Probability

- **Complementary events**

For an event E such that $0 \le P(E) \le 1$ of an experiment, the event $\bar{E}$ represents 'not E' , which is called the complement of the event E. We say, E and $\bar{E}$ are **complementary** events.

$P(E) + P(\bar{E}) = 1$

$\Rightarrow P(\bar{E}) = 1 - P(E)$

Example:
A pair of dice is thrown once. Find the probability of getting a different number on each die.
Solution:
When a pair of dice is thrown, the possible outcomes of the experiment can be listed as:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The number of all possible outcomes = 6 × 6 = 36
Let E be the event of getting the same number on each die.
Then, $\bar{E}$ is the event of getting different numbers on each die.

Now, the number of outcomes favourable to E is 6.

$\therefore P(\bar{E}) = 1 - P(E) = 1 - \frac{6}{36} = \frac{5}{6}$

Thus, the required probability is $\frac{5}{6}$.

- **Algebra of events**
  - **Complementary event:** For every event *A*, there corresponds another event *A′* called the complementary event to *A*. It is also called the event 'not *A*'.

$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$

  - **The event '*A* or *B*':** When sets *A* and *B* are two events associated with a sample space, then the set $A \cup B$ is the event 'either *A* or *B* or both'.

That is, event '*A* or *B*' = $A \cup B = \{\omega: \omega \in A \text{ or } \omega \in B\}$

  - **The event '*A* and *B*':** When sets *A* and *B* are two events associated with a sample space, then the set $A \cap B$ is the event '*A* and *B*'.

That is, event '*A* and *B*' = $A \cap B = \{\omega: \omega \in A \text{ and } \omega \in B\}$

  - **The event '*A* but not *B*':** When sets *A* and *B* are two events associated with a sample space, then the set *A* - *B* is the event '*A* but not *B*'.

That is, event '*A* but not *B*' = $A - B = A \cap B' = \{\omega: \omega \in A \text{ and } \omega \notin B\}$

Example: Consider the experiment of tossing 2 coins. Let *A* be the event 'getting at least one head' and *B* be the event 'getting exactly two heads'. Find the sets representing the events

(i) complement of '*A* or *B*'

(ii) *A* and *B*

(iii) *A* but not *B*

Solution:
Here, *S* = {HH, HT, TH, TT}
A = {HH, HT, TH}, *B* = {HH}

(i) *A* or *B* = $A \cup B$ = {HH, HT, TH}

Hence, complement of *A* or *B* = (*A* or *B*)' = $(A \cup B)' = U - (A \cup B)$= {TT}

(ii) *A* and *B* = $A \cap B$ = {HH}

(iii) A but not $B = A - B$ = {HT, TH}

- **Mutually Exclusive Events**

Two events, A and B, are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e., if they cannot occur simultaneously.

In this case, sets A and B are disjoint i.e., $A \cap B = \emptyset$

If $E_1, E_2, \ldots E_n$ are n events of a sample space S, and if

$\bigcup_{i=1}^{n} E_i = S$, $E_1 \cup E_2 \cup \ldots \cup E_n = \cup_{i=1}^{n} E_i = S$ then $E_1, E_2, \ldots E_n$ are called mutually exclusive and exhaustive events.
In other words, at least one of $E_1, E_2, \ldots E_n$ necessarily occurs whenever the experiment is performed.

The events $E_1, E_2, \ldots E_n$, i.e., n events of a sample space (S) are called mutually exclusive and exhaustive events if

$E_i \cap E_i = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint, and

$\cup_{i=1}^{n} = S$

Example: Consider the experiment of tossing a coin twice. Let A and B be the event of "getting at least one head" and "getting exactly two tails" respectively. Are the events A and B mutually exclusive and exhaustive?
Solution:
Here, S = {HH, HT, TH, TT}
A = {HH, HT, TH}
B = {TT}
Now, $A \cap B = \emptyset$ and $A \cup B$ = {HH, HT, TH, TT} = S
Thus, A and B are mutually exclusive and exhaustive events.

- The number P (ω_i) i.e., the probability of the outcome ω_i, is such that
  - $0 \leq P(\omega_i) \leq 1$
  - $\sum P(\omega_i) = 1$ for all $\omega_i \in S$
  - For any event A, $P(A) = \sum P(\omega_i)$ *for all* $\omega_i \in A$

- For a finite sample space S, with equally likely outcomes, the probability of an event A is denoted as P (A) and it is given by

$$P(A) = \frac{n(A)}{n(S)},$$

- Where, $n(A)$ = Number of elements in set A and $n(S)$ = Number of elements in set S
  - If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  - If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

  - If A is any event, then

$$P(A') = 1 - P(A)$$

Example: Consider the experiment of tossing a die. Let A be the event "getting an even number greater than 2" and B be the event "getting the number 4". Find the probability of

(i) getting an even number greater than 2 or the number 4

(ii) getting a number, which is not the number 4, on the top face of the die

Solution: Here, $S = \{1, 2, 3, 4, 5, 6\}$
$A = \{4, 6\}$, $B = \{4\}$
$A \cap B = \{4\}$
$p(A) = \frac{2}{6}, \quad p(B) = \frac{1}{6}, \quad p(A \cap B) = \frac{1}{6}$

(i) Required probability = P ($A \cup B$)

$$= P(A) + P(B) - P(A \cap B)$$
$$= \frac{2}{6} + \frac{1}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

(ii) $P(B) = \frac{1}{6}$

$\therefore$ P (not B) = 1 – P (B) $= 1 - \frac{1}{6} = \frac{5}{6}$

Hence, the required probability of not getting number 4 on the top face of the die is $\frac{5}{6}$.

Example: 20 cards are selected at random from a deck of 52 cards. Find the probability of getting at least 12 diamonds.

Solution: 20 cards can be selected at random from a deck of 52 cards in ${}^{52}C_{20}$ ways. Hence, Total possible outcomes = ${}^{52}C_{20}$

P (at least 12 diamonds) = P (12 diamonds or 13 diamonds)

= P (12 diamonds) + P (13 diamonds)

$$= \frac{{}^{13}C_{12} \times {}^{39}C_{8}}{{}^{52}C_{20}} + \frac{{}^{13}C_{13} \times {}^{39}C_{7}}{{}^{52}C_{20}}$$

$$= \frac{13 \times {}^{39}C_{8}}{{}^{52}C_{20}} + \frac{{}^{39}C_{7}}{{}^{52}C_{20}}$$

$$= \frac{13 \times {}^{39}C_{8} + {}^{39}C_{7}}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39!}{32! \times 7!}}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39! \times 8}{32 \times 31! \times 7! \times 8}}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{8}{32} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}}$$

$$= \frac{\frac{53}{4} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}}$$

$$= \frac{53}{4} \times \frac{{}^{39}C_{8}}{{}^{52}C_{20}}$$