produce PO towards O in such a way that it intersect circle at R.



$\therefore \quad \angle POR = 2$ right angles

We know that the angles at the centre of a circle are proportional to the length at arcs subtending them

$$\therefore \quad \frac{\angle POQ}{\angle POR} = \frac{\widehat{PQ}}{\widehat{PR}} \Rightarrow \frac{\angle POQ}{\angle POR} = \frac{r}{\pi r} = \frac{1}{\pi}$$

$$\left[\because \widehat{PR} = \frac{1}{2}(2\pi r) = \pi r\right]$$

$$\Rightarrow \quad \angle POQ = \frac{\angle POR}{\pi} = \frac{\text{Straight angle}}{\pi}$$

$$\Rightarrow \quad \angle POQ = \text{constant}$$

[$\because$ a straight angle and π both are constants]

Hence, radian is constant angle.

Relation between Degree and Radian

A circle subtends at its centre an angle whose radian measure is 2π and its degree measure is $360°$. It follows that

$$\boxed{2\pi \text{ radian} = 360°}$$

or

$$\pi \text{ radian} = 180°$$

We can express a radian measure in terms of degree measure and a degree measure in terms of radian measure by using these relations. Using approximate value of π as $\frac{22}{7}$, we have

$$\boxed{1 \text{ rad} = \frac{180°}{\pi} = 57°16'\,22'' \text{ (approx.)}}$$

Also, $\quad 1° = \frac{\pi}{180}$ rad $= 0.01746$ rad (approx.)

Generally, when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $45° = \pi/4$, is written with the understanding that $\pi/4$ is radian measure. Thus,

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Note

If D denotes degree measure and 'R' denotes radian measure then relation between degree and radian is given by $\frac{D}{180} = \frac{R}{\pi}$.

The relation between degree measure and radian measure of some common angles are given in the following table

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

METHOD TO CONVERT DEGREE MEASURE INTO RADIANS

To convert degree measure into radians, *we use the following steps*

***Step* I** Firstly, convert the seconds (if given) into minutes by using the relation

$$1 \text{ second} = \frac{1}{60} \text{ minutes or } 1'' = \left(\frac{1}{60}\right)'$$

***Step* II** Convert the total minutes (given minutes + minutes obtained in step I) into degrees by using the relation

$$1 \text{ minute} = \frac{1}{60} \text{ degree or } 1' = \left(\frac{1}{60}\right)^{\circ}$$

***Step* III** Now, convert the total degrees (given degree + degree obtained in step II) into radians by using the relation

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

Thus, we convert given degree measure into radians.

EXAMPLE |2| Convert the following into radians.

(i) 240° (ii) 520°

(iii) 25° (iv) $-47°\,30'$ **[NCERT]**

Sol. (i) $240° = \frac{\pi}{180} \times 240 \text{ rad} = \frac{4\pi}{3}$ rad

(ii) Required radian measure $= \frac{\pi}{180} \times$ Degree measure

$$= \frac{\pi}{180} \times 520 \text{ rad} = \frac{26}{9}\pi$$

(iii) We know that,

Radian measure $= \frac{\pi}{180} \times$ Degree measure

$\therefore$ Required radian measure

$$= \frac{\pi}{180} \times 25 \text{rad} = \frac{5\pi}{36} \text{rad}$$

(iv) We have,

$$-47°\,30' = -\left[47° + \left(\frac{30}{60}\right)^{\circ}\right] \qquad \left[\because 1' = \left(\frac{1}{60}\right)^{\circ}\right]$$

$$= -\left[47° + \left(\frac{1}{2}\right)^{\circ}\right] = -\left(\frac{95}{2}\right)^{\circ}$$

Now, required radian measure

$$= \frac{\pi}{180} \times \text{Degree measure}$$

$$= -\left(\frac{\pi}{180} \times \frac{95}{2}\right) \text{rad} = \frac{-19\pi}{72} \text{ rad}$$

Hence, radian measure of $-47°\,30'$ is $-\frac{19\pi}{72}$ rad.

EXAMPLE |3| Express the following in radians.

(i) $40° 20'$ (ii) $48° 37' 30''$

Sol. (i) Here, given degree measure have minutes. Firstly, convert minutes into degree.

$$\because \quad 1' = \left(\frac{1}{60}\right)^\circ$$

$$\therefore \quad 20' = \left(\frac{20}{60}\right)^\circ = \left(\frac{1}{3}\right)^\circ$$

Now, total degree to convert $= 40° + \frac{1}{3}^\circ$

$$= \left(\frac{120+1}{3}\right)^\circ = \left(\frac{121}{3}\right)^\circ$$

$\because$ Radian measure $= \frac{\pi}{180} \times$ Degree measure

$\therefore$ Required radian measure $= \frac{\pi}{180} \times \frac{121}{3}$

$$= \frac{121\pi}{540} \text{ radian}$$

(ii) Here, given degree measure have seconds and minutes. So, firstly convert seconds into minutes.

$$\because \quad 1'' = \left(\frac{1}{60}\right)'$$

$$\therefore \quad 30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$$

$\therefore$ Total minutes to convert

$$= 37' + \frac{1}{2}' = \left(\frac{74+1}{2}\right)' = \left(\frac{75}{2}\right)'$$

$$\because \quad 1' = \left(\frac{1}{60}\right)^\circ$$

$$\therefore \quad \left(\frac{75}{2}\right)' = \left(\frac{75}{2 \times 60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$$

$\therefore$ Total degree to convert

$$= 48° + \left(\frac{5}{8}\right)^\circ$$

$$= \left(\frac{384+5}{8}\right)^\circ = \left(\frac{389}{8}\right)^\circ$$

$\because$ Radian measure $= \frac{\pi}{180} \times$ Degree measure

$$\therefore \quad \left(\frac{389}{8}\right)^\circ = \frac{\pi}{180} \times \frac{389}{8}$$

$$= \frac{389\pi}{1440} \text{ rad}$$

Hence, $48° 37' 30''$ is $\frac{389\pi}{1440}$ rad.

METHOD TO CONVERT RADIAN MEASURE INTO DEGREE

To convert given radian into degree, *we use the following steps*

***Step* I** Firstly, convert given radian into degree by using the relation,

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

[use $\pi = 22/7$]

***Step* II** Convert the fraction part obtained in step I into minutes by using the relation

$$1° = 60' \text{ or } 1 \text{ degree } = 60 \text{ min}$$

***Step* III** If we again get a fraction part in step II, then we convert it into second by using the relation

$$1' = 60'' \text{ or } 1 \text{ min } = 60 \text{ seconds}$$

Thus, we convert given radian measure into degree which have minutes and seconds also.

EXAMPLE |4| Find the degree measure corresponding to following radians.

(i) 6 rad (ii) $\frac{\pi}{64}$ rad

Sol. (i) We know that, degree measure $= \frac{180}{\pi} \times$ Radian measure

$\therefore$ Required degree measure $= \left(\frac{180}{\frac{22}{7}} \times 6\right)^\circ \quad \left[\because \pi = \frac{22}{7}\right]$

$$= \left(\frac{1080 \times 7}{22}\right)^\circ = \left(343\frac{7}{11}\right)^\circ$$

$$= 343° + \frac{7}{11} \times 60 \text{ min} \qquad [\because 1° = 60']$$

$$= 343° + 38\frac{2}{11}\text{min} = 343° + 38' + \frac{2}{11}\text{min}$$

$$= 343° + 38' + \frac{2}{11} \times 60'' \qquad [\because 1' = 60'']$$

$$= 343° + 38' + 10.9'' = 343° 38' 11'' \text{ (approx.)}$$

Hence, the degree measure of 6 rad is $343° 38' 11''$ (approx.).

(ii) We know that, degree measure $= \frac{180}{\pi} \times$ Radian measure

$$\therefore \quad \frac{\pi}{64} \text{ rad} = \left(\frac{\pi}{64} \times \frac{180}{\pi}\right)^\circ = \left(\frac{45}{16}\right)^\circ = \left(2\frac{13}{16}\right)^\circ$$

Here, we see that fraction part is $\left(\frac{13}{16}\right)^\circ$. So, convert it into minutes.

$$\because \quad 1° = 60'$$

$\therefore \left(2\frac{13}{16}\right)^\circ = 2^\circ + \left(\frac{13}{16} \times 60\right)' = 2^\circ + \left(\frac{195}{4}\right)' = 2^\circ + \left(48\frac{3}{4}\right)'$

Here, fraction part is $\frac{3}{4}$. So, we convert it into seconds.

$\because \quad 1' = 60''$

$\therefore \quad \left(2\frac{13}{16}\right)^\circ = 2^\circ + 48' + \left(\frac{3}{4} \times 60\right)'' = 2^\circ\, 48'\, 45''$

Hence, the degree measure of $\frac{\pi}{64}$ rad is $2^\circ 48' 45''$.

PROBLEMS BASED ON APPLICATIONS

EXAMPLE |5| Find the length of an arc of a circle of radius 3 cm, if the angle subtended at the centre is 30°. $[\pi = 3.14]$

Sol. Let l be the length of an arc. We know that,

Angle $\theta = \frac{l}{r}$, where θ is in radian. ...(i)

Given, $r = 3$ cm and $\theta = 30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

On putting the values of r and θ in Eq. (i), we get

$$\frac{\pi}{6} = \frac{l}{3} \Rightarrow l = \frac{\pi}{2} = \frac{3.14}{2} = 1.57 \text{ cm}$$

EXAMPLE |6| Find in degrees and radians, the angle between the hour hand and the minute hand of a clock at half past three.

Sol. The angle traced by the hour hand in 12 h = 360°

$\therefore$ The angle traced by the hour hand in 3 h 30 min

i.e. $\frac{7}{2}\text{h} = \left(\frac{360}{12} \times \frac{7}{2}\right)^\circ = 105^\circ$

The angle traced by the minute hand in 60 min = 360°

$\therefore$ The angle traced by the minute hand in 30 min

$$= \left(\frac{360}{60} \times 30\right)^\circ = 180^\circ$$

Hence, the required angle between two hands

$$= 180^\circ - 105^\circ = 75^\circ$$

or $\left(75 \times \frac{\pi}{180}\right) = \frac{5\pi}{12}$ rad

EXAMPLE |7| The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 min? [use $\pi = 3.14$]

Sol. In one complete rotation, the minute hand of a watch rotates through 360° in 60 min.

$\therefore$ Angle traced out by minute hand in 1 min

$$= \left(\frac{360^\circ}{60}\right) = 6^\circ$$

$\therefore$ Angle traced by the minute hand in 40 min

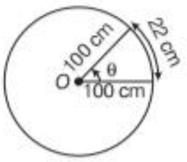
$= (40 \times 6)^\circ = 240^\circ = 240 \times \frac{\pi}{180} = \left(\frac{4\pi}{3}\right)$ rad

We know that, $\theta = \frac{l}{r} \Rightarrow \frac{4\pi}{3} = \frac{l}{1.5}$

$\Rightarrow \quad l = \frac{4\pi}{3} \times 1.5 = 2\pi = 2 \times 3.14 = 6.28$ cm

Hence, tip of the minute hand travels 6.28 cm in 40 min.

EXAMPLE |8| Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm as shown in figure. [use $\pi = 22/7$]



Sol. Given, radius, $r = 100$ cm and arc length, $l = 22$ cm

We know that, $l = r\theta$

$\therefore \quad \theta = \frac{l}{r} = \frac{\text{Arc length}}{\text{Radius}} = \frac{22}{100} = 0.22$ rad

$= 0.22 \times \frac{180}{\pi}$ degree

$= \left(0.22 \times \frac{180 \times 7}{22}\right)^\circ = \left(\frac{22}{100} \times \frac{180 \times 7}{22}\right)^\circ$

$= \left(\frac{126}{10}\right)^\circ = \left(12\frac{6}{10}\right)^\circ$

$= 12^\circ + \frac{6}{10} \times 60' \quad [\because 1^\circ = 60']$

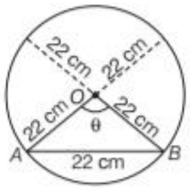
$= 12^\circ + 36' = 12^\circ\, 36'$

Hence, the degree measure of the required angle is $12^\circ 36'$.

EXAMPLE |9| In a circle of diameter 44 cm, the length of chord is 22 cm. Find the length of minor arc of the chord. **[NCERT]**

Sol. Given, diameter = 44 cm

$\therefore \quad$ Radius, $r = \frac{44}{2} = 22$ cm



Let chord $AB = 22$ cm [given]

Since, $OA = OB = AB = 22$ cm

$\therefore$ ΔOAB is equilateral.

$\Rightarrow \quad \theta = \angle AOB = 60^\circ$

Now, we convert 60° into radian measure.

$\therefore \quad 60^\circ = \left(60 \times \frac{\pi}{180}\right)$ rad

$\left[\because \text{radian measure} = \frac{\pi}{180} \times \text{degree measure}\right]$

$\Rightarrow \quad 60^\circ = \frac{\pi}{3}$ radian

$\therefore$ Length of minor arc $AB = r\theta \quad \left[\because \theta = \frac{\text{arc length } (l)}{\text{radius } (r)} \;\; \therefore \text{arc length} = r\theta\right]$

$$= 22 \times \frac{\pi}{3} = \frac{22}{3} \times \frac{22}{7} \qquad \left[\because \pi = \frac{22}{7}\right]$$

$$= \frac{484}{21} \text{ cm}$$

Note

Always remember that in formula $\theta = \frac{l}{r}$, θ is measured in radian. So, first convert the degree in radian whenever use this formula.

EXAMPLE |10| Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye. What is the height of the letters that he can read at a distance of 12 m?

Sol. Let 'h' be the required height in meters. Here, h metre can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

Clearly, $5' = \left(\frac{5}{60}\right)^{\circ} = \left(\frac{1}{12} \times \frac{\pi}{180}\right)^{c}$

Now, $\theta = \frac{l}{r} \Rightarrow \frac{\pi}{12 \times 180} = \frac{h}{12}$

$\Rightarrow \quad h = \left(\frac{\pi}{180}\right) \text{m} = 1.7 \text{ cm}$

EXAMPLE |11| The angles of a triangle are in AP. The number of degrees in the least is to the number of radians in the greatest is as $60 : \pi$. Find the angles in degree.

Sol. Let the three angles in AP be $a - d$, a, $a + d$.

$\therefore \quad (a - d) + a + (a + d) = 180°$

[$\because$ sum of all the angles of a triangle is 180°]

$\Rightarrow \quad 3a = 180° \Rightarrow a = 60°$

So, the angles are $(60 - d)°$, $60°$, $(60 + d)°$.

Number of degrees in least angle $= (60 - d)°$

Number of degrees in greatest angle $= (60 + d)°$

$\therefore$ Number of radians in greatest angle

$$= \frac{\pi}{180} \times (60 + d) \text{ rad}$$

According to the given condition,

$$\frac{(60 - d)}{\frac{\pi}{180} \times (60 + d)} = \frac{60}{\pi}$$

$\Rightarrow \quad \frac{60 - d}{60 + d} \times \frac{180}{\pi} = \frac{60}{\pi} \Rightarrow \frac{60 - d}{60 + d} = \frac{1}{3}$

$\Rightarrow \quad 180 - 3d = 60 + d$

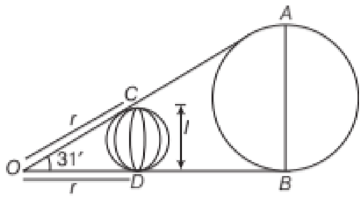
$\Rightarrow \quad 4d = 120 \Rightarrow d = 30$

Hence, angles are $(60° - 30°)$, $60°$ and $(60° + 30°)$

i.e. 30°, 60° and 90°.

EXAMPLE |12| Find the distance from the eye at which a coin of diameter 2 cm should be held so as just to conceal the full moon, whose angular diameter is 31′.

Sol. Let AB be the diameter of the moon and O be the eye of the observer so that $\angle AOB = 31'$. Let CD = 2 cm be the diameter of the coin. The full moon will be just concealed, if the diameter of a coin also subtends the same angle as the diameter of the moon at O, i.e. if $\angle COD = 31'$.



As, $\angle COD$ is small, CD may be treated as the arc of a circle, whose centre is O and radius = OC or OD. Let $OC = OD = r$

Here, l = length of an arc CD = 2 cm

and $\theta = 31' = \left(31 \times \frac{1}{60}\right)^{\circ} \qquad \left[\because 1' = \left(\frac{1}{60}\right)^{\circ}\right]$

$$= \left(\frac{31}{60} \times \frac{\pi}{180}\right) \text{rad} \qquad \left[\because 1° = \frac{\pi}{180} \text{ rad}\right]$$

Now, $r = \frac{l}{\theta} = \frac{2}{\frac{31}{60} \times \frac{\pi}{180}}$

$$= \frac{2 \times 60 \times 180}{31 \times \pi}$$

$$= \frac{21600 \times 7}{31 \times 22} \qquad \left[\because \pi = \frac{22}{7}\right]$$

$$= \frac{75600}{341} \text{ cm} = \frac{756}{341} \text{ m} \qquad [\because 100 \text{ cm} = 1 \text{ m}]$$

$= 2.22$ m

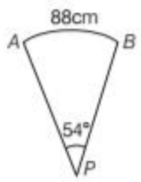
EXAMPLE |13| A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 m when it has traced out 54° at the centre. Find the length of the rope.

Sol. Let the post be at point P and PA be the length of the rope in tight position.

Suppose the horse moves along the arc AB, so that $\angle APB = 54°$ and arc AB of length, $l = 88$ m.

Let r be the length of the rope i.e. $PA = r$ metre.

Now, $\theta = 54° = \left(54 \times \frac{\pi}{180}\right) = \left(\frac{3\pi}{10}\right) \qquad \left[\because 1° = \frac{\pi}{180} \text{ rad}\right]$



$\because \quad \theta = \frac{l}{r}$

$\therefore \quad \frac{3\pi}{10} = \frac{88}{r}$

$\Rightarrow \quad r = \frac{88 \times 10}{3\pi}$

$\Rightarrow \quad r = \frac{880}{9.42} = 93.41 \text{ m} \qquad [\because \pi = 3.14]$

EXAMPLE |14| If the arcs of same length in two circles subtend angle of 60° and 75° at their centres. Find the ratio of their radii.

Sol. Let r_1 and r_2 be the radii of the given circles and let their arcs of same length l subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right) = \left(\frac{\pi}{3}\right)$ rad

$75° = \left(75 \times \frac{\pi}{180}\right) = \left(\frac{5\pi}{12}\right)$ rad

$\Rightarrow \quad \frac{\pi}{3} = \frac{l}{r_1}$ and $\frac{5\pi}{12} = \frac{l}{r_2} \qquad \left[\because \theta = \frac{l}{r}\right]$

$\Rightarrow \quad l = \frac{\pi}{3} r_1$ and $l = \frac{5\pi}{12} r_2$

$\Rightarrow \quad \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2$

$\Rightarrow \quad 4r_1 = 5r_2 \Rightarrow \frac{r_1}{r_2} = \frac{5}{4}$

$\Rightarrow \quad r_1 : r_2 = 5 : 4$

EXAMPLE |15| The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minute and seconds.

Sol. Let 'r' be the radius of the circle and θ be the sector angle. Then,

Perimeter of the sector $= 2r + r\theta \qquad \left[\because \theta = \frac{l}{r}\right]$

length of arc of semi-circle $= \pi r \qquad$ [$\because$ it is a semi-circle]

$2r + r\theta = \pi r \Rightarrow (2 + \theta) = \pi$

$\Rightarrow \quad \theta = (\pi - 2)$ rad

$\Rightarrow \quad \theta = (\pi - 2) \times \frac{180°}{\pi} \qquad \left[\because 1 \text{ rad} = \frac{180°}{\pi}\right]$

$\Rightarrow \quad \theta = \pi \times \frac{180°}{\pi} - \frac{2 \times 180°}{\pi}$

$\Rightarrow \quad \theta = 180° - \frac{360°}{\pi}$

$\Rightarrow \quad \theta = 180° - 114° 32' 44''$

$\Rightarrow \quad \theta = 65° 27' 16''$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 If a rapidly spinning wheel is making an angle of say 15 revolutions per second, then this is an example of

(a) small angle (b) obtuse angle
(c) large angle (d) None of these

2 In a circle of radius r, an arc of length r will subtend an angle of

(a) r radian (b) 1 radian
c) π radian (d) 2π radian

3 Radian measure of 40°20′ is equal to

(a) $\frac{120\pi}{504}$ radian (b) $\frac{121\pi}{540}$ radian
(c) $\frac{121\pi}{3}$ radian (d) None of these

4 The minute hand of a watch is 1.5 cm long. The distance travelled by the minute hand in 40 minutes is equal to

(a) 3.28 cm (b) 4.28 cm
(c) 5.28 cm (d) 6.28 cm

VERY SHORT ANSWER Type Questions

5 Find the radian measure corresponding to the following degree measures.

(i) 75° (ii) 135° (iii) −22° 30′ (iv) −300°

6 Find the degree measure corresponding to the following radian measures. $\left[\text{use } \pi = \frac{22}{7}\right]$

(i) $\frac{\pi}{9}$ rad (ii) $\frac{9\pi}{5}$
(iii) $-\frac{\pi}{3}$ rad (iv) $-\frac{5\pi}{6}$

SHORT ANSWER Type I Questions

7 Find the radian measures corresponding to following degree measures.

(i) 5°37′ 30″ (ii) −14° 20′ 15″

8 Find the degree measure corresponding to following radian measures.

(i) $(-3)^c$ (ii) 11^c

9 Find the length of arc of a circle of radius 5 cm, subtending a central angle measuring 15°.

10 Find the angle in degree subtends at the centre of a circle by an arc whose length is 2.2 times the radius.

11 In a circle of diameter 40 cm, a chord is 20 cm. Find the length of the minor arc of the chord.

12 The minute hand of a watch is 2.2 cm long. How far does its tip move in 50 min.

13 A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describe 70 m when it has traced out 80° at the centre, find the length of the rope.

SHORT ANSWER Type II Questions

14 The Moon's distance from the Earth is 360000 km and its diameter subtend an angle of 31′ at the eye of observer. Find the diameter of the Moon.

15 If the angular diameter of the Moon is 30′, how far from the eye a coin of diameter 2.2 cm be kept to hide the Moon?

16 If the arc of the same length in two circles subtend angles 65° and 110° at the centre, find the ratio of their radii.

17 Find the diameter of the sun in km supposing that it subtends an angle 32′ at the eye of the observer. Given that distance of the sun is 91×10^6 km.

HINTS & ANSWERS

1. (c) **2.** (b)

3. (b) We know that, $180° = \pi$ radian

Hence, $40°20' = 40\frac{1}{3}$ degree

$$= \frac{\pi}{180} \times \frac{121}{3} \text{ radian} = \frac{121\pi}{540} \text{ radian.}$$

Therefore, $40°20' = \frac{121\pi}{540}$ radian.

4. (d) In 60 min, the minute hand of watch completes one revolution. Therefore, in 40 min the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm.}$$

5. (i) $\frac{5\pi}{12}$ (ii) $\theta = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$

(iii) $-(22°30') = -\left[22° + \left(\frac{1}{2}\right)^{\circ}\right] = -\left(\frac{45}{2}\right)^{\circ}$

$$= -\frac{45}{2} \times \frac{\pi}{180} = \left(\frac{45}{2}\right)^{\circ} = -\frac{\pi}{8}$$

(iv) $-\frac{5\pi}{3}$

6. (i) $\frac{\pi}{9} = \left(\frac{\pi}{9} \times \frac{180}{\pi}\right)^{\circ} = 20°$

(ii) $\frac{9\pi}{5} = \frac{9}{5} \times 180° = 324°$

(iii) Solve as part (i). **Ans.** − 60°

(iv) Solve as part (ii). **Ans.** −150°

7. (i) $(30)'' = \left(\frac{1}{2}\right)' \Rightarrow 37' \, 30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^{\circ} = \left(\frac{5}{8}\right)^{\circ}$

$5°37' \, 30'' = 5° + \frac{5°}{8} = \frac{45}{8} \times \frac{\pi}{180}$ **Ans.** (i) $\frac{\pi}{32}$ rad

(ii) Solve as part (i). **Ans.** $\frac{-3\pi}{1600}$

8. (i) $(-3)^c = -\left(3 \times \frac{180}{\pi}\right)^{\circ} = -\left(3 \times \frac{180}{22} \times 7\right)^{\circ}$

$$= -\left(\frac{1890}{11}\right)^{\circ} = \left(171\frac{9}{11}\right)^{\circ}$$

$$= 171° + \left(\frac{9}{11} \times 60\right)' = 171° + \left(49\frac{1}{11}\right)'$$

$$= 171° + 49' + \left(\frac{1}{11} \times 60\right)' = 171° + 49' + 5'' = 171°49'5''$$

(ii) Solve as part (i). **Ans.** 630°

9. $l = r\theta = 5 \times \frac{\pi}{12} = 5 \times \frac{22/7}{12} = 1.30$ cm

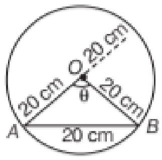
10. $\theta = \frac{l}{r} = \left(\frac{2.2 \times r}{r}\right) \text{rad} = \left(2.2 \times \frac{180}{22/7}\right)^{\circ} = 126°$

11. Given, diameter = 40 cm

$\therefore$ Radius $(r) = \frac{40}{2} = 20$ cm

and length of chord, $AB = 20$ cm

Since, ΔOAB is an equilateral triangle.



$\therefore$ Angle, $\theta = 60° = \frac{\pi}{3}$ rad

arc $AB = \theta \times r$ **Ans.** $\frac{20}{3}\pi$ cm

12. Angle traced by minute hand in 50 min

$$= (50 \times 6)° = 50 \times 6 \times \frac{\pi}{180} = \frac{5\pi}{3}$$

$\therefore$ $\theta = \frac{l}{r} \Rightarrow \frac{5\pi}{3} = \frac{l}{2.2} \Rightarrow l = 11.52$ cm

13. $\theta = 80° = \left(80 \times \frac{\pi}{180}\right)^c = \frac{4\pi}{9}$; $l = 70$ m

$\therefore$ $\theta = \frac{l}{r} \Rightarrow \frac{4\pi}{9} = \frac{70}{r}$; $r = 50.11$ m

14. $\theta = 31' = \left(\frac{31}{60} \times \frac{\pi}{180}\right)$ rad and $r = 360000$ km

$\because$ $\theta = \frac{l}{r} \therefore \frac{31}{60} \times \frac{\pi}{180} = \frac{l}{360000}$

Ans. 3247.62 km

15. Solve as Example 15. **Ans.** 252 cm

16. 22 : 13

17. 847407.4 km

|TOPIC 2|

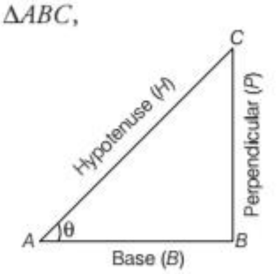
Trigonometric Functions

We have already studied trigonometric ratios for acute angles in earlier classes. In this topic, we will study trigonometric ratios for any angle which are termed as **trigonometric functions**.

TRIGONOMETRIC (OR CIRCULAR) FUNCTIONS

In a right angle triangle, there are actually six possible trigonometric ratios or functions. A greek letter (such as theta θ) will now be used to represent the angle.

In right angled ΔABC,



(i) $\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}\left(\text{or } \dfrac{P}{H}\right) = \dfrac{BC}{AC}$

(ii) $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}}\left(\text{or } \dfrac{B}{H}\right) = \dfrac{AB}{AC}$

(iii) $\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}}\left(\text{or } \dfrac{P}{B}\right) = \dfrac{BC}{AB}$

(iv) $\cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}}\left(\text{or } \dfrac{B}{P}\right) = \dfrac{AB}{BC}$

(v) $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Base}}\left(\text{or } \dfrac{H}{B}\right) = \dfrac{AC}{AB}$

(vi) $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}}\left(\text{or } \dfrac{H}{P}\right) = \dfrac{AC}{BC}$

Note

Here, $\sin\theta$ does not mean the product of sin and θ. The $\sin\theta$ is correctly read as sine of angle θ.

EXAMPLE |1| In ΔABC, $\angle B$ is right angled triangle. If $\tan A = 1$, then show that $2\sin A\cos A = 1$.

Sol. Given, $\angle B$ is right angled in ΔABC

and $\tan A = 1$

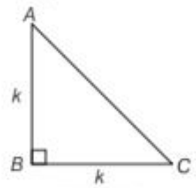
$\therefore \quad \tan A = \dfrac{BC}{AB} = 1 \Rightarrow BC = AB$

Let $AB = BC = k$, where k is a positive real number.

In right angled ΔABC,

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC = \sqrt{AB^2 + BC^2}$$



$$\Rightarrow \quad AC = \sqrt{k^2 + k^2}$$

$$\Rightarrow \quad AC = \sqrt{2k^2} \Rightarrow AC = k\sqrt{2}$$

Now, $\quad \sin A = \dfrac{BC}{AC} = \dfrac{k}{k\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

and $\quad \cos A = \dfrac{AB}{AC} = \dfrac{k}{k\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\therefore \quad \text{LHS} = 2\sin A\cos A$

$$= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 2 \times \frac{1}{2} = 1 = \text{RHS}$$

Quadrantal Angles and Related Trigonometric Functions

All angles which are integral multiples of $\dfrac{\pi}{2}$ are called **quadrantal angles**.

If θ increase (or decrease) by any integral multiple of 2π, then the values of sine and cosine functions do not change.Thus,

$$\sin(2n\pi + \theta) = \sin\theta, n \in Z$$

and $\quad \cos(2n\pi + \theta) = \cos\theta, n \in Z$

Further, if θ is an **integral multiples** of π i.e. $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$, then $\sin\theta = 0$

and if θ is an **odd multiples of** $\dfrac{\pi}{2}$,

i.e. $\theta = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \dots$, then $\cos\theta = 0$.

Thus, we can say that,

> $\sin\theta = 0 \Rightarrow \theta = n\pi$, where n is any integer..
>
> and $\cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2}$, where n is any integer.

Trigonometric Functions of Some Standard Angles

Angle θ	0°	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$	120° $\left(\frac{2\pi}{3}\right)$	135° $\left(\frac{3\pi}{4}\right)$	150° $\left(\frac{5\pi}{6}\right)$	180° (π)	270° $\left(\frac{3\pi}{2}\right)$	360° (2π)
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	−1	0	1
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0	∞	0
cotθ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	∞	0	∞
secθ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	−1	∞	1
cosec θ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−1	∞

Note Here, the symbol '∞' is used for not defined.

EXAMPLE |2| Find the value of

$$\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{4}.$$

Sol. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{4}=\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2$

$$=\frac{1}{4}+\frac{3}{4}+\frac{1}{2}=\frac{1+3+2}{4}=\frac{6}{4}=\frac{3}{2}$$

Trigonometric Functions in Terms of sine and cosine Functions

Given below are trigonometric functions defined in terms of sine and cosine functions.

$$\operatorname{cosec} x=\frac{1}{\sin x},\ x\neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x=\frac{1}{\cos x},\ x\neq(2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x=\frac{\sin x}{\cos x},\ x\neq(2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x=\frac{\cos x}{\sin x},\ x\neq n\pi, \text{ where } n \text{ is any integer.}$$

Trigonometric Identities

The relation involving trigonometric functions which are true for all values of the angles involved are called trigonometric identities.

$$\cos^2\theta+\sin^2\theta=1;\ 1+\tan^2\theta=\sec^2\theta;\ 1+\cot^2\theta=\operatorname{cosec}^2\theta$$

With the help of these identities we can write one trigonometric function in terms of all other trigonometric function, *which are given in the following table*

Transformation of One Trigonometric Function to Another Trigonometric Function

Trigonometric function	sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
sin θ	sin θ	$\pm\sqrt{(1-\cos^2\theta)}$	$\pm\frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$	$\pm\frac{1}{\sqrt{(1+\cot^2\theta)}}$	$\pm\frac{\sqrt{(\sec^2\theta-1)}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
cos θ	$\pm\sqrt{(1-\sin^2\theta)}$	cos θ	$\pm\frac{1}{\sqrt{(1+\tan^2\theta)}}$	$\pm\frac{\cot\theta}{\sqrt{(1+\cot^2\theta)}}$	$\frac{1}{\sec\theta}$	$\pm\frac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$
tan θ	$\pm\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$	$\pm\frac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$	tan θ	$\frac{1}{\cot\theta}$	$\pm\sqrt{(\sec^2\theta-1)}$	$\pm\frac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
cot θ	$\pm\frac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}$	$\pm\frac{\cos\theta}{\sqrt{(1-\cos^2\theta)}}$	$\frac{1}{\tan\theta}$	cot θ	$\pm\frac{1}{\sqrt{(\sec^2\theta-1)}}$	$\pm\sqrt{(\operatorname{cosec}^2\theta-1)}$
sec θ	$\pm\frac{1}{\sqrt{(1-\sin^2\theta)}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{(1+\tan^2\theta)}$	$\pm\frac{\sqrt{(1+\cot^2\theta)}}{\cot\theta}$	sec θ	$\pm\frac{\operatorname{cosec}\theta}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
cosec θ	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{(1-\cos^2\theta)}}$	$\pm\frac{\sqrt{(1+\tan^2\theta)}}{\tan\theta}$	$\pm\sqrt{(1+\cot^2\theta)}$	$\pm\frac{\sec\theta}{\sqrt{(\sec^2\theta-1)}}$	cosec θ

EXAMPLE |3| If $\operatorname{cosec} A = \frac{x}{y}$, then find the value of $\cot A$.

Sol. Given, $\operatorname{cosec} A = \frac{x}{y}$

We know that, $\cot^2 A = \operatorname{cosec}^2 A - 1$

$$\cot^2 A = \left(\frac{x}{y}\right)^2 - 1 = \frac{x^2}{y^2} - 1 = \frac{x^2 - y^2}{y^2}$$

$$\Rightarrow \quad \cot A = \pm \frac{\sqrt{x^2 - y^2}}{y}$$

EXAMPLE |4| Prove that

$$\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta.$$

Sol. LHS $= \tan^2\theta - \sin^2\theta$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \qquad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$= \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta}$$

$[\because 1 - \cos^2\theta = \sin^2\theta]$

$$= \frac{\sin^2\theta \cdot \sin^2\theta}{\cos^2\theta} = \tan^2\theta \cdot \sin^2\theta \qquad \left[\because \frac{\sin\theta}{\cos\theta} = \tan\theta\right]$$

= LHS **Hence proved.**

EXAMPLE |5| If $a\cos\theta + b\sin\theta = x$ and $a\sin\theta - b\cos\theta = y$, prove that $a^2 + b^2 = x^2 + y^2$.

Sol. RHS $= x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$

$$= (a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta) + (a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta)$$

$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$

$$= a^2 + b^2 = \text{LHS} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

Hence proved.

EXAMPLE |6| Prove that

$$\frac{1 + \tan^2 A}{1 + \cot^2 A} = \left(\frac{1 - \tan A}{1 - \cot A}\right)^2 = \tan^2 A.$$

Sol. LHS $= \frac{1 + \tan^2 A}{1 + \cot^2 A}$

$$= \frac{1 + \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\cos^2 A}{\sin^2 A}} \qquad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}\right]$$

$$= \frac{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}{\frac{\sin^2 A + \cos^2 A}{\sin^2 A}} = \frac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1}{\cos^2 A} \times \frac{\sin^2 A}{1} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A = \text{RHS}$$

$$\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\text{Mid term} = \left(\frac{1 - \tan A}{1 - \cot A}\right)^2 = \left(\frac{1 - \frac{\sin A}{\cos A}}{1 - \frac{\cos A}{\sin A}}\right)^2$$

$$\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ or } \cot\theta = \frac{\cos\theta}{\sin\theta}\right]$$

$$= \frac{\frac{(\cos A - \sin A)^2}{\cos^2 A}}{\frac{(\sin A - \cos A)^2}{\sin^2 A}} = \frac{(\cos A - \sin A)^2 \times \sin^2 A}{(\sin A - \cos A)^2 \times \cos^2 A}$$

$$= \frac{\sin^2 A}{\cos^2 A} = \tan^2 A = \text{RHS}$$

$\therefore$ LHS = Mid term = RHS **Hence proved.**

EXAMPLE |7| Prove the following identities.

(i) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$

(ii) $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$

Sol. (i) LHS $= (1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$

$$= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\sin\theta + \cos\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$$ **Hence proved.**

(ii) LHS $= \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$$

$[\because \sec^2\theta - \tan^2\theta = 1]$

$$= \frac{(\tan\theta + \sec\theta)\ \{1 - (\sec\theta - \tan\theta)\}}{\tan\theta - \sec\theta + 1}$$

$[\because a^2 - b^2 = (a - b)(a + b)]$

$$= \frac{(\tan\theta + \sec\theta)(\tan\theta - \sec\theta + 1)}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \frac{\sin\theta + 1}{\cos\theta} = \text{RHS}$$ **Hence proved.**

EXAMPLE |8| If $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$

and $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0.$

Prove that $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Sol. Given, $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$ ... (i)

and $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$ ... (ii)

From Eq. (i), we have

$$ax\sin^3\theta - by\cos^3\theta = 0$$

$$\Rightarrow \quad \frac{\sin^3\theta}{by} = \frac{\cos^3\theta}{ax}$$

$$\Rightarrow \quad \left(\frac{\sin^3\theta}{by}\right)^{2/3} = \left(\frac{\cos^3\theta}{ax}\right)^{2/3}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{(by)^{2/3} + (ax)^{2/3}}$$

$$\left[\because \frac{a}{b} = \frac{c}{d} \Rightarrow \text{each ratio} = \frac{a+c}{b+d}\right]$$

$$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{1}{(by)^{2/3} + (ax)^{2/3}}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$\therefore \quad \sin^2\theta = \frac{(by)^{2/3}}{(by)^{2/3} + (ax)^{2/3}}$$

$$\text{and} \quad \cos^2\theta = \frac{(ax)^{2/3}}{(by)^{2/3} + (ax)^{2/3}}$$

$$\Rightarrow \quad \sin\theta = \frac{(by)^{1/3}}{\sqrt{(by)^{2/3} + (ax)^{2/3}}}$$

$$\text{and} \quad \cos\theta = \frac{(ax)^{1/3}}{\sqrt{(by)^{2/3} + (ax)^{2/3}}}$$

Substituting the value of $\sin\theta$ and $\cos\theta$ in Eq. (ii), we have

$$(ax)^{2/3}\sqrt{(by)^{2/3} + (ax)^{2/3}} + (by)^{2/3}\sqrt{(by)^{2/3} + (ax)^{2/3}} = a^2 - b^2$$

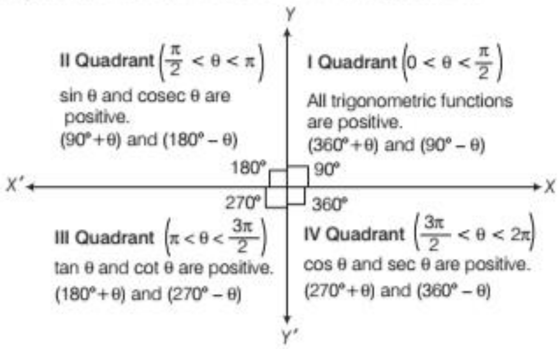
$$\Rightarrow \quad [(ax)^{2/3} + (by)^{2/3}]^{3/2} = a^2 - b^2$$

$$\Rightarrow \quad (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$ **Hence proved.**

Sign of Trigonometric Functions in Different Quadrants

If we draw two mutually perpendicular (intersecting) lines in the plane of paper, then these lines divide the plane of paper into four parts, known as quadrants.

In anti-clockwise order, these quadrants are numbered as Ist, IInd, IIIrd and IVth. All angles from 0° to 90° are taken in Ist quardant, 90° to 180° in IInd quardant, 180° to 270° in IIIrd quadrant and 270° to 360° in IVth quadrant.



To remember the signs of trigonometrical functions in different quadrants, we use the following four word phrase 'After School To College'. Here,

After
Indicate that **all trigonometric functions** are positive in I quadrant.

School
Indicate that **sine and its reciprocal** are positive in II quadrant.

To
Indicate that **tan and its reciprocal** are positive in III quadrant.

College
Indicate that **cosine and its reciprocal** are positive in IV quadrant.

EXAMPLE |9| Find the value of $\cos\theta$ and $\tan\theta$, if $\sin\theta = -\dfrac{3}{5}$ and $\pi < \theta < \dfrac{3\pi}{2}$.

Sol. We know that, $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1 - \sin^2\theta}$

Given that, $\pi < \theta < \dfrac{3\pi}{2} \Rightarrow \theta$ lies in third quadrant.

In third quadrant, $\cos\theta$ is negative.

$$\therefore \quad \cos\theta = -\sqrt{1 - \sin^2\theta}\ \theta = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$$

Now, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}\ \theta = -\left(\dfrac{3}{5}\right) \times \left(-\dfrac{5}{4}\right) = \dfrac{3}{4}$

EXAMPLE |10| If $\sec x = \frac{13}{5}$, where x lies in fourth quadrant, then find the values of other five trigonometric functions. **[NCERT]**

(i) Use the transformation table for converting one trigonometric function to another trigonometric function.
(ii) Use the sign table for marking the sign.

Sol. Given, $\sec x = \frac{13}{5}$ and x lies in fourth quadrant.

i.e. $\frac{3\pi}{2} < x < 2\pi$

Now, $\cos x = \frac{1}{\sec x} = \frac{1}{13/5} = \frac{5}{13}$

We know that, $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169}$$

$$= \frac{169-25}{169} = \frac{144}{169} = \left(\frac{12}{13}\right)^2$$

$$\Rightarrow \sin x = \pm \frac{12}{13}$$

Since, x lies in fourth quadrant and $\sin x$ is negative in fourth quadrant.

$$\therefore \sin x = -\frac{12}{13} \Rightarrow \operatorname{cosec} x = \frac{1}{\sin x} = \frac{-13}{12}$$

Now, $\tan x = \frac{\sin x}{\cos x} = \frac{-\left(\frac{12}{13}\right)}{\left(\frac{5}{13}\right)} = -\frac{12}{5}$

and $\cot x = \frac{1}{\tan x} = -\frac{5}{12}$

EXAMPLE |11| If $\cos\theta = -\frac{1}{2}$, θ lies in III quadrant, then find other five trigonometric functions. **[NCERT]**

Sol. We have, $\cos\theta = -\frac{1}{2}$ and θ lies in III quadrant.

We know that, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

In III quadrant, $\sin\theta$ is negative, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(-\frac{1}{2}\right)^2}$$

$$= -\sqrt{1-\frac{1}{4}} = -\sqrt{\frac{4-1}{4}}$$

$$\Rightarrow \sin\theta = \frac{-\sqrt{3}}{2}$$

Now, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{-\sqrt{3}}{2}\right)} = \frac{-2}{\sqrt{3}} \quad \left[\because \sin\theta = \frac{-\sqrt{3}}{2}\right]$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(-\frac{1}{2}\right)} = -2 \quad \left[\because \cos\theta = -\frac{1}{2}\right]$$

and $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$ $[\because \tan\theta = \sqrt{3}]$

Hence, other five trigonometric functions are

$\sin\theta = \frac{-\sqrt{3}}{2}$, $\operatorname{cosec}\theta = \frac{-2}{\sqrt{3}}$,

$\sec\theta = -2$, $\tan\theta = \sqrt{3}$ and $\cot\theta = \frac{1}{\sqrt{3}}$.

EXAMPLE |12| If $\sin\theta = -\frac{3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $\frac{\sec\theta - \cot\theta}{\tan\theta - \operatorname{cosec}\theta}$.

Sol. Here, $\pi < \theta < \frac{3\pi}{2}$, it means θ lies in third quadrant.

We have, $\sin\theta = -\frac{3}{5}$, then $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{-3/5} = -\frac{5}{3}$

Now, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \left(-\frac{3}{5}\right)^2 + \cos^2\theta = 1$

$$\Rightarrow \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow \cos\theta = \pm\sqrt{\frac{16}{25}}$$

$$\Rightarrow \cos\theta = \pm\frac{4}{5}$$

Since, θ lies in third quadrant. So, $\cos\theta$ is negative.

$$\therefore \cos\theta = -\frac{4}{5} \Rightarrow \sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-3/5}{-4/5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4}$$

and $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{3/4} = \frac{4}{3}$

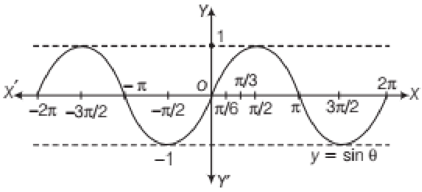
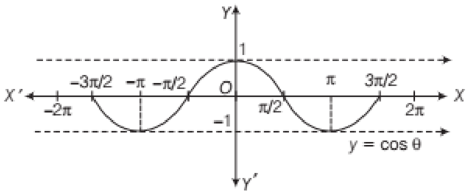
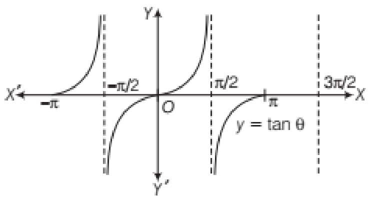
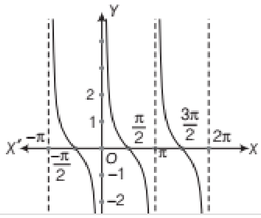
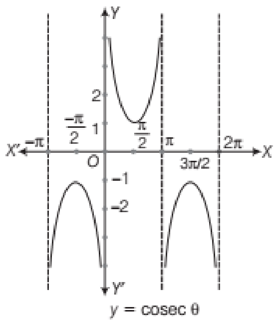
On putting these values in given expression, we get

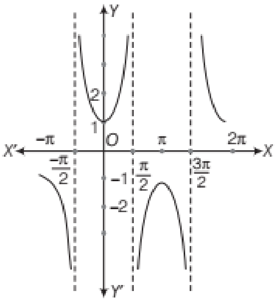
$$\frac{\sec\theta - \cot\theta}{\tan\theta - \operatorname{cosec}\theta} = \frac{\left(-\frac{5}{4} - \frac{4}{3}\right)}{\left(\frac{3}{4} + \frac{5}{3}\right)} = \frac{\left(\frac{-15-16}{12}\right)}{\left(\frac{9+20}{12}\right)} = -\frac{31}{29}$$

DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

The domain of a trigonometric function consists of all the values that you can use as input into the angles i.e. the values for which trigonometric function is defined. The range of trigonometric functions consists of all its output value i.e. the values which are obtained by trigonometric functions. e.g. $\sin x$ is define for all real values of x, so its domain is R and obtained value lies between 1 and -1, so its range is the interval $[-1, 1]$.

The domain, range and graph of each one of the six trigonometric functions are given below

T-function	Domain	Range	Graph
$\sin\theta$	R	$[-1, 1]$	
$\cos\theta$	R	$[-1, 1]$	
$\tan\theta$	$\{x : x \in R \text{ and } x \neq (2n+1)\,\pi/2,\ n \in I\}$	R	
$\cot\theta$	$\{x : x \in R \text{ and } x \neq n\pi,\ n \in I\}$	R	
$\operatorname{cosec}\theta$	$\{x : x \in R \text{ and } x \neq n\pi,\ n \in I\}$	$R - (-1, 1)$	

T-function	Domain	Range	Graph
$\sec\theta$	$\{x : x \in R \text{ and } x \neq (2n+1)\ \pi/2, n \in I\}$	$R - (-1, 1)$	 $y = \sec\theta$

EXAMPLE |13| Which of the following is not correct?

[NCERT Exemplar]

(i) $\sin\theta = -\frac{1}{5}$ (ii) $\cos\theta = 1$

(iii) $\sec\theta = \frac{1}{2}$ (iv) $\tan\theta = 20$

Verify through the range of trigonometric function.

Sol. (i) $\sin\theta = -\frac{1}{5} \in [-1, 1]$. Therefore, the given equation is correct.

(ii) $\cos\theta = 1 \in [-1, 1]$. Therefore, the given equation is correct.

(iii) Since, $\sec\theta \notin (-1, 1)$

$\therefore \quad \sec\theta \neq \frac{1}{2}$,

$\Rightarrow \sec x = \frac{1}{2}$ is not correct.

(iv) $\tan\theta = 20 \in (-\infty, \infty)$. Therefore, the given equation is correct.

EXAMPLE |14| If x is any non-zero real number, show that $\cos\theta$ and $\sin\theta$ can never be equal to $x + \frac{1}{x}$.

Sol. Given x is any non-zero real number, then x may be greater than zero or less than zero.

Here, two cases arise

(i) When $x > 0$, then

$$x + \frac{1}{x} = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2 \times \sqrt{x} \times \frac{1}{\sqrt{x}} + 2 \times \sqrt{x} \times \frac{1}{\sqrt{x}}$$

$$x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \geq 2$$

(ii) When $x < 0$, then

$$x = -y,\ y > 0 \text{ (let)}$$

$$\therefore \quad -y - \frac{1}{y} = -\left(y + \frac{1}{y}\right)$$

$$\Rightarrow \quad -\left(y + \frac{1}{y}\right) \leq -2 \ \Rightarrow\ x + \frac{1}{x} \leq -2$$

$\therefore\ x + \frac{1}{x} \geq 2$ for $x > 0$ and $x + \frac{1}{x} \leq -2$ for $x < 0$.

But $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$ for all θ.

$\therefore\ x + \frac{1}{x}$ can not be equal to $\sin\theta$ or $\cos\theta$.

Behaviour of Trigonometric Functions in Different Quadrants

Behaviour of trigonometric functions in different quadrants i.e. the variation in the value of trigonometric ratio in different quadrants, can be understood from the table given below

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	Increases from 0 to 1	Decreases from 1 to 0	Decreases from 0 to −1	Increases from −1 to 0
cos	Decreases from 1 to 0	Decreases from 0 to −1	Increases from −1 to 0	Increases from 0 to 1
tan	Increases from 1 to 0	Increases from −∞ to 0	Increases from 0 to ∞	Increases from −∞ to 0
cot	Decreases from ∞ to 0	Decreases from 0 to −∞	Decreases from ∞ to 0	Decreases from 0 to −∞
sec	Increases from 0 to ∞	Increases from −∞ to −1	Decreases from −1 to −∞	Decreases from ∞ to 1
cosec	Decreases from ∞ to 1	Increases from 1 to ∞	Increases from −∞ to −1	Decreases from −1 to −∞

TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES

Two angles are said to be **allied** when their sum or difference is either zero or a multiple of $\pi/2$. The angles $-\theta$, $\frac{\pi}{2}\pm\theta$, $\pi\pm\theta$, $2\pi\pm\theta$ etc., are angles allied to the angle θ, if θ is measured in radians.

Note If θ is measured in degrees, then the angles allied to θ are $-\theta, 90°\pm\theta, 180°\pm\theta, 270°\pm\theta$, etc.

The trigonometric functions changes at allied angles which are given in the following table

Allied angle/ Trigonometric function	$\sin\theta$	$\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$\pi/2-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$\pi/2+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$\pi-\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$\pi+\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$3\pi/2-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$3\pi/2+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$2\pi-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$2\pi+\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$

Note

(i) Trigonometric functions for $(2\pi-\theta)$ and $(2n\pi-\theta)$ are same as those for$(-\theta)$, where $n\in Z$.

(ii) Trigonometric functions for $(2\pi+\theta)$ and $(2n\pi+\theta)$ are same as those for θ, where $n\in Z$.

> ***Knowledge Plus***
>
> **Even Function** A function $f(x)$ is said to be an even function, if $f(-x)=f(x)$ for all x in its domain.
>
> e.g. $\cos x$ is an even function, as $\cos(-x)=\cos x$.
>
> **Odd Function** A function $f(x)$ is said to be an odd function, if $f(-x)=-f(x)$ for all x in its domain.
>
> e.g. $\sin x$ is an odd function, as $\sin(-x)=-\sin x$.

METHOD TO FIND THE VALUES OF TRIGONOMETRICAL FUNCITONS FOR ANY ANGLE

(i) Check whether the given angle is positive or negative. If it is negative, then make it positive by using the following formulae

$\sin(-\theta)=-\sin\theta, \cos(-\theta)=\cos\theta, \tan(-\theta)=-\tan\theta$, etc.

(ii) If the given angle is greater than $360°$ (or 2π), then split it in the integral multiple of $360°$ (or 2π) and then use the results mention in the above note.

(iii) If the given angle is less than $360°$ (or 2π), then split it as $90°\pm\theta, 180°\pm\theta, 270°\pm\theta, 360°-\theta$ $\left(\text{or } \frac{\pi}{2}\pm\theta, \pi\pm\theta, \frac{3\pi}{2}\pm\theta, 2\pi-\theta\right)$, where θ is an acute angle.

EXAMPLE |15| Evaluate

(i) $\sin(-1125°)$ (ii) $\sin 930°$

(iii) $\cos(-870°)$ (iv) $\tan(-2025°)$

(i) Firstly, use the following formulae $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$ and $\tan(-\theta)=-\tan\theta$

(ii) Split the given angle in the integral multiple of 360° and further solve it.

Sol. (i) $\sin(-1125°)=-\sin(1125°)$ $[\because \sin(-\theta)=-\sin\theta]$

$=-\sin(1080°+45°)$

$=-\sin(3\times 360°+45°)$

$=-\sin 45°$ $[\because \sin(n\times 360°+\theta)=\sin\theta]$

$=-\frac{1}{2}$

(ii) $\sin(930°)=\sin(3\times 60°-150°)$

$=-\sin 150°$ $[\because \sin(n\times 360°-\theta)=-\sin\theta, n\in Z]$

$=-\sin(180°-30°)$

$=-\sin 30°$ $[\because \sin(180°-\theta)]=\sin\theta$

$=-\frac{1}{2}$

(iii) $\cos(-870°)=\cos 870°$ $[\because \cos(-\theta)=\cos\theta]$

$=\cos(720°+150°)=\cos(2\times 360°+150°)$

$=\cos 150°$ $[\because \cos(n\times 360°+\theta)=\cos\theta, n\in Z]$

$=\cos(180°-30°)=-\cos 30°$ $[\because \cos(180°-\theta)=-\cos\theta]$

$=-\frac{\sqrt{3}}{2}$

(iv) $\tan(-2025°)=-\tan(2025°)$ $[\because \tan(-\theta)=-\tan\theta]$

$=-\tan(6\times 360°-135°)$

$=\tan 135°$ $[\because \tan(n\times 360°-\theta)=-\tan\theta, n\in Z]$

$=\tan(90°+45°)=-\cot 45°$ $[\because \tan(90°+\theta)=-\cot\theta]$

$=-1$

EXAMPLE |16| Find the value of

(i) $\sin\left(\frac{25\pi}{3}\right)$ (ii) $\cos\left(\frac{41\pi}{4}\right)$

Sol. (i) $\sin\left(\frac{25\pi}{3}\right)=\sin\left(8\pi+\frac{\pi}{3}\right)=\sin\left(2\pi\times 4+\frac{\pi}{3}\right)=\sin\frac{\pi}{3}$

$[\because \sin(2n\pi+\theta)=\sin\theta, n\in Z]$

$=\frac{\sqrt{3}}{2}$

(ii) $\cos\left(\frac{41\pi}{4}\right)=\cos\left(10\pi+\frac{\pi}{4}\right)=\cos\left(2\pi\times 5+\frac{\pi}{4}\right)=\cos\frac{\pi}{4}$

$[\because \cos(2n\pi+\theta)=\cos\theta, n\in Z]$

$=\frac{1}{\sqrt{2}}$

EXAMPLE |17| Prove that

$$\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}=0.$$

Sol. $\text{LHS} = \cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}$

$= \cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\left(\pi-\frac{3\pi}{8}\right)+\cos\left(\pi-\frac{\pi}{8}\right)$

$= \cos\frac{\pi}{8}+\cos\frac{3\pi}{8}-\cos\frac{3\pi}{8}-\cos\frac{\pi}{8}$

$[\because \cos(\pi-\theta)=-\cos\theta]$

$= 0 = \text{RHS}$ **Hence proved.**

EXAMPLE |18| Find the value of $\sin 135° \text{ cosec } 225° \tan 150° \cot 315°$.

Sol. $\sin 135° \text{ cosec } 225° \tan 150° \cot 315°$

$= \sin(180°-45°)\text{ cosec }(180°+45°)\tan(180°-30°)\cot(360°-45°)$

$= \sin 45°(-\text{cosec } 45°)(-\tan 30°)(-\cot 45°)$

$\left[\because \sin(180°-\theta)=\sin\theta,\ \text{cosec}(180°+\theta)=-\text{cosec }\theta,\ \tan(180°-\theta)=-\tan\theta \text{ and } \cot(360°-\theta)=-\cot\theta\right]$

$= -\sin 45° \text{ cosec } 45° \tan 30° \cot 45°$

$= -\frac{1}{\sqrt{2}}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{3}}\cdot 1 = -\frac{1}{\sqrt{3}}$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 If $f(x)=\cos^2 x+\sec^2 x$, then **[NCERT Exemplar]**

(a) $f(x)<1$ (b) $f(x)=1$

(c) $2<f(x)<1$ (d) $f(x)\geq 2$

2 Which of the following is not correct? **[NCERT Exemplar]**

(a) $\sin\theta=-\frac{1}{5}$ (b) $\cos\theta=1$

(c) $\sec\theta=\frac{1}{2}$ (d) $\tan\theta=20$

3 The value of $\tan 1° \tan 2° \tan 3° \dots \tan 89°$ is **[NCERT Exemplar]**

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) Not defined

4 If $\tan\theta=3$ and θ lies in third quadrant, then the value of $\sin\theta$ is **[NCERT Exemplar]**

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$

(c) $\frac{-3}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$

5 The value of $\tan 75° - \cot 75°$ is **[NCERT Exemplar]**

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) 1

VERY SHORT ANSWER Type Questions

6 If $\sin\theta+\text{cosec }\theta=2$, then find the value of $\sin^2\theta+\text{cosec}^2\theta$. **[NCERT Exemplar]**

7 If $\tan x=\frac{3}{4}$ and x lies in III quadrant, then find the value of $\sec x$.

8 Evaluate the value of the following.

(i) $\sec\frac{15\pi}{4}$ (ii) $\sin 390°$

(iii) $\sin 690°$ (iv) $\text{cosec}(-1170°)$

(v) $\cos 135°$ (vi) $\sec 120°$

(vii) $\text{cosec } 150°$

(viii) $\tan(-945°)$

(ix) $\cot 1215°$.

9 Which of the following are correct?

(i) $\sin 1° > \sin 1$ (ii) $\sin 1° < \sin 1$

(iii) $\sin 1° = \sin 1$ (iv) $\sin 1° = \frac{\pi}{180}\sin 1$

[NCERT Exemplar]

SHORT ANSWER Type I Questions

10 Prove that $\sec^2\theta+\text{cosec}^2\theta\geq 4$. **[NCERT Exemplar]**

11 If $\cos\theta+\sin\theta=\sqrt{2}\cos\theta$, then prove that $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$.

12 Prove that

$$\frac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}=-1.$$

13 Prove that

$$2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}.$$

SHORT ANSWER Type II Questions

14 If $\cot x=-\frac{5}{12}$, x lies in II quadrant, then find the value of other five trigonometric functions. **[NCERT]**

15 If $\sec x=-2$ and $\pi<x<\frac{3\pi}{2}$, then find the values of all other five trigonometric functions.

16 Prove that

$$\cos 510° \cos 330° + \sin 390° \cos 120° = -1.$$

17 Prove that
$\sin(-420°)\cos(390°)+\cos(-660°)\sin(330°)=-1.$

18 If $\tan\theta+\sin\theta=m$ and $\tan\theta-\sin\theta=n$, show that $m^2-n^2=4\sqrt{mn}$.

19 If $10\sin^4\alpha+15\cos^4\alpha=6$. Find the value of $27\operatorname{cosec}^6\alpha+8\sec^6\alpha$.

20 If $\cos\theta=\frac{-3}{5}$ and $\pi<\theta<\frac{3\pi}{2}$, then find the values of remaining trigonometric functions and hence evaluate $\frac{\operatorname{cosec}\theta+\cot\theta}{\sec\theta-\tan\theta}$.

21 Prove that
$$\sin150°\cos120°+\cos330°\sin660°=-1.$$

HINTS & ANSWERS

1. (d) Given that, $f(x)=\cos^2 x+\sec^2 x$

We know that, AM ≥ GM

$$\frac{\cos^2 x+\sec^2 x}{2}\geq\sqrt{\cos^2 x\cdot\sec^2 x}$$

$\Rightarrow\quad \cos^2 x+\sec^2 x\geq 2 \quad [\because \cos x\cdot\sec x=1]$

$\Rightarrow\quad f(x)\geq 2$

2. (c) We know that, the range of $\sec\theta$ is $R-(-1,1)$.

Hence, $\sec\theta$ cannot be equal to $\frac{1}{2}$.

3. (b) Given expression, $\tan1°\tan2°\tan3°\ldots\tan89°$

$=\tan1°\tan2°\ldots\tan45°\cdot\tan(90°-44°)\tan(90°-43°)\ldots\tan(90°-1°)$

$=\tan1°\cdot\cot1°\cdot\tan2°\cdot\cot2°\ldots\tan89°\cdot\cot89°$

$=1\cdot1\ldots1\cdot1=1$

4. (c) Given that, $\tan\theta=3$

$\Rightarrow\quad \sec^2\theta=1+\tan^2\theta$

$\Rightarrow\quad \sec\theta=\pm\sqrt{1+9}=\pm\sqrt{10}$

$\Rightarrow\quad \sec\theta=\pm\sqrt{10}$

$\Rightarrow\quad \cos\theta=\pm\frac{1}{\sqrt{10}}$

$\Rightarrow\quad \sin\theta=\pm\sqrt{1-\frac{1}{10}}$

$=\pm\sqrt{\frac{9}{10}}=\pm\frac{3}{\sqrt{10}}$

$\therefore\quad \sin\theta=-\frac{3}{\sqrt{10}}$

[since, θ lies in third quadrant]

5. (a) Given expression, $\tan75°-\cot75°$

$$=\frac{\sin75°}{\cos75°}-\frac{\cos75°}{\sin75°}$$

$$=\frac{\sin^2 75°-\cos^2 75°}{\sin75°\cdot\cos75°}$$

$$=\frac{-2\cos150°}{\sin150°}$$

$$=\frac{-2\cos(90°+60°)}{\sin(90°+60°)}$$

$$=\frac{+2\sin60°}{\cos60°}$$

$$=\frac{2\cdot\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2\sqrt{3}$$

6. $(\sin\theta+\operatorname{cosec}\theta)^2=4\Rightarrow\sin^2\theta+\operatorname{cosec}^2\theta+2=4$

$\Rightarrow\quad \sin^2\theta+\operatorname{cosec}^2\theta=2$

7. Since, $\sec^2 x=1+\tan^2 x=1+\left(\frac{3}{4}\right)^2=\frac{25}{16}$

$\therefore\quad \sec x=-\frac{5}{4}$

8. (i) $\sec\frac{15\pi}{4}=\sec\left(4\pi-\frac{\pi}{4}\right)=\sec\frac{\pi}{4}=\sqrt{2}$

(ii) $\sin390°=\sin(360°+30°)=\sin30°=\frac{1}{2}$

(iii) $\sin690°=\sin(2\times360°-30°)=-\sin30°=-\frac{1}{2}$

(iv) Solve as part (iii). **Ans.** −1

(v) $\cos135°=\cos(90°+45°)=-\sin45°=-\frac{1}{\sqrt{2}}$

(vi) Solve as part (v). **Ans.** − 2

(vii) Solve as part (v). **Ans.** 2

(viii) $\tan(-945°)=-\tan(720°+225°)$
$=-\tan(225°)=-1$

(ix) −1

9. (ii) $\sin1°<\sin1$

10. LHS $=\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}$

$=\frac{4}{(\sin2\theta)^2}=4\operatorname{cosec}^2 2\theta.$

$\because\quad \operatorname{cosec}^2\phi\geq1$, so $4\operatorname{cosec}^2 2\theta\geq4$

11. $\because\quad (\cos\theta+\sin\theta)^2=(\sqrt{2}\cos\theta)^2$

$\Rightarrow\quad 1+2\sin\theta\cos\theta=2\cos^2\theta$

$\Rightarrow\quad 1+2\sin\theta\cos\theta=2-2\sin^2\theta$

$\Rightarrow\quad 2\sin^2\theta=1-2\sin\theta\cos\theta$

$\Rightarrow\quad 2\sin^2\theta=\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta$

14. $\tan x = \dfrac{1}{\cot x}$; $\sec x = -\sqrt{1+\tan^2 x}$; $\cos x = \dfrac{1}{\sec x}$

$$\sin x = \sqrt{1-\cos^2 x}$$

and $\operatorname{cosec} x = \dfrac{1}{\sin x}$

Ans. $\tan x = \dfrac{-12}{5}$, $\sec x = \dfrac{-13}{5}$, $\cos x = \dfrac{-5}{13}$,

$\sin x = \dfrac{12}{13}$, $\operatorname{cosec} x = \dfrac{13}{12}$

15. $\cos x = \dfrac{1}{\sec x}$; $\tan x = +\sqrt{\sec^2 x - 1}$;

$\cot x = \dfrac{1}{\tan x}$; $\sin x = -\sqrt{1-\cos^2 x}$

and $\operatorname{cosec} x = \dfrac{1}{\sin x}$

$\therefore\quad \sin x = -\dfrac{\sqrt{3}}{2}$, $\cos x = -\dfrac{1}{2}$, $\tan x = \sqrt{3}$, $\cot x = \dfrac{1}{\sqrt{3}}$

and $\operatorname{cosec} x = \dfrac{-2}{\sqrt{3}}$

18. LHS $= m^2 - n^2$

$= (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2$

$= (\tan\theta + \sin\theta + \tan\theta - \sin\theta)$

$(\tan\theta + \sin\theta - \tan\theta + \sin\theta)$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$= (2\tan\theta)(2\sin\theta)$

$\Rightarrow m^2 - n^2 = 4\tan\theta\sin\theta$ ... (i)

RHS $= 4\sqrt{mn}$

$= 4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)}$

$= 4\sqrt{\tan^2\theta - \sin^2\theta}$ $[\because (a+b)(a-b) = (a^2 - b^2)]$

$= 4\sqrt{\sin^2\theta\left(\dfrac{1}{\cos^2\theta} - 1\right)}$

$= 4\sqrt{\sin^2\theta(\sec^2\theta - 1)}$

$= 4\sqrt{\sin^2\theta\tan^2\theta}$

$\Rightarrow\quad 4\sqrt{mn} = 4\sin\theta\tan\theta$ ... (ii)

From Eqs. (i) and (ii),

LHS = RHS **Hence proved.**

19. Given, $10\sin^4\alpha + 15\cos^4\alpha = 6$

$\Rightarrow\quad 10\sin^4\alpha + 15\cos^4\alpha = 6(\sin^2\alpha + \cos^2\alpha)^2$

$[\because \sin^2\alpha + \cos^2\alpha = 1]$

$\Rightarrow\quad 4\tan^4\alpha + 9 - 12\tan^2\alpha = 0$

$\Rightarrow (2\tan^2\alpha)^2 + (3)^2 - 2\times 3\times 2\tan^2\alpha = 0$

$\Rightarrow\quad (2\tan^2\alpha - 3)^2 = 0$

$\Rightarrow\quad \tan^2\alpha = \dfrac{3}{2}$

$\therefore\quad 27\operatorname{cosec}^6\alpha + 8\sec^6\alpha = \operatorname{cosec}^6\alpha(27 + 8\tan^6\alpha)$

$= (1+\cot^2\alpha)^3\left(27 + 8\times\dfrac{27}{8}\right)$

$= \left(1+\dfrac{2}{3}\right)^3(54)$

$= \left(\dfrac{5}{3}\right)^3 \times 54 = \dfrac{125}{27}\times 54 = 250$

20. We have,

$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\dfrac{3}{5}\right)^2 = \dfrac{16}{25}$

$\Rightarrow\quad \sin\theta = \pm\dfrac{4}{5}$; but $\sin\theta$ is $-$ve in third quadrant.

$\therefore\quad \sin\theta = -\dfrac{4}{5}$; $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{-5}{3}$;

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{-5}{4}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-4/5}{-3/5} = \dfrac{4}{3}$

and $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\left(\dfrac{4}{3}\right)} = \dfrac{3}{4}$

Now, $\dfrac{\operatorname{cosec}\theta + \cot\theta}{\sec\theta - \tan\theta} = \dfrac{-\dfrac{5}{4}+\dfrac{3}{4}}{-\dfrac{5}{3}-\dfrac{4}{3}}$

$= \dfrac{-2/4}{-9/3} = \dfrac{1}{6}$

21. LHS $= \sin 150° \cos 120° + \cos 330° \sin 660°$

$= \sin(90° + 60°)\cos(180° - 60°)$

$+ \cos(360° - 30°)\sin(2\times 360° - 60°)$

$= \cos 60°(-\cos 60°) + (\cos 30°)(-\sin 60°)$

$= \dfrac{1}{2}\cdot\left(-\dfrac{1}{2}\right) + \dfrac{\sqrt{3}}{2}\cdot\left(-\dfrac{\sqrt{3}}{2}\right)$

$= -\dfrac{1}{4} - \dfrac{3}{4}$

$= -1 =$ RHS **Hence proved.**

|TOPIC 3|
Trigonometric Functions of Compound Angles

The algebraic sum of two or more angles are generally called **compound angles** and the angles are known as the **constituent angles.**

e.g. If A, B and C are three angles, then $A \pm B$, $A+B+C$, $A-B+C$ etc., are **compound angles.**

Now, some formulae or identities which will express the trigonometric ratios of compound angles in terms of trigonometric ratios of constituent angles are given below.

Formula 1. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Formula 2. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Formula 3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Formula 4. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Formula 5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, where A, B and $(A+B)$ are not odd multiples of $\dfrac{\pi}{2}$.

Formula 6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, where A, B and $(A-B)$ are not odd multiples of $\dfrac{\pi}{2}$.

Formula 7. $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$, where A, B and $(A+B)$ are not multiples of π

Formula 8. $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$, where A, B and $(A-B)$ are not multiples of π

Some Important Results

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

EXAMPLE |1| Evaluate $\cos 47° \cos 13° - \sin 47° \sin 13°$

Sol. $\cos 47° \cos 13° - \sin 47° \sin 13° = \cos(47° + 13°)$

$[\because \cos A \cos B - \sin A \sin B = \cos(A+B)]$

$= \cos 60° = \dfrac{1}{2}$

EXAMPLE |2| Find the value of sin 75°.

Sol. $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}$

$\left[\because \sin 30° = \dfrac{1}{2}, \cos 30° = \dfrac{\sqrt{3}}{2} \text{ and } \sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}\right]$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE |3| Find the value of $\sin 23° \cos 67° + \cos 23° \sin 67°$.

Sol. $\sin 23° \cos 67° + \cos 23° \sin 67°$

$= \sin(23° + 67°)$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \sin 90° = 1$

EXAMPLE |4| Evaluate $\sin 78° \cos 18° - \cos 78° \sin 18°$.

Sol. $\sin 78° \cos 18° - \cos 78° \sin 18° = \sin(78° - 18°)$

$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$= \sin 60° = \dfrac{\sqrt{3}}{2}$

EXAMPLE |5| Prove that

$$\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} = 0.$$

Sol. $\text{LHS} = \dfrac{\sin(A-B)}{\cos A \cos B} + \dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A}$

$= \dfrac{\cos C \sin(A-B) + \cos A \sin(B-C) + \cos B \sin(C-A)}{\cos A \cos B \cos C}$

$$= \frac{\begin{bmatrix} \cos C(\sin A \cos B - \cos A \sin B) + \cos A(\sin B \cos C \\ - \cos B \sin C) + \cos B(\sin C \cos A - \cos C \sin A) \end{bmatrix}}{\cos A \cos B \cos C}$$

$$= \frac{\begin{bmatrix} \sin A \cos B \cos C - \cos A \cos C \sin B + \cos A \\ \cos C \sin B - \cos A \cos B \sin C + \cos B \cos A \\ \sin C - \cos B \cos C \sin A \end{bmatrix}}{\cos A \cos B \cos C}$$

$$= \frac{0}{\cos A \cos B \cos C}$$

$= 0 =$ RHS **Hence proved.**

EXAMPLE |6| Find the value of $\tan 105°$.

Sol. $\tan 105° = \tan(60° + 45°)$

$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°} \quad \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} \quad [\because \tan 45° = 1 \text{ and } \tan 60° = \sqrt{3}]$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{(1+\sqrt{3})}{(1-\sqrt{3})} \times \frac{(1+\sqrt{3})}{(1+\sqrt{3})} \quad \text{[on rationalisation]}$$

$$= \frac{(\sqrt{3}+1)^2}{1-3} \quad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{3+1+2\sqrt{3}}{1-3} = \frac{4+2\sqrt{3}}{-2} = \frac{2(2+\sqrt{3})}{-2} = -(2+\sqrt{3})$$

EXAMPLE |7| If $\tan A = \frac{1}{4}$ and $\tan B = \frac{1}{5}$, then find the value of $\tan(A+B)$.

Sol. Given, $\tan A = \frac{1}{4}$ and $\tan B = \frac{1}{5}$

We know that, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{4}\times\frac{1}{5}} = \frac{\frac{5+4}{20}}{\frac{20-1}{20}} = \frac{\frac{9}{20}}{\frac{19}{20}} = \frac{9}{19}$$

EXAMPLE |8| Prove that $\frac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°} = \sqrt{3}$.

Sol. LHS $= \frac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°} = \frac{\frac{\cos 15°}{\cos 15°} + \frac{\sin 15°}{\cos 15°}}{\frac{\cos 15°}{\cos 15°} - \frac{\sin 15°}{\cos 15°}}$

[dividing numerator and denominator by $\cos 15°$]

$$= \frac{1 + \tan 15°}{1 - \tan 15°} \quad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$= \frac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°} \quad [\because \tan 45° = 1]$$

$$= \tan(45° + 15°) \quad \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$= \tan 60° = \sqrt{3} =$ RHS **Hence proved.**

EXAMPLE |9| Find the value of $\tan 15°$.

Sol. $\tan 15° = \tan(60° - 45°)$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°} \quad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1} \quad [\because \tan 45° = 1 \text{ and } \tan 60° = \sqrt{3}]$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \quad \text{[on rationalisation]}$$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - (1)^2} \quad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{(\sqrt{3})^2 + (1)^2 - 2\sqrt{3}}{3-1} \quad [\because (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= \frac{3+1-2\sqrt{3}}{3-1} = \frac{4-2\sqrt{3}}{2} = \frac{2(2-\sqrt{3})}{2} = (2-\sqrt{3})$$

EXAMPLE |10| Find the value of $\frac{\cot 47° \cot 43° - 1}{\cot 47° + \cot 43°}$.

Sol. $\frac{\cot 47° \cot 43° - 1}{\cot 47° + \cot 43°} = \cot(47° + 43°)$

$$\left[\because \cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}\right]$$

$= \cot 90° = 0 \quad [\because \cot 90° = 0]$

EXAMPLE |11| Show that
$\sin(40° + \theta)\cos(10° + \theta) - \cos(40° + \theta)\sin(10° + \theta) = \frac{1}{2}$.

[NCERT]

Use the formula, $\sin A \cos B - \cos A \sin B = \sin(A - B)$ and then simplify it.

Sol. LHS

$= \sin(40° + \theta)\cos(10° + \theta) - \cos(40° + \theta)\cdot\sin(10° + \theta)$

$= \sin\{(40° + \theta) - (10° + \theta)\}$

$[\because \sin(x - y) = \sin x \cdot \cos y - \cos x \cdot \sin y]$

$= \sin 30° = \frac{1}{2} \quad \left[\because \sin 30° = \frac{1}{2}\right]$

= RHS **Hence proved.**

EXAMPLE |12| Prove that

$$\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2.$$

[NCERT]

Sol. LHS $= \dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \dfrac{\left(\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\cdot\tan x}\right)}{\left(\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right)}$

$\left[\because \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\cdot\tan y} \text{ and } \tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\cdot\tan y}\right]$

$= \dfrac{\left(\dfrac{1+\tan x}{1-\tan x}\right)}{\left(\dfrac{1-\tan x}{1+\tan x}\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |13| If $A+B=\dfrac{\pi}{4}$, then prove that

(i) $(1+\tan A)(1+\tan B)=2$

(ii) $(\cot A-1)(\cot B-1)=2$.

Sol. (i) We have, $A+B=\dfrac{\pi}{4}$

$\Rightarrow \tan(A+B)=\tan\dfrac{\pi}{4}$

$\Rightarrow \dfrac{\tan A+\tan B}{1-\tan A\tan B}=1$

$\Rightarrow \tan A+\tan B=1-\tan A\tan B$

$\Rightarrow \tan A+\tan B+\tan A\tan B=1$ ...(i)

On adding 1 both sides of Eq. (i), we get

$1+\tan A+\tan B+\tan A\tan B=2$

$\Rightarrow (1+\tan A)+\tan B(1+\tan A)=2$

$\Rightarrow (1+\tan A)(1+\tan B)=2$

Hence proved.

(ii) On dividing both sides of Eq. (i) by $\tan A\tan B$, we get

$\dfrac{\tan A+\tan B+\tan A\tan B}{\tan A\tan B}=\dfrac{1}{\tan A\tan B}$

$\Rightarrow \cot B+\cot A+1=\cot A\cot B$

$\Rightarrow \cot A\cot B-\cot A-\cot B=1$

$\Rightarrow \cot A\cot B-\cot A-\cot B+1=2$

$\Rightarrow \cot A(\cot B-1)-(\cot B-1)=2$

$\Rightarrow (\cot B-1)(\cot A-1)=2$ **Hence proved.**

EXAMPLE |14| If $3\tan A\tan B=1$, then prove that $2\cos(A+B)=\cos(A-B)$.

Sol. Given, $3\tan A\tan B=1$

$\Rightarrow \dfrac{3\sin A\sin B}{\cos A\cos B}=1 \Rightarrow \dfrac{\cos A\cos B}{\sin A\sin B}=\dfrac{3}{1}$

$\Rightarrow \dfrac{\cos A\cos B+\sin A\sin B}{\cos A\cos B-\sin A\sin B}=\dfrac{3+1}{3-1}$

[applying componendo and dividendo]

$\Rightarrow \dfrac{\cos(A-B)}{\cos(A+B)}=\dfrac{4}{2}$

$\left[\because \cos(A-B)=\cos A\cos B+\sin A\sin B \text{ and } \cos(A+B)=\cos A\cos B-\sin A\sin B\right]$

$\Rightarrow \dfrac{\cos(A-B)}{\cos(A+B)}=2$

$\Rightarrow \cos(A-B)=2\cos(A+B)$

Hence proved.

EXAMPLE |15| Prove that

$\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}$. **[NCERT]**

Sol. LHS $=\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$

$=\dfrac{\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}}$

[dividing numerator and denominator by $\cos x\cos y$]

$=\dfrac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{\dfrac{\sin x}{\cos x}-\dfrac{\sin y}{\cos y}}=\dfrac{\tan x+\tan y}{\tan x-\tan y}=\text{RHS}$ **Hence proved.**

EXAMPLE |16| If $\tan A-\tan B=x$ and $\cot B-\cot A=y$, then prove that $\cot(A-B)=\dfrac{1}{x}+\dfrac{1}{y}$.

Sol. Given, $\tan A-\tan B=x$ and $\cot B-\cot A=y$

$\Rightarrow \dfrac{\tan A-\tan B}{\tan A\tan B}=y$

$\Rightarrow \dfrac{x}{\tan A\tan B}=y \Rightarrow \tan A\tan B=\dfrac{x}{y}$

$\therefore \cot(A-B)=\dfrac{1}{\tan(A-B)}=\dfrac{1+\tan A\tan B}{\tan A-\tan B}$

$=\dfrac{1+\dfrac{x}{y}}{x}=\dfrac{x+y}{xy}=\dfrac{1}{x}+\dfrac{1}{y}$ **Hence proved.**

EXAMPLE |17| If $\sin A=\dfrac{4}{5}$ and $\cos B=\dfrac{5}{13}$, where $0<A, B<\dfrac{\pi}{2}$, then find the values of the following.

(i) $\sin(A+B)$ (ii) $\cos(A+B)$

(iii) $\sin(A-B)$ (iv) $\cos(A-B)$

Sol. Given, $\sin A = \frac{4}{5}, 0 < A < \frac{\pi}{2}$

$\therefore \quad \cos A = \sqrt{1 - \sin^2 A}$ [$\because A$ lies in Ist quadrant]

$= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}}$

$\Rightarrow \quad \cos A = \sqrt{\frac{9}{25}} = \frac{3}{5}$ and $\cos B = \frac{5}{13}, \ 0 < B < \frac{\pi}{2}$

$\therefore \quad \sin B = \sqrt{1 - \cos^2 B}$ [$\because B$ lies in Ist quadrant]

$= \sqrt{1 - \left(\frac{5}{13}\right)^2} = \sqrt{1 - \frac{25}{169}}$

$\Rightarrow \quad \sin B = \sqrt{\frac{144}{169}} = \frac{12}{13}$

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$= \frac{3}{5} \times \frac{5}{13} - \frac{4}{5} \times \frac{12}{13} = \frac{15 - 48}{65} = \frac{-33}{65}$

(iii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$= \frac{4}{5} \times \frac{5}{13} - \frac{3}{5} \times \frac{12}{13} = \frac{20 - 36}{65} = \frac{-16}{65}$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15 + 48}{65} = \frac{63}{65}$

EXAMPLE |18| If x and y are acute angles such that $\cos x = \frac{13}{14}$ and $\cos y = \frac{1}{7}$, then prove that $x - y = -\frac{\pi}{3}$.

Sol. Given, $0 < x, y < \frac{\pi}{2}, \cos x = \frac{13}{14}$ and $\cos y = \frac{1}{7}$

$\because \quad 0 < x < \frac{\pi}{2}$

$\therefore \sin x = \sqrt{1 - \cos^2 x}$ [$\because$ sine is +ve in first quadrant]

$= \sqrt{1 - \frac{169}{196}} = \sqrt{\frac{196 - 169}{196}}$

$= \sqrt{\frac{27}{196}} = \frac{3\sqrt{3}}{14}$

Similarly, $\sin y = \sqrt{1 - \cos^2 y}$

[$\because 0 < y < \pi/2$ and sine is +ve in first quadrant]

$= \sqrt{1 - \frac{1}{49}} = \sqrt{\frac{48}{49}}$

$= \frac{4\sqrt{3}}{7}$

Now, consider $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$= \frac{13}{14} \times \frac{1}{7} + \frac{3\sqrt{3}}{14} \times \frac{4\sqrt{3}}{7}$

$= \frac{13 + 36}{14 \times 7} = \frac{49}{14 \times 7} = \frac{1}{2}$ ...(i)

Since, $\cos x > \cos y$, therefore $y > x$

[$\because \cos x$ is a decreasing function]

$\Rightarrow \quad x - y < 0$

Now, as $\cos\left(\frac{-\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$

$\therefore \quad x - y = \frac{-\pi}{3}$ **Hence proved.**

EXAMPLE |19| Find the value of $\tan(\alpha + \beta)$, if $\cot \alpha = \frac{1}{2}, \alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec \beta = \frac{-5}{3}, \beta \in \left(\frac{\pi}{2}, \pi\right)$.

Sol. We have, $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$

$\left[\because \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}\right]$

Given, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$ $\left[\because \tan\alpha = \frac{1}{\cot\alpha}\right]$

Also, $\sec\beta = \frac{-5}{3}$

Then, $\tan\beta = \pm\sqrt{\sec^2\beta - 1}$

$\Rightarrow \quad \tan\beta = \pm\sqrt{\frac{25}{9} - 1} = \pm\sqrt{\frac{16}{9}}$ $\left[\because \sec\beta = \frac{-5}{3}\right]$

$\therefore \quad \tan\beta = \pm\frac{4}{3}$

But, $\quad \tan\beta \neq \frac{4}{3}$

$\left[\because \beta \in \left(\frac{\pi}{2}, \pi\right) \text{and } \tan\beta \text{ is } -\text{ve in IInd quadrant}\right]$

$\therefore \quad \tan\beta = -\frac{4}{3}$

$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \frac{2 + \left(-\frac{4}{3}\right)}{1 - (2)\left(\frac{-4}{3}\right)}$

$\left[\because \tan\alpha = 2 \text{ and } \tan\beta = \frac{-4}{3}\right]$

$= \frac{\left(2 - \frac{4}{3}\right)}{\left(1 + \frac{8}{3}\right)} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{11}{3}\right)} = \frac{2}{11}$

EXAMPLE |20| If

$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$, then prove that

$$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0.$$

Sol. Given, $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$

$\Rightarrow \quad 2\cos(\alpha-\beta)+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)=-3$

$\Rightarrow (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)$
$+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)$
$+1+1+1=0$

$\Rightarrow (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)$
$+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma)+2\sin\gamma\sin\alpha)$
$+(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)$
$+(\cos^2\gamma+\sin^2\gamma)=0$

$\Rightarrow (\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta$
$+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)$
$+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma$
$+2\sin\gamma\sin\alpha)=0$

$\Rightarrow (\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2=0$

$\Rightarrow \quad \cos\alpha+\cos\beta+\cos\gamma=0$

and $\quad \sin\alpha+\sin\beta+\sin\gamma=0$ **Hence proved.**

EXAMPLE |21| If

$a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$, where $\alpha\neq\beta$.

Prove that $a\cos\beta=b\cos\alpha$.

Sol. Given, $a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$,

where $\alpha\neq\beta$

$\Rightarrow a\left\{\tan\alpha-\tan\left(\frac{\alpha+\beta}{2}\right)\right\}=b\left\{\tan\left(\frac{\alpha+\beta}{2}\right)-\tan\beta\right\}$

$$\Rightarrow \quad \frac{a\sin\left(\alpha-\frac{\alpha+\beta}{2}\right)}{\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{b\sin\left(\frac{\alpha+\beta}{2}-\beta\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\beta}$$

$$\left[\because \tan A-\tan B=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}=\frac{\sin(A-B)}{\cos A\cos B}\right]$$

$$\Rightarrow \quad \frac{a\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\alpha}=\frac{b\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\beta}$$

$\Rightarrow \quad a\cos\beta=b\cos\alpha$

$$\left[\because \alpha\neq\beta,\text{ so }\sin\left(\frac{\alpha-\beta}{2}\right)\neq 0\right]$$

Hence proved.

EXAMPLE |22| If $\tan\alpha=\frac{1}{\sqrt{x(x^2+x+1)}}$,

$\tan\beta=\frac{\sqrt{x}}{\sqrt{x^2+x+1}}$ and $\tan\gamma=\sqrt{x^{-3}+x^{-2}+x^{-1}}$,

then prove that $\alpha+\beta=\gamma$.

Sol. We know that,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{\frac{1}{\sqrt{x(x^2+x+1)}}+\frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1-\frac{\sqrt{x}}{\sqrt{x(x^2+x+1)}\sqrt{x^2+x+1}}}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{\frac{1+x}{\sqrt{x(x^2+x+1)}}}{1-\frac{1}{x^2+x+1}}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{1+x}{\sqrt{x(x^2+x+1)}}\times\frac{x^2+x+1}{x^2+x+1-1}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{(1+x)(x^2+x+1)}{x(x+1)\sqrt{x(x^2+x+1)}}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{\sqrt{x(x^2+x+1)}}{x^2}$$

$$=\sqrt{\frac{x^3+x^2+x}{x^4}}$$

$$=\sqrt{x^{-1}+x^{-2}+x^{-3}}$$

$\Rightarrow \quad \tan(\alpha+\beta)=\tan\gamma$

$\therefore \quad \alpha+\beta=\gamma$ **Hence proved.**

EXAMPLE |23| Prove that

$\sin^2 A=\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B$.

Sol. RHS $=\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B$

$=\cos^2 B+\cos^2(A-B)-2\cos(A-B)\cos A\cos B$

$=\cos^2 B+\cos(A-B)\{\cos(A-B)-2\cos A\cos B\}$

$=\cos^2 B+\cos(A-B)\{\cos A\cos B$
$+\sin A\sin B-2\cos A\cos B\}$

$=\cos^2 B+\cos(A-B)[\sin A\sin B-\cos A\cos B]$

$=\cos^2 B+\cos(A-B)[-(\cos A\cos B-\sin A\sin B)]$

$=\cos^2 B-\cos(A-B)\cos(A+B)$

$=\cos^2 B-(\cos^2 A-\sin^2 B)$

$=\cos^2 B+\sin^2 B-\cos^2 A$

$=1-\cos^2 A=\sin^2 A=$ LHS **Hence proved.**

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$ is **[NCERT Exemplar]**

(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

2 The value of $\sin(45° + \theta) - \cos(45° - \theta)$ is **[NCERT Exemplar]**

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) 1 (d) 0

3 The value of $\cot\left(\frac{\pi}{4} + \theta\right)\cot\left(\frac{\pi}{4} - \theta\right)$ is **[NCERT Exemplar]**

(a) -1 (b) 0 (c) 1 (d) Not defined

4 The value of $\tan 3A - \tan 2A - \tan A$ is **[NCERT Exemplar]**

(a) $\tan 3A \tan 2A \tan A$
(b) $-\tan 3A \tan 2A \tan A$
(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$
(d) None of the above

5 If $\theta + \phi = \alpha$ and $\tan\theta = p\tan\phi$, then $\sin(\theta - \phi)$ is equal to

(a) $\frac{p-1}{p+1}\sin\alpha$ (b) $\frac{p-1}{p+1}\cos\alpha$
(c) $\frac{p-1}{p+1}$ (d) $\frac{p+1}{p-1}$

VERY SHORT ANSWER Type Questions

6 Find the value of the following trigonometric functions.

(i) $\sin(105°)$ (ii) $\cos(-75°)$
(iii) $\tan(435°)$ (iv) $\sec(105°)$
(v) $\sin 15°$

7 Evaluate the following.

(i) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$
(ii) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$

8 Find the value of $\sin(45° + \theta) - \cos(45° - \theta)$. **[NCERT Exemplar]**

9 If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, then find the value of $\alpha + \beta$. **[NCERT Exemplar]**

10 Find the value of $\tan 3A - \tan 2A - \tan A$. **[NCERT Exemplar]**

11 Prove that $\frac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$.

12 Find the values of the following.

(i) $\tan 75°$ (ii) $\tan\frac{13\pi}{12}$

13 Prove that $\frac{\sqrt{3}\cos 23° - \sin 23°}{2} = \cos 53°$.

SHORT ANSWER Type I Questions

14 Prove that $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cdot \cos(n+2)x = \cos x$. **[NCERT]**

15 Show that

$$\cos\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \phi\right) - \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} - \phi\right) = \sin(\theta + \phi).$$ **[NCERT]**

16 Prove that $\tan 15° + \cot 15° = 4$.

17 Prove that

$$\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x.$$

18 Prove that $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

19 Prove that

$$\cos 2\alpha \cos 2\beta + \sin^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2(\alpha + \beta).$$

SHORT ANSWER Type II Questions

20 If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$; $\frac{3\pi}{2} < A, B < 2\pi$, then find the values of the following.

(i) $\cos(A + B)$ (ii) $\sin(A - B)$

21 If $\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2}$ and $\cos B = -\frac{12}{13}, \pi < B < \frac{3\pi}{2}$, then find the following.

(i) $\sin(A - B)$ (ii) $\cos(A + B)$
(iii) $\tan(A - B)$

22 If x and y are acute angles such that $\sin x = \frac{1}{\sqrt{5}}$ and $\sin y = \frac{1}{\sqrt{10}}$, then prove that $x + y = \frac{\pi}{4}$.

23 If $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, then prove that $\tan 2\alpha=\frac{56}{33}$.

24 If $\theta+\phi=\alpha$ and $\tan\theta=k\tan\phi$, then prove that $\sin(\theta-\phi)=\frac{k-1}{k+1}\sin\alpha$.

25 If $\cos(\theta+\phi)=m\cos(\theta-\phi)$, then find the value of $\frac{1-m}{1+m}\cot\phi$. **[NCERT Exemplar]**

HINTS & ANSWERS

1. (d) Use $\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\cdot\tan\phi}$.

2. (d) Use formula, $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and $\cos(A-B)=\cos A\cos B+\sin A\sin B$

3. (a) Use formulae,

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

and $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot A-\cot B}$$

4. (a) $\tan 3A=\tan(2A+A)$

$$\Rightarrow \tan 3A=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}$$

5. (a) Given, $\tan\theta=p\tan\phi$

$$\Rightarrow \frac{\tan\theta}{\tan\phi}=\frac{p}{1}\Rightarrow\frac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\frac{p+1}{p-1}$$

[applying componendo and dividendo]

$$\Rightarrow \frac{\sin\theta\cdot\cos\phi+\cos\theta\sin\phi}{\sin\theta\cdot\cos\phi-\cos\theta\sin\phi}=\frac{p+1}{p-1}$$

$$\Rightarrow \frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}=\frac{p+1}{p-1}$$

$$\Rightarrow \sin(\theta-\phi)=\frac{p-1}{p+1}\sin\alpha$$

6. (i) $\sin(105^\circ)=\sin(60^\circ+45^\circ)$

$$=\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos(-75^\circ)=\cos 75^\circ=\cos(45^\circ+30^\circ)$

$$=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) $\tan(435^\circ)=\tan(360^\circ+75^\circ)=\tan 75^\circ$

$$=\tan(45^\circ+30^\circ)$$

$$=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

(iv) $\because \cos(105^\circ)=\cos(60^\circ+45^\circ)$

$$=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$$

$$=\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$\therefore \sec 105^\circ=\frac{2\sqrt{2}}{1-\sqrt{3}}$$

(v) $\sin 15^\circ=\sin(45^\circ-30^\circ)$

$$=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

7. (i) Use formula $\sin(A+B)$. **Ans.** $\frac{\sqrt{3}}{2}$

(ii) Use formula $\sin(A-B)$. **Ans.** $\frac{\sqrt{3}}{2}$

8. $\sin(45^\circ+\theta)-\cos(45^\circ-\theta)$

$$=(\sin 45^\circ\cos\theta+\cos 45^\circ\sin\theta)-(\cos 45^\circ\cos\theta+\sin 45^\circ\sin\theta)$$

$$=0$$

9. $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\times\frac{1}{2m+1}}$$

$$\Rightarrow \tan(\alpha+\beta)=1\therefore\alpha+\beta=\frac{\pi}{4}$$

10. $\tan 3A=\tan(2A+A)$

$$\Rightarrow \tan 3A=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}$$

$\therefore \tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A$

11. Solve as Example 8.

12. (i) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

(ii) $\tan\frac{13\pi}{12}=\tan\left(\pi+\frac{\pi}{12}\right)=\tan\frac{\pi}{12}=\tan 15^\circ$

$$=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

13. $\text{LHS}=\frac{\sqrt{3}\cos 23^\circ-\sin 23^\circ}{2}=\frac{\sqrt{3}}{2}\cdot\cos 23^\circ-\frac{1}{2}\cdot\sin 23^\circ$

$=\cos 30^\circ\cdot\cos 23^\circ-\sin 30^\circ\cdot\sin 23^\circ$

$=\cos(30^\circ+23^\circ)$

$=\cos 53^\circ=\text{RHS}$ **Hence proved.**

14. $\text{LHS}=\sin(n+1)x\cdot\sin(n+2)x+\cos(n+1)x\cdot\cos(n+2)x$

$=\cos(n+1)x\cdot\cos(n+2)x+\sin(n+1)x\cdot\sin(n+2)x$

$=\cos[(n+1)x-(n+2)x]$

15. Use the formula of $\cos(A+B)$.

16. $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$\therefore \cot 15° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

17. $\cos\left(\dfrac{3\pi}{4} + x\right) - \cos\left(\dfrac{3\pi}{4} - x\right)$

$= \left(-\dfrac{1}{\sqrt{2}}\cos x - \dfrac{1}{\sqrt{2}}\sin x\right) - \left(-\dfrac{1}{\sqrt{2}}\cos x + \dfrac{1}{\sqrt{2}}\sin x\right)$

$= -\sqrt{2}\sin x$

18. Use the formula,
$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$ and then simplify it.

19. $\cos 2\alpha \cos 2\beta + \sin(\alpha - \beta + \alpha + \beta)\sin[\alpha - \beta - (\alpha + \beta)]$

$= \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin(-2\beta)$

$= \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

20. Given, $\cos A = \dfrac{4}{5}, \dfrac{3\pi}{2} < A < 2\pi$

$\therefore \quad \sin A = -\sqrt{1 - \cos^2 A} = -\dfrac{3}{5}$

[$\because$ A lies in IVth quadrant]

and $\quad \cos B = \dfrac{12}{13}, \dfrac{3\pi}{2} < B < 2\pi$

$\therefore \quad \sin B = -\sqrt{1 - \cos^2 B} = -\dfrac{5}{13}$

[$\because$ B lies in IVth quadrant]

(i) $\cos(A+B) = \dfrac{4}{5} \times \dfrac{12}{13} - \left(-\dfrac{3}{5}\right)\left(-\dfrac{5}{13}\right)$

$= \dfrac{48}{65} - \dfrac{15}{65} = \dfrac{33}{65}$

(ii) $\sin(A-B) = \left(-\dfrac{3}{5}\right) \times \dfrac{12}{13} - \dfrac{4}{5} \times \left(-\dfrac{5}{13}\right)$

$= -\dfrac{36}{65} + \dfrac{20}{65} = -\dfrac{16}{65}$

21. $\cos A = \sqrt{1 - \dfrac{9}{25}} = \dfrac{4}{5},$

$\sin B = -\sqrt{1 - \dfrac{144}{169}} = -\dfrac{5}{13}$

(i) $-\dfrac{16}{65}$ (ii) $-\dfrac{33}{65}$ (iii) $\dfrac{16}{63}$

22. Solve as Example 18.

23. Since, α, β lie between 0 and $\dfrac{\pi}{4}$.

$\therefore \quad -\dfrac{\pi}{4} < \alpha - \beta < \dfrac{\pi}{4}$ and $0 < \alpha + \beta < \dfrac{\pi}{2}$

$\Rightarrow \cos(\alpha - \beta)$ and $\sin(\alpha + \beta)$ are positive.

Now, $\sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)} = \dfrac{3}{5}$

and $\quad \cos(\alpha - \beta) = \sqrt{1 - \sin^2(\alpha - \beta)} = \dfrac{12}{13}$

$\therefore \quad \tan(\alpha + \beta) = \dfrac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$

and $\quad \tan(\alpha - \beta) = \dfrac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \dfrac{5/13}{12/13} = \dfrac{5}{12}$

Now, $\tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$= \dfrac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \dfrac{\dfrac{3}{4} + \dfrac{5}{12}}{1 - \dfrac{3}{4} \times \dfrac{5}{12}} = \dfrac{56}{33}$

Hence proved.

24. Given, $\quad \tan\theta = k\tan\phi$

$\Rightarrow \quad \dfrac{\tan\theta}{\tan\phi} = \dfrac{k}{1} \Rightarrow \dfrac{\tan\theta + \tan\phi}{\tan\theta - \tan\phi} = \dfrac{k+1}{k-1}$

[applying componendo and dividendo]

$\Rightarrow \quad \dfrac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\phi}{\cos\phi}}{\dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\phi}{\cos\phi}} = \dfrac{k+1}{k-1}$

$\Rightarrow \quad \dfrac{\sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi}{\sin\theta \cdot \cos\phi - \cos\theta \cdot \sin\phi} = \dfrac{k+1}{k-1}$

$\Rightarrow \dfrac{\sin(\theta + \phi)}{\sin(\theta - \phi)} = \dfrac{k+1}{k-1} \Rightarrow \dfrac{\sin(\theta - \phi)}{\sin(\theta + \phi)} = \dfrac{k-1}{k+1}$

$\Rightarrow \quad \sin(\theta - \phi) = \dfrac{k-1}{k+1} \cdot \sin\alpha$ **Hence proved.**

25. We have, $\cos(\theta + \phi) = m\cos(\theta - \phi)$

$\Rightarrow \quad \dfrac{1}{m} = \dfrac{\cos(\theta - \phi)}{\cos(\theta + \phi)}$

Now, $\quad \dfrac{1-m}{1+m} = \dfrac{\cos(\theta - \phi) - \cos(\theta + \phi)}{\cos(\theta - \phi) + \cos(\theta + \phi)} = \tan\theta \tan\phi$

Ans. $\tan\theta$

|TOPIC 4|

Transformation Formulae

In this topic, we deals with mainly two types of transformation

(i) Transformation of product into sum or difference.

(ii) Transformation of sum or difference into product.

TRANSFORMATION OF PRODUCT INTO SUM OR DIFFERENCE

Here, we transform the products of two sines or two cosines or one sine and one cosine of constituent angles into the sum or difference of two sines or two cosines of compound angles.

(i) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(ii) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(iii) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(iv) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

EXAMPLE |1| Find the value of the following functions.

(i) $2\cos 45°\sin 15°$ (ii) $2\sin 15°\cos 75°$

(iii) $\cos 315°\cos 75°$

Sol. (i) $2\cos 45°\sin 15°$

$= \sin(45°+15°) - \sin(45°-15°)$ $[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$

$= \sin 60° - \sin 30°$

$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$ $\left[\because \sin 60° = \frac{\sqrt{3}}{2} \text{ and } \sin 30° = \frac{1}{2}\right]$

(ii) $2\sin 15°\cos 75° = \sin(15°+75°) + \sin(15°-75°)$

$[\because 2\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)]$

$= \sin 90° + \sin(-60°)$

$= \sin 90° - \sin 60°$ $[\because \sin(-\theta) = -\sin\theta]$

$= 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$

(iii) $\cos 315°\cos 75° = \frac{1}{2}(2\cos 315°\cos 75°)$

$= \frac{1}{2}\{\cos(315°+75°) + \cos(315°-75°)\}$

$[\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$= \frac{1}{2}\{\cos 390° + \cos 240°\}$

$= \frac{1}{2}\{\cos(360°+30°) + \cos(180°+60°)\}$

$= \frac{1}{2}\{\cos 30° + (-\cos 60°)\}$

$[\because \cos(360°+\theta) = \cos\theta \text{ and } \cos(180°+\theta) = -\cos\theta]$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ $\left[\because \cos 30° = \frac{\sqrt{3}}{2} \text{ and } \cos 60° = \frac{1}{2}\right]$

$= \frac{1}{2}\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\sqrt{3}-1}{4}$

EXAMPLE |2| Prove that

$$\sin 7A\cos 3A = \frac{1}{2}(\sin 10A + \sin 4A).$$

Sol. LHS $= \sin 7A\cos 3A = \frac{1}{2}(2\sin 7A\cos 3A)$

$= \frac{1}{2}[\sin(7A+3A) + \sin(7A-3A)]$

$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$= \frac{1}{2}(\sin 10A + \sin 4A) =$ RHS **Hence proved.**

EXAMPLE |3| Find the value of $2\sin 52\frac{1°}{2}\sin 7\frac{1°}{2}$.

Sol. $2\sin 52\frac{1°}{2}\sin 7\frac{1°}{2} = 2\sin\frac{105°}{2}\sin\frac{15°}{2}$

$= \cos\left(\frac{105°}{2} - \frac{15°}{2}\right) - \cos\left(\frac{105°}{2} + \frac{15°}{2}\right)$

$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$

$= \cos\left(\frac{90°}{2}\right) - \cos\left(\frac{120°}{2}\right) = \cos 45° - \cos 60°$

$= \frac{1}{\sqrt{2}} - \frac{1}{2}$ $\left[\because \cos 45° = \frac{1}{\sqrt{2}} \text{ and } \cos 60° = \frac{1}{2}\right]$

$= \frac{2-\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2}-1)}{2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$ $[\because \sqrt{2}\times\sqrt{2} = 2]$

EXAMPLE |4| Find the value of

$$\cos 33°\cos 27° - \cos 57°\cos 63°.$$

Sol. $\cos 33°\cos 27° - \cos 57°\cos 63°$

$= \frac{1}{2}(2\cos 33°\cos 27° - 2\cos 57°\cos 63°)$

$= \frac{1}{2}[\{\cos(33°+27°) + \cos(33°-27°)\}$

$- \{\cos(63°+57°) + \cos(63°-57°)\}]$

$[\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$= \frac{1}{2}[(\cos 60° + \cos 6°) - (\cos 120° + \cos 6°)]$

$$= \frac{1}{2}(\cos 60° - \cos 120°) = \frac{1}{2}[\cos 60° - \cos (180° - 60°)]$$

$$= \frac{1}{2}(\cos 60° + \cos 60°) \qquad [\because \cos (180° - \theta) = -\cos \theta]$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} \qquad \left[\because \cos 60° = \frac{1}{2}\right]$$

EXAMPLE |5| Find the value of

$$\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right).$$

Sol. $\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right)$

$$= \frac{1}{2}\left[2\cos\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right)\right]$$

$$= \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{6} + x\right) + \left(\frac{\pi}{6} - x\right)\right\} + \cos\left\{\left(\frac{\pi}{6} + x\right) - \left(\frac{\pi}{6} - x\right)\right\}\right]$$

$[\because 2\cos A \cos B = \cos (A + B) + \cos (A - B)]$

$$= \frac{1}{2}\left(\cos\frac{\pi}{3} + \cos 2x\right) = \frac{1}{2}\left(\frac{1}{2} + \cos 2x\right)\left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x$$

EXAMPLE |6| Prove that

$$\cos 2A \cos\frac{A}{2} - \cos 3A \cos\frac{9A}{2} = \sin 5A \sin\frac{5A}{2}.$$

Sol. $\text{LHS} = \cos 2A \cos\frac{A}{2} - \cos 3A \cos\frac{9A}{2}$

$$= \frac{1}{2}\left[2\cos 2A \cos\frac{A}{2}\right] - \frac{1}{2}\left[2\cos\frac{9A}{2}\cos 3A\right]$$

$$= \frac{1}{2}\left[\cos\left(2A + \frac{A}{2}\right) + \cos\left(2A - \frac{A}{2}\right)\right] - \frac{1}{2}\left[\cos\left(\frac{9A}{2} + 3A\right) + \cos\left(\frac{9A}{2} - 3A\right)\right]$$

$[\because 2\cos A \cos B = \cos (A + B) + \cos (A - B)]$

$$= \frac{1}{2}\left[\cos\left(\frac{4A + A}{2}\right) + \cos\left(\frac{4A - A}{2}\right)\right] - \frac{1}{2}\left[\cos\left(\frac{9A + 6A}{2}\right) + \cos\left(\frac{9A - 6A}{2}\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{5A}{2} + \cos\frac{3A}{2}\right) - \frac{1}{2}\left(\cos\frac{15A}{2} + \cos\frac{3A}{2}\right)$$

$$= \frac{1}{2}\cos\frac{5A}{2} + \frac{1}{2}\cos\frac{3A}{2} - \frac{1}{2}\cos\frac{15A}{2} - \frac{1}{2}\cos\frac{3A}{2}$$

$$= \frac{1}{2}\cos\frac{5A}{2} - \frac{1}{2}\cos\frac{15A}{2} = \frac{1}{2}\left(\cos\frac{5A}{2} - \cos\frac{15A}{2}\right)$$

$$\text{RHS} = \sin 5A \sin\frac{5A}{2} = \frac{1}{2}\left(2\sin 5A \sin\frac{5A}{2}\right)$$

$$= \frac{1}{2}\left[\cos\left(5A - \frac{5A}{2}\right) - \cos\left(5A + \frac{5A}{2}\right)\right]$$

$[\because 2\sin A \sin B = \cos (A - B) - \cos (A + B)]$

$$= \frac{1}{2}\left[\cos\left(\frac{10A - 5A}{2}\right) - \cos\left(\frac{10A + 5A}{2}\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{5A}{2} - \cos\frac{15A}{2}\right)$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |7| Prove that

$$4\cos 12° \cos 48° \cos 72° = \cos 36°.$$

Sol. $\text{LHS} = 4\cos 12° \cos 48° \cos 72°$

$$= 2(2\cos 12° \cos 48°)\cos 72°$$

$$= 2[\cos (12° + 48°) + \cos (12° - 48°)]\cos 72°$$

$[\because 2\cos A \cos B = \cos (A + B) + \cos (A - B)]$

$$= 2[\cos 60° + \cos (-36°)]\cos 72°$$

$$= 2[\cos 60° + \cos 36°]\cos 72° \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= 2\left[\frac{1}{2}\cos 72° + \cos 36° \cos 72°\right]$$

$$= \cos 72° + 2\cos 36° \cos 72°$$

$$= \cos 72° + \cos (36° + 72°) + \cos (36° - 72°)$$

$$= \cos 72° + \cos 108° + \cos 36°$$

$$= \cos 72° + \cos (180° - 72°) + \cos 36°$$

$$= \cos 72° - \cos 72° + \cos 36° = \cos 36°$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |8| Prove that

$$\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}.$$

Sol. $\text{LHS} = \sin 10° \sin 30° \sin 50° \sin 70°$

$$= \sin 30°(\sin 50° \sin 10°)\sin 70°$$

$$= \frac{1}{2} \times \frac{1}{2}(2\sin 50° \sin 10°)\sin 70°$$

$$= \frac{1}{4}[\cos (50° - 10°) - \cos (50° + 10°)]\sin 70°$$

$[\because 2\sin A \sin B = \cos (A - B) - \cos (A + B)]$

$$= \frac{1}{4}[\cos 40° - \cos 60°]\sin 70°$$

$$= \frac{1}{4}[\sin 70° \cos 40° - \sin 70° \cos 60°]$$

$$= \frac{1}{4}\left[\sin 70° \cos 40° - \frac{1}{2}\sin 70°\right]$$

$$= \frac{1}{8}[2\sin 70° \cos 40° - \sin 70°]$$

$$= \frac{1}{8}[\sin (70° + 40°) + \sin (70° - 40°) - \sin 70°]$$

$[\because 2\sin A \cos B = \sin (A + B) + \sin (A - B)]$

$$= \frac{1}{8}[\sin 110° + \sin 30° - \sin 70°]$$

$$= \frac{1}{8}[\sin (180° - 70°) + \frac{1}{2} - \sin 70°]$$

$$= \frac{1}{8}[\sin 70° + \frac{1}{2} - \sin 70°] = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |9| Prove that

$\tan\theta \tan(60° - \theta)\tan(60° + \theta) = \tan 3\theta$.

Sol. LHS $= \tan\theta\tan(60° - \theta)\tan(60° + \theta)$

$$= \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin(60° - \theta)}{\cos(60° - \theta)} \cdot \frac{\sin(60° + \theta)}{\cos(60° + \theta)}$$

$$= \frac{\sin\theta[2\sin(60° - \theta)\sin(60° + \theta)]}{\cos\theta[2\cos(60° - \theta)\cos(60° + \theta)]}$$

$$= \frac{\sin\theta(\cos 2\theta - \cos 120°)}{\cos\theta(\cos 120° + \cos 2\theta)}$$

$\left[\begin{array}{l}\because 2\sin A\sin B = \cos(A - B) - \cos(A + B) \\ \text{and } 2\cos A\cos B = \cos(A + B) + \cos(A - B)\end{array}\right]$

$$= \frac{\sin\theta\left(\cos 2\theta + \frac{1}{2}\right)}{\cos\theta\left(\cos 2\theta - \frac{1}{2}\right)} \quad \left[\begin{array}{l}\because \cos 120° = \cos(180° - 60°) \\ = -\cos 60° = -\frac{1}{2}\end{array}\right]$$

$$= \frac{\sin\theta\cos 2\theta + \frac{1}{2}\sin\theta}{\cos\theta\cos 2\theta - \frac{1}{2}\cos\theta} = \frac{2\sin\theta\cos 2\theta + \sin\theta}{2\cos\theta\cos 2\theta - \cos\theta}$$

$$= \frac{\sin(\theta + 2\theta) + \sin(-\theta) + \sin\theta}{\cos(\theta + 2\theta) + \cos(-\theta) - \cos\theta}$$

$\left[\begin{array}{l}\because 2\sin A\cos B = \sin(A + B) + \sin(A - B) \\ \text{and } 2\cos A\cos B = \cos(A + B) + \cos(A - B)\end{array}\right]$

$$= \frac{\sin 3\theta - \sin\theta + \sin\theta}{\cos 3\theta + \cos\theta - \cos\theta} \quad \left[\begin{array}{l}\because \sin(-\theta) = -\sin\theta \\ \text{and} \cos(-\theta) = \cos\theta\end{array}\right]$$

$$= \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta = \text{RHS}$$ **Hence proved.**

EXAMPLE |10| If $\frac{\tan(\theta + \alpha)}{a} = \frac{\tan(\theta + \beta)}{b} = \frac{\tan(\theta + \gamma)}{c}$, then prove that

$\frac{a + b}{a - b}\sin^2(\alpha - \beta) + \frac{b + c}{b - c}\sin^2(\beta - \gamma) + \frac{c + a}{c - a}\sin^2(\gamma - \alpha) = 0.$

Sol. Given, $\frac{\tan(\theta + \alpha)}{a} = \frac{\tan(\theta + \beta)}{b}$

$$\Rightarrow \quad \frac{a}{b} = \frac{\tan(\theta + \alpha)}{\tan(\theta + \beta)}$$

$$\Rightarrow \quad \frac{a + b}{a - b} = \frac{\tan(\theta + \alpha) + \tan(\theta + \beta)}{\tan(\theta + \alpha) - \tan(\theta + \beta)}$$

[applying componendo and dividendo]

$$\Rightarrow \quad \frac{a + b}{a - b} = \frac{\sin(\theta + \alpha + \theta + \beta)}{\sin(\theta + \alpha - \theta - \beta)}$$

$$\left[\because \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}} = \frac{\sin(A + B)}{\sin(A - B)}\right]$$

$$\Rightarrow \quad \frac{a + b}{a - b} = \frac{\sin(2\theta + \alpha + \beta)}{\sin(\alpha - \beta)}$$

$$\Rightarrow \left(\frac{a + b}{a - b}\right)\sin(\alpha - \beta) = \sin(2\theta + \alpha + \beta)$$

$$\Rightarrow \left(\frac{a + b}{a - b}\right)\sin^2(\alpha - \beta) = \sin(2\theta + \alpha + \beta)\sin(\alpha - \beta)$$

$$\Rightarrow \left(\frac{a + b}{a - b}\right)\sin^2(\alpha - \beta) = \frac{1}{2}[2\sin(2\theta + \alpha + \beta)\sin(\alpha - \beta)]$$

$$\Rightarrow \left(\frac{a + b}{a - b}\right)\sin^2(\alpha - \beta) = \frac{1}{2}[\cos(2\theta + 2\beta) - \cos(2\theta + 2\alpha)]$$

$[\because 2\sin A\sin B = \cos(A - B) - \cos(A + B)]$

Similarly, $\frac{b + c}{b - c}\sin^2(\beta - \gamma) = \frac{1}{2}[\cos(2\theta + 2\gamma) - \cos(2\theta + 2\beta)]$

and $\frac{c + a}{c - a}\sin^2(\gamma - \alpha) = \frac{1}{2}[\cos(2\theta + 2\alpha) - \cos(2\theta + 2\gamma)]$

Now, $\left(\frac{a + b}{a - b}\right)\sin^2(\alpha - \beta) + \left(\frac{b + c}{b - c}\right)\sin^2(\beta - \gamma) + \left(\frac{c + a}{c - a}\right)\sin^2(\gamma - \alpha)$

$$= \frac{1}{2}[\cos(2\theta + 2\beta) - \cos(2\theta + 2\alpha) + \cos(2\theta + 2\gamma) - \cos(2\theta + 2\beta) + \cos(2\theta + 2\alpha) - \cos(2\theta + 2\gamma)]$$

$$= \frac{1}{2} \times 0 = 0$$

$\therefore$ LHS = RHS **Hence proved.**

TRANSFORMATION OF SUM OR DIFFERENCE INTO PRODUCT

Here, we transform the sum and difference of two sine or cosine of two constituent angles into product of sine and cosine of compound angles.

(i) $\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right)$

(ii) $\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right) \cdot \sin\left(\frac{A - B}{2}\right)$

(iii) $\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right)$

(iv) $\sin A - \sin B = 2\cos\left(\frac{A + B}{2}\right) \cdot \sin\left(\frac{A - B}{2}\right)$

EXAMPLE |11| Find the value of $\sin 75° + \sin 15°$.

Sol $\sin 75° + \sin 15° = 2\sin\left(\frac{75°+15°}{2}\right)\cos\left(\frac{75°-15°}{2}\right)$

$$\left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$= 2\sin\frac{90°}{2}\cos\frac{60°}{2} = 2\sin 45°\cos 30°$

$= 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}$ $\quad\left[\because \sin 45° = \frac{1}{\sqrt{2}} \text{ and } \cos 30° = \frac{\sqrt{3}}{2}\right]$

$= \frac{\sqrt{3}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

EXAMPLE |12| Find the value of $\sin 10° + \sin 50° - \sin 70°$.

Sol. $\sin 10° + \sin 50° - \sin 70°$

$= \sin 10° - (\sin 70° - \sin 50°)$

$= \sin 10° - \left[2\cos\left(\frac{70°+50°}{2}\right)\sin\left(\frac{70°-50°}{2}\right)\right]$

$$\left[\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$= \sin 10° - \left[2\cos\left(\frac{120°}{2}\right)\sin\left(\frac{20°}{2}\right)\right]$

$= \sin 10° - (2\cos 60°\sin 10°)$

$= \sin 10° - \left(2\cdot\frac{1}{2}\cdot\sin 10°\right)$ $\quad\left[\because \cos 60° = \frac{1}{2}\right]$

$= \sin 10° - \sin 10° = 0$

EXAMPLE |13| Prove

$$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}.$$

Sol. LHS $= (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \left(-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\right)^2 + \left(2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right)^2$

$$\left[\begin{array}{l}\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\\ \text{and } \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\end{array}\right]$$

$= 4\sin^2\frac{A+B}{2}\sin^2\frac{A-B}{2} + 4\cos^2\frac{A+B}{2}\sin^2\frac{A-B}{2}$ $\quad[\because \sin^2(-\theta) = \sin^2\theta]$

$= 4\sin^2\frac{A-B}{2}\left(\sin^2\frac{A+B}{2} + \cos^2\frac{A+B}{2}\right)$

$= 4\sin^2\frac{A-B}{2}\times 1$ $\quad[\because \sin^2\theta + \cos^2\theta = 1]$

$= 4\sin^2\frac{A-B}{2} =$ RHS **Hence proved.**

EXAMPLE |14| Prove that $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$. **[NCERT]**

Sol. LHS $= \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$

$= \frac{2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}$ [by formulae]

$= \frac{\sin 4x\cos x}{\cos 4x\cos x} = \tan 4x =$ RHS **Hence proved.**

EXAMPLE |15| If $m\sin\theta = n\sin(\theta + 2\alpha)$, then prove that $\tan(\theta+\alpha)\cot\alpha = \frac{m+n}{m-n}$. **[NCERT Exemplar]**

Sol. Given, $m\sin\theta = n\sin(\theta + 2\alpha)$

$\Rightarrow \quad \frac{m}{n} = \frac{\sin(\theta+2\alpha)}{\sin\theta}$

Applying componendo and dividendo rule, we get

$\frac{m+n}{m-n} = \frac{\sin(\theta+2\alpha) + \sin\theta}{\sin(\theta+2\alpha) - \sin\theta}$

$= \frac{2\sin\left(\frac{\theta+2\alpha+\theta}{2}\right)\cos\left(\frac{\theta+2\alpha-\theta}{2}\right)}{2\cos\left(\frac{\theta+2\alpha+\theta}{2}\right)\sin\left(\frac{\theta+2\alpha-\theta}{2}\right)}$

$$\left[\begin{array}{l}\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \text{and } \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\end{array}\right]$$

$= \frac{2\sin(\theta+\alpha)\cos\alpha}{2\cos(\theta+\alpha)\sin\alpha} = \tan(\theta+\alpha)\cot\alpha$ **Hence proved.**

EXAMPLE |16| If

$$\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta),$$

then prove that $\cot\alpha\cot\beta\cot\gamma = \cot\delta$.

Sol. Given, $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$

$\Rightarrow \quad \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$

$\Rightarrow \quad \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)}$

[applying componendo and dividendo]

$\Rightarrow \quad \frac{2\cos\alpha\cos(-\beta)}{-2\sin\alpha\sin(-\beta)} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta}$

$\Rightarrow \quad \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta}$

$[\because \cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta]$

$\Rightarrow \quad \cot\alpha\cot\beta = \tan\gamma\cot\delta$

$\Rightarrow \quad \cot\alpha\cot\beta\cot\gamma = \cot\delta$ **Hence proved.**

EXAMPLE |17| Prove that

$$\sin x+\sin y+\sin z-\sin(x+y+z)=4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{y+z}{2}\right)\cdot\sin\left(\frac{z+x}{2}\right)$$

Sol. LHS $=(\sin x+\sin y)+[\sin z-\sin(x+y+z)]$

$$=2\sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)+2\cos\left(\frac{z+x+y+z}{2}\right)\cdot\sin\left(\frac{z-x-y-z}{2}\right)$$

$$\left[\because \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) \text{ and } \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)\right]$$

$$=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+2\cos\left(\frac{x+y+2z}{2}\right)\cdot\sin\left[-\left(\frac{x+y}{2}\right)\right]$$

$$=2\sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)-2\cos\left(\frac{x+y+2z}{2}\right)\cdot\sin\left(\frac{x+y}{2}\right) \quad [\because \sin(-\theta)=-\sin\theta]$$

$$=2\sin\left(\frac{x+y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y+2z}{2}\right)\right]$$

$$=2\sin\left(\frac{x+y}{2}\right)\left[-2\sin\frac{\left(\frac{x-y}{2}+\frac{x+y+2z}{2}\right)}{2}\sin\frac{\left(\frac{x-y}{2}-\frac{x+y+2z}{2}\right)}{2}\right]$$

$$\left[\because \cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right]$$

$$=-4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x+z}{2}\right)\cdot\sin\left(\frac{-y-z}{2}\right)$$

$$=-4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x+z}{2}\right)\left[-\sin\left(\frac{y+z}{2}\right)\right] \quad [\because \sin(-\theta)=-\sin\theta]$$

$$=4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x+z}{2}\right)\cdot\sin\left(\frac{y+z}{2}\right)$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |18| If $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$, then show that $\frac{\tan x}{\tan y}=\frac{a}{b}$. **[NCERT Exemplar]**

Sol. Given, $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$

Using componendo and dividendo rule, we get

$$\frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)}=\frac{a+b+a-b}{a+b-a+b}$$

$$\Rightarrow \frac{2\sin\left(\frac{x+y+x-y}{2}\right)\cdot\cos\left(\frac{x+y-x+y}{2}\right)}{2\cos\left(\frac{x+y+x-y}{2}\right)\cdot\sin\left(\frac{x+y-x+y}{2}\right)}=\frac{2a}{2b}$$

$$\left[\because \sin A+\sin B=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2} \text{ and } \sin A-\sin B=2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}\right]$$

$$\Rightarrow \frac{\sin x\cdot\cos y}{\cos x\cdot\sin y}=\frac{a}{b} \Rightarrow \frac{\tan x}{\tan y}=\frac{a}{b}$$ **Hence proved.**

EXAMPLE |19| Prove that $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$. **[NCERT]**

Sol. LHS $=\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}$

$$=\frac{(\cos 4x+\cos 2x)+\cos 3x}{(\sin 4x+\sin 2x)+\sin 3x}$$

$$=\frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\sin 3x}$$

$$=\frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x}=\frac{\cos 3x(2\cos x+1)}{\sin 3x(2\cos x+1)}$$

$$=\cot 3x$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |20| Prove that

$$\frac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cdot\cot 5A.$$

Sol. LHS $=\frac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cdot\cos 10A}{\sin 4A\sin 3A-\sin 2A\cdot\sin 5A+\sin 4A\sin 7A}$

$$=\frac{2\cos 2A\cos 3A-2\cos 7A\cos 2A+2\cos 10A\cos A}{2\sin 4A\sin 3A-2\sin 5A\sin 2A+2\sin 7A\sin 4A}$$

$$=\frac{[\cos(3A+2A)+\cos(3A-2A)-[\cos(7A+2A)+\cos(7A-2A)]+[\cos(10A+A)+\cos(10A-A)]]}{[\cos(4A-3A)-\cos(4A+3A)-[\cos(5A-2A)-\cos(5A+2A)]+[\cos(7A-4A)-\cos(7A+4A)]]}$$

$$=\frac{\cos 5A+\cos A-\cos 9A-\cos 5A+\cos 11A+\cos 9A}{\cos A-\cos 7A-\cos 3A+\cos 7A+\cos 3A-\cos 11A}$$

$$=\frac{\cos A+\cos 11A}{\cos A-\cos 11A}$$

$$= \frac{2\cos\left(\frac{A+11A}{2}\right)\cos\left(\frac{A-11A}{2}\right)}{2\sin\left(\frac{A+11A}{2}\right)\sin\left(\frac{11A-A}{2}\right)}$$

$$\left[\begin{array}{l} \because \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \text{and } \cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right) \\ \qquad = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right) \end{array}\right]$$

$$= \frac{\cos 6A \cdot \cos 5A}{\sin 6A \cdot \sin 5A} \qquad [\because \cos(-\theta) = \cos\theta]$$

$= \cot 6A \cdot \cot 5A =$ RHS **Hence proved.**

EXAMPLE |21| Prove that

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

Sol. $\text{LHS} = \left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$

$$= \left[\frac{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}\right]^n + \left[\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}\right]^n$$

$$= \left[\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}\right]^n + \left[-\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}\right]^n$$

$$\Rightarrow \text{LHS} = \cot^n\left(\frac{A-B}{2}\right) + (-1)^n\cot^n\left(\frac{A-B}{2}\right)$$

Here, two cases arise.

***Case* I** When n is even, then

$$\cot^n\left(\frac{A-B}{2}\right) + \cot^n\left(\frac{A-B}{2}\right) \qquad [\because (-1)^n = 1, \text{ if } n \text{ is even}]$$

$$= 2\cot^n\left(\frac{A-B}{2}\right)$$

***Case* II** When n is odd, then

$$\cot^n\left(\frac{A-B}{2}\right) - \cot^n\left(\frac{A-B}{2}\right) \qquad [\because (-1)^n = -1, \text{ if } n \text{ is odd}]$$

$$= 0$$

$$\text{Hence, LHS} = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

EXAMPLE |22| If $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$, prove that $\sin 3\theta + \sin 3\phi = 0$.

Sol. Given,

$$\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$$

$$\Rightarrow 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = 2\sqrt{3}\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \qquad [\because \cos(-\theta) = \cos\theta]$$

$$\Rightarrow 2\sin\left(\frac{\theta+\phi}{2}\right)\left[\cos\left(\frac{\theta-\phi}{2}\right) - \sqrt{3}\sin\left(\frac{\theta-\phi}{2}\right)\right] = 0$$

$$\Rightarrow \sin\left(\frac{\theta+\phi}{2}\right) = 0$$

$$\text{or} \quad \cos\left(\frac{\theta-\phi}{2}\right) - \sqrt{3}\sin\left(\frac{\theta-\phi}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\theta+\phi}{2}\right) = 0$$

$$\text{or} \quad \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{\sqrt{3}} \qquad \left[\because \sin 0 = 0 \text{ and } \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow \left(\frac{\theta+\phi}{2}\right) = 0 \quad \text{or} \quad \left(\frac{\theta-\phi}{2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \theta = -\phi \quad \text{or} \quad \theta - \phi = \frac{\pi}{3}$$

Here, two cases arise.

***Case* I** When $\theta = -\phi$, then

$$\sin 3\theta + \sin 3\phi = \sin 3(-\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

***Case* II** When $\theta - \phi = \frac{\pi}{3}$, then $3\theta - 3\phi = \pi$

$$\Rightarrow 3\theta = \pi + 3\phi$$

$$\sin 3\theta + \sin 3\phi = \sin(\pi + 3\phi) + \sin 3\phi$$
$$= -\sin 3\phi + \sin 3\phi \qquad [\because \sin(\pi+\theta) = \sin\theta]$$
$$= 0$$

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 The value of $\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is **[NCERT Exemplar]**

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) $\frac{1}{8}$

2 The value of $\sin 50° - \sin 70° + \sin 10°$ is **[NCERT Exemplar]**

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) 2

3 The value of $\sin\frac{\pi}{18} + \sin\frac{\pi}{9} + \sin\frac{2\pi}{9} + \sin\frac{5\pi}{18}$ is **[NCERT Exemplar]**

(a) $\sin\frac{7\pi}{18} + \sin\frac{4\pi}{9}$ (b) 1

(c) $\cos\frac{\pi}{6} + \cos\frac{3\pi}{7}$ (d) $\cos\frac{\pi}{9} + \sin\frac{\pi}{9}$

4 $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$ is equal to **[NCERT Exemplar]**

(a) $\sin 2x$ (b) $\cos 3x$ (c) $\tan 3x$ (d) $\cot 3x$

5 The value of $\cos\theta \cdot \cos\frac{\theta}{2} - \cos 3\theta \cdot \cos\frac{9\theta}{2}$ is equal to

(a) $\sin 4\theta . \sin\frac{7\theta}{2}$ (b) $\sin 8\theta$

(c) $\sin 7\theta + \sin 8\theta$ (d) $\sin 7\theta \cdot \sin 8\theta$

SHORT ANSWER Type I Questions

17 Prove that $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$. **[NCERT]**

18 Show that $\tan(60° + \theta)\tan(60° - \theta) = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$.

19 Prove that $\cos 55° + \cos 65° + \cos 75° = 2\cos 40° \cos 35°$.

20 Prove that $\sin 65° + \cos 65° = \sqrt{2}\cos 20°$.

21 Prove that $\sin 47° + \cos 77° = \cos 17°$.

22 Prove that $\sin 50° \cos 85° = \frac{1 - \sqrt{2}\sin 35°}{2\sqrt{2}}$.

23 Prove that

(i) $\frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)$.

(ii) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$.

24 Prove that (i) $\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$.

(ii) $\frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)$.

SHORT ANSWER Type II Questions

25 Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

26 Prove that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-12) *Convert each of the following products into the sum or difference of sines and cosines.*

6 $2\cos 22\frac{1°}{2} \cdot \cos 67\frac{1°}{2}$.

7 $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$.

8 $2\sin 5\theta \cos\theta$

9 $2\cos 4\theta \cos 3\theta$

10 $2\sin 3\theta \sin\theta$

11 $\sin 75° \cos 15°$

12 $\cos 75° \cos 15°$

Directions (Q. Nos. 13-16) *Express each of the following as a product*

13 $\sin 4\theta + \sin 2\theta$

14 $\sin 6\theta - \sin 2\theta$

15 $\cos 4\theta + \cos 8\theta$

16 $\cos 6\theta - \cos 8\theta$

Directions (Q. Nos. 27-31) *Prove each of the following.*

27 $\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$

28 $\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

29 $\frac{\sin 5A\cos 2A - \sin 6A\cos A}{\sin A\sin 2A - \cos 2A\cos 3A} = \tan A$

30 $\frac{\sin A\sin 2A + \sin 3A \cdot \sin 6A}{\sin A \cdot \cos 2A + \sin 3A \cdot \cos 6A} = \tan 5A$

31 $\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$

32 If $\cos x + \cos y = \frac{1}{3}$ and $\sin x + \sin y = \frac{1}{4}$, then prove that $\tan\left(\frac{x+y}{2}\right) = \frac{3}{4}$.

LONG ANSWER Type Questions

33 If $\frac{\cos(A-B)}{\cos(A+B)} + \frac{\cos(C+D)}{\cos(C-D)} = 0$, then prove that $\tan A \tan B \tan C \tan D = -1$.

34 Prove that
$4\sin A \cdot \sin(60° - A) \cdot \sin(60° + A) = \sin 3A$.
Hence, deduce that
$\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$.

35 Prove tha
$\sin^2 A = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)$
$\cos A \cos B$.

36 If $\sin B = 3\sin(2A+B)$, then prove that
$2\tan A + \tan(A+B) = 0$.

37 If $\cos(\theta+\phi) = m\cos(\theta-\phi)$, then prove that
$\tan\theta = \frac{1-m}{1+m}\cot\phi$.
[NCERT Exemplar]

38 If $a\sin\theta = b\sin\left(\theta + \frac{2\pi}{3}\right) = c\sin\left(\theta + \frac{4\pi}{3}\right)$, then prove that $ab + bc + ca = 0$.

39 If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, then show that $xy + yz + zx = 0$.

HINTS & ANSWERS

1. (c) Use formula

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$$

and $\cos A - \cos B = -2\sin\frac{A+B}{2}\cdot\sin\frac{A-B}{2}$

to solve this problem.

2. (b) Use formula

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

and $\sin(-\theta) = -\sin\theta$.

3. (a) Use formula

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

4. (d)

$$= \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x}$$

$$= \frac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x}$$

$$= \cot 3x$$

5. (a) Use formula

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

6. $2\cos 22\frac{1°}{2} \cdot \cos 67\frac{1°}{2}$

$$= \cos\left(22\frac{1}{2} + 67\frac{1}{2}\right)^\circ + \cos\left(22\frac{1}{2} - 67\frac{1}{2}\right)^\circ = \frac{1}{\sqrt{2}}$$

7. $\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12}\right]$

$$= \frac{1}{2}\left[\cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) - \cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right) = \frac{1}{4}$$

8. $\sin 6\theta + \sin 4\theta$

9. $\cos 7\theta + \cos\theta$

10. $\cos 2\theta - \cos 4\theta$

11. $\frac{1}{2}[\sin 90° + \sin 60°]$

12. $\frac{1}{2}[\cos 90° + \cos 60°]$

13. $2\sin 3\theta \cos\theta$

14. $2\cos 4\theta \sin 2\theta$

15. $2\cos 6\theta \cos 2\theta$

16. $2\sin 7\theta \sin\theta$

17. LHS $= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + 2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$

$$= 2\cos\left(\frac{\pi}{13}\right)\left[\cos\frac{9\pi}{13} + \cos\frac{4\pi}{13}\right]$$

$$= 2\cos\left(\frac{\pi}{13}\right) \times 2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}$$

$$= 4\cos\frac{\pi}{13} \times 0 \times \cos\frac{5\pi}{26}$$

18. LHS $= \frac{\sin(60°+\theta)\sin(60°-\theta)}{\cos(60°+\theta)\cos(60°-\theta)} \times \frac{2}{2}$

$$= \frac{\cos(2\theta) - \cos(120°)}{\cos 120° + \cos 2\theta} = \frac{\cos 2\theta + \frac{1}{2}}{\left(-\frac{1}{2}\right) + \cos 2\theta}$$

19. $(\cos 55° + \cos 65°) + \cos 75°$

$$= 2\cos 60° \cos 5° + \cos 75°$$
$$= \cos 5° + \cos 75° = 2\cos 40° \cos 35°$$

20. $\sin 65° + \cos 65° = \sin 65° + \cos(90° - 25°)$

$$= \sin 65° + \sin 25° = 2\sin 45° \cos 20°$$

21. Solve as Q. 20.

22. LHS $= \frac{1}{2}[2\sin 50° \cos 85°] = \frac{1}{2}[\sin 135° - \sin 35°]$

$$= \frac{1}{2}[\sin(90° + 45°) - \sin 35°]$$

$$= \frac{1}{2}[\cos 45° - \sin 35°]$$

23. (i) LHS $= \dfrac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}$

(ii) LHS $= \dfrac{2\sin 2A\cos A}{2\sin 2A\sin A}$

24. Solve as Q. 23.

25. LHS $= \frac{1}{2}\cos 60°\cos 20°(2\cos 40°\cos 80°)$

$= \frac{1}{2}\times\frac{1}{2}\cos 20°(\cos 120° + \cos 40°)$

$= \frac{1}{4}\left[-\frac{\cos 20°}{2} + \frac{1}{2}(\cos 60° + \cos 20°)\right]$

$= \frac{1}{4}\left[\frac{1}{2}\times\frac{1}{2}\right] = \frac{1}{16}$

26. $\frac{1}{2}\sin 60°\sin 20°(2\sin 40°\sin 80°)$

$= \frac{1}{2}\times\frac{\sqrt{3}}{2}\sin 20°[\cos(-40°) - \cos(120°)]$

$= \frac{\sqrt{3}}{4}\sin 20°\left(\cos 40° + \frac{1}{2}\right)$

$= \frac{\sqrt{3}}{4}\left[\frac{1}{2}(2\sin 20°\cos 40°) + \frac{1}{2}\sin 20°\right]$

27. LHS $= \dfrac{\cos 3A + \cos 7A + 2\cos 5A}{\cos A + \cos 5A + 2\cos 3A}$

$= \dfrac{2\cos 5A\cos 2A + 2\cos 5A}{2\cos 3A\cos 2A + 2\cos 3A} = \dfrac{2\cos 5A}{2\cos 3A}$

28. LHS $= \dfrac{(\sin 3A + \sin 9A) + (\sin 5A + \sin 7A)}{(\cos 3A + \cos 9A) + (\cos 5A + \cos 7A)}$

$= \dfrac{2\sin 6A\cos 3A + 2\sin 6A\cos A}{2\cos 6A\cos 3A + 2\cos 6A\cos A} = \dfrac{2\sin 6A}{2\cos 6A}$

29. LHS $= \dfrac{2\sin 5A\cos 2A - 2\sin 6A\cos A}{2\sin A\sin 2A - 2\cos 2A\cos 3A}$

$= \dfrac{(\sin 7A + \sin 3A) - (\sin 7A + \sin 5A)}{\cos(A) - \cos 3A - (\cos 5A + \cos A)}$

$= \dfrac{\sin 3A - \sin 5A}{-(\cos 3A + \cos 5A)} = \dfrac{2\cos 4A\sin(-A)}{-2\cos 4A\cos A} = \tan A$

30. Solve as Q. 29.

31. LHS $= \dfrac{\sin(\theta+\phi) + \sin(\theta-\phi) - 2\sin\theta}{\cos(\theta+\phi) + \cos(\theta-\phi) - 2\cos\theta}$

$= \dfrac{2\sin\theta\cos\phi - 2\sin\theta}{2\cos\theta\cos\phi - 2\cos\phi} = \dfrac{2\cos\theta(\cos\phi - 1)}{2\cos\theta(\cos\phi - 1)}$

32. $\because \cos x + \cos y = \frac{1}{3} \Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{3}$

and $\sin x + \sin y = \frac{1}{4} \Rightarrow 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{4}$

33. $\because \dfrac{\cos(A-B)}{\cos(A+B)} + \dfrac{\cos(C+D)}{\cos(C-D)} = 0$

$\Rightarrow \dfrac{\cos(A-B)}{\cos(A+B)} = \dfrac{-\cos(C+D)}{\cos(C-D)}$

$\Rightarrow \dfrac{\cos(A-B) + \cos(A+B)}{\cos(A-B) - \cos(A+B)} = \dfrac{-\cos(C+D) + \cos(C-D)}{-[\cos(C+D) - \cos(C-D)]}$

[applying componendo and dividendo]

$\Rightarrow \dfrac{\cos(A+B) + \cos(A-B)}{\cos(A-B) - \cos(A+B)} = \dfrac{\cos(C-D) - \cos(C+D)}{-\cos(C+D) + \cos(C-D)}$

$\Rightarrow \dfrac{2\cos A\cos B}{2\sin A\sin B} = -\dfrac{\sin C\sin D}{2\cos C\cos D}$

$\Rightarrow \dfrac{1}{\tan A\tan B} = -\tan C\tan D$

$\Rightarrow \tan A\tan B\tan C\tan D = -1$

34. LHS $= \sin A\sin(60° - A)\sin(60° + A)$

$= \frac{1}{2}\sin A[2\sin(60° - A)\sin(60° + A)]$

$= \frac{1}{2}\sin A[\cos\{(60° - A) - (60° + A)\} - \cos\{(60° - A) + (60° + A)\}]$

$= \frac{1}{2}\sin A[\cos(-2A) - \cos 120°]$

$= \frac{1}{2}\sin A\left\{\cos 2A + \frac{1}{2}\right\} = \frac{1}{2}\sin A\cos 2A + \frac{1}{4}\sin A$

$= \frac{1}{4}[\sin(A + 2A) + \sin(A - 2A)] + \frac{1}{4}\sin A$

$= \frac{1}{4}[\sin 3A + \sin(-A)] + \frac{1}{4}\sin A$

$= \frac{1}{4}\sin 3A =$ RHS **Hence proved.**

35. RHS $= \cos(A-B)[\cos(A-B) - 2\cos A\cos B] + \cos^2 B$

$= \cos(A-B)[\cos A\cos B + \sin A\sin B - 2\cos A\cos B] + \cos^2 B$

$= \cos(A-B)[\sin A\sin B - \cos A\cos B] + \cos^2 B$

$= -\cos(A-B)\cos(A+B) + \cos^2 B$

$= \cos^2 B - (\cos^2 B - \sin^2 A) = \sin^2 A$

36. Given, $\sin B = 3\sin(2A + B)$

$\Rightarrow \dfrac{\sin(2A+B)}{\sin B} = \dfrac{1}{3} \Rightarrow \dfrac{\sin\{(A+B)+A\}}{\sin\{(A+B)-A\}} = \dfrac{1}{3}$

$\Rightarrow \dfrac{\sin\{(A+B)+A\} + \sin\{(A+B)-A\}}{\sin\{(A+B)+A\} - \sin\{(A+B)-A\}} = \dfrac{4}{-2}$

[applying componendo and dividendo]

$\Rightarrow \dfrac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = -2$

$\Rightarrow \tan(A+B) = -2\tan A$

$\Rightarrow 2\tan A + \tan(A+B) = 0$ **Hence proved.**

37. Given, $\cos(\theta+\phi) = m\cos(\theta-\phi)$

$$\Rightarrow \quad \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = \frac{m}{1} \Rightarrow \frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = \frac{1}{m}$$

Using componendo and dividendo rule, we get

$$\frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{\cos(\theta-\phi)+\cos(\theta+\phi)} = \frac{1-m}{1+m}$$

$$\Rightarrow \frac{-2\sin\left(\frac{\theta-\phi+\theta+\phi}{2}\right)\cdot\sin\left(\frac{\theta-\phi-\theta-\phi}{2}\right)}{2\cos\left(\frac{\theta-\phi+\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi-\theta-\phi}{2}\right)} = \frac{1-m}{1+m}$$

$$\Rightarrow \quad \frac{\sin\theta\cdot\sin\phi}{\cos\theta\cdot\cos\phi} = \frac{1-m}{1+m} \quad \left[\begin{array}{l}\because \ \sin(-\theta) = -\sin\theta \\ \text{and } \cos(-\theta) = \cos\theta\end{array}\right]$$

$$\Rightarrow \quad \tan\theta\cdot\tan\phi = \frac{1-m}{1+m}$$

$$\Rightarrow \quad \tan\theta = \left(\frac{1-m}{1+m}\right)\cot\phi$$ **Hence proved.**

38. Let $a\sin\theta = b\sin\left(\theta+\frac{2\pi}{3}\right) = c\sin\left(\theta+\frac{4\pi}{3}\right) = k$ (say)

$$\Rightarrow \sin\theta = \frac{k}{a}, \sin\left(\theta+\frac{2\pi}{3}\right) = \frac{k}{b} \text{ and } \sin\left(\theta+\frac{4\pi}{3}\right) = \frac{k}{c}$$

On adding these, we get

$$\frac{k}{a}+\frac{k}{b}+\frac{k}{c} = \sin\theta + \sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c} = \left[\sin\left(\theta+\frac{4\pi}{3}\right)+\sin\theta\right] + \sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c} = -\sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0 \qquad [\because k \neq 0]$$

$$\Rightarrow \quad \frac{bc+ca+ab}{abc} = 0 \quad \therefore ab+bc+ca = 0$$

Hence proved.

39. Let $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$ (say)

$$\Rightarrow \frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cos\theta + \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta+\frac{4\pi}{3}\right)$$

Simplify it to obtain the required result.

|TOPIC 5|

Trigonometric Functions of Multiple, Sub-multiple Angles and Trigonometric Equations

TRIGONOMETRIC FUNCTIONS OF MULTIPLE ANGLES

Some important trigonometric functions of angles $2x, 3x, \ldots$ etc, in terms of the trigonometric functions of angle x are given below

Formula 1. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$$= 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

Formula 2. $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

Formula 3. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

Formula 4. $\cos 3x = 4\cos^3 x - 3\cos x$

Formula 5. $\sin 3x = 3\sin x - 4\sin^3 x$

Formula 6. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

EXAMPLE |1| If $\cos A = \frac{2}{3}$, then find the value of $\cos 2A$.

Sol. Given, $\cos A = \frac{2}{3}$

$$\therefore \quad \cos 2A = 2\cos^2 A - 1 = 2\left(\frac{2}{3}\right)^2 - 1$$

$$= 2\times\frac{4}{9} - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$$

EXAMPLE |2| Find the value of $\tan 2A$, if $\tan A = \frac{5}{12}$.

Sol. Given, $\tan A = \frac{5}{12}$

$$\therefore \quad \tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \frac{\frac{5}{6}}{1-\frac{25}{144}}$$

$$= \frac{\frac{5}{6}}{\frac{144-25}{144}} = \frac{5}{6}\times\frac{144}{119} = \frac{5\times 24}{119} = \frac{120}{119}$$

EXAMPLE |3| Find the value of $2\sin A\cos A$, if $A = 15°$.

Sol. Given, $A = 15°$

$\therefore \quad 2\sin A\cos A = \sin 2A \quad [\because \sin 2A = 2\sin A\cos A]$

$= \sin 2\times 15°$

$= \sin 30° = \frac{1}{2} \quad \left[\because \sin 30° = \frac{1}{2}\right]$

EXAMPLE |4| If $a\sin\theta = b\cos\theta$, then find the value of $\sin 2\theta$.

Sol. Given, $a\sin\theta = b\cos\theta$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{b}{a} \Rightarrow \tan\theta = \frac{b}{a} \quad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$

$\therefore \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\times\frac{b}{a}}{1+\left(\frac{b}{a}\right)^2}$

$= \frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}} = \frac{\frac{2b}{a}}{\frac{a^2+b^2}{a^2}} = \frac{2b}{a}\times\frac{a^2}{a^2+b^2} = \frac{2ab}{a^2+b^2}$

EXAMPLE |5| If $x = a\tan\theta$, then find the value of $\cos 2\theta$.

Sol. Given, $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$ ...(i)

$\therefore \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\left(\frac{x}{a}\right)^2}{1+\left(\frac{x}{a}\right)^2} = \frac{1-\frac{x^2}{a^2}}{1+\frac{x^2}{a^2}} = \frac{\frac{a^2-x^2}{a^2}}{\frac{a^2+x^2}{a^2}}$

$= \frac{a^2-x^2}{a^2}\times\frac{a^2}{a^2+x^2} = \frac{a^2-x^2}{a^2+x^2}$

EXAMPLE |6| Prove that $\cos 4x = 1-8\sin^2 x\cos^2 x$. **[NCERT]**

Sol. LHS $= \cos 4x = 1-2\sin^2 2x \quad [\because \cos 2x = 1-2\sin^2 x]$

$= 1-2(\sin 2x)^2$

$= 1-2(2\sin x\cdot\cos x)^2 \quad [\because \sin 2x = 2\sin x\cos x]$

$= 1-8\sin^2 x\cdot\cos^2 x =$ RHS **Hence proved.**

EXAMPLE |7| Prove that

$\sin A\sin(60°-A)\sin(60°+A) = \frac{1}{4}\sin 3A$.

Sol. LHS $= \sin A\sin(60°-A)\sin(60°+A)$

$= \sin A(\sin^2 60°-\sin^2 A)$

$[\because \sin(A+B)\sin(A-B) = \sin^2 A-\sin^2 B]$

$= \sin A\left(\frac{3}{4}-\sin^2 A\right) \quad \left[\because \sin 60° = \frac{\sqrt{3}}{2}\right]$

$= \frac{1}{4}\sin A(3-4\sin^2 A)$

$= \frac{1}{4}(3\sin A-4\sin^3 A) = \frac{1}{4}\sin 3A =$ RHS **Hence proved.**

EXAMPLE |8| Prove that

$\cot A+\cot(60°+A)-\cot(60°-A) = 3\cot 3A$.

Sol. LHS $= \cot A+\cot(60°+A)-\cot(60°-A)$

$= \frac{1}{\tan A}+\frac{1}{\tan(60°+A)}-\frac{1}{\tan(60°-A)}$

$= \frac{1}{\tan A}+\frac{1}{\frac{\tan 60°+\tan A}{1-\tan 60°\tan A}}-\frac{1}{\frac{\tan 60°-\tan A}{1+\tan 60°\tan A}}$

$= \frac{1}{\tan A}+\frac{1-\sqrt{3}\tan A}{\sqrt{3}+\tan A}-\frac{1+\sqrt{3}\tan A}{\sqrt{3}-\tan A}$

$\left[\because \tan(A\pm B) = \frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \text{ and } \tan 60° = \sqrt{3}\right]$

$= \frac{1}{\tan A}-\frac{8\tan A}{3-\tan^2 A} = \frac{3-9\tan^2 A}{3\tan A-\tan^3 A}$

$= 3\left(\frac{1-3\tan^2 A}{3\tan A-\tan^3 A}\right) = \frac{3}{\tan 3A} = 3\cot 3A =$ RHS

Hence proved.

EXAMPLE |9| If $\sin A = \frac{3}{5}$ and A is in I quadrant, then find the values of $\sin 2A$, $\cos 2A$ and $\tan 2A$.

Sol. We have, $\sin A = \frac{3}{5} \Rightarrow \cos A = +\sqrt{1-\sin^2 A}$

$[\because \sin^2 A+\cos^2 A = 1 \Rightarrow \cos^2 A = 1-\sin^2 A]$

$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\tan A = \frac{\sin A}{\cos A} = \frac{3/5}{4/5} = \frac{3}{4}$

Now, $\sin 2A = 2\sin A\cdot\cos A = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

$\cos 2A = 1-2\sin^2 A = 1-2\times\frac{9}{25} = 1-\frac{18}{25} = \frac{7}{25}$

and $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{3}{4}}{1-\frac{9}{16}} = \frac{24}{7}$

EXAMPLE |10| If $\cos\alpha+\cos\beta = 0 = \sin\alpha+\sin\beta$, then find the value of $\cos 2\alpha+\cos 2\beta$. **[NCERT Exemplar]**

Sol. Given, $\cos\alpha+\cos\beta = 0$ and $\sin\alpha+\sin\beta = 0$

On squaring both equations, we get

$(\cos\alpha+\cos\beta)^2 = 0$ ...(i)

and $(\sin\alpha+\sin\beta)^2 = 0$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0$$

$$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta) - (\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta) = 0$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta - \sin^2\alpha - \sin^2\beta - 2\sin\alpha\sin\beta = 0$$

$$\Rightarrow \quad (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + 2[\cos\alpha\cos\beta - \sin\alpha\sin\beta] = 0$$

$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta) = 0$$

$$\left[\begin{array}{l}\because \cos 2x = \cos^2 x - \sin^2 x \text{ and} \\ \cos A\cos B - \sin A\sin B = \cos(A+B)\end{array}\right]$$

$$\therefore \quad \cos 2\alpha + \cos 2\beta = -2\cos(\alpha+\beta)$$

EXAMPLE |11| Prove that

$$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A.$$

Sol. LHS $= \cos 5A = \cos(3A + 2A)$

$$= \cos 3A \cdot \cos 2A - \sin 3A \cdot \sin 2A$$

$$[\because \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B]$$

$$= (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3\sin A - 4\sin^3 A)(2\sin A \cdot \cos A)$$

$$[\because \cos 3x = 4\cos^3 x - 3\cos x, \ \sin 3x = 3\sin x - 4\sin^3 x, \ \cos 2x = 2\cos^2 x - 1 \text{ and } \sin 2x = 2\sin x \cdot \cos x]$$

$$= (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3 - 4\sin^2 A)(2\sin^2 A \cdot \cos A)$$

$$= (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - [3 - 4(1 - \cos^2 A)] \times [2(1 - \cos^2 A) \cdot \cos A]$$

$$[\because \sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x]$$

$$= (8\cos^5 A - 4\cos^3 A - 6\cos^3 A + 3\cos A) - (3 - 4 + 4\cos^2 A) \times 2\cos A(1 - \cos^2 A)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(1 - \cos^2 A) \times (4\cos^2 A - 1)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(4\cos^2 A - 1 - 4\cos^4 A + \cos^2 A)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(5\cos^2 A - 4\cos^4 A - 1)$$

$$= 8\cos^5 A - 10\cos^3 A + 3\cos A - 10\cos^3 A + 8\cos^5 A + 2\cos A$$

$$= 16\cos^5 A - 20\cos^3 A + 5\cos A$$

= RHS **Hence proved.**

EXAMPLE |12| Prove that

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1.$$ **[NCERT]**

Sol. We know that, $\cos 3x = 4\cos^3 x - 3\cos x$

On replacing x by $2x$, we get

$$\cos 3(2x) = 4\cos^3(2x) - 3\cos 2x$$

$$\Rightarrow \quad \cos 6x = 4(2\cos^2 x - 1)^3 - 3(2\cos^2 x - 1)$$

$$[\because \cos 2x = 2\cos^2 x - 1]$$

$$= 4[8\cos^6 x - 12\cos^4 x + 6\cos^2 x - 1] - 6\cos^2 x + 3$$

$$[\because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3]$$

$$= 32\cos^6 x - 48\cos^4 x + 24\cos^2 x - 4 - 6\cos^2 x + 3$$

$$\Rightarrow \quad \cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

Hence proved.

EXAMPLE |13| Show that

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8\theta}}} = 2\cos\theta.$$

Sol. LHS $= \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8\theta}}}$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(1 + \cos 8\theta)}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(1 + 2\cos^2 4\theta - 1)}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{4\cos^2 4\theta}}} = \sqrt{2 + \sqrt{2 + 2\cos 4\theta}}$$

$$= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = \sqrt{2 + \sqrt{2(1 + 2\cos^2 2\theta - 1)}}$$

$$= \sqrt{2 + \sqrt{4\cos^2 2\theta}} = \sqrt{2 + 2\cos 2\theta}$$

$$= \sqrt{2(1 + \cos 2\theta)} = \sqrt{2(1 + 2\cos^2\theta - 1)}$$

$$= \sqrt{4\cos^2\theta} = 2\cos\theta$$ **Hence proved.**

EXAMPLE |14| Prove that

$$\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \cdot \sin 4x.$$ **[NCERT]**

Sol. LHS $= \sin 2x + 2\sin 4x + \sin 6x$

$$= \sin 2x + \sin 6x + 2\sin 4x$$

$$= 2\sin\left(\frac{2x + 6x}{2}\right)\cos\left(\frac{2x - 6x}{2}\right) + 2\sin 4x$$

$$\left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\sin 4x \cdot \cos(-2x) + 2\sin 4x$$

$$= 2\sin 4x \cdot \cos 2x + 2\sin 4x \quad [\because \cos(-\theta) = \cos\theta]$$

$$= 2\sin 4x(\cos 2x + 1)$$

$$= 2\sin 4x(2\cos^2 x) = 4\sin 4x \cdot \cos^2 x$$

$$[\because \cos 2x = 2\cos^2 x - 1]$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |15| Prove that
$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=\frac{1}{2}(\tan 27x-\tan x).$$

Sol. LHS $=\dfrac{\sin x}{\cos 3x}+\dfrac{\sin 3x}{\cos 9x}+\dfrac{\sin 9x}{\cos 27x}$

$$=\frac{1}{2}\left[\frac{2\sin x\cos x}{\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x}+\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right]$$
$$=\frac{1}{2}\left[\frac{\sin 2x}{\cos 3x\cos x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 18x}{\cos 27x\cos 9x}\right]$$
$$=\frac{1}{2}\left[\frac{\sin(3x-x)}{\cos 3x\cos x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right]$$
$$=\frac{1}{2}\left[\frac{\sin 3x\cos x-\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x-\cos 9x\sin 3x}{\cos 9x\cos 3x}+\frac{\sin 27x\cos 9x-\cos 27x\sin 9x}{\cos 27x\cos 9x}\right]$$
$$=\frac{1}{2}(\tan 3x-\tan x+\tan 9x-\tan 3x+\tan 27x-\tan 9x)$$
$$=\frac{1}{2}(\tan 27x-\tan x)$$

Hence proved.

EXAMPLE |16| If $\tan\alpha=\frac{1}{7}$, $\sin\beta=\frac{1}{\sqrt{10}}$. Prove that $\alpha+2\beta=\frac{\pi}{4}$, where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$.

Sol. Given, $\sin\beta=\dfrac{1}{\sqrt{10}}, 0<\beta<\dfrac{\pi}{2}$ and $\tan\alpha=\dfrac{1}{7}$

$\Rightarrow \quad \cos\beta=+\sqrt{1-\sin^2\beta}$ $\quad$ [$\because$ β lies in I quadrant]

$\Rightarrow \quad \cos\beta=\sqrt{1-\sin^2\beta}$

$\Rightarrow \quad \cos\beta=\sqrt{1-\dfrac{1}{10}} \quad\Rightarrow\quad \cos\beta=\dfrac{3}{\sqrt{10}}$

$$\tan\beta=\frac{\sin\beta}{\cos\beta}=\frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}}=\frac{1}{3}$$

Now, $\tan 2\beta=\dfrac{2\tan\beta}{1-\tan^2\beta}$

$$\tan 2\beta=\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}=\frac{2/3}{8/9}=\frac{2}{3}\times\frac{9}{8}=\frac{3}{4}$$

$$\therefore\ \tan(\alpha+2\beta)=\frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}}=\frac{25}{25}=1$$

$\Rightarrow \quad \tan(\alpha+2\beta)=1\Rightarrow \tan(\alpha+2\beta)=\tan\dfrac{\pi}{4}$

$\therefore \quad (\alpha+2\beta)=\dfrac{\pi}{4}$ **Hence proved.**

EXAMPLE |17| Prove that
$$\cos A\cdot\cos 2A\cdot\cos 2^2A\cdot\cos 2^3A\ldots\cos 2^{n-1}A=\frac{\sin 2^nA}{2^n\sin A}.$$

Sol. LHS $=\cos A\cdot\cos 2A\cdot\cos 2^2A\cdot\cos 2^3A\ldots\cos 2^{n-1}A$

$$=\frac{1}{2\sin A}[(2\sin A\cos A)\cos 2A\cos 2^2A\cdot\cos 2^3A\ldots\cos 2^{n-1}A]$$
$$=\frac{1}{2\sin A}[(\sin 2A\cos 2A)\cdot\cos 2^2A\cos 2^3A\ldots\cos 2^{n-1}A]$$
$$=\frac{1}{2^2\sin A}[(2\sin 2A\cos 2A)\cos 2^2A\cdot\cos 2^3A\ldots\cos 2^{n-1}A]$$
$$=\frac{1}{2^2\sin A}[(\sin 2^2A\cos 2^2A)\cos 2^3A\ldots\cos 2^{n-1}A]$$
$$=\frac{1}{2^3\sin A}[(2\sin 2^2A\cos 2^2A)\cos 2^3A\ldots\cos 2^{n-1}A]$$
$$=\frac{1}{2^3\sin A}\quad[\sin 2^3A\cos 2^3A\ldots\cos 2^{n-1}A]$$

Proceeding in this manner, we get

LHS $=\dfrac{1}{2^{n-1}\sin A}[\sin 2^{n-1}A\cdot\cos 2^{n-1}A]$

$$=\frac{1}{2^n\sin A}[2\sin 2^{n-1}A\cos 2^{n-1}A]$$
$$=\frac{1}{2^n\sin A}\sin 2^nA=\text{RHS}$$

Hence proved.

EXAMPLE |18| If $\tan\beta=\dfrac{\tan\alpha+\tan\gamma}{1+\tan\alpha\tan\gamma}$, then prove that $\sin 2\beta=\dfrac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$.

Sol. Given, $\tan\beta=\dfrac{\tan\alpha+\tan\gamma}{1+\tan\alpha\cdot\tan\gamma}=\dfrac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\gamma}{\cos\gamma}}{1+\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\gamma}{\cos\gamma}}$

$$=\frac{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma}{\cos\alpha\cos\gamma+\sin\alpha\sin\gamma}$$

$\Rightarrow \quad \tan\beta=\dfrac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}$

Now, $\sin 2\beta=\dfrac{2\tan\beta}{1+\tan^2\beta}$

$$=\frac{2\times\frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1+\frac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}}$$

$$= \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$= \frac{\sin 2\alpha + \sin 2\gamma}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)]$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{2\cos^2(\alpha-\gamma)+2\sin^2(\alpha+\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{1+\cos 2(\alpha-\gamma)+1-\cos 2(\alpha+\gamma)}$$

$$\left[\begin{array}{l}\because 2\cos^2 A = 1+\cos 2A \\ \text{and } 2\sin^2 A = 1-\cos 2A\end{array}\right]$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+\cos 2(\alpha-\gamma)-\cos 2(\alpha+\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+2\sin 2\alpha \sin 2\gamma} = \frac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

EXAMPLE |19| Prove that

$$\tan 4x = \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}.$$

[NCERT]

Sol. $\text{LHS} = \tan 4x = \tan 2(2x) = \dfrac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\dfrac{2\tan x}{1-\tan^2 x}}{1-\left(\dfrac{2\tan x}{1-\tan^2 x}\right)^2}$$

$$= \frac{4\tan x}{1-\tan^2 x}\times\frac{(1-\tan^2 x)^2}{(1-\tan^2 x)^2-4\tan^2 x}$$

$$= \frac{4\tan x(1-\tan^2 x)}{1+\tan^4 x-2\tan^2 x-4\tan^2 x}$$

$$= \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x} = \text{RHS}$$

$\therefore$ LHS = RHS **Hence proved.**

EXAMPLE |20| Prove that

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$$

[NCERT Exemplar]

Sol. $\text{LHS} = \cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{16\pi}{15}$

$$= \cos 24^\circ \cos 48^\circ \cos 96^\circ \cos 192^\circ$$

$$= \frac{1}{16\sin 24^\circ}[(2\sin 24^\circ\cos 24^\circ)(2\cos 48^\circ)(2\cos 96^\circ)(2\cos 192^\circ)]$$

$[\because 2\sin\theta\cos\theta = \sin 2\theta]$

$$= \frac{1}{16\sin 24^\circ}[2\sin 48^\circ\cos 48^\circ(2\cos 96^\circ)(2\cos 192^\circ)]$$

$$= \frac{1}{16\sin 24^\circ}[(2\sin 96^\circ\cos 96^\circ)(2\cos 192^\circ)]$$

$$= \frac{1}{16\sin 24^\circ}(2\sin 192^\circ\cos 192^\circ)$$

$$= \frac{1}{16\sin 24^\circ}\sin 384^\circ = \frac{\sin(360^\circ+24^\circ)}{16\sin 24^\circ}$$

$$= \frac{1}{16} = \text{RHS}$$

Hence proved.

EXAMPLE |21| Prove that $\dfrac{\tan 3x}{\tan x}$ never lies between $\dfrac{1}{3}$ and 3.

Sol. Let $y = \dfrac{\tan 3x}{\tan x}$

$\Rightarrow \quad y = \dfrac{3\tan x - \tan^3 x}{\tan x(1-3\tan^2 x)}$

$\Rightarrow \quad y = \dfrac{3-\tan^2 x}{1-3\tan^2 x}$

$\Rightarrow \quad y - 3y\tan^2 x = 3 - \tan^2 x$

$\Rightarrow \quad (1-3y)\tan^2 x = 3 - y$

$\Rightarrow \quad \tan^2 x = \dfrac{3-y}{1-3y}$

$\Rightarrow \quad \dfrac{3-y}{1-3y} \ge 0 \;\Rightarrow\; \dfrac{y-3}{3y-1} \ge 0 \quad [\because \tan^2 x \ge 0 \text{ for all } x]$

Case **I** When $y-3\ge 0$ and $3y-1>0$

$\therefore \quad y \ge 3$ and $y > \dfrac{1}{3} \;\therefore\; y \ge 3$

Case **II** When $y-3 \le 0$ and $3y-1<0$

$\therefore \quad y \le 3$ and $y < \dfrac{1}{3}$

$\Rightarrow \quad y < \dfrac{1}{3}$

$\Rightarrow \quad y \in \left(-\infty, \dfrac{1}{3}\right) \cup [3, \infty)$

Hence, y does not lie between $\dfrac{1}{3}$ and 3.

TRIGONOMETRIC FUNCTIONS OF SUB-MULTIPLE ANGLES

Some important trigonometric functions x of angle in terms of trigonometric function of sub-multiple angles $\dfrac{x}{2}$ and $\dfrac{x}{3}$ are given below

Trigonometic ratios of the angle x in terms of $\frac{x}{2}$

(i) $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$= 1 - 2\sin^2\left(\frac{x}{2}\right)$

$\cos x = \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

(ii) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

(iii) $\tan x = \dfrac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$

Trigonometric ratios of the angle x in terms of $\frac{x}{3}$

(i) $\sin x = 3\sin\left(\frac{x}{3}\right) - 4\sin^3\left(\frac{x}{3}\right)$

(ii) $\cos x = 4\cos^3\left(\frac{x}{3}\right) - 3\cos\left(\frac{x}{3}\right)$

(iii) $\tan x = \dfrac{3\tan\left(\frac{x}{3}\right) - \tan^3\left(\frac{x}{3}\right)}{1 - 3\tan^2\left(\frac{x}{3}\right)}$

EXAMPLE |22| Find the value of $\sin 22\frac{1°}{2}$.

Sol. We know that, $\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$

$\therefore \sin\frac{45°}{2} = \sqrt{\frac{1 - \cos 45°}{2}}$ $\left[\because \frac{45°}{2} \text{ lies in I quadrant}\right]$

$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\frac{\sqrt{2} - 1}{\sqrt{2}}}{2}}$ $\left[\because \cos 45° = \frac{1}{\sqrt{2}}\right]$

$\Rightarrow \sin\frac{45°}{2} = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}} = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}} = \sqrt{\frac{2 - \sqrt{2}}{4}}$

$\therefore \sin 22\frac{1°}{2} = \frac{\sqrt{2 - \sqrt{2}}}{2}$

EXAMPLE |23| Prove that $\dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan\dfrac{A}{2}$.

Sol. LHS $= \dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \dfrac{1 - \cos A + \sin A}{1 + \cos A + \sin A}$

$= \dfrac{1 - \left(1 - 2\sin^2\frac{A}{2}\right) + \sin A}{1 + \left(2\cos^2\frac{A}{2} - 1\right) + \sin A}$

$\left[\because \cos A = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1\right]$

$= \dfrac{1 - 1 + 2\sin^2\frac{A}{2} + \sin A}{1 + 2\cos^2\frac{A}{2} - 1 + \sin A} = \dfrac{2\sin^2\frac{A}{2} + \sin A}{2\cos^2\frac{A}{2} + \sin A}$

$= \dfrac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}$

$\left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$

$= \dfrac{2\sin\frac{A}{2}\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)}{2\cos\frac{A}{2}\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)}$

$= \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \tan\dfrac{A}{2}$ $\left[\because \frac{\sin\theta}{\cos\theta} = \tan\theta\right]$

= RHS **Hence proved.**

EXAMPLE |24| If $\tan\frac{\theta}{2} = \sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi}{2}$, then prove that $\cos\phi = \dfrac{\cos\theta - e}{1 - e\cos\theta}$.

Sol. Given, $\tan\frac{\theta}{2} = \sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi}{2} \Rightarrow \tan\frac{\phi}{2} = \sqrt{\frac{1 + e}{1 - e}}\tan\frac{\theta}{2}$

We know that, $\cos\phi = \dfrac{1 - \tan^2\frac{\phi}{2}}{1 + \tan^2\frac{\phi}{2}}$

$\Rightarrow \quad \cos\phi = \dfrac{1 - \frac{1 + e}{1 - e}\tan^2\frac{\theta}{2}}{1 + \frac{1 + e}{1 - e}\tan^2\frac{\theta}{2}}$

$\Rightarrow \quad \cos\phi = \dfrac{(1 - e) - (1 + e)\tan^2\frac{\theta}{2}}{(1 - e) + (1 + e)\tan^2\frac{\theta}{2}}$

$$\Rightarrow \quad \cos\phi = \frac{(1-e)\cos^2\frac{\theta}{2} - (1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2} + (1+e)\sin^2\frac{\theta}{2}}$$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - e\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) - e\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}$$

$$= \frac{\cos\theta - e}{1 - e\cos\theta}$$ **Hence proved.**

EXAMPLE |25| If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the values of $\sin\frac{x}{2}, \cos\frac{x}{2}, \tan\frac{x}{2}$. **[NCERT]**

Sol. Given, $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$

$$\because \quad \cos x = \pm\frac{1}{\sqrt{1+\tan^2 x}}$$

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} \qquad [\because x \text{ lies in III quadrant}]$$

$$\Rightarrow \quad \cos x = -\frac{1}{\sqrt{1+\frac{9}{16}}} = -\frac{4}{\sqrt{25}}$$

$$\Rightarrow \quad \cos x = -\frac{4}{5}$$

Now, $$\cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$$

$$\left[\because \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \cos x < 0, \sin x > 0\right]$$

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+4/5}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

and $$\tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2} = \frac{3/\sqrt{10}}{-1/\sqrt{10}} = -3$$

EXAMPLE |26| If $\cos\theta = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, then prove that $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

Sol. Given, $\cos\theta = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$

$$\Rightarrow \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta}{1-\cos\alpha\cos\beta} \left[\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\theta}{2} + 1 + \tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2} - 1 - \tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta + 1 - \cos\alpha\cos\beta}{\cos\alpha - \cos\beta - 1 + \cos\alpha\cos\beta}$$

[applying componendo and dividendo]

$$\Rightarrow -\frac{2}{2\tan^2\frac{\theta}{2}} = \frac{(\cos\alpha+1)(1-\cos\beta)}{(\cos\alpha-1)(\cos\beta+1)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{(1+\cos\alpha)(1-\cos\beta)}{(1-\cos\alpha)(1+\cos\beta)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{\left(1+2\cos^2\frac{\alpha}{2}-1\right)\left(1-1+2\sin^2\frac{\beta}{2}\right)}{\left(1-1+2\sin^2\frac{\alpha}{2}\right)\left(1+2\cos^2\frac{\beta}{2}-1\right)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{4\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}} \Rightarrow \tan^2\frac{\theta}{2} = \frac{\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}}{\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2}$$

$$\therefore \quad \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$ **Hence proved.**

EXAMPLE |27| If $\cos\theta = \cos\alpha\cos\beta$, then prove that $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}$.

Sol. Given, $\cos\theta = \cos\alpha\cos\beta$

$$\Rightarrow \quad \cos\beta = \frac{\cos\theta}{\cos\alpha} \Rightarrow \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \frac{1-\tan^2\frac{\beta}{2} + 1 + \tan^2\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2} - 1 - \tan^2\frac{\beta}{2}} = \frac{\cos\theta + \cos\alpha}{\cos\theta - \cos\alpha}$$

[applying componendo and dividendo]

$$\Rightarrow \frac{2}{-2\tan^2\frac{\beta}{2}} = \frac{2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)}{-2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)}$$

$$\Rightarrow \quad \frac{1}{\tan^2\frac{\beta}{2}} = \cot\left(\frac{\theta+\alpha}{2}\right)\cot\left(\frac{\theta-\alpha}{2}\right)$$

$$\Rightarrow \quad \tan^2\frac{\beta}{2} = \tan\left(\frac{\theta+\alpha}{2}\right)\tan\left(\frac{\theta-\alpha}{2}\right)$$ **Hence proved.**

EXAMPLE |28| If α and β are distinct roots of $a\cos\theta + b\sin\theta = c$, then prove that

$$\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}.$$

Sol. Given that α and β are distincts root of

$$a\cos\theta + b\sin\theta = c$$

$\therefore$ $a\cos\alpha + b\sin\alpha = c$ and $a\cos\beta + b\sin\beta = c$

$\Rightarrow$ $a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$

$\Rightarrow$ $a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$

$$\Rightarrow \quad -2a\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + 2b\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 0$$

$$\Rightarrow 2\sin\left(\frac{\alpha-\beta}{2}\right)\left[-a\sin\left(\frac{\alpha+\beta}{2}\right) + b\cos\left(\frac{\alpha+\beta}{2}\right)\right] = 0$$

$$\Rightarrow b\cos\left(\frac{\alpha+\beta}{2}\right) = a\sin\frac{(\alpha+\beta)}{2}$$

$$\left[\because \alpha \neq \beta, \text{ therefore } \sin\left(\frac{\alpha-\beta}{2}\right) \neq 0\right]$$

$$\Rightarrow \quad \tan\frac{(\alpha+\beta)}{2} = \frac{b}{a}$$

$$\therefore \quad \sin(\alpha+\beta) = \frac{2\tan\frac{(\alpha+\beta)}{2}}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\times\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{\frac{2b}{a}}{\frac{a^2+b^2}{a^2}}$$

$$= \frac{2ab}{a^2+b^2}$$

Hence proved.

Trigonometric Ratios of Some More Angles

(i) $\sin 18° = \cos 72° = \dfrac{\sqrt{5}-1}{4}$

(ii) $\cos 18° = \sin 72° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(iii) $\cos 36° = \sin 54° = \dfrac{\sqrt{5}+1}{4}$

(iv) $\sin 36° = \cos 54° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

EXAMPLE |29| Prove that

$$\tan 6° \tan 42° \tan 66° \tan 78° = 1.$$

Sol. LHS $= \tan 6° \cdot \tan 42° \cdot \tan 66° \cdot \tan 78°$

$$= \frac{\sin 6°}{\cos 6°}\cdot\frac{\sin 42°}{\cos 42°}\cdot\frac{\sin 66°}{\cos 66°}\cdot\frac{\sin 78°}{\cos 78°}$$

$$= \frac{(2\sin 66° \sin 6°)}{(2\cos 66° \cos 6°)} \times \frac{(2\sin 78° \sin 42°)}{(2\cos 78° \cos 42°)}$$

$$= \frac{\cos 60° - \cos 72°}{\cos 60° + \cos 72°} \times \frac{\cos 36° - \cos 120°}{\cos 36° + \cos 120°}$$

$$= \frac{\cos 60° - \cos(90°-18°)}{\cos 60° + \cos(90°-18°)} \times \frac{\cos 36° - \cos(90°+30°)}{\cos 36° + \cos(90°+30°)}$$

$$= \frac{\cos 60° - \sin 18°}{\cos 60° + \sin 18°} \times \frac{\cos 36° + \sin 30°}{\cos 36° - \sin 30°}$$

$$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)} \times \frac{\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)}$$

$$= \frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)}$$

$$= \frac{9-5}{5-1} = 1 = \text{RHS}$$

Hence proved.

EXAMPLE |30| Prove that

$$16\cos\frac{2\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{8\pi}{15}\cdot\cos\frac{14\pi}{15} = 1.$$

Sol. LHS $= 16\cos\dfrac{2\pi}{15}\cdot\cos\dfrac{4\pi}{15}\cdot\cos\dfrac{8\pi}{15}\cdot\cos\dfrac{14\pi}{15}$

$$= 4(2\cos 24° \cos 96°)(2\cos 48° \cdot \cos 168°)$$

$$= 4(\cos 120° + \cos 72°)(\cos 216° + \cos 120°)$$

$$= 4(-\sin 30° + \sin 18°)(-\cos 36° - \sin 30°)$$

$$= 4\left(-\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right) = 4\left(\frac{\sqrt{5}-3}{4}\right)\left(\frac{-\sqrt{5}-3}{4}\right)$$

$$= 4\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right) = \frac{9-5}{4} = 1$$

Hence proved.

EXAMPLE |31| Find the value of

$$\cos 12° + \cos 84° + \cos 156° + \cos 132°.$$

[NCERT Exemplar]

Sol. We have, $\cos 12° + \cos 84° + \cos 156° + \cos 132°$

$$= \cos 12° + \cos 156° + \cos 84° + \cos 132°$$

$$= 2\cos\left(\frac{12° + 156°}{2}\right)\cdot\cos\left(\frac{12° - 156°}{2}\right) + 2\cos\left(\frac{84° + 132°}{2}\right)\cdot\cos\left(\frac{84° - 132°}{2}\right)$$

$$= 2\cos 84° \cos 72° + 2\cos 108° \cdot \cos 24°$$

$$[\because \cos(-72°) = \cos 72° \text{ and } \cos(-24°) = \cos 24°]$$

$$= 2\cos 84° \cos(90° - 18°) + 2\cos(90° + 18°)\cdot\cos 24°$$

$$= 2\cos 84° \sin 18° - 2\sin 18° \cdot \cos 24°$$

$= 2\sin 18°(\cos 84° - \cos 24°)$

$= 2\sin 18°\left[-2\sin\left(\frac{84° + 24°}{2}\right)\cdot \sin\left(\frac{84° - 24°}{2}\right)\right]$

$= -4\sin 18° \cdot \sin 54° \sin 30°$

$= -4\left(\frac{\sqrt{5}-1}{4}\right)\cdot \cos 36° \cdot \frac{1}{2}$ $[\because \sin(90° - 36°) = \cos 36°]$

$= -(\sqrt{5}-1)\left(\frac{\sqrt{5}+1}{4}\right)\cdot \frac{1}{2}$

$= -\left(\frac{5-1}{8}\right)$

$= \frac{-4}{8} = \frac{-1}{2}$

TRIGONOMETRIC EQUATIONS

Equations which involve trigonometric functions of unknown angles (i.e. variable) are known as **trigonometric equations.**

e.g. $\sin x = \frac{3}{4}$, $\tan x + \sec x = -\sqrt{3}$, $\cos x + 4\sin x = 1$ etc.

Solution of a Trigonometric Equation

A solution of a trigonometric equation is the value of the unknown angle (i.e. variable) that satisfies the equation. We know that, the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π.

Solutions of trigonometric equation is of two types

PRINCIPAL SOLUTION

The solution of a trigonometric equation for which the value of unknown angle say x lies between 0 and 2π, i.e. $0 \le x < 2\pi$, is called its **principal solution.**

EXAMPLE |32| Find the principal solutions of the following equations.

(i) $\sin x = \frac{\sqrt{3}}{2}$

(ii) $\tan x = \frac{-1}{\sqrt{3}}$

(iii) $\sec x = -2$ **[NCERT]**

Sol. (i) We have, $\sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = \sin\frac{\pi}{3}$ $\left[\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$

$\therefore x = \frac{\pi}{3}$, which lies in I quadrant.

and $\sin x = \sin\left(\pi - \frac{\pi}{3}\right)$ $[\because \sin\theta = \sin(\pi - \theta)]$

$\therefore x = \frac{2\pi}{3}$, which lies in II quadrant.

Hence, the principal solutions are $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$.

(ii) We have, $\tan x = -\frac{1}{\sqrt{3}}$

Here, the value of $\tan x$ is negative. So, x lies in II and IV quadrants.

We know that, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Thus, $\tan\left(\pi - \frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}} \Rightarrow \tan\frac{5\pi}{6} = \frac{-1}{\sqrt{3}}$

$\therefore x = \frac{5\pi}{6}$, which lies in II quadrant.

Also, $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} \Rightarrow \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

$\therefore x = \frac{11\pi}{6}$, which lies in IV quadrant.

Hence, the principal solutions are $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.

(iii) We have,

$\sec x = -2 \Rightarrow \cos x = -\frac{1}{2}$

Here, the value of $\cos x$ is negative.
So, x lies in II and III quadrants.

We know that $\cos\frac{\pi}{3} = \frac{1}{2}$

Thus $\cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2} \Rightarrow \cos\frac{2\pi}{3} = -\frac{1}{2}$

$\therefore x = \frac{2\pi}{3}$, which lies in II quadrant.

Also $\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{1}{2} \Rightarrow \cos\frac{4\pi}{3} = -\frac{1}{2}$

$\therefore x = \frac{4\pi}{3}$, which lies in III quadrant.

Hence, the principal solutions are $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is **[NCERT Exemplar]**

(a) 1 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2

2 If $\sin\theta+\cos\theta=1$, then the value of $\sin 2\theta$ is **[NCERT Exemplar]**

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) -1

3 If $\tan\theta=\frac{a}{b}$, then $b\cos 2\theta+a\sin 2\theta$ is equal to **[NCERT Exemplar]**

(a) a (b) b

(c) $\frac{a}{b}$ (d) None of these

4 The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{16}$

5 $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$ is equal to

(a) $\cos\theta$ (b) $2\cos\theta$ (c) $3\cos\theta$ (d) None of these

6 The number of solutions of equation $\tan x+\sec x=2\cos x$ lying in the interval $[0, 2\pi]$ is **[NCERT Exemplar]**

(a) 0 (b) 1 (c) 2 (d) 3

7 The minimum value of $3\cos x+4\sin x+8$ is **[NCERT Exemplar]**

(a) 5 (b) 9 (c) 7 (d) 3

VERY SHORT ANSWER Type Questions

8 Find the value of $2\sin A\cos A$, if $A=22\frac{1}{2}°$.

9 If $\cos A=\frac{1}{3}$, then find the value of $\cos 2A$.

10 If $\tan A=\frac{2}{5}$, then find the value of $\tan 2A$.

Directions (Q.Nos. 11-13) *Find the value of the following*

11 $\tan x=\frac{1}{\sqrt{3}}$ **[NCERT]**

12 $\operatorname{cosec} x=-2$

13 $\tan x=\sqrt{3}$

14 If θ is the positive acute angle, then solve the equation $4\cos^2\theta-4\sin\theta=1$.

15 If $3\sin\theta=2\cos\theta$, then find the value of $\sin 2\theta$.

16 Find the value of trigonometric ratio $\cos 22\frac{1}{2}°$.

17 Find the value of trigonometric ratio $\tan 22\frac{1}{2}°$.

SHORT ANSWER Type I Questions

18 Prove that $\frac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}=2\sin x$.

19 Prove that $\frac{\sin 2\theta}{1+\cos 2\theta}=\tan\theta$.

20 Prove that $\frac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta$.

21 Prove that $\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta}=\cot\theta$.

22 Prove that $\frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A}=\tan A$.

23 Prove that $\frac{\cos 2\theta}{1+\sin 2\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$.

24 Prove that $\frac{\sec 8\theta-1}{\sec 4\theta-1}=\frac{\tan 8\theta}{\tan 2\theta}$.

25 Find the value of $\sin 7\frac{1}{2}°$.

SHORT ANSWER Type II Questions

26 Show that $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$.

27 Prove that $\sin 4A=4\sin A\cos^3 A-4\cos A\sin^3 A$.

28 Prove that $\cos 4A=1-8\cos^2 A+8\cos^4 A$.

29 Prove that $4\cos\theta\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right)=\cos 3\theta$.

30 Prove that $4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right)=\sin 3\theta$.

31 Prove that

$(\cos x-\cos y)^2+(\sin x-\sin y)^2=4\sin^2\left(\frac{x-y}{2}\right)$.

32 If $2\cos\theta=x+\frac{1}{x}$, then prove that

$2\cos 3\theta=x^3+\frac{1}{x^3}$.

33 If $\tan x=\frac{b}{a}$, then find the value of

$\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$. **[NCERT Exemplar]**

LONG ANSWER Type Questions

34 Prove that $\cot 7\frac{1^\circ}{2} = \tan 82\frac{1^\circ}{2} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$.

35 Find the value of the expression

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}.$$

[NCERT Exemplar]

36 Prove that

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}.$$

37 Prove that

$$\cos^3 A+\cos^3(120^\circ+A)+\cos^3(240^\circ+A)=\frac{3}{4}\cos 3A.$$

38 Prove that

$$\cot\theta\cot 2\theta+\cot 2\theta\cot 3\theta+2=\cot\theta(\cot\theta-\cot 3\theta).$$

HINTS & ANSWERS

1. (c) Use formula, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Put $\theta = 15^\circ$

2. (c) Given that, $\sin\theta+\cos\theta=1$

On squaring both sides, we get

$$\sin^2\theta+\cos^2\theta+2\sin\theta\cdot\cos\theta=1$$

$$\Rightarrow \quad 1+\sin 2\theta = 1$$

$$\therefore \quad \sin 2\theta = 0$$

$$=\frac{1}{16\sin 24^\circ}[(2\sin 96^\circ\cos 96^\circ)(2\cos 192^\circ)]$$

$$=\frac{1}{16\sin 24^\circ}(2\sin 192^\circ\cos 192^\circ)$$

$$=\frac{1}{16\sin 24^\circ}\sin 384^\circ$$

$$=\frac{\sin(360^\circ+24^\circ)}{16\sin 24^\circ}$$

$$=\frac{1}{16}$$

5. (b) We have, $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$

$$=\sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$$

$$=\sqrt{2+\sqrt{2+\sqrt{2(1+2\cos^2 4\theta-1)}}}$$

$$=\sqrt{2+\sqrt{2+\sqrt{4\cos^2 4\theta}}}$$

$$=\sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$=\sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$=\sqrt{2+\sqrt{2(1+\cos^2 2\theta-1)}}$$

$$=\sqrt{2+\sqrt{4\cos^2 2\theta}}$$

$$=\sqrt{2+2\cos 2\theta}$$

$$=\sqrt{2(1+\cos 2\theta)}$$

$$=\sqrt{2(1+2\cos^2\theta-1)}$$

$$=\sqrt{4\cos^2\theta}$$

$$=2\cos\theta$$

3. (b) We have,

$$b\cos 2\theta + a\sin 2\theta = b\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)+a\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=b\left(\frac{1-\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}\right)+a\left(\frac{\frac{2a}{b}}{1+\frac{a^2}{b^2}}\right)\quad\left[\because \tan\theta=\frac{a}{b}\right]$$

$$=b$$

4. (d) We have,

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$$

$$=\cos 24^\circ\cos 48^\circ\cos 96^\circ\cos 192^\circ$$

$$=\frac{1}{16\sin 24^\circ}[(2\sin 24^\circ\cos 24^\circ)(2\cos 48^\circ)(2\cos 96^\circ)(2\cos 192^\circ)]$$

$$[\because 2\sin\theta\cos\theta=\sin 2\theta]$$

$$=\frac{1}{16\sin 24^\circ}[2\sin 48^\circ\cos 48^\circ(2\cos 96^\circ)(2\cos 192^\circ)]$$

6. (c) Given equation, $\tan x+\sec x = 2\cos x$

$$\Rightarrow \quad \frac{\sin x}{\cos x}+\frac{1}{\cos x}=2\cos x$$

$$\Rightarrow \quad 1+\sin x = 2\cos^2 x$$

$$\Rightarrow \quad 1+\sin x = 2(1-\sin^2 x)$$

$$\Rightarrow \quad 2\sin^2 x+\sin x-1=0$$

$$\Rightarrow \quad (\sin x+1)(2\sin x-1)=0$$

$$\Rightarrow \quad \sin x=-1,\ \sin x=\frac{1}{2}$$

$$\therefore \quad x=\frac{3\pi}{2},\ x=\frac{\pi}{6}$$

Hence, only two solutions possible.

7. (d) Given expression, $3\cos x+4\sin x+8$

Let $\quad y=3\cos x+4\sin x+8$

$$\Rightarrow \quad y-8=3\cos x+4\sin x$$

$\therefore$ Minimum value of

$$y-8=-\sqrt{9+16}$$

$$\Rightarrow \quad y-8=-5 \Rightarrow y=-5+8$$

$$\therefore \quad y=3$$

Hence, the minimum value of $3\cos x+4\sin x+8$ is 3.

8. Use the formula, $\sin 2A = 2\sin A\cos A$ Ans. $\frac{1}{\sqrt{2}}$

9. Use formula, $\cos 2A = 2\cos^2 A - 1$. **Ans.** $-\frac{7}{9}$

10. Use formula, $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$. **Ans.** $\frac{20}{21}$

11. $\tan x = \frac{1}{\sqrt{3}} \Rightarrow \tan x = \tan\frac{\pi}{6}, \tan\frac{7\pi}{6}$ **Ans.** $x = \frac{\pi}{6}, \frac{7\pi}{6}$

12. $\operatorname{cosec} x = -2 \Rightarrow \sin x = -\frac{1}{2}$ **Ans.** $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

13. $\frac{\pi}{3}, \frac{2\pi}{3}$

14. We have, $4\cos^2\theta - 4\sin\theta = 1$

$\Rightarrow 4(1-\sin^2\theta) - 4\sin\theta = 1$

$\therefore \sin\theta = \frac{-4 \pm \sqrt{(4)^2 - 4\cdot 4(-3)}}{2\cdot 4} = \frac{-4 \pm \sqrt{16+48}}{8}$

$= \frac{-4 \pm \sqrt{64}}{8} = \frac{-4 \pm 8}{8} = \frac{1}{2}$ or $-\frac{3}{2}$

$\therefore \sin\theta = \frac{1}{2}$ **Ans.** $\theta = 30°$

15. $\tan\theta = \frac{2}{3}$; then $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{12}{13}$

16. $\cos 22\frac{1}{2}^{\circ} = \sqrt{\frac{\cos 45° + 1}{2}} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

17. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\therefore \tan 45° = \frac{2\tan\frac{45°}{2}}{1-\tan^2\frac{45°}{2}}$

$\Rightarrow 1 - \tan^2\frac{45°}{2} = 2\tan\left(\frac{45°}{2}\right)$

$\Rightarrow \tan^2\frac{45°}{2} + 2\tan\left(\frac{45°}{2}\right) - 1 = 0$

$\Rightarrow \tan\frac{45°}{2} = \frac{-2 \pm \sqrt{(2)^2 - 4\times(-1)}}{2\times 1} = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$

Since, $\left(\frac{45°}{2}\right)$ is in Ist quadrant, so we take '+ve' sign.

Ans. $-1+\sqrt{2}$

18. LHS $= \frac{2\cos 2x \sin(-x)}{-\cos 2x} = 2\sin x$

19. Use the formulae, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$.

20. Use the formulae, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$.

21. LHS $= \frac{1+\cos 2\theta + \sin 2\theta}{1-\cos 2\theta + \sin 2\theta}$

$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta} = \frac{2\cos\theta(\cos\theta + \sin\theta)}{2\sin\theta(\sin\theta + \cos\theta)} = \cot\theta$

22. Solve as Q. 21.

23. LHS $= \frac{\sin\left(\frac{\pi}{2} - 2\theta\right)}{1+\cos\left(\frac{\pi}{2} - 2\theta\right)} = \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cdot\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4} - 2\theta\right)}$

$= \tan\left(\frac{\pi}{4} - \theta\right)$

24. LHS $= \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \frac{(1-\cos 8\theta)\cos 4\theta}{(1-\cos 4\theta)\cos 8\theta} = \frac{2\sin^2 4\theta\cos 4\theta}{2\sin^2 2\theta\cos 8\theta}$

$= \frac{\sin 4\theta\,(2\sin 4\theta\cos 4\theta)}{2\sin^2 2\theta\cos 8\theta} = \frac{2\sin 2\theta\cos 2\theta\cdot\sin 8\theta}{2\sin^2 2\theta\cos 8\theta}$

25. $\sin 7\frac{1°}{2} = \sin\frac{15°}{2} = \sqrt{\frac{1-\cos 15°}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}+1}{2\sqrt{2}}}{2}}$

$= \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$

26. LHS $= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$

$= \sqrt{\frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}} = \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$

$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\theta}{2}}$

27. RHS $= 4\sin A\cos A(\cos^2 A - \sin^2 A)$

$= 2\sin 2A(\cos 2A) = \sin 4A$

28. RHS $= 1 - 8\cos^2 A(1-\cos^2 A) = 1 - 8\cos^2 A\sin^2 A$

$= 1 - 2(2\sin A\cos A)^2 = 1 - 2\sin^2 2A = \cos 4A$

29. LHS $= 2\cos\theta\left[2\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right)\right]$

$= 2\cos\theta\left[\cos\left(\frac{2\pi}{3}\right) + \cos(2\theta)\right]$

$= 2\cos\theta\left(-\frac{1}{2} + \cos 2\theta\right) = 2\cos\theta\left(\frac{-1}{2} + 2\cos^2\theta - 1\right)$

$= 2\cos\theta\left(\frac{-3+4\cos^2\theta}{2}\right) = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$

30. Solve as Q. 29.

31. LHS $= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2(\cos x\cos y + \sin x\sin y)$

$= 1 + 1 - 2\cos(x-y) = 2[1 - \cos(x-y)]$

32. $2\cos 3\theta = 2(4\cos^3\theta - 3\cos\theta)$

$$= 8\cos^3\theta - 6\cos\theta = (2\cos\theta)^3 - 3(2\cos\theta)$$

$$= \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)$$

33. $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{2a}{\sqrt{a^2-b^2}} = \frac{2}{\sqrt{1-\tan^2 x}}$

$$= \frac{2\cos x}{\sqrt{\cos^2 x - \sin^2 x}} = \frac{2\cos x}{\sqrt{\cos 2x}}$$

34. LHS $= \tan 82\frac{1^\circ}{2} = \tan\left(90^\circ - 7\frac{1^\circ}{2}\right) = \cot 7\frac{1^\circ}{2} = \cot A$ (say)

where, $A = 7\frac{1^\circ}{2}$

Now, $\cot A = \frac{\cos A}{\sin A} = \frac{\cos A\,(2\cos A)}{\sin A\,(2\cos A)} = \frac{1+\cos 2A}{\sin 2A}$

$$\therefore \cot 7\frac{1^\circ}{2} = \frac{1+\cos 2\left(\frac{15}{2}\right)^\circ}{\sin 2\left(\frac{15}{2}\right)^\circ} = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\cos(45^\circ - 30^\circ)}{\sin(45^\circ - 30^\circ)}$$

$$= \frac{1+(\cos 45^\circ\cdot\cos 30^\circ + \sin 45^\circ\cdot\sin 30^\circ)}{(\sin 45^\circ\cdot\cos 30^\circ - \cos 45^\circ\cdot\sin 30^\circ)}$$

$$= \frac{1+\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)}{\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)} = (\sqrt{2}+1)(\sqrt{3}+\sqrt{2})$$

35. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\left(\pi - \frac{3\pi}{8}\right) + \cos^4\left(\pi - \frac{\pi}{8}\right)$$

$$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$$

$$= 2\left[\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right] = 2\left[\cos^4\frac{\pi}{8} + \cos^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right]$$

$$= 2\left[\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}\right]$$

$$= 2\left[\left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8}\cdot\sin^2\frac{\pi}{8}\right]$$

$$= 2\left[1 - 2\cos^2\frac{\pi}{8}\cdot\sin^2\frac{\pi}{8}\right]$$

$$= 2 - \left(2\sin\frac{\pi}{8}\cdot\cos\frac{\pi}{8}\right)^2$$

$$= 2 - \left(\sin\frac{2\pi}{8}\right)^2 = \frac{3}{2}$$

36. Solve as Q. 35.

37. LHS $= \cos^3 A + \cos^3(120^\circ + A) + \cos^3(240^\circ + A)$

$$= \left[\frac{3}{4}\cos A + \frac{1}{4}\cos 3A\right] + \left[\frac{3}{4}\cos(120^\circ + A) + \frac{1}{4}\cos 3(120^\circ + A)\right] + \left[\frac{3}{4}\cos(240^\circ + A) + \frac{1}{4}\cos 3(240^\circ + A)\right]$$

$$= \left[\frac{3}{4}\cos A + \frac{1}{4}\cos 3A\right] + \left[\frac{3}{4}\cos(120^\circ + A) + \frac{1}{4}\cos(360^\circ + 3A)\right] + \left[\frac{3}{4}\cos(240^\circ + A) + \frac{1}{4}\cos(720^\circ + 3A)\right]$$

$$= \frac{3}{4}[\cos A + \cos(120^\circ + A) + \cos(240^\circ + A)] + \frac{1}{4}[\cos 3A + \cos(360^\circ + 3A) + \cos(360^\circ \times 2 + 3A)]$$

$$= \frac{3}{4}[\cos A + \cos(120^\circ + A) + \cos(240^\circ + A)] + \frac{1}{4}[\cos 3A + \cos 3A + \cos 3A]$$

$$= \frac{3}{4}[\cos A + \cos(120^\circ + A) + \cos(240^\circ + A)] + \frac{1}{4}\times 3(\cos 3A)$$

$$= \frac{3}{4}[\cos A + 2\cos(180^\circ + A)\cdot\cos(-60^\circ)] + \frac{3}{4}\cos 3A$$

$$= \frac{3}{4}\left[\cos A - 2\cos A\cdot\frac{1}{2}\right] + \frac{3}{4}\cos 3A$$

$$= \frac{3}{4}[\cos A - \cos A] + \frac{3}{4}\cos 3A = \frac{3}{4}\cos 3A$$

38. RHS $= \cot\theta\left(\cot\theta - \left(\frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}\right)\right)$

$$= \cot^2\theta\left(\frac{2\cot^2\theta + 2}{3\cot^2\theta - 1}\right) = \frac{2\cot^2\theta(\cot^2\theta + 1)}{3\cot^2\theta - 1}$$

LHS $= \cot\theta\left(\frac{\cot^2\theta - 1}{2\cot\theta}\right) + \left(\frac{\cot^2\theta - 1}{2\cot\theta}\right)\left(\frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}\right) + 2$

$$= \frac{\cot^2\theta - 1}{2} + \left(\frac{\cot^2\theta - 1}{2}\right)\left(\frac{\cot^2\theta - 3}{3\cot^2\theta - 1}\right) + 2$$

$$= \frac{(\cot^2\theta - 1)(3\cot^2\theta - 1) + (\cot^2\theta - 1)(\cot^2\theta - 3) + 4(3\cot^2\theta - 1)}{2(3\cot^2\theta - 1)}$$

$$= \frac{4\cot^4\theta + 4\cot^2\theta}{2(3\cot^2\theta - 1)}$$

$$= \frac{2\cot^2\theta(\cot^2\theta + 1)}{3\cot^2\theta - 1}$$

|TOPIC 6|

Applications of Sine and Cosine Formulae

Sine Rule

In any triangle, the sides are proportional to the sines of the opposite angles.

in ΔABC, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

The above rule may also expressed as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

The sine rule is very useful tool to express side of triangle in terms of the sines of angles and *vice-versa* in the following manner.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=K$$

$\Rightarrow \quad a=K\sin A,\ b=K\sin B,\ c=K\sin C$

Similarly, $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}=\lambda$

$\Rightarrow \sin A=a\lambda$, $\sin B=b\lambda$ and $\sin C=c\lambda$

EXAMPLE |1| In ΔABC, if $a=2$, $b=3$ and $\sin A=\dfrac{2}{3}$, then find $\angle B$.

Sol. Given, $a=2, b=3, \sin A=\dfrac{2}{3}$

By sine rule, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}\Rightarrow \frac{2}{2/3}=\frac{3}{\sin B}$$

$$\Rightarrow \quad 3=\frac{3}{\sin B}\Rightarrow \sin B=1$$

$$\Rightarrow \quad \sin B=\sin\frac{\pi}{2}\Rightarrow B=\frac{\pi}{2}$$

EXAMPLE |2| In any ΔABC, prove that

(i) $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$

(ii) $\dfrac{b^2-c^2}{\cos B+\cos C}+\dfrac{c^2-a^2}{\cos C+\cos A}+\dfrac{a^2-b^2}{\cos A+\cos B}=0$

Sol. Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=K$

Then, $a=K\sin A, b=K\sin B, c=K\sin C$

(i) $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{K\sin A\sin(B-C)}{K^2\sin^2 B-K^2\sin^2 C}$

$=\dfrac{K\sin[\pi-(B+C)]\sin(B-C)}{K^2(\sin^2 B-\sin^2 C)}$ $[\because A+B+C=180°]$

$=\dfrac{K\sin(B+C)\sin(B-C)}{K^2(\sin^2 B-\sin^2 C)}=\dfrac{K(\sin^2 B-\sin^2 C)}{K^2(\sin^2 B-\sin^2 C)}$

$=\dfrac{1}{K}$

Similarly, $\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{1}{K}$ and $\dfrac{c\sin(A-B)}{a^2-b^2}=\dfrac{1}{K}$

$\therefore \dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$

(ii) Now $\dfrac{b^2-c^2}{\cos B+\cos C}=\dfrac{K^2\sin^2 B-K^2\sin^2 C}{\cos B+\cos C}$

$=\dfrac{K^2(\sin^2 B-\sin^2 C)}{\cos B+\cos C}$

$=\dfrac{K^2(1-\cos^2 B-1+\cos^2 C)}{\cos B+\cos C}$

$=\dfrac{K^2(\cos^2 C-\cos^2 B)}{\cos B+\cos C}$

$=\dfrac{K^2[(\cos B+\cos C)(\cos C-\cos B)]}{(\cos B+\cos C)}$

$=K^2[\cos C-\cos B]$

Similarly, $\dfrac{c^2-a^2}{\cos C+\cos A}=K^2[\cos A-\cos C]$

and $\dfrac{a^2-b^2}{\cos A+\sin B}=K^2[\cos B-\cos A]$

$\therefore$ LHS $=K^2[\cos C-\cos B+\cos A-\cos C+\cos B-\cos A]$

$=K^2\times 0=0$ **Hence proved.**

EXAMPLE |3| In any ΔABC, prove that

(i) $\dfrac{\sin(B-C)}{\sin(B+C)}=\dfrac{b^2-c^2}{a^2}$

(ii) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$. **[NCERT]**

Sol. By sine rule, we have $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=K$

$\Rightarrow \quad a=K\sin A, b=K\sin B, c=K\sin C$ ...(i)

(i) RHS $=\dfrac{b^2-c^2}{a^2}=\dfrac{K^2\sin^2 B-K^2\sin^2 C}{K^2\sin^2 A}$

$=\dfrac{\sin^2 B-\sin^2 C}{\sin^2 A}=\dfrac{\sin(B+C)\sin(B-C)}{\sin^2 A}$

$$= \frac{\sin(\pi - A)\sin(B-C)}{\sin^2 A} = \frac{\sin A \sin(B-C)}{\sin^2 A}$$

$$[\because A+B+C=\pi \Rightarrow B+C=\pi-\pi]$$

$$= \frac{\sin(B-C)}{\sin A} = \frac{\sin(B-C)}{\sin A}$$

$$= \frac{\sin(B-C)}{\sin(\pi-(B+C))} = \frac{\sin(B-C)}{\sin(B+C)}$$

$$[\because A+B+C=\pi,\ A=\pi-(B+C)]$$

$= \text{LHS}$

(ii) $\text{LHS} = a\sin(B-C) + b\sin(C-A) + C\sin(A-B)$

$$= K\sin A\sin(B-C) + K\sin B\sin(C-A) + K\sin C\sin(A-B)$$

$$= K[\sin(\pi-(B+C)\sin(B-C) + \sin(\pi-(A+C)) \sin(C-A) + \sin(\pi-(A+B)\sin(A-B)]$$

$$[\because A+B+C=\pi]$$

$$= K[\sin(B+C)\sin(B-C) + \sin(C+A) \sin(C-A) + \sin(A+B)\sin(A-B)]$$

$$= K[\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B]$$

$= K[0] = 0 = \text{RHS}$ **Hence proved.**

EXAMPLE |4| In any ΔABC, prove that

$$\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}.$$

[NCERT]

Sol. $\text{RHS} = \left(\frac{b-c}{a}\right)\cos\frac{A}{2} = \left(\frac{K\sin B - K\sin C}{K\sin A}\right)\cos\frac{A}{2}$

$$= \left(\frac{\sin B - \sin C}{\sin A}\right)\cos\frac{A}{2}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}} \times \cos\frac{A}{2}$$

$$= \frac{\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left[\frac{\pi}{2} - \left(\frac{B+C}{2}\right)\right]}$$

$$\left[\because A+B+C=\pi \Rightarrow \frac{A}{2} = \frac{\pi}{2} - \frac{(B+C)}{2}\right]$$

$$= \frac{\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} = \sin\left(\frac{B-C}{2}\right)$$

Hence proved.

EXAMPLE |5| In a ΔABC, $\dfrac{a^2-b^2}{a^2+b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$, prove that it is either a right angled or an isosceles triangle.

Sol. Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = K$

$\Rightarrow \quad a = K\sin A,\ b = K\sin B$ and $c = K\sin C$

Now, $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{a^2+b^2} = \dfrac{K^2\sin^2 A - K^2\sin^2 B}{K^2\sin^2 A + K^2\sin^2 B}$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin(A+B)} = \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin(\pi-C)} = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$[\because A+B=\pi-C]$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin C\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} - \frac{\sin C\sin(A-B)}{\sin^2 A + \sin^2 B} = 0$$

$$\Rightarrow \quad \sin(A-B)\left[\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right] = 0$$

$$\Rightarrow \sin(A-B) = 0 \text{ or } \frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B} = 0$$

$$\Rightarrow \quad A = B \text{ or } \sin^2 A + \sin^2 B - \sin^2 C = 0$$

$$\Rightarrow \quad A = B \text{ or } \frac{a^2}{K^2} + \frac{b^2}{K^2} - \frac{c^2}{K^2} = 0$$

$$\Rightarrow \quad A = B \text{ or } a^2 + b^2 = c^2$$

$\therefore$ Either the triangle is isosceles or right angled.

Hence proved.

Cosine Rule

Let a, b, c be the length of sides of ΔABC opposite to $\angle A$, $\angle B$ and $\angle C$, respectively. Then,

(i) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ (ii) $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

(iii) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

EXAMPLE |6| In ΔABC, if $a = 18$, $b = 24$, $c = 30$, then find $\cos A$, $\cos B$, $\cos C$.

Sol. By cosine formula, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{(24)^2 + (30)^2 - (18)^2}{2\times 24\times 30}$$

$$= \frac{576 + 900 - 324}{1440} = \frac{1152}{1440} = \frac{4}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(30)^2 + (18)^2 - (24)^2}{2 \times 30 \times 18}$$

$$= \frac{900 + 324 - 576}{1080} = \frac{648}{1080} = \frac{3}{5}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(18)^2 + (24)^2 - (30)^2}{2 \times 18 \times 24}$$

$$= \frac{324 + 576 - 900}{1440} = 0$$

Hence, $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$ and $\cos C = 0$

EXAMPLE |7| If the sides of a ΔABC are $a = 4$, $b = 6$ and $c = 8$, show that $4 \cos B + 3 \cos C = 2$

Sol. Given, $a = 4, b = 6, c = 8$

$$\therefore \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 64 - 36}{2 \times 4 \times 8} = \frac{44}{64} = \frac{11}{16}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 36 - 64}{2 \times 4 \times 6} = \frac{-12}{48} = -\frac{1}{4}$$

$$\therefore \quad 4 \cos B + 3 \cos C = 4 \times \frac{11}{16} - \frac{3}{4} = \frac{11}{4} - \frac{3}{4} = 2$$

EXAMPLE |8| With usual notations, if in a ΔABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

Sol. Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = K$

$\Rightarrow b + c = 11k, c + a = 12k, a + b = 13k$

$\Rightarrow 2(a + b + c) = 36k$

$\Rightarrow a + b + c = 18k$

Now, $b + c = 11k$ and $a + b + c = 18k \Rightarrow a = 7k$

$c + a = 12k$ and $a + b + c = 18k \Rightarrow b = 6k$

$a + b = 13k$ and $a + b + c = 18k \Rightarrow c = 5k$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36K^2 + 25K^2 - 49K^2}{60K^2} = \frac{12}{60} = \frac{1}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{25K^2 + 49K^2 - 36K^2}{70K^2} = \frac{38}{70} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49K^2 + 36K^2 - 25K^2}{84K^2}$$

$$= \frac{60}{84} = \frac{5}{7}$$

$$\therefore \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$$

or $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$ **Hence proved.**

EXAMPLE |9| In a ΔABC, prove that

(i) $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$

(ii) $\left(\frac{b^2 - c^2}{a^2}\right) \sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) \sin 2B + \left(\frac{a^2 - b^2}{c^2}\right) \sin 2C = 0.$ **[NCERT]**

Sol. (i) LHS $= (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C$

$$= (b^2 - c^2) \frac{\cos A}{\sin A} + (c^2 - a^2) \frac{\cos B}{\sin B} + (a^2 - b^2) \frac{\cos C}{\sin C}$$

$$= \frac{b^2 - c^2}{Ka}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{c^2 - a^2}{Kb}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + \left(\frac{a^2 - b^2}{Kc}\right)\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

[using sine rule, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = b$ (say)

$\Rightarrow \sin A = aK, \sin B = bK$ and $\sin C = cK$]

$$= \frac{1}{2Kabc}[(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)]$$

$$= \frac{1}{2Kabc}[(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) + (c^2 - a^2)(c^2 + a^2) - b^2(c^2 - a^2) + (a^2 - b^2)(a^2 + b^2) - c^2(a^2 - b^2)]$$

$$= \frac{1}{2Kabc}[(b^2 - c^2)(b^2 + c^2) + (c^2 - a^2)(c^2 + a^2) + (a^2 - b^2)(a^2 + b^2) - a^2(b^2 - c^2) - b^2(c^2 - a^2) - c^2(a^2 - b^2)]$$

$$= \frac{1}{2Kabc}[(b^4 - c^4) + (c^4 - a^4) + (a^4 - b^4) - (a^2b^2 - a^2c^2) - (b^2c^2 - b^2a^2) - (c^2a^2 - c^2b^2)]$$

$$= \frac{1}{2Kabc} \times 0 = 0 = \text{RHS}$$ **Hence proved.**

(ii) LHS $= \left(\frac{b^2 - c^2}{a^2}\right) \sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) \sin 2B + \left(\frac{a^2 - b^2}{c^2}\right) \sin 2c$

$$= \left(\frac{b^2 - c^2}{a^2}\right) 2 \sin A \cos A + \left(\frac{c^2 - a^2}{b^2}\right) 2 \sin B \cos B + \left(\frac{a^2 - b^2}{c^2}\right) 2 \sin C \cos C$$

$$= \left(\frac{b^2 - c^2}{a^2}\right) 2Ka\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \left(\frac{c^2 - a^2}{b^2}\right)$$

[using sine rule, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = b$ (say)

$\Rightarrow \sin A = aK, \sin B = bK$ and $\sin C = CK$]

$$2Kb\left(\frac{a^2+c^2-b^2}{2ac}\right)+\left(\frac{a^2-b^2}{c^2}\right)2Kc\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$=\frac{K}{abc}[(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)$$

$$(c^2+a^2-b^2)+(a^2-b^2)(a^2+b^2-c^2)]$$

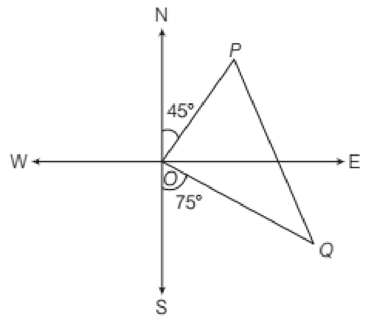
$$=\frac{K}{abc}\times 0=0=\text{RHS}$$ **Hence proved.**

Applications of Sine and Cosine Formulae at in Heights and Distances

Sometimes while solving the problem of Heights and Distances, we use sine and cosine formula.

EXAMPLE |10| Two ships leave a port at the same time. One goes 24 km/h in the direction N 45°E and other travels 32 km/h in the direction S75°E. Find the distance between the ships at the end of 3 h.

Sol. Let P and Q be the position of two ships at the end of 3 h.



Then, $OP = 3\times 24 = 72$ km and $OQ = 3\times 32 = 96$ km

Using cosine formula in ΔOPQ, we have

$$PQ^2 = OP^2 + OQ^2 - 2OP\times OQ\cos 60^\circ$$

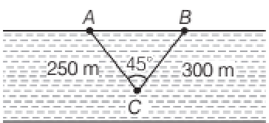
$$\Rightarrow \quad PQ^2 = (72)^2 + (96)^2 - 2\times 72\times 96\times\frac{1}{2}$$

$$\Rightarrow \quad PQ^2 = 5184 + 9216 - 6912 = 7488$$

$$\Rightarrow \quad PQ = \sqrt{7488} = 86.53 \text{ km}$$

EXAMPLE |11| Two trees A and B are on the same side of a river. From a point C in the river the distance of trees A and B are 250 m and 300 m, respectively of the angle C is 45°, find the distance between the trees. [use $\sqrt{2} = 1.44$]

Sol. According to the given information, we have the following figure



In ΔABC, by cosine rule, we have

$$AB^2 = AC^2 + BC^2 - 2AC\cdot BC\cos\frac{\pi}{4}$$

$$\therefore\ AB = \sqrt{(250)^2 + (300)^2 - 2\times 250\times 300\times\frac{1}{\sqrt{2}}}$$

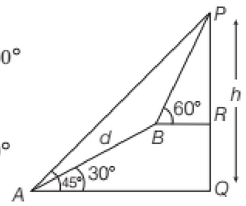
$$= \sqrt{62500 + 90000 - 75000\sqrt{2}}$$

$$= \sqrt{152500 - 75000\times 1.44}$$

$$= \sqrt{152500 - 108000} = \sqrt{44500} = 210.95 \text{ m}$$

EXAMPLE |12| The angle of elevation of the top point P, of the vertical tower PQ of height h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the points A, measured along the line AB which makes an angle 30° with AQ. Prove that $d = (\sqrt{3}-1)h$.

Sol. It is given that,



$\angle PAQ = 45^\circ$ and $\angle BAQ = 30^\circ$

$\therefore\ \angle BAP = 15^\circ$

In ΔAQP,

$\angle PAQ = 45^\circ$ and $\angle PQA = 90^\circ$

$\therefore\ \angle APQ = 45^\circ$

In ΔBRP, we have

$\angle PBR = 60^\circ$ and $\angle PRB = 90^\circ$

$\therefore\ \angle BPR = 30^\circ$

Now, $\angle APQ = 45^\circ$ and $\angle BPR = 30^\circ \Rightarrow \angle BPA = 15^\circ$

In ΔABP, we have, $\angle PAB = 15^\circ$ and $\angle BPA = 15^\circ$

$\therefore\ \angle ABP = 150^\circ$

Using sine rule in ΔABP, we get

$$\frac{AB}{\sin(\angle APB)} = \frac{BP}{\sin(\angle PAB)} = \frac{AP}{\sin(\angle ABP)}$$

$$\Rightarrow \quad \frac{d}{\sin 15^\circ} = \frac{BP}{\sin 15^\circ} = \frac{AP}{\sin 150^\circ}$$

$$\Rightarrow \quad \frac{d}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{AP}{\frac{1}{2}} \Rightarrow AP = \frac{\sqrt{2}d}{\sqrt{3}-1} \qquad \text{...(i)}$$

Using sine rule in ΔAQP, we get

$$\frac{AP}{\sin(\angle AQP)} = \frac{AQ}{\sin(\angle APQ)} = \frac{PQ}{\sin(\angle PAQ)}$$

$$\Rightarrow \quad \frac{AP}{\sin 90^\circ} = \frac{PQ}{\sin 45^\circ}$$

$$\Rightarrow \quad AP = \sqrt{2}\,PQ$$

$$\Rightarrow \quad AP = \sqrt{2}h \qquad \text{...(ii)}$$

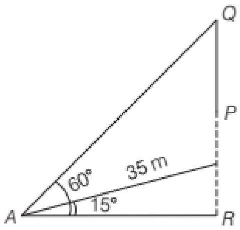
From Eqs. (i) and (ii), we get

$$\sqrt{2}h = \frac{\sqrt{2}d}{\sqrt{3}-1}$$

$$\Rightarrow \quad d = (\sqrt{3}-1)h$$ **Hence proved.**

EXAMPLE |13| A tree stand vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35 m down the hill form the base of the tree. The angle of elevation of the top of the tree is 60°. Find the height of the tree. **[NCERT]**

Sol. Let PQ be the tree on the hill which makes an angle of 15° with the horizontal AR, where A is a point on the ground 35 m down the hill form the base P of the tree.



In ΔARQ, we have

$$\angle RAQ = 60° \text{ and } \angle ARQ = 90°$$

$$\therefore \quad \angle AQP = 30°$$

In ΔAPQ, we have

$$\angle PAQ = 45° \text{ and } \angle AQP = 30°$$

In ΔAPQ by since rule, we have

$$\frac{AP}{\sin(\angle AQP)} = \frac{PQ}{\sin(\angle PAQ)}$$

$$\Rightarrow \quad \frac{35}{\sin 30°} = \frac{PQ}{\sin 45°}$$

$$\Rightarrow \quad \frac{35}{\frac{1}{2}} = \frac{PQ}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad PQ = \frac{70}{\sqrt{2}}$$

$$\Rightarrow \quad PQ = 35\sqrt{2} \text{ m}$$

Napier's Analogies (Law's of Tangent)

In ΔABC, we have

(i) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

(ii) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$

EXEMPLE |14| Solve the triangle in which

$$a = (\sqrt{3}+1), b = (\sqrt{3}-1) \text{ and } \angle C = 60°.$$

Sol. Using, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot\frac{C}{2}$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{2}{2\sqrt{3}}\cot 30° = \frac{1}{\sqrt{3}} \times \sqrt{3} = 1$$

$$\Rightarrow \quad \frac{A-B}{2} = 45° \Rightarrow A - B = 90° \quad \text{...(i)}$$

Also, $\quad A + B + C = 180°$

$$\Rightarrow \quad A + B = 120° \quad \text{...(ii)}$$

On solving (i) and (ii), we get

$$A = 105°, B = 15°$$

Also, $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \quad \cos 60° = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 - c^2}{2(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{(3+1+2\sqrt{3})+(3+1-2\sqrt{3})-c^2}{2(3-1)}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{8-c^2}{4}$$

$$\Rightarrow \quad 8 - c^2 = 2$$

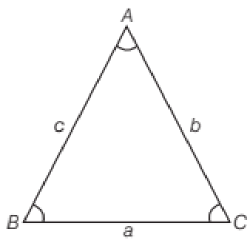
$$\Rightarrow \quad c = \sqrt{6}$$

Hence, $c = \sqrt{6}$, $\angle A = 105°$ and $\angle B = 15°$.

TOPIC PRACTICE 6

OBJECTIVE TYPE QUESTIONS

1 Consider the figure given below



Based on above figure, which among the following statements is true?

I. $\frac{\sin A}{a} = \frac{\sin B}{b}$

II. $\frac{\sin B}{b} = \frac{\sin C}{c}$

III. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

(a) I is true (b) II is true
(c) III is true (d) All are true

2 Let A, B and C be angles of a triangle and a, b and c be lengths of sides opposite to angles A, B and C respectively, then which of the following is/are correct?

I. $a^2 = b^2 + c^2 - 2bc\cos A$

II. $b^2 = c^2 + a^2 - 2ca\cos B$

III. $c^2 = a^2 + b^2 - 2ab\cos C$

(a) I is correct (b) II is correct
(c) III is correct (d) All are correct

3 Which among the following is/are called Napier's Analogy in a ΔABC?

(a) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(b) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(c) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

(d) All of the above

4 Angles of a triangle are in the ratio 4 : 1 : 1. The ratio between its greatest side and perimeter is

(a) $\frac{3}{2+\sqrt{3}}$ (b) $\frac{1}{2+\sqrt{3}}$ (c) $\frac{\sqrt{3}}{\sqrt{3}+2}$ (d) $\frac{2}{2+\sqrt{3}}$

5 The sides of a triangle are respectively 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm, then the smallest angle of the triangle is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{5}$

SHORT ANSWER Type I Questions

6 In ΔABC, if $a = 3$, $b = 5$ and $\sin A = \frac{3}{5}$, find $\angle B$.

7 In ΔABC, if $A = 45°$, $B = 60°$, and $C = 75°$, find the ratio of its sides.

8 In a ΔABC, if $a = 3$, $b = 4$ and $c = 5$, then find angle C.

9 In any trigangle, if angle are in the ratio 1 : 2 : 3, then find their corresponding sides.

10 In ΔABC, if $\cos A = \frac{\sin B}{2\sin C}$, show that the triangle is isosceles.

11 In a ΔABC, if $a = 4$ cm $b = 5$ cm, $c = 6$ cm, then find $\cos A$, $\cos B$ and $\cos C$.

12 In ΔABC, if $\angle A = 30°$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

SHORT ANSWER Type II Questions

13 In any ΔABC, prove that $\frac{\sin(C-A)}{\sin(C+A)} = \frac{c^2 - a^2}{b^2}$.

14 In any ΔABC, prove that $\frac{a-b}{c}\cos\frac{C}{2} = \sin\frac{A-B}{2}$.

15 In any ΔABC, prove that $\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$.

16 In any ΔABC, prove that

$$a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}.$$

17 In any ΔABC, prove that

$$(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2.$$

18 In ΔABC, prove that

$$\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0.$$

LONG ANSWER Type Questions

19 In any ΔABC, prove that

$$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C.$$

20 In ΔABC, prove that

$$a^3\sin(B-C) + b^3\sin(C-A) + c^3\sin(A-B) = 0.$$

21 In a ΔABC, prove that

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0.$$

22 A lamp-post is situated at the middle point M of the side AC of a triangular plot ABC with $BC = 7$m $CA = 8$ m and $AB = 9$ m. The lamp-post subtend an angle $\tan^{-1} 3$ at the point B. Determine the height of the lamp-post.

HINTS & ANSWERS

1. (d) **2.** (d) **3.** (d)

4. (c) Assume that angles are $4x$, x and x.

As $4x + x + x = 180°$ $[\because \angle A + \angle B + \angle C = 180°]$

$\Rightarrow$ $x = 30°$

$\therefore$ Angles are 120°, 30° and 30°.

Ratio of sides $= \sin A : \sin B : \sin C$

$= \sin 120° : \sin 30° : \sin 30° = \sqrt{3} : 1 : 1$

$\therefore$ Required ratio $= \frac{\sqrt{3}}{1+1+\sqrt{3}} = \frac{\sqrt{3}}{2+\sqrt{3}}$

5. (a) Let $a = 7$ cm, $b = 4\sqrt{3}$ cm and $c = \sqrt{13}$ cm

Here, we see that the smallest side is c.

Therefore, the smallest angle will be C.

$$\therefore \quad \cos C = \frac{(7)^2 + (4\sqrt{3})^2 - (\sqrt{13})^2}{2 \times 7 \times 4\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \angle C = \frac{\pi}{6}$$

6. $B = 90°$

7. $2 : \sqrt{6} : (\sqrt{3} + 1)$

8. $\angle C = 90°$

9. Here, $x + 2x + 3x = 180° \Rightarrow x = 30°$

$\therefore \angle A = 30°, \angle B = 60°, \angle C = 90°$

$\therefore \quad \angle A : \angle B : \angle C = 1 : \sqrt{3} : 2$

10. $\cos A = \dfrac{\sin B}{2\sin C} \Rightarrow \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{kb}{2kc}$

$\Rightarrow \; b^2 + c^2 - a^2 = b^2 \;\Rightarrow\; c^2 = a^2 \Rightarrow c = a$

11. Use the cosine rule.

Ans. $\cos A = \dfrac{3}{4}, \cos B = \dfrac{9}{16}, \cos C = \dfrac{1}{8}$

12. $\tan\left(\dfrac{B - C}{2}\right) = \dfrac{2 - \sqrt{3}}{2 + \sqrt{3}} \cot \dfrac{30°}{2}$

Now, $\quad \tan 2 \times 15° = \dfrac{2\tan 15°}{1 - \tan^2 15°}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{2\tan 15°}{1 - \tan^2 15°}$$

$$\Rightarrow \quad \tan^2 15° + 2\sqrt{3}\tan 15° - 1 = 0$$

$$\Rightarrow \quad \tan 15° = \sqrt{3} + 2$$

Ans. $B = 90°$

13. $\text{RHS} = \dfrac{k^2\sin^2 C - k^2\sin^2 A}{k^2\sin^2 B} = \dfrac{\sin(C + A)\sin(C - A)}{\sin^2 B}$

$$= \frac{\sin B \sin(C - A)}{\sin^2 B} = \frac{\sin(C - A)}{\sin(C + A)}$$

14. $\text{LHS} = \dfrac{k\sin A - k\sin B}{k\sin C}\cos\dfrac{C}{2}$

$$= \frac{2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2}} \times \cos\frac{C}{2}$$

$$= \frac{\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\sin\left(\frac{A - B}{2}\right)}{\sin\frac{C}{2}}$$

$$= \sin\left(\frac{A - B}{2}\right)$$

15. $\text{LHS} = \dfrac{k\sin B - k\sin C}{k\sin B + k\sin C}$

$$= \frac{2\cos\left(\frac{B + C}{2}\right)\sin\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{B + C}{2}\right)\cos\left(\frac{B - C}{2}\right)} = \frac{\tan\left(\frac{B - C}{2}\right)}{\tan\left(\frac{B + C}{2}\right)}$$

16. $\text{LHS} = k\sin A(\cos C - \cos B)$

$$= k\sin A\left[2\sin\left(\frac{C + B}{2}\right)\sin\left(\frac{B - C}{2}\right)\right]$$

$$= 2k\sin A\left[\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\right]\sin\frac{B - C}{2}$$

$$= 2k\sin A\cos\frac{A}{2}\sin\left(\frac{B - C}{2}\right)$$

$$= 2k\cos^2\frac{A}{2}\left(2\sin\frac{A}{2}\sin\frac{B - C}{2}\right)$$

$$= 2k\cos^2\frac{A}{2}\left[\left(\cos\frac{A - B + C}{2}\right) - \cos\left(\frac{A + B - C}{2}\right)\right]$$

$$= 2\cos^2\frac{A}{2}(k\sin B - k\sin C)$$

17. $\text{LHS} = a^2\left(\cos^2\dfrac{C}{2} + \sin^2\dfrac{C}{2}\right) + b^2\left(\cos^2\dfrac{C}{2} + \sin^2\dfrac{C}{2}\right) - 2ab\left(\cos^2\dfrac{C}{2} - \sin^2\dfrac{C}{2}\right)$

$$= a^2 + b^2 - 2ab(\cos C) = c^2$$

19. $\text{LHS} = k\sin A\cos A + k\sin B\cos B + k\sin C\cos C$

$$= \frac{k}{2}[\sin 2A + \sin 2B + 2\sin C\cos C]$$

$$= \frac{k}{2}[2\sin(A + B)\cos(A - B) + 2\sin C\cos C]$$

$$= \frac{k}{2}[2\sin C\cos(A - B) + 2\sin C\cos C]$$

$$= \frac{k}{2} \times 2\sin C[\cos(A - B) + \cos C]$$

$$= k\sin C[\cos(A - B) - \cos(A + B)]$$

$$= k\sin C[-2\sin A\sin(-B)]$$

$$= 2k\sin A\sin B\sin C = 2a\sin B\sin C$$

22. $\cos C = \dfrac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8} = \dfrac{2}{7}$

Now $BM^2 = BC^2 + CM^2 - 2BC \times CM\cos C$

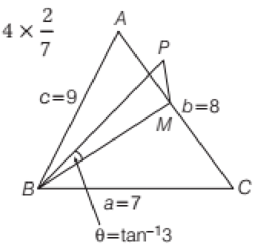
$$= 7^2 + 4^2 - 2 \times 7 \times 4 \times \frac{2}{7}$$

$\Rightarrow \quad BM = 7$

In right ΔBMP,

$$\tan\theta = \frac{PM}{BM} = \frac{h}{7}$$

Ans. 21m



SUMMARY

1. $1° = 60'$; $1' = 60''$

2. $\theta = \frac{l}{r} = \frac{\text{length of arc}}{\text{radius}}$ or $l = r\,\theta$, where θ is radian.

3. **Relation between Degree and Radian**
   - Radian measure $= \frac{\pi}{180} \times$ Degree measure
   - Degree measure $= \frac{180}{\pi} \times$ Radian measure

4. A. $\sin(2n\pi + \theta) = \sin\theta,\ n \in Z$

   B. $\cos(2n\pi + \theta) = \cos\theta,\ n \in Z$

5. **Trigonometric Identities**

   i. $1 = \cos^2\theta + \sin^2\theta$

   ii. $\sec^2\theta = 1 + \tan^2\theta$

   iii. $\text{cosec}^2\theta = 1 + \cot^2\theta$

6. **Domain, Range *and* Period of Trigonometric Functions**

✓ $y = \sin x$	*a.* Domain $= R$ *b.* Range $= [-1, 1]$ *c.* Period $= 2\pi$
✓ $y = \cos x$	*a.* Domain $= R$ *b.* Range $= [-1, 1]$ *c.* Period $= 2\pi$
✓ $y = \tan x$	*a.* Domain $= R \sim (2n+1)\frac{\pi}{2}, n \in I$ *b.* Range $= (-\infty, \infty)$ *i.e.*, R. *c.* Period $= \pi$
✓ $y = \cot x$	*a.* Domain $= R \sim (n\pi), n \in I$ *b.* Range $= (-\infty, \infty)$ *i.e.*, R. *c.* Period $= \pi$
✓ $y = \sec x$	*a.* Domain $= R \sim (2n+1)\frac{\pi}{2}, n \in I$ *b.* Range $= (-\infty, -1] \cup [1, \infty)$ or $R \sim]-1, 1[$ *c.* Period $= 2\pi$
✓ $y = \text{cosec}\, x$	*a.* Domain $= R \sim n\pi, n \in I$ *b.* Range $= (-\infty, -1] \cup [1, \infty)$ or $R \sim]-1, 1[$ *c.* Period $= 2\pi$

7. **Sum and Difference of Two Angles**
   - $\cos(A+B) = \cos A \cos B - \sin A \sin B$
   - $\cos(A-B) = \cos A \cos B + \sin A \sin B$
   - $\sin(A+B) = \sin A \cos B + \cos A \sin B$
   - $\sin(A-B) = \sin A \cos B - \cos A \sin B$
   - $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$,
     where A, B *and* $(A+B)$ *are not odd multiples of* $\frac{\pi}{2}$.
   - $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$,
     where A, B *and* $(A-B)$ *are not odd multiples of* $\frac{\pi}{2}$.
   - $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$,
     where A, B *and* $(A+B)$ *are not multiples of* π.
   - $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$,
     where, A, B *and* $(A-B)$ *are not multiples of* π.

8. **Some Important Results**
   - $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
   - $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
   - $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
   - $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
   - $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

9. **Transformation of Product into Sum or Difference**

   I. $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

   II. $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

   III. $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

   IV. $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

10. Transformation of Sum or Difference into Product

A. $\cos A + \cos B = 2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$

B. $\cos A - \cos B = -2\sin\frac{A+B}{2}\cdot\sin\frac{A-B}{2}$

C. $\sin A + \sin B = 2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$

D. $\sin A - \sin B = 2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}$

11. Trigonometric Functions of Multiples of Angles

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$
- $\sin 2x = 2\sin x\cos x = \frac{2\tan x}{1+\tan^2 x}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

12. Trigonometric Functions of Sub-multiples of Angles

i. $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$= 1 - 2\sin^2\left(\frac{x}{2}\right) = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

ii. $\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

iii. $\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$

13. Maximum and Minimum Value of $a\sin\theta + b\cos\theta$

- Maximum value of
  $a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}$
- Minimum value of
  $a\sin\theta + b\cos\theta = -\sqrt{a^2 + b^2}$

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. The radian of $48° \; 37' \; 30''$ is

(a) $\frac{389\pi}{1440}$ rad (b) $\frac{389}{1440}$ rad

(c) $\frac{752\,\pi}{1440}$ rad (d) $\frac{752}{1440}$ rad

2. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is **[NCERT Exemplar]**

(a) $\frac{1}{\sqrt{2}}$ (b) 0 (c) 1 (d) –1

3. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is **[NCERT Exemplar]**

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) 1

4. If $\tan\theta = \frac{1}{\sqrt{7}}$, then $\left(\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}\right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{5}{4}$ (d) (2)

5. If $x = h + a\sec\theta$ and $y = k + b\operatorname{cosec}\theta$. Then,

(a) $\frac{a^2}{(x+h)^2} - \frac{b^2}{(y+k)^2} = 1$

(b) $\frac{a^2}{(x-h)^2} + \frac{b^2}{(y-k)^2} = 1$

(c) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

(d) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

6. If $\tan\theta = \frac{-4}{3}$, then $\sin\theta$ is **[NCERT Exemplar]**

(a) $\frac{-4}{5}$ but not $\frac{4}{5}$ (b) $\frac{-4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these

7. If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, then $\alpha + \beta$ is equal to **[NCERT Exemplar]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

8. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$ is equal to

(a) $\tan 3x$ (b) $\cot 3x$

(c) $\tan 6x$ (d) $\cot 6x$

9. In any ΔABC, $\left(\frac{b-c}{a}\right)\cos\frac{A}{2}$ is equal to

(a) $\sin\left(\frac{B-C}{2}\right)$

(b) $\cos\left(\frac{B-C}{2}\right)$

(c) $\sin\left(\frac{A-C}{2}\right)$

(d) $\cos\left(\frac{A-C}{2}\right)$

VERY SHORT ANSWER Type Questions

10. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm. **[NCERT]**

11. Find the radius measure corresponding the degree measure $125° \; 30'$.

12. Find the degree measure corresponding the the radius measure -2^c.

13. Find the value of the following trigonomitric ratios.

(i) $\sin\left(\frac{-11\pi}{3}\right)$ **[NCERT]**

(ii) $\cot\left(\frac{-15\pi}{4}\right)$ **[NCERT]**

(iii) $\operatorname{cosec}(-1200°)$

14. Find the principal solution of the following trigonometric equations.

(i) $\sec\theta = -\frac{2}{\sqrt{3}}$

(ii) $\cot x = \frac{1}{\sqrt{3}}$

(iii) $\sin x = -\frac{\sqrt{3}}{2}$

SHORT ANSWER Type I Questions

15. The minute hand of a watch is 1.8 cm long. How far does its tip move in 30 min?

16. A rail road curve is to be laid out on a circle. What radius should be used, if the track is to change direction by 30° is a distance of 50 m?

17. Find $\sin\theta$ and $\tan\theta$, if $\cos\theta = -\frac{3}{5}$ and θ lies in the third quadrant.

18. Prove the following result. **[NCERT]**

(i) $2\sin^2\frac{\pi}{6} + \text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

(ii) $\cot^2\frac{\pi}{6} + \text{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

(iii) $\tan 720° - \cos 70° - \sin 150° \cos 120° = \frac{1}{4}$

19. In any quadilateral $ABCD$, prove that

$$\cos(A+B) = \cos(C+D).$$

20. If $\text{cosec}\theta + \cot\theta = \frac{11}{2}$, then find the value of $\tan\theta$.

21. If θ is an acute angle and $\tan\theta = \frac{1}{\sqrt{7}}$, then find the value of $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$.

22. Prove that

(i) $\frac{1-\cos 4x}{\cot x - \tan x} = \frac{1}{2}\sin 4x$

(ii) $8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} = 1$

(iii) $108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 18$

23. If $0 \geq x \geq \pi$ and x lies in the II quadrant such that $\sin x = \frac{1}{4}$. Find the value of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

24. In a ΔABC, prove that

$a^2 + b^2 + c^2 = 2(bc\cos A + ca\cos B + ab\cos C)$.

25. The angle of a triangle are in AP. The number of grades in the least, is to the number of radians in the greatest as 40%, find the angle in degrees.

26. If $a\cos\theta - b\sin\theta = c$, then show that

$$a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}.$$

27. If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p.

28. Prove that $2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta$

$$= \frac{1-\tan^8\theta}{\tan^4\theta}.$$

29. Prove the following identities.

(i) $\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 2$

(ii) $\text{cosec}\,\theta(\sec\theta - 1) - \cot\theta(1-\cos\theta) = \tan\theta - \sin\theta$

(iii) $\frac{\tan^3\theta}{1+\tan^2\theta} + \frac{\cot^3\theta}{1+\cot^2\theta} = \frac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$

(iv) $1 - \frac{\sin^2\theta}{1+\cot\theta} - \frac{\cos^2\theta}{1+\tan\theta} = \sin\theta\cos\theta$

30. Prove that

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta - \tan\theta, & \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\sec\theta + \tan\theta, & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}.$$

31. If $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ and $\cos B = -\frac{12}{13}$,

$\pi < B < \frac{3\pi}{2}$, find $\tan(A-B)$.

32. If $A + B = \frac{\pi}{4}$, then prove that $(\cot A - 1)(\cot B - 1) = 2$.

33. Prove that

$$\frac{\sin(x+\theta)}{\sin(x+\phi)} = \cos(\theta-\phi) + \cot(x+\phi)\sin(\theta-\phi).$$

34. If $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$ show that $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$.

SHORT ANSWER Type II Questions

35. Find the maximum and minimum values of the trigonometrical expression $12\sin\theta - 5\cos\theta$.

36. Express $3\cos\theta - 4\sin\theta$ as sine and cosine of a single expression.

37. Prove that

(i) $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$

(ii) $\cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B)$
$= \sin^2(A+B)$

(iii) $4\cos 12°\cos 48°\cos 72° = \cos 36°$

(iv) $4\cos A\cos(60°-A)\cos(60°+A) = \cos 3A$

(v) $\tan 20°\tan 40°\tan 60°\tan 80° = 3$

(vi) $\frac{\cos 8A\cos 5A - \cos 12A\cos 9A}{\sin 8A\cos 5A + \cos 12A\sin 9A} = \tan 4A$

38. If $2\alpha + 2\beta = 90°$, find the maximum and minimum values of $\sin 2\alpha \sin 2\beta$.

39. Prove that

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4\cos 4A \cos 5A \cos 6A$

(ii) $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$

(iii) $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$

(iv) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$

40. If in a ΔABC, $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, prove that $c = 60°$.

41. In a ΔABC, if $\cos A = \frac{\sin B}{2\sin C}$, show that the triangle is isosceles.

42. Prove that

$$\cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right) = \frac{3}{2}.$$

43. If in ΔABC, $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$, prove that $A = 60°$.

44. A tree stands vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35m down the hill from the base of the tree, the angle of elevation of the hop of tree is 60°. Find the height of the tree.

45. In a ΔABC, prove that

$$\frac{\cos^2\left(\frac{B-C}{2}\right)}{(b+c)^2} + \frac{\sin^2\left(\frac{(B-C)}{2}\right)}{(b-c)^2} = a^{-2}.$$

46. In any ΔABC, prove that $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$.

CASE BASED Questions

47. Consider $T_n = \sin^n \theta + \cos^n \theta$

Answer the following questions

(i) Find $T_3 - T_5$

(a) $\sin^2\theta \cdot \cos^2\theta(\sin\theta + \cos\theta)$

(b) $\sin\theta\cos\theta(\sin\theta - \cos\theta)$

(c) $\sin\theta(\sin^2\theta + \cos\theta)$

(d) $\cos\theta(\cos^2\theta + \sin\theta)$

(ii) Find $\frac{T_3 - T_5}{T_1}$

(a) $\sin^2\theta$ (b) $\sin^2\theta\cos^2\theta$

(c) $\sin\theta\cos\theta$ (d) $\tan^2\theta$

(iii) Find $T_5 - T_7$

(a) $\sin^3\theta\cos^3\theta(\sin\theta + \cos\theta)$

(b) $\sin\theta\cos\theta(\sin^3\theta + \cos^3\theta)$

(c) $\sin^2\theta\cos^2\theta(\sin^3\theta + \cos^3\theta)$

(d) $\sin^3\theta\cos^3\theta(\sin^2\theta + \tan^2\theta)$

(iv) Find $\frac{T_5 - T_7}{T_3}$

(a) $\sin^2\theta\cos^2\theta$ (b) $\sin 3\theta\cos 3\theta$

(c) $\sin 2\theta$ (d) $\cos 2\theta$

(v) Find the value of T_3 if $\theta = \pi$.

(a) 1 (b) -1 (c) 2 (d) -2

48. Consider $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where $0 < A$ and $B < \frac{\pi}{2}$.

Based on the above information answer the following questions.

(i) Find the value of $\cos A + \sin B$.

(a) $\frac{5}{13}$ (b) $\frac{99}{65}$ (c) $\frac{20}{65}$ (d) $\frac{9}{13}$

(ii) Find the value of $\sin(A + B)$.

(a) $\frac{17}{13}$ (b) $\frac{29}{65}$ (c) $\frac{56}{65}$ (d) $\frac{-13}{17}$

(iii) Find value of $\cos(A + B)$.

(a) $-\frac{11}{19}$ (b) $-\frac{33}{65}$ (c) $-\frac{19}{11}$ (d) $-\frac{65}{33}$

(iv) Find value of $\sin(A - B)$.

(a) $\frac{-16}{65}$ (b) $\frac{-39}{40}$ (c) $\frac{-65}{16}$ (d) $\frac{-40}{39}$

(v) Find the value of $\cos(A - B)$.

(a) $\frac{65}{73}$ (b) $\frac{63}{65}$ (c) $\frac{73}{65}$ (d) $\frac{72}{65}$

49. Consider $\sin(A + B) = 1$ and $\sin(A - B) = \frac{1}{2}$ where $A, B \in \left[0, \frac{\pi}{2}\right]$.

Based on the above information, answer the following questions.

(i) What is the value of A?

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

(ii) What is the value of B?

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

(iii) What is the value of $\tan(A + 2B)\tan(2A + B)$?

(a) -1 (b) 0 (c) 1 (d) 2

(iv) What is the value of $\sin^2 A - \sin^2 B$?

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 1

(v) What is the value of $\cos 2A$?

(a) -2 (b) $-\frac{1}{4}$ (c) $-\frac{1}{3}$ (d) $-\frac{1}{2}$

HINTS & ANSWERS

1. (a) We have,
Angle in degree is $48° 37' 30''$

in $30'' = \left(\frac{1}{2}\right)'$

$\therefore\ 48° 37' 30'' = 48° \left(37\frac{1}{2}\right)' = 48° \left(\frac{75}{2}\right)'$ in $1' = \left(\frac{1}{60}\right)°$

$$\therefore\ 48° \left(\frac{75}{2}\right)' = 48° + \left(\frac{75}{2} \times \frac{1}{60}\right)°$$
$$= 48° + \left(\frac{5}{8}\right)° = \left(\frac{389}{8}\right)°$$

we know that in $1° = \frac{\pi}{180}$ radian

$$\therefore\quad \left(\frac{389}{8}\right)° = \frac{389}{8} \times \frac{\pi}{180} \text{ radian} = \frac{389\pi}{1440} \text{ radian}$$

2. (b) We have,
$$\cos 1° \cdot \cos 2° \cdot \cos 3° \ldots \cos 179°$$
$$= 0 \qquad [\because \cos 90° = 0]$$

3. (c) $\sin\frac{\pi}{10}\sin\left(\pi + \frac{3\pi}{10}\right)$
$$= -\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$$
$$= -\sin 18° \cdot \sin 54°$$
$$= -\sin 18° \cdot \cos 36°$$
$$= -\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$
[since, put this value here]

4. (b) Use formulae $\text{cosec}^2\theta = 1 + \cot^2\theta$
and $\sec^2\theta = 1 + \tan^2\theta$

5. (b) Given equations can be rewritten as
$$\cos\theta = \frac{a}{x-h},\ \sin\theta = \frac{b}{y-k}$$
Now, $\frac{a^2}{(x-h)^2} + \frac{b^2}{(y-k)^2} = 1 \qquad [\because \cos^2\theta + \sin^2\theta = 1]$

6. (b) Since, $\tan\theta = -\frac{4}{3}$ is negative, θ lies either in second quadrant or in fourth quardant. Thus, $\sin\theta = \frac{4}{5}$ if θ lies in the second quadrant or $\sin\theta = -\frac{4}{5}$, if θ lies in the fourth quadrant.

7. (d) Use formula $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

8. (c) Use formulae $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
and $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

9. (a) By sine rule the given equation can be written as
$$\left(\frac{\sin B - \sin C}{\sin A}\right)\cos\frac{A}{2}$$
$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}} \times \cos\frac{A}{2}$$
$$= \frac{\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left[\frac{\pi}{2} - \left(\frac{B+C}{2}\right)\right]}$$
$$\left[\because A + B + C = \pi \Rightarrow \frac{A}{2} = \frac{\pi}{2} - \frac{(B+C)}{2}\right]$$
$$= \sin\left(\frac{B-C}{2}\right)$$

10. If a circle of radius r and an arc of length l subtends an angle θ radian at the centre, we use the formula, $\theta = \frac{l}{r}$.
Ans. $12°36'$

11. Let $\theta = 125°30'$
$$= 125° + \left(\frac{30}{60}\right)° \qquad \left[\because 1° = 60' \Rightarrow 1' = \left(\frac{1}{60}\right)°\right]$$
$$= 125° + \left(\frac{1}{2}\right) = \left(\frac{251}{2}\right)°$$
$\therefore$ Radian measure $= \frac{\pi}{180} \times$ Degree measure
$$= \frac{\pi}{180} \times \left(\frac{251}{2}\right) = \frac{251\pi}{360}$$

12. Let $\theta = (-2)^c = \left(\frac{180}{\pi} \times (-2)\right)°$
$$= \left(\frac{180}{\pi} \times 7 \times -2\right)° = \left(\frac{180}{22} \times 7 \times -2\right)°$$
$$= \left(-114\frac{6}{11}\right)° = \left[-114°\left(\frac{6}{11} \times 60\right)'\right]$$
$$= -\left[114\left(32\frac{8}{11}\right)'\right] = -\left[114°32'\left(\frac{8}{11} \times 60\right)''\right]$$
$$= -[114°32'44'']$$

13. (i) $\sin\left(\frac{-11\pi}{3}\right) = -\sin\left(\frac{11\pi}{3}\right) \qquad [\because \sin(-\theta) = -\sin\theta]$
$$= -\sin\left(4\pi - \frac{\pi}{3}\right)$$
$$= -\left(-\sin\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$[\because \sin(2n\pi - \theta) = -\sin\theta]$

(ii) Solve as part (i). **Ans.** 1

(iii) Solve as part (ii). **Ans.** $\frac{-2}{\sqrt{3}}$

14. (i) Solve as Example 32 (iii) of Topic 5. **Ans.** $\frac{5\pi}{6}, \frac{7\pi}{6}$

(ii) Solve as Example 32 (ii) of Topic 5. **Ans.** $\frac{\pi}{3}, \frac{4\pi}{3}$

(iii) Solve as Example 32 (i) of Topic 5. **Ans.** $\frac{4\pi}{3}, \frac{5\pi}{3}$

15. In 60 min, the minute hand of a watch completes one rotation i.e. it rotates through 360°.

$\therefore$ Angle traced by the minute hand in 1 min $= \left(\frac{360}{60}\right)^\circ = 6^\circ$

$\therefore$ Angle traced by the minute hand in 30 min

$$= (30 \times 6)^\circ = 180^\circ$$
$$= \left(180 \times \frac{\pi}{180}\right)^c = \pi^c$$

$\because \quad \theta = \frac{\text{Arc}}{\text{Radius}} \Rightarrow \pi = \frac{\text{Arc}}{1.8}$

$\therefore \quad \text{Arc} = \frac{22}{7} \times 1.8 = 5.66$ cm

16. Here, $\theta = 30^\circ = \left(30 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{6}\right)^c$ and arc = 50 m

$\because \quad \theta = \frac{\text{Arc}}{\text{Radius}} \Rightarrow \frac{\pi}{6} = \frac{50}{\text{Radius}}$

$\therefore$ Radius $= \frac{50 \times 6 \times 7}{22} = 95.45$ m

17. $\sin\theta = \frac{-4}{5}$ and $\tan\theta = \frac{3}{4}$

18. (i) LHS $= 2\sin^2\frac{\pi}{6} + \text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}$

$$= 2\left(\frac{1}{2}\right)^2 + \text{cosec}^2\left(\pi + \frac{\pi}{6}\right)\cos^2\frac{\pi}{3}$$
$$= \frac{1}{2} + \left(-\text{cosec}\frac{\pi}{6}\right)^2 \cos^2\frac{\pi}{3}$$

[$\because \text{cosec}(\pi + \theta) = -\text{cosec}\theta$]

25. Let the angles of a triangle be $(a - d)$, a and $(a + d)$.

Then, $(a - d) + a + (a + d) = 180^\circ \Rightarrow a = 60^\circ$

So, the angle are $(60 - d)^\circ, 60^\circ, (60 + d)^\circ$. Clearly, $(60 - d)^\circ$ is the least angle and $(60 + d)^\circ$ is the greatest angle.

Now, greatest angle $= (60 + d)^\circ = \left[(60 + d)\frac{\pi}{180}\right]^c$

According to the question,

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{60 - d}{(60 + d)\frac{\pi}{180}} = \frac{60}{\pi}$$

$$\Rightarrow \quad 3(60 - d) = (60 + d) \Rightarrow d = 30^\circ$$

Hence, the angles are 30°, 60° and 90°.

26. Given, $a\cos\theta - b\sin\theta = c$

$\Rightarrow a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta = c^2$

[squaring both sides]

$\Rightarrow a^2(1 - \sin^2\theta) + b^2(1 - \cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$

$\Rightarrow (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

27. We know that, $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$\Rightarrow \quad (\sec\theta - \tan\theta)p = 1$

[$\because \sec\theta + \tan\theta = p$, given]

$\Rightarrow \quad \sec\theta - \tan\theta = \frac{1}{p}$

Now, $(\sec\theta + \tan\theta) + (\sec\theta - \tan\theta) = p + \frac{1}{p}$

$$= \frac{1}{2} + (-2)^2\left(\frac{1}{2}\right)^2 = \frac{1}{2} + 1 = \frac{3}{2} = \text{RHS}$$ **Hence proved.**

(ii) Solve as part (i).

(iii) Solve as part (i).

19. It is given that $ABCD$ is a quadrilateral.

$\therefore A + B + C + D = 360^\circ$

[$\because$ sum of all angles of a quadrilateral is 360°]

$\Rightarrow \quad A + B = 360^\circ - (C + D)$

On taking cosine both sides, we get

$\cos(A + B) = \cos[360^\circ - (C + D)]$

$\Rightarrow \cos(A + B) = \cos(C + D)$ [$\because \cos(360^\circ - \theta) = \cos\theta$]

Hence proved.

20. $\frac{44}{117}$

21. $\frac{3}{4}$

23. $\sqrt{\frac{4 - \sqrt{15}}{8}}, \sqrt{\frac{4 + \sqrt{15}}{8}}$ and $4 + \sqrt{15}$

$\Rightarrow \quad 2\sec\theta = \frac{p^2 + 1}{p} \Rightarrow \sec\theta = \frac{p^2 + 1}{2p}$ ...(i)

and $(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = p - \frac{1}{p}$

$\Rightarrow \quad 2\tan\theta = \frac{p^2 - 1}{p} \Rightarrow \tan\theta = \frac{p^2 - 1}{2p}$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get $\sin\theta = \frac{p^2 - 1}{p^2 + 1}$

28. LHS $= 2\sec^2\theta - \sec^4\theta - 2\text{cosec}^2\theta + \text{cosec}^4\theta$

$= 2(1 + \tan^2\theta) - (1 + \tan^2\theta)^2 - 2(1 + \cot^2\theta) + (1 + \cot^2\theta)^2$

$= 2(1 + \tan^2\theta - 1 - \cot^2\theta) + (1 + 2\cot^2\theta + \cot\theta)$
$\qquad - (1 + 2\tan^2\theta + \tan^4\theta)$

$= 2(\tan^2\theta - \cot^2\theta) + (2\cot^2\theta - 2\tan^2\theta) + (\cot^4\theta - \tan^4\theta)$

$= \frac{1}{\tan^4\theta} - \tan^4\theta = \frac{1 - \tan^8\theta}{\tan^4\theta} = \text{RHS}$

Hence proved.

29. (i) LHS $= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x}$

$= \frac{\sin x + \cos x(\sin^2 x + \cos^2 x - \sin x \cos x)}{(\sin x + \cos x)}$

$+ \frac{(\sin x - \cos x)(\sin^2 x + \cos^2 x + \sin x \cos x)}{(\sin x - \cos x)}$

$\left[\begin{array}{l} \because a^3 + b^3 = (a+b)(a^2 + b^2 - ab) \\ \text{and } a^3 - b^3 = (a-b)(a^2 + b^2 + ab) \end{array}\right]$

$= 1 - \sin x \cos x + 1 + \sin x \cos x \quad [\because \sin^2\theta + \cos^2\theta = 1]$

$= 2 = \text{RHS}$ **Hence proved.**

(ii) LHS $= \operatorname{cosec}\theta(\sec\theta - 1) - \cot\theta(1 - \cos\theta)$

$= \frac{1}{\sin\theta}\left(\frac{1}{\cos\theta} - 1\right) - \frac{\cos\theta}{\sin\theta}(1 - \cos\theta)$

$= \frac{1}{\sin\theta}\left(\frac{1 - \cos\theta}{\cos\theta}\right) - \frac{\cos\theta}{\sin\theta}(1 - \cos\theta)$

$= (1 - \cos\theta)\left(\frac{1}{\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)$

$= (1 - \cos\theta)\left(\frac{1 - \cos^2\theta}{\sin\theta\cos\theta}\right) = (1 - \cos\theta)\left(\frac{\sin^2\theta}{\sin\theta\cos\theta}\right)$

$= (1 - \cos\theta)\left(\frac{\sin\theta}{\cos\theta}\right)$

$= (\tan\theta - \sin\theta) = \text{RHS}$ **Hence proved.**

(iii) LHS $= \frac{\tan^3\theta}{1 + \tan^2\theta} + \frac{\cot^3\theta}{1 + \cot^2\theta}$

$= \frac{\frac{\sin^3\theta}{\cos^3\theta}}{1 + \frac{\sin^2\theta}{\cos^2\theta}} + \frac{\frac{\cos^3\theta}{\sin^3\theta}}{1 + \frac{\cos^2\theta}{\sin^2\theta}}$

$= \frac{\sin^3\theta}{\cos^3\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)} + \frac{\cos^3\theta}{\sin^3\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right)}$

$= \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta} \quad [\because \sin^2\theta + \cos^2\theta = 1]$

$= \frac{\sin^4\theta + \cos^4\theta}{\sin\theta\cos\theta} = \frac{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\cos^2\theta}{\sin\theta\cos\theta}$

$= \frac{1 - 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta} = \text{RHS}$ **Hence proved.**

(iv) LHS $= 1 - \frac{\sin^2\theta}{1 + \cot\theta} - \frac{\cos^2\theta}{1 + \tan\theta}$

$= 1 - \frac{\sin^2\theta}{1 + \frac{\cos\theta}{\sin\theta}} - \frac{\cos^2\theta}{1 + \frac{\sin\theta}{\cos\theta}}$

$= 1 - \frac{\sin^3\theta}{(\sin\theta + \cos\theta)} - \frac{\cos^3\theta}{(\sin\theta + \cos\theta)}$

$= 1 - \left[\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}\right]$

$= 1 - \frac{(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)}{(\sin\theta + \cos\theta)}$

$= 1 - (1 - \sin\theta\cos\theta)$

$= \sin\theta\cos\theta = \text{RHS}$ **Hence proved.**

30. $\sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}} = \sqrt{\frac{(1 - \sin\theta)(1 - \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)}}$

$= \sqrt{\frac{(1 - \sin\theta)^2}{1 - \sin^2\theta}} = \frac{1 - \sin\theta}{\sqrt{\cos^2\theta}}$

$[\because -1 \le \sin\theta \le 1, \text{ therefore } 1 - \sin\theta \ge 0]$

$= \frac{1 - \sin\theta}{|\cos\theta|}$

$= \begin{cases} \frac{1 - \sin\theta}{\cos\theta}, \text{ if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \frac{1 - \sin\theta}{-\cos\theta}, \text{ if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$

31. Solve as Example 19 of Topic 3.

32. $\cos A = \sqrt{1 - \sin^2 A} \quad \left[\because 0 < A < \frac{\pi}{2}\right]$

$= \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \left[\because \sin A = \frac{3}{5}\right]$

$\Rightarrow \tan A = \frac{3}{4}$

Also, $\sin B = -\sqrt{1 - \cos^2 B} \quad \left[\because \pi < B < \frac{3\pi}{2}\right]$

$= -\sqrt{1 - \frac{144}{169}} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}$

$\Rightarrow \tan B = \frac{5}{12}$

Now, $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{16}{63}$

and $\cot(A + B) = 1$

$\Rightarrow \frac{\cot A \cot B - 1}{\cot B + \cot A} = 1$

$\Rightarrow \cot A \cot B - 1 = \cot A + \cot B$

$\Rightarrow \cot A \cot B - \cot A - \cot B = 1$

$\Rightarrow \cot A \cot B - \cot A - \cot B + 1 = 2$

$\Rightarrow \cot A(\cot B - 1) - (\cot B - 1) = 2$

$\Rightarrow (\cot B - 1)(\cot A - 1) = 2$

33. RHS $= \cos(\theta - \phi) + \frac{\cos(x + \phi)}{\sin(x + \phi)}\sin(\theta - \phi)$

$= \frac{\cos(\theta - \phi)\sin(x + \phi) + \cos(x + \phi)\sin(\theta - \phi)}{\sin(x + \phi)}$

$= \frac{\sin(x + \phi + \theta - \phi)}{\sin(x + \phi)} = \frac{\sin(x + \theta)}{\sin(x + \phi)}$

34. We have, $\dfrac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)}=\dfrac{n}{1}$

$$\Rightarrow \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\dfrac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)}+\dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}{\dfrac{\cos(\alpha+\theta)}{\sin(\alpha+\theta)}-\dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\sin(\alpha-\theta)\cos(\alpha+\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta)-\cos(\alpha+\theta)\sin(\alpha-\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(\alpha+\theta+\alpha-\theta)}{\sin(\alpha+\theta-\alpha+\theta)}=\frac{n+1}{n-1} \Rightarrow \frac{\sin 2\alpha}{\sin 2\theta}=\frac{n+1}{n-1}$$

35. We know that the maximum and minimum values of $a\cos\theta+b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$, respectively.

Hence, the maximum and minimum values of $(-5\cos\theta+12\sin\theta)$ are

$\sqrt{(-5)^2+(12)^2}$ and $-\sqrt{(-5)^2+(12)^2}$, respectively

i.e. 13 and −13, respectively.

36. Let $f(\theta)=3\cos\theta-4\sin\theta$

On multiplying and dividing by $\sqrt{3^2+(-4)^2}=5$, we get

$$f(\theta)=5\left(\frac{3}{5}\cos\theta-\frac{4}{5}\sin\theta\right)$$

$=5[\sin\alpha\cos\theta-\cos\alpha\sin\theta]$ where, $\sin\alpha=\dfrac{3}{5}$ and

$\cos\alpha=\dfrac{4}{5} \Rightarrow f(\theta)=5\sin(\alpha-\theta)$, where $\tan\alpha=\dfrac{3}{4}$

Again, $f(\theta)=5\left(\dfrac{3}{5}\cos\theta-\dfrac{4}{5}\sin\theta\right)$

$=5(\cos\alpha\cos\theta-\sin\alpha\sin\theta)$

where, $\cos\alpha=\dfrac{3}{5}$ and $\sin\alpha=\dfrac{4}{5}$

$=5\cos(\alpha+\theta)$, where $\tan\alpha=\dfrac{4}{3}$

37. (i) Use formula $\sin(A+B)=\sin A\cos B+\cos A\sin B$,

$\sin(A-B)=\sin A\cos B-\cos A\sin B$,

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

(ii) LHS

$=\cos^2 A+\cos^2 B-[\cos(A+B)+\cos(A-B)]\cos(A+B)$

$=\cos^2 A+\cos^2 B-\cos(A+B)-\cos(A-B)\cos(A+B)$

$=\cos^2 A+\cos^2 B-\cos^2(A+B)-\cos^2 A+\sin^2 B$

$=1-\sin^2 B-\cos^2(A+B)+\sin^2 B$

$=1-\cos^2(A+B)=\sin^2(A+B)$

(iii) LHS $=4\cos 12^\circ\cos 48^\circ\cos 72^\circ$

$=2\cos 12^\circ(2\cos 48^\circ\cos 72^\circ)$

$=2\cos 12^\circ(\cos 120^\circ+\cos 24^\circ)$

$=2\cos 12^\circ\left(-\dfrac{1}{2}+\cos 24^\circ\right)=-\cos 12^\circ+2\cos 12^\circ\cos 24^\circ$

$=-\cos 12^\circ+\cos 36^\circ+\cos 12^\circ=\cos 36^\circ$

(iv) LHS $=2\cos A[2\cos(60^\circ-A)\cos(60^\circ+A)]$

$=2\cos A[\cos(120^\circ)+\cos 2A]$

$=2\cos A\left(-\dfrac{1}{2}+\cos 2A\right)=-\cos A+2\cos A\cos 2A$

$=-\cos A+\cos 3A+\cos A=\cos 3A$

(v) LHS $=\sqrt{3}\left(\dfrac{\sin 20^\circ}{\cos 20^\circ}\cdot\dfrac{\sin 40^\circ}{\cos 40^\circ}\cdot\dfrac{\sin 80^\circ}{\cos 80^\circ}\right)$

$=\dfrac{\sqrt{3}(2\sin 20^\circ\sin 40^\circ)\sin 80^\circ}{(2\cos 20^\circ\cos 40^\circ)\cos 80^\circ}$

$=\dfrac{\sqrt{3}(\cos 20^\circ-\cos 60^\circ)\sin 80^\circ}{(\cos 60^\circ+\cos 20^\circ)\cos 80^\circ}$

$=\dfrac{\sqrt{3}(2\cos 20^\circ\sin 80^\circ-\sin 80^\circ)}{(2\cos 20^\circ\cos 80^\circ+\cos 80^\circ)}$

$=\dfrac{\sqrt{3}(\sin 100^\circ+\sin 60^\circ-\sin 80^\circ)}{(\cos 100^\circ+\cos 60^\circ+\cos 80^\circ)}$

$=\dfrac{\sqrt{3}[\sin(180^\circ-80^\circ)+\sin 60^\circ-\sin 80^\circ]}{[\cos(180^\circ-80^\circ)+\cos 60^\circ+\cos 80^\circ]}$

(vi) LHS $=\dfrac{2\cos 8A\cos 5A-2\cos 12\cos 9A}{2\sin 8A\cos 5A+2\cos 12A\sin 9A}$

$=\dfrac{\cos 13A+\cos 3A-\cos 21A-\cos 3A}{\sin 13A+\sin 3A+\sin 21A-\sin 3A}$

$=\dfrac{\cos 13A-\cos 21A}{\sin 13A+\sin 21A}=\dfrac{2\sin 17A\sin 4A}{2\sin 17A\cos 4A}$

38. $\sin 2\alpha\sin 2\beta=\dfrac{1}{2}(2\sin 2\alpha\sin 2\beta)$

$=\dfrac{1}{2}[\cos(2\alpha-2\beta)-\cos(2\alpha+2\beta)]$

$=\dfrac{1}{2}[\cos 2(\alpha-\beta)-\cos 90^\circ]=\dfrac{1}{2}\cos 2(\alpha-\beta)$

$\because \quad -1\le\cos 2(\alpha-\beta)\le 1$

$\therefore \quad \dfrac{-1}{2}\le\dfrac{1}{2}\cos 2(\alpha-\beta)\le\dfrac{1}{2}$

So, the maximum value of $\sin 2\alpha\sin 2\beta$ is $\dfrac{1}{2}$ and the minimum value is $-\dfrac{1}{2}$.

39. (i) LHS $=(\cos 3A+\cos 7A)+(\cos 5A+\cos 15A)$

$=2\cos 5A\cos 2A+2\cos 10A\cos 5A$

$=2\cos 5A(\cos 2A+2\cos 10A)$

$=2\cos 5A(2\cos 6A\cos 4A)$

$=4\cos 4A\cos 5A\cos 6A$

(ii) LHS $=\dfrac{1}{2}\left(2\sin\dfrac{\theta}{2}\sin\dfrac{7\theta}{2}+2\sin\dfrac{3\theta}{2}\sin\dfrac{11\theta}{2}\right)$

$=\dfrac{1}{2}(\cos 3\theta-\cos 4\theta+\cos 4\theta-\cos 7\theta)$

$=\dfrac{1}{2}(\cos 3\theta-\cos 7\theta)=\sin 5\theta\sin 2\theta$

(iii) Let $A = \frac{\pi}{7}$, then LHS $= \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{2^2\pi}{7} = \frac{\sin 2^3 A}{2^3 \sin A}$

$$= \frac{\sin 8A}{8\sin A} = \frac{\sin(7A + A)}{8\sin A}$$

$$= \frac{\sin(\pi + A)}{8\sin A} = \frac{-\sin A}{8\sin A} \qquad \left[\because A = \frac{\pi}{7}\right]$$

(iv) LHS $= \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{\pi}{8}$

$$= 2\left(\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8}\right) = 2\cos^2\frac{\pi}{8} + 2\cos^2\frac{3\pi}{8}$$

$$= 1 + \cos 2\cdot\frac{\pi}{8} + 1 + \cos 2\cdot\frac{3\pi}{8} = 1 + \cos\frac{\pi}{4} + 1 + \cos\frac{3\pi}{4}$$

$$= 1 + \cos\frac{\pi}{4} + 1 - \cos\frac{\pi}{4} = 2$$

40. We have, $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$$\Rightarrow \frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c} \Rightarrow \frac{a+b+2c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

$$\Rightarrow (a+b+2c)(a+b+c) = 3(a+c)(b+c)$$

$$\Rightarrow a^2 + b^2 - c^2 = ab$$

$$\Rightarrow \frac{a^2+b^2-c^2}{2ab} = \frac{ab}{2ab} \qquad \text{[dividing both sides by } 20b]$$

$$\Rightarrow \cos C = \frac{1}{2} = \cos 60^\circ \therefore \angle C = 60^\circ$$

41. Given, $\cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{b^2+c^2-a^2}{2bc} = \frac{b}{2c}$

[using cosine and sine formulae]

$$\Rightarrow b^2 + c^2 - a^2 = b^2$$

$$\Rightarrow c^2 = a^2 \Rightarrow c = a$$

$\therefore \Delta ABC$ is isosceles.

42. LHS $= \frac{1}{2}\left[2\cos^2 A + 2\cos^2\left(A + \frac{2\pi}{3}\right) + 2\cos^2\left(A - \frac{2\pi}{3}\right)\right]$

$$= \frac{1}{2}\left[1 + \cos 2A + \left\{1 + \cos 2\left(A + \frac{2\pi}{3}\right)\right\} + \left\{1 + \cos 2\left(A - \frac{2\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left[1 + \cos 2A + 1 + \cos\left(2A + \frac{4\pi}{3}\right) + 1 + \cos\left(2A - \frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2A + \cos\left(2A + \frac{4\pi}{3}\right) + \cos\left(2A - \frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2A + 2\cos 2A\cos\frac{4\pi}{3}\right]$$

$[\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$

$$= \frac{1}{2}\left[3 + \cos 2A + 2\cos 2A\left(\frac{-1}{2}\right)\right]$$

$= \frac{3}{2} =$ RHS **Hence proved.**

43. Given equation can be rewritten as

$$\frac{\sin(A-B)}{\sin(A+B)} = \frac{\sin C - \sin B}{\sin C} \Rightarrow \frac{\sin(A-B)}{\sin(A+B)} = \frac{\sin C - \sin B}{\sin(A+B)}$$

$[\because \sin C = \sin\{180^\circ - (A+B)\}]$

$$\Rightarrow \sin(A-B) = \sin C - \sin B \qquad \begin{bmatrix}\because 0^\circ < A + B + 180^\circ \\ \therefore \sin(A+B) \neq 0\end{bmatrix}$$

$$\Rightarrow \sin(A-B) = \sin(A+B) - \sin B$$

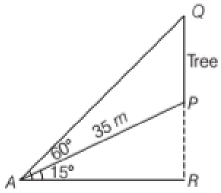
$$\Rightarrow \sin(A+B) - \sin(A-B) = \sin B \Rightarrow 2\cos A\sin B = \sin B$$

$$\Rightarrow 2\cos A = 1 \qquad [\because 0^\circ < B < 180^\circ \Rightarrow \sin B \neq 0]$$

$$\Rightarrow \cos A = \frac{1}{2} = \cos 60^\circ$$

$\therefore \quad A = 60^\circ$ **Hence proved.**

44. According to given information, we have the following figure



In ΔARQ, we have

$\angle RAQ = 60^\circ$ and

$\angle ARQ = 90^\circ$

$\therefore \quad \angle AQR = 30^\circ$

Now, in ΔAQP, we have

$\angle PAQ = 45^\circ$ and $\angle AQP = 30^\circ$

Using sine rule in ΔAPQ, we get

$$\frac{AP}{\sin\angle AQP} = \frac{PQ}{\sin\angle PAQ}$$

$$\Rightarrow PQ = 35\sqrt{2}\text{ m}$$

45. LHS $= \frac{\cos^2\left(\frac{B-C}{2}\right)}{(b+c)^2} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{(b-c)^2}$

$$= \frac{\cos^2\left(\frac{B-C}{2}\right)}{(k\sin B + k\sin C)^2} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{(k\sin B - k\sin C)^2}$$

$$= \frac{\cos^2\left(\frac{B-C}{2}\right)}{k^2(\sin B + \sin C)^2} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{k^2(\sin B - \sin C)^2}$$

$$= \frac{1}{k^2}\left[\frac{\cos^2\left(\frac{B-C}{2}\right)}{4\sin^2\left(\frac{B+C}{2}\right)\cos^2\left(\frac{B-C}{2}\right)} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{4\cos^2\left(\frac{B+C}{2}\right)\sin^2\left(\frac{B-C}{2}\right)}\right]$$

$$= \frac{1}{k^2}\left[\frac{1}{4\sin^2\left(\frac{B+C}{2}\right)} + \frac{1}{4\cos^2\left(\frac{B+C}{2}\right)}\right]$$

$$= \frac{1}{k^2}\left[\frac{\cos^2\left(\frac{B+C}{2}\right)+\sin^2\left(\frac{B+C}{2}\right)}{\left\{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right)\right\}^2}\right]$$

$$= \frac{1}{k^2 \sin^2(B+C)}$$

$$= \frac{1}{k^2 \sin^2 A} = a^{-2} = \text{RHS}$$ **Hence proved.**

46. $\text{LHS} = k^3 \sin^3 A \sin(B-C) + k^3 \sin^3 B \sin(C-A) + k^3 \sin^3 C \sin(A-B)$

$= k^3[\sin^2 A \sin A \sin(B-C) + \sin^2 B \sin B \sin(C-A) + \sin^2 C \sin C \sin(A-B)]$

$= k^3[\sin^2 A \sin(B+C)\sin(B-C) + [\sin^2 B \sin(C+A)\sin(C-A) + \sin^2 C \sin(A+B)\sin(A-B)]$

$= k^3[\sin^2 A(\sin^2 B - \sin^2 C) + \sin^2 B(\sin^2 C - \sin^2 A) + \sin^2 C(\sin^2 A - \sin^2 B)]$

$= k^3 \times 0 = \text{RHS}$ **Hence proved.**

47. (i) (a) $T_3 - T_5 = (\sin^3\theta + \cos^3\theta) - (\sin^5\theta + \cos^5\theta)$

$= \sin^3\theta(1-\sin^2\theta) + \cos^3\theta(1-\cos^2\theta)$

$= \sin^3\theta \cdot \cos^2\theta + \cos^3\theta \cdot \sin^2\theta$

$= \sin^2\theta \cdot \cos^2\theta\,(\sin\theta + \cos\theta)$

(ii) (b) $\frac{T_3 - T_5}{T_1} = \frac{\sin^2\theta\cos^2\theta(\sin\theta+\cos\theta)}{(\sin\theta+\cos\theta)}$

$= \sin^2\theta \cdot \cos^2\theta$

(iii) (c) $T_5 - T_7 = (\sin^5\theta + \cos^5\theta) - (\sin^7\theta + \cos^7\theta)$

$= \sin^5\theta(1-\sin^2\theta) + \cos^5\theta\,(1-\cos^2\theta)$

$= \sin^5\theta \cdot \cos^2\theta + \cos^5\theta \cdot \sin^2\theta$

$= \sin^2\theta \cdot \cos^2\theta\,(\sin^3\theta + \cos^3\theta)$

(iv) (a) $\frac{T_5 - T_7}{T_3} = \frac{\sin^2\theta\cdot\cos^2\theta(\sin^3\theta+\cos^3\theta)}{(\sin^3\theta+\cos^3\theta)} = \sin^2\theta\cdot\cos^2\theta$

(v) (b) $T_3 = \sin^3\theta + \cos^3\theta$

at $\theta = \pi$, $T_3 = (\sin\pi)^3 + (\cos\pi)^3 = (0)^3 + (-1)^3 = -1$

48. (i) (b) Given, $\sin A = \frac{4}{5}, 0 < A < \frac{\pi}{2}$

$\therefore \quad \cos A = \sqrt{1-\sin^2 A}$ [$\because A$ lies in 1st quadrant]

$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$

$\Rightarrow \quad \cos A = \sqrt{\frac{9}{25}} = \frac{3}{5}$

and $\quad \cos B = \frac{5}{13}, 0 < B < \frac{\pi}{2}$

$\therefore \quad \sin B = \sqrt{1-\cos^2 B}$ [$\because B$ lies in Ist quadrant]

$= \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}}$

$\Rightarrow \quad \sin B = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\therefore \quad \cos A + \sin B = \frac{3}{5} + \frac{12}{13} = \frac{39+60}{65} = \frac{99}{65}$

(ii) (c) $\sin(A+B) = \sin A\cos B + \cos A\sin B.$

$= \frac{4}{5}\times\frac{5}{13} + \frac{3}{5}\times\frac{12}{13} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$

(iii) (b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$= \frac{3}{5}\times\frac{5}{13} - \frac{4}{5}\times\frac{12}{13} = \frac{15-48}{65} = \frac{-33}{65}$

(iv) (a) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= \frac{4}{5}\times\frac{5}{13} - \frac{3}{5}\times\frac{12}{13} = \frac{20-36}{65} = \frac{-16}{65}$

(v) (b) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \frac{3}{5}\times\frac{5}{13} + \frac{4}{5}\times\frac{12}{13} = \frac{15+48}{65} = \frac{63}{65}$

49. (i) (b) Given that, $\sin(A+B) = 1$

$\Rightarrow \quad \sin(A+B) = \sin\frac{\pi}{2}$

$\Rightarrow \quad A + B = \frac{\pi}{2}$ ...(i)

and $\quad \sin(A-B) = \frac{1}{2}$

$\Rightarrow \quad \sin(A-B) = \sin\frac{\pi}{6}$

$\Rightarrow \quad A - B = \frac{\pi}{6}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2A = \frac{2\pi}{3} \Rightarrow A = \frac{\pi}{3}$

(ii) (a) Put $A = \frac{\pi}{3}$ in Eq. (i), we get

$\frac{\pi}{3} + B = \frac{\pi}{2} \Rightarrow B = \frac{\pi}{6}$

(iii) (c) $\tan(A+2B)\cdot\tan(2A+B)$

$= \tan\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\tan\left(\frac{2\pi}{3}+\frac{\pi}{6}\right)$

$= \tan\left(\frac{2\pi}{3}\right)\tan\left(\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\tan\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$

$= \left(-\cot\frac{\pi}{6}\right)\left(-\cot\frac{\pi}{3}\right) = (-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right) = 1$

(iv) (b) $\sin^2 A - \sin^2 B$

$= \sin^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{6}\right)$

$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

(v) (d) $\cos 2A = 2\cos^2 A - 1$

$= 2\cos^2\left(\frac{\pi}{3}\right) - 1$

$= 2\left(\frac{1}{2}\right)^2 - 1 = \frac{2}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$