Chapter 1

INTRODUCTION

: SOME ALGEBRAIC RESULTS

Quadratic Equations

1. The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

may easily be shewn to be

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

They are therefore real and unequal, equal, or imaginary, according as the quantity b^2-4ac is positive, zero, or negative,

i.e. according as $b^2 \gtreqless 4ac$.

2. *Relations between the roots of any algebraic equation and the coefficients of the terms of the equation.*

If any equation be written so that the coefficient of the highest term is unity, it is shewn in any treatise on Algebra that

(1) the sum of the roots is equal to the coefficient of the second term with its sign changed,

(2) the sum of the products of the roots, taken two at a time, is equal to the coefficient of the third term,

(3) the sum of their products, taken three at a time, is equal to the coefficient of the fourth term with its sign changed,

and so on.

Ex. 1. If α and β be the roots of the equation

$$ax^2+bx+c=0, \ \ i.e. \ \ x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

we have $$\alpha+\beta=-\frac{b}{a} \ \text{ and } \ \alpha\beta=\frac{c}{a}.$$

Ex. 2. If α, β, and γ be the roots of the cubic equation

$$ax^3+bx^2+cx+d=0,$$

i.e. of $$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0,$$

we have $$\alpha+\beta+\gamma=-\frac{b}{a},$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\frac{c}{a},$$

and $$\alpha\beta\gamma=-\frac{d}{a}.$$

3. It can easily be shewn that the solution of the equations

$$a_1x+b_1y+c_1z=0,$$

and $$a_2x+b_2y+c_2z=0,$$

is $$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Determinant Notation

4. The quantity $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for the quantity $a_1b_2-a_2b_1$, so that

$$\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}=a_1b_2-a_2b_1.$$

Exs. (i) $\begin{vmatrix} 2, & 3 \\ 4, & 5 \end{vmatrix}=2\times5-4\times3=10-12=-2$;

(ii) $\begin{vmatrix} -3, & -4 \\ -7, & -6 \end{vmatrix}=-3\times(-6)-(-7)\times(-4)=18-28=-10.$

5. The quantity $\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$ ....(1)

is called a determinant of the third order and stands for the quantity

$$a_1 \times \begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1, & b_2 \\ c_1, & c_2 \end{vmatrix} \text{ .... (2),}$$

i.e. by Art. 4, for the quantity

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1),$$

i.e. $$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

6. A determinant of the third order is therefore reduced to three determinants of the second order by the following rule:

Take in order the quantities which occur in the first row of the determinant; multiply each of these in turn by the determinant which is obtained by erasing the row and column to which it belongs; prefix the sign + and − alternately to the products thus obtained and add the results.

Thus, if in (1) we omit the row and column to which a_1 belongs, we have left the determinant $\begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix}$ and this is the coefficient of a_1 in (2).

Similarly, if in (1) we omit the row and column to which a_2 belongs, we have left the determinant $\begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix}$ and this with the − sign prefixed is the coefficient of a_2 in (2).

7. Ex. The determinant $\begin{vmatrix} 1, & -2, & -3 \\ -4, & 5, & -6 \\ -7, & 8, & -9 \end{vmatrix}$

$$= 1 \times \begin{vmatrix} 5, & -6 \\ 8, & -9 \end{vmatrix} - (-2) \times \begin{vmatrix} -4, & -6 \\ -7, & -9 \end{vmatrix} + (-3) \times \begin{vmatrix} -4, & 5 \\ -7, & 8 \end{vmatrix}$$

$$= \{5 \times (-9) - 8 \times (-6)\} + 2 \times \{(-4)(-9) - (-7)(-6)\}$$
$$- 3 \times \{(-4) \times 8 - (-7) \times 5\}$$

$$= \{-45 + 48\} + 2\{36 - 42\} - 3\{-32 + 35\}$$

$$= 3 - 12 - 9 = -18.$$

8. The quantity $\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}$

is called a determinant of the fourth order and stands for the quantity

$$a_1 \times \begin{vmatrix} b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \\ d_2, & d_3, & d_4 \end{vmatrix} - a_2 \times \begin{vmatrix} b_1, & b_3, & b_4 \\ c_1, & c_3, & c_4 \\ d_1, & d_3, & d_4 \end{vmatrix}$$
$$+ a_3 \times \begin{vmatrix} b_1, & b_2, & b_4 \\ c_1, & c_2, & c_4 \\ d_1, & d_2, & d_4 \end{vmatrix} - a_4 \times \begin{vmatrix} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix},$$

and its value may be obtained by finding the value of each of these four determinants by the rule of Art. 6.

The rule for finding the value of a determinant of the fourth order in terms of determinants of the third order is clearly the same as that for one of the third order given in Art. 6.

Similarly for determinants of higher orders.

9. A determinant of the second order has two terms. One of the third order has 3×2, *i.e.* 6, terms. One of the fourth order has $4 \times 3 \times 2$, *i.e.* 24, terms, and so on.

10. Exs. Prove that

(1) $\begin{vmatrix} 2, & -3 \\ 4, & 8 \end{vmatrix} = 28.$ (2) $\begin{vmatrix} -6, & 7 \\ -4, & -9 \end{vmatrix} = 82.$ (3) $\begin{vmatrix} 5, & -3, & 7 \\ -2, & 4, & -8 \\ 9, & 3, & -10 \end{vmatrix} = -98.$

(4) $\begin{vmatrix} 9, & 8, & 7 \\ 6, & 5, & 4 \\ 3, & 2, & 1 \end{vmatrix} = 0.$ (5) $\begin{vmatrix} -a, & b, & c \\ a, & -b, & c \\ a, & b, & -c \end{vmatrix} = 4abc.$

(6) $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$

Elimination

11. Suppose we have the two equations

$$a_1x + a_2y = 0 \text{ .... } (1),$$
$$b_1x + b_2y = 0 \text{ .... } (2),$$

between the two unknown quantities x and y. There must be some relation holding between the four coefficients a_1, a_2, b_1, and b_2. For, from (1), we have

$$\frac{x}{y} = -\frac{a_2}{a_1},$$

and, from (2), we have $\frac{x}{y} = -\frac{b_2}{b_1}$.

Equating these two values of $\frac{x}{y}$ we have

$$\frac{b_2}{b_1} = \frac{a_2}{a_1},$$

i.e. $$a_1b_2 - a_2b_1 = 0 \text{ .... } (3).$$

The result (3) is the condition that both the equations (1) and (2) should be true for the same values of x and y. The process of finding this condition is called the eliminating of x and y from the equations (1) and (2), and the result (3) is often called the eliminant of (1) and (2).

Using the notation of Art. 4, the result (3) may be written in the form $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} = 0$.

This result is obtained from (1) and (2) by taking the coefficients of x and y in the order in which they occur in the equations, placing them in this order to form a determinant, and equating it to zero.

12. Suppose, again, that we have the three equations

$$a_1x + a_2y + a_3z = 0 \text{ .... } (1),$$
$$b_1x + b_2y + b_3z = 0 \text{ .... } (2),$$

and $$c_1x + c_2y + c_3z = 0 \text{ .... } (3),$$

between the three unknown quantities x, y, and z.

By dividing each equation by z we have three equations between the two unknown quantities $\frac{x}{z}$ and $\frac{y}{z}$. Two of these will be sufficient to determine these quantities. By substituting their values in the third equation we shall obtain a relation between the nine coefficients.

Or we may proceed thus. From the equations (2) and (3) we have

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{b_3c_1 - b_1c_3} = \frac{z}{b_1c_2 - b_2c_1}.$$

Substituting these values in (1), we have

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0 \ldots (4).$$

This is the result of eliminating x, y, and z from the equations (1), (2), and (3).

But, by Art. 5, equation (4) may be written in the form

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0.$$

This eliminant may be written down as in the last article, *viz.* by taking the coefficients of x, y, and z in the order in which they occur in the equations (1), (2), and (3), placing them to form a determinant, and equating it to zero.

13. Ex. *What is the value of a so that the equations*

$$ax + 2y + 3z = 0, \quad 2x - 3y + 4z = 0,$$

and

$$5x + 7y - 8z = 0$$

may be simultaneously true?

Eliminating x, y, and z, we have

$$\begin{vmatrix} a, & 2, & 3 \\ 2, & -3, & 4 \\ 5, & 7, & -8 \end{vmatrix} = 0,$$

i.e. $a[(-3)(-8) - 4 \times 7] - 2[2 \times (-8) - 4 \times 5] + 3[2 \times 7 - 5 \times (-3)] = 0,$

i.e.
$$a[-4] - 2[-36] + 3[29] = 0,$$

so that
$$a = \frac{72 + 87}{4} = \frac{159}{4}.$$

14. If again we have the four equations

$$a_1x + a_2y + a_3z + a_4u = 0,$$

$$b_1x + b_2y + b_3z + b_4u = 0,$$

$$c_1x + c_2y + c_3z + c_4u = 0,$$

and $$d_1x + d_2y + d_3z + d_4u = 0,$$

it could be shewn that the result of eliminating the four quantities x, y, z, and u is the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = 0.$$

A similar theorem could be shewn to be true for n equations of the first degree, such as the above, between n unknown quantities.

It will be noted that the right-hand member of each of the above equations is zero.