ISC Boards 2023

Mathematics

1. In subparts (i) to (x) choose the correct options and in subparts (xi) to (xv), answer the questions as instructed.

i. A relation R on $\{1, 2, 3\}$ is given by $R = \{(1, 1), (2, 2), (1, 2), (3,3), (2, 3)\}$. Then the relation R is:
A. Reflexive
B. Symmetric
C. Transitive
D. Symmetric and transitive.

Answer (A)
Sol.
Let $A = \{1, 2, 3\}$
$R = \{(1, 1), (2, 2), (1, 2), (3,3), (2, 3)\}$.
For all $a \in A$, we have $(a, a) \in R$
i.e., $1,2,3 \in A$, we have $(1, 1), (2, 2), (3, 3) \in R$
So, R is reflexive.
Since $(1, 2) \in R$ but $(2, 1) \notin R$
Hence, R is not symmetric.
Similarly, $(1, 2), (2,3) \in R$ but $(1, 3) \notin R$
Here, R is not transitive.

ii. If A is a square matrix of order 3, then $|2A|$ is equal to:
A. $2|A|$
B. $4|A|$
C. $8|A|$
D. $6|A|$

Answer (C)

Sol.
We know that,
$|KA| = K^n|A|$, where n is order of matrix.
$\therefore |2A| = 2^3|A|$
$= 8|A|$

iii. If the following function is continuous at $x = 2$ then the value of k will be:

$$f(x) = \begin{cases} 2x + 1, if\ x < 2 \\ k, \quad if\ x = 2 \\ 3x - 1, if\ x > 2 \end{cases}$$

A. 2
B. 3
C. 5
D. -1

Answer (C)

Sol.

$$f(x) = \begin{cases} 2x+1, if\ x<2 \\ k, \quad\quad if\ x=2 \\ 3x-1, if\ x>2 \end{cases}$$

$f(x)$ is continuous at $x = 2$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow 2(2) + 1 = 3(2) - 1 = k$

$\Rightarrow k = 5$

iv. An edge of a variable cube is increasing at the rate of 10cm/sec. How fast will the volume of the cube increase if the edge is 5 cm long?

A. $75\ cm^3/sec$
B. $750\ cm^3/sec$
C. $7500\ cm^3/sec$
D. $1250\ cm^3/sec$

Answer (B)

Sol.

Let side of the cube is x
Volume = x^3
Given $\frac{dx}{dt} = 10\ cm/sec$
$V = x^3$
Differentiate w.r.t t
$\frac{dv}{dt} = 3x^2 . \frac{dx}{dt}$
$= 3(5)^2 \times 10$
$= 750\ cm^3/sec$

v. Let $f(x) = x^3$ be a function with domain {0, 1, 2, 3}. Then domain of f^{-1} is:

A. $\{3, 2, 1, 0\}$
B. $\{0, -1, -2, -3\}$
C. $\{0, 1, 8, 27\}$
D. $\{0, -1, -8, -27\}$

Answer (C)

Sol.

$f(x) = x^3$
Domain = $\{0, 1, 2, 3\}$
$\Rightarrow f(x) = \{(0,0), (1,1), (2,8), (3,27)\}$
$\Rightarrow f^{-1}(x) = \{(0,0), (1,1), (8,2), (27,3)\}$
$\therefore$ Domain of $f^{-1} = \{0, 1, 8, 27\}$

vi. For the curve $y^2 = 2x^3 - 7$, the slope of the normal at (2, 3) is:

A. 4
B. $\frac{1}{4}$
C. -4
D. $\frac{-1}{4}$

Answer (D)

Sol.

$y^2 = 2x^3 - 7$
Differentiate w.r.t x

$$2y\frac{dy}{dx} = 6x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y}$$
At (2, 3), $\frac{dy}{dx}|_{(2.3)} = \frac{3(2)^2}{3} = 4$

$\therefore$ Slope of Normal $= \frac{-1}{4}$

vii. Evaluate: $\int \frac{x}{x^2+1} dx$

A. $2\log(x^2+1) + c$
B. $\frac{1}{2}\log(x^2+1) + c$
C. $e^{x^2+1} + c$
D. $\log x + \frac{x^2}{2} + c$

Answer (B)

Sol.

$$I = \int \frac{x}{x^2+1} dx$$
Put $x^2 + 1 = t$
$$\Rightarrow 2xdx = dt$$
$$\Rightarrow I = \frac{1}{2}\int \frac{dt}{t}$$
$$= \frac{1}{2}\ln t + C$$
$$= \frac{1}{2}\ln(x^2+1) + C$$

viii. The derivative of $\log x$ with respect to $\frac{1}{x}$ is:

A. $\frac{1}{x}$
B. $\frac{-1}{x^3}$
C. $-\frac{1}{x}$
D. $-x$

Answer (D)

Sol.

Let $\frac{1}{x} = t$
$$\Rightarrow -\frac{1}{x^2} = \frac{dt}{dx}$$
$$\Rightarrow \frac{dx}{dt} = -x^2$$
Now, $y = \log x$
$$\frac{dy}{dt} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right)$$
$$= \left(\frac{1}{x}\right)(-x^2)$$
$$= -x$$

ix. The interval in which the function $f(x) = 5 + 36x - 3x^2$ increases will be:

A. $(-\infty, 6)$
B. $(6, \infty)$
C. $(-6, 6)$
D. $(0, -6)$

Answer (B)

Sol.

$\boldsymbol{f(x) = 5 + 36x - 3x^2}$

$\Rightarrow \boldsymbol{f'(x) = 36 - 6x = -6(x - 6)}$

$\Rightarrow \boldsymbol{f'(x) > 0}$ for $\boldsymbol{x < 6}$

$\therefore$ function is increasing in $(-\infty, \boldsymbol{6})$

x. Evaluate $\int_{-1}^{1} x^{17} \cos^4 x \ dx$

A. ∞

B. 1

C. -1

D. 0

Answer (D)

Sol.

$$I = \int_{-1}^{1} x^{17} \cos^4 x \ dx$$

Since, $x^{17} \cos^4 x$ is an odd function.

And we know that $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function.

$\therefore I = 0$

xi. Solve the differential equation: $\frac{dy}{dx} = \operatorname{cosec} y$

Sol.

$$\frac{dy}{dx} = \operatorname{cosec} y \Rightarrow \frac{dy}{\operatorname{cosec} y} = dx$$

$\Rightarrow \sin y \ dy = dx$

Integrating both sides

$$\Rightarrow \int \sin y \ dy = \int dx$$

$\Rightarrow -\cos y = x + C$

xii. For what value of k the matrix $\begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix}$ is a skew symmetric matrix?

Sol.

$A = \begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix}$

For skew symmetric matrix,

$A = -A^T$

$\Rightarrow \begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix} = -\begin{bmatrix} \begin{bmatrix} 0 & -6 \\ k & 0 \end{bmatrix} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & k \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 6 \\ -k & 0 \end{bmatrix} \end{bmatrix}$

$\therefore k = 6$

xiii. Evaluate $\int_0^1 |2x + 1| dx$

Sol.

$$I = \int_0^1 |2x + 1| dx$$

$I = \int_0^1 (2x + 1) dx \qquad [\because 2x + 1 > 0 \ \forall \ x \in (0, 1)]$

$= [x^2 + x]_0^1$

$= 2$

xiv. Evaluate $\int \frac{1 + \cos x}{\sin^2 x} dx$

Sol.

$$I = \int \frac{1 + \cos x}{\sin^2 x}$$
$$= \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$$
$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cdot \cot x) dx$$
$$= -\cot x - \operatorname{cosec} x + C$$

xv. A bag contains 19 tickets, numbered from 1 to 19. Two tickets are drawn randomly in succession with replacement. Find the probability that both the tickets drawn are even numbers.

Sol.

No. of even numbers from 1 to 19 is 9.

∴ Required probability is,

$$P = \left(\frac{9}{19}\right) \times \left(\frac{9}{19}\right)$$
$$= \frac{81}{361}$$

2.

i. If $f(x) = [4 - (x - 7)^3]^{\frac{1}{5}}$ is a real invertible function, then find $f^{-1}(x)$

Sol.

$f(x) = [4 - (x - 7)^3]^{\frac{1}{5}}$

Let $y = [4 - (x - 7))^3]^{\frac{1}{5}}$

$\Longrightarrow y^5 = 4 - (x - 7))^3$

$\Longrightarrow (x - 7)^3 = 4 - y^5$

$\Longrightarrow x - 7 = (4 - y^5)^{\frac{1}{3}}$

$\Rightarrow x = 7 + (4 - y^5)^{\frac{1}{3}}$

Now, replace x by y

$\Longrightarrow y = 7 + (4 - x^5)^{\frac{1}{3}}$

$\therefore f^{-1}(x) = 7 + (4 - x^5)^{\frac{1}{3}}$

OR

ii. Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \{1\}$. If $f; A \to B$ is a function defined by $f(x) = \frac{x-1}{x-2}$ then show that f is a one-one and an onto function.

Sol.

Given, $\boldsymbol{A} = \mathbb{R} - \{\boldsymbol{2}\}, \boldsymbol{B} = \mathbb{R} - \{\boldsymbol{1}\}$

$$\boldsymbol{f(x) = \frac{x - 1}{x - 2}}$$

For one-one function

$\boldsymbol{f(x_1) = f(x_2)}$

$$\Longrightarrow \boldsymbol{\frac{x_1 - 1}{x_1 - 2} = \frac{x_2 - 1}{x_2 - 2}}$$

$\Longrightarrow \boldsymbol{x_1x_2 - 2x_1 - x_2 + 2 = x_1x_2 - x_1 - 2x_2 + 2}$

$\Rightarrow \boldsymbol{x_1 = x_2}$

∴ $\boldsymbol{f(x)}$ is one – one function.

For onto function $\boldsymbol{y \in B}$

$$\Longrightarrow \boldsymbol{y = \frac{x - 1}{x - 2}}$$

$\Longrightarrow \boldsymbol{xy - 2y = x - 1}$

$\Longrightarrow \boldsymbol{x(y - 1) = 2y - 1}$

$$\Longrightarrow \boldsymbol{x = \frac{2y - 1}{y - 1}}$$

$\Longrightarrow \boldsymbol{x \in A \;\forall\; y \in B}$

∴ $\boldsymbol{f(x)}$ is onto function.

3. Evaluate the following determinant without expanding.

$$\begin{vmatrix} 5 & 5 & 5 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

Sol.

$$\Delta = \begin{vmatrix} 5 & 5 & 5 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$\Delta = \begin{vmatrix} 5 & 5 & 5 \\ a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \end{vmatrix}$$

$$\Delta = 5(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b+c & c+a & a+b \end{vmatrix}$$

Since R_1 & R_2 are identical.

$\therefore \Delta = 0$

4. The probability of the event A occurring is $\frac{1}{3}$ and of the event B occurring is $\frac{1}{2}$. If A and B are independent events, then find the probability of neither A nor B occurring.

Sol.

$P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$

$\Rightarrow\ P(A \cap B) = P(A) \cdot P(B)$

$= \frac{1}{3} \times \frac{1}{2}$

$= \frac{1}{6}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6}$

$= \frac{2}{3}$

Required probability is $P(A \cup B)'$

$\Rightarrow\ P(A \cup B)' = 1 - P(A \cup B)$

$= 1 - \frac{2}{3}$

$= \frac{1}{3}$

5. Solve for x: $5\tan^{-1} x + 3\cot^{-1} x = 2\pi$

Sol.

$5\tan^{-1} x\ + 3\cot^{-1} x = 2\,\pi$

$\Rightarrow\ 2\tan^{-1} x\ + 3(\tan^{-1} x\ + \cot^{-1} x) = 2\pi$

We know that $\tan^{-1} x\ + \cot^{-1} x = \frac{\pi}{2}$

$\Rightarrow\ 2\tan^{-1} x + \frac{3\pi}{2} = 2\,\pi$

$\Rightarrow 2\ \tan^{-1} x = \frac{\pi}{2}$

$\Rightarrow\ \tan^{-1} x = \frac{\pi}{4}$

$\Rightarrow\ x = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow\ x = 1$

6. **(i)** Evaluate: $\int \cos^{-1}(\sin x)\ dx$

Sol.

$I = \int \cos^{-1}(\sin x) dx$

We know that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\Rightarrow I = \int \left(\frac{\pi}{2} - \sin^{-1}(\sin x)\right) dx$

$= \int \left(\frac{\pi}{2} - x\right) dx$ where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= \frac{\pi x}{2} - \frac{x^2}{2} + c$

$= \frac{\pi}{2}(\pi - x) + c$

(ii) If $\int x^5 \cos(x^6)\, dx = k \sin(x^6) + c$, find the value of k

Sol.

$\int x^5 \cos(x^6)\, dx = k \sin(x^6) + c$

Let $I = \int x^5 \cos(x^6) dx$

Put $x^6 = t \Rightarrow 6x^5 dx = dt$

$\Rightarrow I = \frac{1}{6}\int \cos t\ dt$

$\Rightarrow I = \frac{\sin t}{6} + c = \frac{\sin(x^6)}{6} + c$

Given $\frac{\sin(x^6)}{6} + c = k \sin(x^6) + c$

$\therefore k = \frac{1}{6}$

7. If $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$ then prove that $24x^2 - 23x - 12 = 0$

Sol.

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

We know that,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\therefore \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \frac{(2x+1)(x-1) + (2x-1)(x+1)}{(x+1)(2x+1) - (x-1)(2x-1)} = \frac{23}{36}$$

$$\Rightarrow \frac{2x^2 - 2x + x - 1 + 2x^2 + 2x - x - 1}{2x^2 + x + 2x + 1 - 2x^2 + x + 2x - 1} = \frac{23}{36}$$

$\Rightarrow \frac{4x^2 - 1}{6x} = \frac{23}{36}$

$\Rightarrow 6(4x^2 - 1) = 23x$

$\Rightarrow 24x^2 - 2x - 6 = 0$ (proved)

8. If $y = e^{ax} \cos bx$, then prove that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\, y = 0$

Sol.

$y = e^{ax} \cos bx \;\; \ldots (i)$

$\frac{dy}{dx} = ae^{ax} \cos bx - be^{ax} \sin bx$

$\frac{dy}{dx} = e^{ax}(a \cos bx - b \sin bx) \ldots (ii)$

$\frac{d^2y}{dx^2} = ae^{ax}(a \cos bx - b \sin bx\ ) + e^{ax}(-a\, b \sin bx - b^2 \cos bx)$

$= e^{ax}\{(a^2 - b^2) \cos bx - 2\ a\ b \sin bx\} \ldots (iii)$

Now,

$\frac{d^2y}{dx^2} - 2\ a\frac{dy}{dx} = e^{ax}\{(a^2 - b^2) \cos bx - 2ab \sin bx\} - 2\ ae^{ax}(a \cos bx - b \sin bx)$

$\Rightarrow \frac{d^2y}{dx^2} - 2\ a\frac{dy}{dx} = e^{ax}\{-a^2 \cos bx - b^2 \cos bx\}$

$\Rightarrow \frac{d^2y}{dx^2} - 2\ a\frac{dy}{dx} = -e^{ax} \cos bx\ (a^2 + b^2)$

$\Rightarrow \frac{d^2y}{dx^2} - 2\ a\frac{dy}{dx} = -y(a^2 + b^2)$ [From $eq\ (i)$]

$\Rightarrow \frac{d^2y}{dx^2} - 2\ a\frac{dy}{dx} + (a^2 + b^2)y = 0$ (Proved)

9. **(i)**

In a company, 15% of the employees are graduates and 85% of the employees are non-graduates. As per the annual report of the company, 80% of the graduate employees and 10% of the non-graduate employees are in Administrative positions. Find the probability that an employee selected at random from those working in administrative positions will be a graduate.

Sol.

Let G, G' and A denote employees who are graduates, non-graduates and are in the administrative position respectively.

Given, $P(G) = \frac{15}{100}$, $P(G') = \frac{85}{100}$

$P(A/\ G) = \frac{80}{100}$, $P(A/G')\ = \frac{10}{100}$

By using Bayes' theorem

$P\left(\frac{G}{A}\right) = \frac{P(G)\cdot P(A\ /\ G)}{P(G)\cdot P(A/G) + P(G')\cdot P(A/G')}$

$= \frac{\frac{15}{100}\times\frac{80}{100}}{\frac{15}{100}\times\frac{80}{100}+\frac{85}{100}\times\frac{10}{100}}$

$= \frac{15\times80}{15\times80+85\times100}$

$= \frac{15\times80}{5\times10(3\times8+17)}$

$P\left(\frac{G}{A}\right) = \frac{24}{41}$

∴ The probability that an employee in an administrative position will be a graduate is $\frac{24}{41}$.

OR

(ii)

A problem in Mathematics is given to three students A, B and C. Their chances of solving the problem are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that

(a) exactly two students will solve the problem.

(b) at least two of them will solve the problem.

Sol.

Let's define the events:

A: The first student solves the problem.

B: The second student solves the problem.

C: The third student solves the problem.

Given, $P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{4}$

(a) Probability that exactly two students will solve the problem:

$P(E) = P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A' \cap B \cap C)$

Since, events are independent.

∴ $P(E) = P(A)P(B)P(C') + P(A)P(B')P(C) + P(A')P(B)P(C)$ ... (i)

Now, $P(A') = 1 - P(A)$

$= 1 - \frac{1}{2} = \frac{1}{2}$

Similarly, $P(B') = 1 - \frac{1}{3} = \frac{2}{3}$

And $P(C') = 1 - \frac{1}{4} = \frac{3}{4}$

From eq (i)

$P(E) = \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$

$= \frac{6}{2 \times 3 \times 4}$

∴ $P(E) = \frac{1}{4}$

(b) Probability that at least two of them will solve the problem:

The Probability that at least two of them will solve the problem is equivalent to Probability that exactly two of them solve the problem plus probability that all of them solve the problem.

∴ Required Probability $= P(E) + P(A \cap B \cap C)$

$= \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$

$= \frac{1}{4}\left(1 + \frac{1}{6}\right)$

$= \frac{7}{24}$

10. (i) Solve the differential equation:
$(1+y^2)\,dx = (\tan^{-1} y - x)dy$

Sol.

$(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{\tan^{-1} y}{1+y^2} - \dfrac{x}{1+y^2}$

$\Rightarrow \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1} y}{1+y^2}$

The given linear differential equation is of the form:

$\dfrac{dx}{dy} + x\,P(y) = Q(y)$

$\therefore I.F. = e^{\int \frac{1}{1+y^2} dy}$

$= e^{\tan^{-1} y}$

Now, solution of above differential equation is.

$x(I.F.) = \int Q\,(I.F.)\,dy + C$

$\Rightarrow x \cdot e^{\tan^{-1} y} = \displaystyle\int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\,dy + c \quad ...(i)$

Put $t = \tan^{-1} y$

$\Rightarrow dt = \dfrac{1}{1+y^2} dy$

$\therefore$ eq(i) becomes

$xe^t = \int te^t\,dt + c$

$\Rightarrow xe^t = t\displaystyle\int e^t\,dt - \int \left(\frac{dt}{dt}\int e^t\,dt\right) dt + c$

$\Rightarrow xe^t = te^t - \int e^t\,dt + c$

$\Rightarrow xe^t = te^t - e^t + c$

$\Rightarrow xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

$\Rightarrow x = (\tan^{-1} y - 1) + c \cdot e^{-\tan^{-1} y}$

(ii) Solve the differential equation:
$(x^2 - y^2)\,dx + 2xydy = 0$

Sol.

$(x^2 - y^2)\,dx + 2\,xy\,dy = 0$

$\Rightarrow 2xy\,dy = (y^2 - x^2)\,dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy} \quad ...(i)$

Put $y = vx$

$\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx} ... \ (ii)$

From eq (i) & (ii)

$v + x\dfrac{dv}{dx} = \dfrac{v^2x^2 - x^2}{2\,vx^2}$

$\Rightarrow v + x\,\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v}$

$\Rightarrow x\,\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v} - v$

$\Rightarrow x\,\dfrac{dv}{dx} = \dfrac{-v^2 - 1}{2v}$

$\Rightarrow \dfrac{dx}{x} = -\dfrac{2v}{1+v^2}\,dv$

Integrating both sides.

$$\Rightarrow \int \frac{dx}{x} = -\int \frac{2v}{1+v^2}\, dv$$

$$\Rightarrow \ln x = -\ln(1+v^2) + C$$

$$\Rightarrow \ln x + \ln\left(1+\frac{y^2}{x^2}\right) = C$$

$$\Rightarrow \ln\left[x \times \left(\frac{x^2+y^2}{x^2}\right)\right] = C$$

$$\Rightarrow \ln\left(\frac{x^2+y^2}{x^2}\right) = \ln k \text{ (Let } c = ln\,k)$$

$$\Rightarrow \frac{x^2+y^2}{x^2} = k$$

$$\Rightarrow x^2+y^2 = kx$$

11. Use matrix method to solve the following system of equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Sol.

Let $\frac{1}{x} = p; \frac{1}{y} = q; \frac{1}{z} = r$

$$2p + 3q + 10r = 4$$

$$4p - 6q + 5r = 1$$

$$6p + 9q - 20r = 2$$

$$AX = B \Longrightarrow X = A^{-1}B$$

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad X = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2(120-45) - 3(-80-30) + 10(36+36) = 1200$$

Co factor of $A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$

$$A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 0 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 0 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

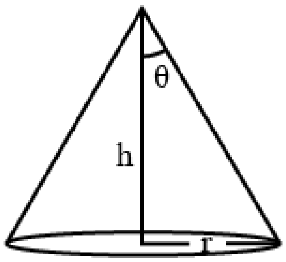
$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$\Rightarrow x = 2, y = 3, z = 5$

12. (i) Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}\sqrt{2}$.

Sol.



Let:

Radius of the base $= r$,

Height $= h$,

Slant height $= l$,

Volume $= V$,

Curved surface area $= S$

Now, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow h = \frac{3V}{\pi r^2}$

Also, the slant height, $l = \sqrt{h^2 + r^2}$

$$= \sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}$$

$$\Rightarrow l = \frac{\sqrt{9V^2 + \pi^2 r^6}}{\pi r^2}$$

And $S = \pi r l$

$$\Rightarrow S(r) = \frac{\sqrt{9V^2 + \pi^2 r^6}}{r}$$

$$\Rightarrow S'(r) = \frac{r \times \frac{6\pi^2 r^5}{2\sqrt{9V^2 + \pi^2 r^6}} - \sqrt{9V^2 + \pi^2 r^6}}{r^2}$$

$$= \frac{\left[\frac{3\pi^2 r^6 - (9V^2 + \pi^2 r^6)}{\sqrt{9V^2 + \pi^2 r^6}}\right]}{r^2}$$

$$= \frac{3\pi^2 r^6 - 9V^2 - \pi^2 r^6}{r^2\sqrt{9V^2 + \pi^2 r^6}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{9V^2 + \pi^2 r^6}}$$

For maxima or minima, $S'(r) = 0$

$$\Rightarrow \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{9V^2 + \pi^2 r^6}} = 0$$

$\Rightarrow 2\,\pi^2 r^6 - 9V^2 = 0$

$\Rightarrow 2\,\pi^2 r^6 = 9V^2$

$\Rightarrow V^2 = \dfrac{2\,\pi^2 r^6}{9}$

$\Rightarrow V = \dfrac{\pi r^3\sqrt{2}}{3}$ or $r = \left(\dfrac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$

So, $h = \dfrac{3}{\pi r^2} \times \dfrac{\pi r^3\sqrt{2}}{3}$

$\Rightarrow h = r\sqrt{2} \Rightarrow \dfrac{h}{r} = \sqrt{2}$

$\Rightarrow \cot\theta = \sqrt{2}$

$\therefore \theta = \cot^{-1}(\sqrt{2})$

Also,

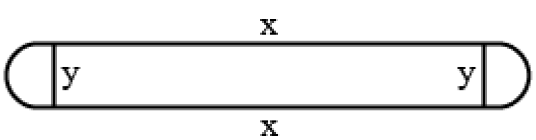
Since, for $r < \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$, $S'(r) < 0$ and for $r > \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$, $S'(r) > 0$

So, the curved surface for $r = \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$ or $V = \frac{\pi r^3\sqrt{2}}{3}$ is the least.

Hence, the semi-vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}\sqrt{2}$.

12. **(ii)** A running track of 440 m is to be laid out enclosing a football field. The football field is in the shape of a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the length of its sides. Also calculate the area of the football field.

Sol.



Let us consider the length and breadth of the rectangle are x and y respectively.

The perimeter of the rectangle $= 2x + 2y$

But here, we have two semi-circles also present, as we know the perimeter of a semi-circle is πr.

The radius of the semi-circle is $\frac{y}{2}$ (Since, breadth is y).

The perimeter of football field $= 2x + \pi\left(\frac{y}{2}\right) + \pi\left(\frac{y}{2}\right)$

$\Rightarrow P = 2x + \pi y = 440$ (since the total perimeter of the running track is 440 m)

$\Rightarrow y = \frac{440 - 2x}{\pi}$

Now, the area of the rectangle is given by,

Area $= x \cdot y$

$\Rightarrow$ Area $= x\left(\dfrac{440 - 2x}{\pi}\right) = \dfrac{440x - 2x^2}{\pi}$

To maximize the area of the rectangle portion, we need to calculate x when $\frac{dA}{dx} = 0$

On differentiating area with respect to x, we get

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{440x - 2x^2}{\pi}\right) = 0$

$\Rightarrow \frac{1}{\pi}(440-4x)=0$

$\Rightarrow 440-4x=0$

$\Rightarrow x=110$

Substitute the value of x in y, we get

$$y=\frac{440-2(110)}{\frac{22}{7}}$$

$$\Rightarrow y=\frac{7\,(220)}{22}=70$$

Therefore, length of the rectangular field is 110 m and breadth is 70 m.

Area of football field $=\left(\frac{1}{2}\right)\pi\left(\frac{y}{2}\right)^2+xy+\left(\frac{1}{2}\right)\pi\left(\frac{y}{2}\right)^2=\pi\left(\frac{y}{2}\right)^2+xy$

$=\frac{22}{7}\cdot 35^2+110\cdot 70$

$=3850+7700=11550$ sq. meter

13. (i) Evaluate: $\int \frac{3e^{2x}-2e^{x}}{e^{2x}+2e^{x}-8}\,dx$

Sol.

$I=\int \frac{3e^{2x}-2e^{x}}{e^{2x}+2e^{x}-8}dx$ substitute $u=e^x \Rightarrow du=e^x dx$

$\Rightarrow I=\int \frac{3u-2}{u^2+2u-8}\,du$

$=\int \frac{3u-2}{(u-2)(u+4)}du$

By using partial fraction, we have

Let $\frac{3u-2}{(u-2)(u+4)}=\frac{A}{(u+4)}+\frac{B}{(u-2)}$

$\Rightarrow 3u-2=A(u-2)+B(u+4)$

$\Rightarrow 3u-2=u\,(A+B)+(4B-2A)$

$\Rightarrow A+B=3$ and $4B-2A=\ -2$

On solving, we get $A=\frac{7}{3}, B=\frac{2}{3}$

$\Rightarrow I=\int \frac{7}{3(u+4)}+\frac{2}{3(u-2)}\,du$

$\Rightarrow I=\frac{7}{3}\ln|u+4|+\frac{2}{3}\ln|u-2|+C$

$\therefore I=\frac{7}{3}\ln|e^x+4|+\frac{2}{3}\ln|e^x-2|+C$

Or

(ii) Evaluate: $\int \frac{2}{(1-x)(1+x^2)}\,dx$

Sol.

$I=\int \frac{2}{(1-x)(1+x^2)}\,dx$

By using partial fraction, we have

Let $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

$\Rightarrow 2=A\,(1+x^2)+(Bx+C)(1-x)$

$\Rightarrow 2 = A + Ax^2 + Bx - Bx^2 + C + Cx$

Equating the coefficient of x^2, x, and constant term, we obtain

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these questions, we get

$A = 1, B = 1$, and $C = 1$

$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$

$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}\, dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2}\, dx + \int \frac{1}{1+x^2}\, dx$

$\Rightarrow I = -\int \frac{1}{x-1} dx + \frac{1}{2}\int \frac{2x}{1+x^2}\, dx + \int \frac{1}{1+x^2} dx$

$\Rightarrow I = -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$

14. A box contains 30 fruits, out of which 10 are rotten. Two fruits are selected at random one by one without replacement from the box. Find the probability distribution of the number of unspoiled fruits. Also find the mean of the probability distribution.

Sol.

Let X be the random variable representing the number of unspoiled fruits.
There are two possible outcomes for each selection: spoiled fruit or unspoiled fruit.
Let S represents a spoiled fruit and U represents an unspoiled fruit.

Now, the probability distribution of X as follows:

$P(X = 0) = P(\text{spoiled and spoiled}) = P(S) \times P(S) = \left(\frac{10}{30}\right) \times \left(\frac{9}{29}\right) = \frac{9}{87}$

$P(X = 1) = P(\text{unspoiled and spoiled}) \text{ or } P(\text{spoiled and unspoiled})$

$= P(U) \times P(S) + P(S) \times P(U) = \left(\frac{20}{30}\right) \times \left(\frac{10}{29}\right) + \left(\frac{10}{30}\right) \times \left(\frac{20}{29}\right) = \frac{40}{87}$

$P(X = 2) = P(\text{unspoiled and unspoiled}) = P(U) \times P(U) = \left(\frac{20}{30}\right) \times \left(\frac{19}{29}\right) = \frac{38}{87}$

So, the probability distribution of X is:

X	0	1	2
$P(X)$	$\frac{9}{87}$	$\frac{40}{87}$	$\frac{38}{87}$

The mean of the probability distribution is given by,

$E(X) = \mu = \sum\left(x_i \times P(x_i)\right)$, where x_i are the possible values of X and $P\ (x_i)$ are their respective probabilities.

$\therefore \mu = 0 \times \frac{9}{87} + 1 \times \frac{40}{87} + 2 \times \frac{38}{87} = \frac{116}{87}$

Therefore, the mean of the probability distribution is $\frac{116}{87}$.

15. **(i)** If $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then the angle between $\vec{a}$ and $\vec{b}$ will be:

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{2}$

Sol. Answer: (**B**)

It is given that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Given, $\vec{a} \times \vec{b}$ is a unit vector.

$\Rightarrow |\vec{a} \times \vec{b}| = 1$

$\Rightarrow \left||\vec{a}||\vec{b}| \sin\theta\, \hat{n}\right| = 1$

$\Rightarrow \left||\vec{a}||\vec{b}| \sin\theta\right| = 1$

$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times |\sin\theta| = 1$

$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Hence, from the given options, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

(ii) The distance of the point $2\hat{\imath} + \hat{\jmath} - \hat{k}$ from the plane $\vec{r}.\left(\hat{\imath} - 2\hat{\jmath} + 4\hat{k}\right) = 9$ will be:

(a) 13

(b) $\frac{13}{\sqrt{21}}$

(c) 21

(d) $\frac{21}{\sqrt{13}}$

Sol. Answer: (**B**)

We have formula,

Distance $= \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$

Given, $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$, $\vec{n} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$ and $d = 9$

Distance $= \left|\frac{(2\hat{\imath}+\hat{\jmath}-\hat{k}).(\hat{\imath}-2\hat{\jmath}+4\hat{k})-9}{\hat{\imath}-2\hat{\jmath}+4\hat{k}}\right|$

$= \frac{|2-2-4-9|}{\sqrt{21}}$

$= \frac{13}{\sqrt{21}}$

(iii) Find the area of the parallelogram whose diagonals are $\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\hat{\imath} + \hat{\jmath} + \hat{k}$.

Sol.

Let $\vec{a} = \hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k}$ be the diagonals.

Area of the parallelogram $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\left|\left(\hat{\imath} - 3\hat{\jmath} + \hat{k}\right) \times \left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\right|$

$= \frac{1}{2}\left|-4\hat{\imath} + 0\hat{\jmath} + 4\hat{k}\right|$

$= \frac{1}{2}\left(4\sqrt{2}\right) = 2\sqrt{2}$ sq. units

(iv) Write the equation of the plane passing through the point $(2,4,6)$ and making equal intercepts on the coordinate axes.

Sol.

We know that the equation of the plane whose intercepts on the coordinate axes are a, b and c is:

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Given that the plane makes equal intercepts on the co-ordinate axes.

So, $a=b=c$

The equation of the plane become $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$

$\Longrightarrow x+y+z=a$ …. (1)

This plane passes through the point $(2,4,6)$.

Substituting points in equation (1), we get

$\Rightarrow 2+4+6=a$

$\Rightarrow a=12$

Substituting this in (1), we get

$\Rightarrow x+y+z=12$

(v) If the two vectors $3\hat{\imath}+\alpha\hat{\jmath}+\hat{k}$ and $2\hat{\imath}-\hat{\jmath}+8\hat{k}$ are perpendicular to each other, then find the value of a.

Sol.

Let $\vec{a}+3\hat{\imath}+\alpha\hat{\jmath}+\hat{k}$ and $\vec{b}=2\hat{\imath}-\hat{\jmath}+8\hat{k}$ be the given vectors.

$\vec{a}$ and $\vec{b}$ are perpendicular.

$\Rightarrow \vec{a}.\vec{b}=0$

$\Rightarrow (3\hat{\imath}+\alpha j+\hat{k}).(2\hat{\imath}-\hat{\jmath}+8\hat{k})=0$

$\Rightarrow 6-\alpha+8=0$

$\Rightarrow \alpha=14$

16.

(i) If $\boldsymbol{A}(\mathbf{1},\mathbf{2},-\mathbf{3})$ and $\boldsymbol{B}(-\mathbf{1},-\mathbf{2},\mathbf{1})$ are the end points of a vector $\overrightarrow{\boldsymbol{AB}}$ then find the unit vector in the direction of $\overrightarrow{\boldsymbol{AB}}$.

Sol.

We have, $\vec{A}\,(1,2,-3)$ and $\vec{B}\,(-1,-2,1)$

Then $\overrightarrow{AB}$ = (Position vector of B) − (Position vector of A)

$= (-\hat{\imath}-2\hat{\jmath}+\hat{k})-(i+2\hat{\jmath}-3\hat{k})$

$= -2\hat{\imath}-4\hat{\jmath}+4\hat{k}$

$\therefore |\overrightarrow{AB}|=\sqrt{(-2)^2+(-4)^2+4^2}=\sqrt{4+16+16}=\sqrt{36}=6$

Unit vector $\widehat{AB}=\frac{-2\hat{\imath}-4\hat{\jmath}+4\hat{k}}{6}$

$=-\frac{1}{3}\hat{\imath}-\frac{2}{3}\hat{\jmath}+\frac{2}{3}\hat{k}$

(ii) If $\hat{a}$ is unit vector and $(2\vec{x}-3\hat{a})(2\vec{x}+3\hat{a})=91$, find the value of $|\vec{x}|$

Sol.

$(2\vec{x}-3\hat{a})\cdot(2\vec{x}+3\hat{a})=91$

$\Rightarrow 4|\vec{x}|^2 + 6\vec{x}.\hat{a} - 6\hat{a}.\vec{x} - 9|\hat{a}|^2 = 91$

$\Rightarrow 4|\vec{x}|^2 - 9|\hat{a}|^2 = 91$

$\hat{a}$ is a unit vector $\Rightarrow |\hat{a}| = 1$

$\Rightarrow 4|\vec{x}|^2 = 100$

$\Rightarrow |\vec{x}|^2 = 25$

$\therefore |\vec{x}| = 5$

17.

(i) Find the equation of the plane passing through the point $(\mathbf{1}, \mathbf{1}, -\mathbf{1})$ and perpendicular to the planes

$x + 2y + 3z = 7$ and $2x - 3y + 4z = 0$

Sol.

Vector equation of a line passing through a point and parallel to a vector is $\vec{r} = \vec{a} + \lambda\vec{b}$ where $\lambda \in R$

The equation of the plane containing the given point is $A(x-1) + B(y-1) + C(z+1) = 0 \cdots (1)$

Applying the condition of perpendicularity to the plane (1) with the given planes $x + 2y + 3z = 7$ and $2x - 3y + 4z = 0$, we get

$A + 2B + 3C = 7$ and $2A - 3B + 4C = 0$

Solving the equation, we get

$$\frac{A}{\begin{vmatrix} 2 & 3 \\ -3 & 4 \end{vmatrix}} = \frac{B}{\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}} = \frac{C}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}}$$

$$\Rightarrow \frac{A}{8+9} = \frac{B}{6-4} = \frac{C}{-3-4}$$

$$\Rightarrow \frac{A}{17} = \frac{B}{2} = \frac{C}{-7} = \lambda(\text{say})$$

$\therefore A = 17\lambda, b = 2\lambda$ and $C = -7\lambda$

Substituting these values in eq(1) we get,

$17\lambda(x-1) + 2\lambda(y-1) - 7\lambda(z+1) = 0$

$\Rightarrow \lambda[17(x-1) + 2(y-1) - 7(z+1)] = 0$

$\Rightarrow 17x - 17 + 2y - 2 - 7z - 7 = 0, \ (\lambda \neq 0)$

$\Rightarrow 17x + 2y - 7z = 26$

This is the required solution to the plane.

OR

(ii) A line passes through the point $(2, -1, 3)$ and is perpendicular to the lines

$\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + \lambda(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + \mu(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$.

Obtain its equation.

Sol.

Line L is passing through point $= (2\hat{\imath} - \hat{\jmath} + 3\hat{k})$

If $L_1 \Rightarrow \vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + \lambda(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$L_2 \Rightarrow \vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + \mu(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

Given that line is perpendicular to L_1 and L_2

Let the line $L = (a_1, a_2, a_3)$

The equation of L in vector form $\Rightarrow$ $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + p(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

Where p is any constant.

So, by condition that L is perpendicular to L_1, we have

$2a_1 - 2a_2 + a_3 = 0 \ldots.(1)$

As $L \perp L_2$

So, $a_1 + 2a_2 + 2a_3 = 0 \ldots.(2)$

On solving (1) and (2), we get

$3a_1 + 3a_2 = 0$

$\Rightarrow a_3 = -a_1$

Put it in (1)

$2a_1 - 2a_2 - a_1 = 0$

$\Rightarrow a_2 = \frac{a_1}{2}$

So, $L \equiv \left(a_1, \frac{a_1}{2}, -a_1\right)$

So, DR of $L = \left(1, \frac{1}{2}, -1\right)$

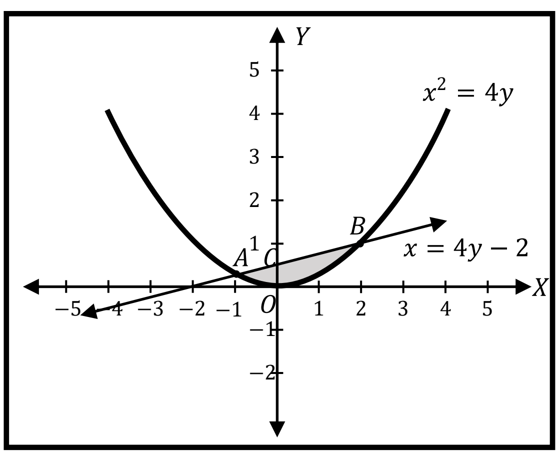
So, equation of L in vector form:

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + k\left(\hat{i} + \frac{\hat{j}}{2} - \hat{k}\right)$

Cartesian form is $\frac{x-2}{1} = \frac{y+1}{\frac{1}{2}} = \frac{z-3}{-1}$.

18. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Sol.



The area bounded by the curve, $x^2 = 4y$ and line, $x = 4y - 2$, is represented by the shaded area $OBAO$.

Let A and B be the points of intersection of the line and parabola.

On solving $x^2 = 4y$ and $x = 4y - 2$, we get

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$

Coordinates of point B are $(2, 1)$.

Required area = $\int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$

$= \int_{-1}^{2}\frac{x+2}{4}\,dx - \int_{-1}^{2}\frac{x^2}{4}dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$

$= \frac{1}{4}\left[2 + 4 - \frac{1}{2} + 2\right] - \frac{1}{4}\left[\frac{8}{3} + \frac{1}{3}\right]$

$= \frac{15}{8} - \frac{3}{4} = \frac{9}{8}$ sq. units

19.

(i) If the demand function is given by $p = 1500 - 2x - x^2$ then find the marginal revenue when $x = 10$

(a) 1160
(b) 1600
(c) 1100
(d) 1200

Sol.

Answer **(a)**

⇒ We have, $p = 1500 - 2x - x^2$
⇒ Revenue function, $R = p \times x$
∴ Marginal revenue = $\frac{d}{dx}R$
⇒ Marginal revenue = $\frac{d}{dx}(1500x - 2x^2 - x^3)$
⇒ Marginal revenue = $1500 - 4x - 3x^2$
⇒ Now, substitute $x = 10$.
⇒ Marginal revenue = $1500 - 4(10) - 3(10)^2 = 1160$
∴ Marginal revenue is 1160.

(ii) If the two regression coefficients are 0.8 and 0.2, then the value of coefficient of correlation r will be:

(a) ± 0.4
(b) ± 0.16
(c) 0.4
(d) 0.16

Sol.

Answer **(c)**

The two regression coefficient between x and y are 0.8 and 0.2.
The correlation coefficient will be positive because both the coefficients are positive.
And the correlation coefficient is the geometric mean of both the coefficients.
So, the correlation coefficient is:

$r = \sqrt{0.8 \times 0.2}$
$\Rightarrow r = \sqrt{0.16}$
$\Rightarrow r = 0.4$

(iii) Out of the two regression lines $x + 2y - 5 = 0$ and $2x + 3y = 8$, find the line of regression of y on x.

Sol.

Let assume $x + 2y - 5 = 0$ is the equation of the regression line of x on y and $2x + 3y = 8$ is the equation of the regression line of y on x.
The above two equations can be written as,
$x = -2y + 5$ and $y = -\frac{2}{3}x + \frac{8}{3}$
Now, the regression coefficient x on y is, $b_{xy} = -2$
And the regression coefficient y on x is, $b_{yx} = -\frac{2}{3}$

Now, the coefficient of correlation r is:
$r^2 = b_{xy}b_{yx}$
$\Rightarrow r^2 = \frac{4}{3} > 1$

Which is not possible because coefficient of correlation r lies between -1 and $+1$ i.e., $-1 \le r \le 1$.
Hence, our assumption is wrong.
Therefore, the regression line x on y is $2x + 3y - 8 = 0$ and the regression line y on x is $x + 2y - 5 = 0$.

(iv) The cost function $C(x) = 3x^2 - 6x + 5$. Find the average cost when $x = 2$

Sol.

$C(2) = 3 \times 2^2 - 6 \times 2 + 5$
$\Rightarrow C(2) = 5$
Average cost = Total cost of the units/Number of units
$\therefore$ Average cost = $\frac{5}{2} = 2.5$

(v) The fixed cost of a product is ₹ 30,000 and its variable cost per unit is ₹ 800. If the demand function is $p(x) = 4500 - 100x$, find the break – even values.

Sol.

Total cost = fixed cost + variable cost
$= 30000 + 800x$
Revenue = $p(x) \cdot x = 4500x - 100x^2$
The break-even point is reached when total cost and revenue are equal.
$\therefore 4500x - 100x^2 = 30000 + 800x$
$\Rightarrow x^2 - 37x + 300 = 0$
$\Rightarrow (x - 25)(x - 12) = 0$
Break even values are $x = 25$ and 12

20.

(i) The total cost function for x units is given by $C(x) = \sqrt{6x + 5} + 2500$. Show that the marginal cost decreases as the output x increases.

Sol.

$C(x) = \sqrt{6x + 5} + 2500$

Marginal cost is given as,

$MC = \frac{d}{dx} C(x)$

$= \frac{d}{dx}\left((6x + 5)^{1/2} + 2500\right)$

$= \frac{1}{2}(6x + 5)^{-\frac{1}{2}} \times 6$

$MC = \frac{3}{\sqrt{6x+5}} \Rightarrow MC \propto \frac{1}{\sqrt{x}}$

Hence, the marginal cost decreases as the output x increases.

(ii) The average revenue function is given by $AR = 25 - \frac{x}{4}$. Find total revenue function and marginal revenue function.

Sol.

Total revenue function $= p(x) \cdot x = 25x - \frac{x^2}{4}$

Marginal revenue function $= \frac{d}{dx}\left(25x - \frac{x^2}{4}\right) = 25 - \frac{x}{2}$

21. Solve the following Linear Programming Problem graphically.

Maximize $Z = 5x + 2y$ subject to:
$x - 2y \le 2,$
$3x + 2y \le 12,$
$-3x + 2y \le 3,$
$x \ge 0, y \ge 0$

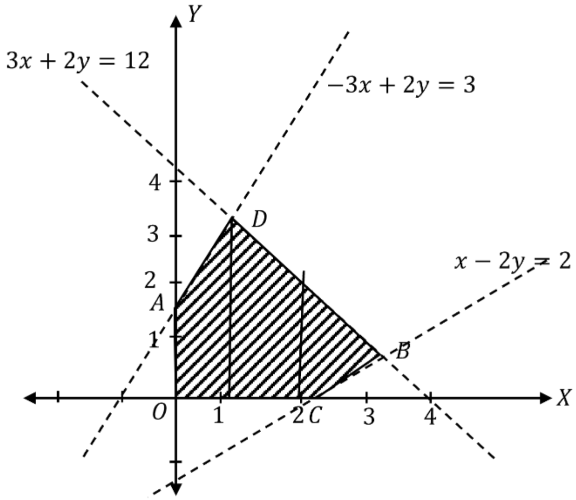
Sol.

$x - 2y \le 2,$
$3x + 2y \le 12,$

$-3x + 2y \leq 3,$
$x \geq 0, y \geq 0$

Converting the inequations into equations, we obtain the lines.

$x - 2y = 2$ .....(i)
$3x + 2y = 12$ ....(ii)
$-3x + 2y = 3$ .....(iii)



From the graph, we get the corner points as
$A(0, 1.5), B(3.5, 0.75), C(2,0), D(1.5, 3.75), O(0, 0)$
The values of the objective function are:

Point (x, y)	Values of the objective function $Z = 5x + 2y$
$A(0, 1.5)$	$5\times 0 + 2 \times 1.5 = 3$
$B(3.5, 0.75)$	$5 \times 3.5 + 2 \times 0.75 = 19$ (Maximum)
$C(2, 0)$	$5 \times 2 + 2 \times 0 = 10$
$D(1.5, 3.75)$	$5 \times 1.5 + 2 \times 3.75 = 15$
$O(0,0)$	$5 \times 0 + 2 \times 0 = 0$ (Minimum)

The maximum value of Z is 19 at $(3.5, 0.75)$.

22.

(i) The following table shows the Mean, the Standard Deviation, and the coefficient of correlation of two variables x and y.

Series	x	y
Mean	8	6
Standard deviation	12	4
Coefficient of correlation	0.6	

Calculate:

(a) The regression coefficient b_{xy} and b_{yx}
(b) The probable value of y when $x = 20$

Sol.

$\bar{x} = 8, \quad \sigma_x = 12$

$\bar{y} = 6, \quad \sigma_y = 4$

$r = 0.6$

$\Rightarrow b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.6\left(\frac{4}{12}\right) = 0.2$

$\Rightarrow b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.6\left(\frac{12}{4}\right) = 1.8$

Regression equation of y on x

$y - \bar{y} = b_{yx}(x - \bar{x})$

$\Rightarrow y - 6 = 0.2\,(x - 8)$

$\Rightarrow y = 0.2x + 4.4$

Putting $x = 20$ gives

$y = 0.2(20) + 4.4 = 8.4$

So, probable value of y when x is 20 is 8.4.

(ii)

An analyst analysed 102 trips of a travel company. He studied the relation between travel expenses (y) and the duration (x) of these trips. He found that the relation between x and y was linear. Given the following data, find the regression equation of y on x.

$$\sum x = 510,\ \sum y = 7140,\ \sum x^2 = 4150,\ \sum y^2 = 740200,\ \sum xy = 54900$$

Sol.

Given that,

$\sum x = 510 \qquad \sum x^2 = 4150 \qquad \sum xy = 54900$

$\sum y = 7140 \qquad \sum y^2 = 740200 \qquad n = 102$

Mean $\bar{x} = \frac{\sum x}{n}$

$= \frac{510}{102} = 5$

Mean $\bar{y} = \frac{\sum y}{n} = \frac{7140}{102}$

$= 70$

Regression co-efficient of y on x

$b_{yx} = \frac{n.\sum xy - \sum x.\sum y}{n\sum x^2 - (\sum x)^2}$

$= \frac{102\times 54900 - 510\times 7140}{102\times 4150 - (510)^2}$

$= \frac{5599800 - 3641400}{423300 - 260100}$

$= \frac{1958400}{163200} = 12$

Regression equation y on x is,

$y - \bar{y} = b_{yx}(x - \bar{x})$

$\Rightarrow y - 70 = 12(x - 5)$

$\Rightarrow y = 12x - 60 + 70$

$\Rightarrow y = 12x + 10$