CHAPTER

02

Definite Integral

Learning Part

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Session 1

Integration Basics, Geometrical Interpretation of Definite Integral, Evaluation of Definite Integrals by Substitution

Integration Basics

What is Definite Integral ?

Let f be a function of x defined in the closed interval $[a, b]$ and ϕ be another function, such that $\phi'(x) = f(x)$ for all x in the domain of f, then

$$\int_a^b f(x)\, dx = [\phi(x) + c]_a^b = \phi(b) - \phi(a)$$

is called the definite integral of the function $f(x)$ over the interval $[a, b]$, a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Remark

In definite integrals constant of integration is never present.

Working Rules

To evaluate definite integral $\int_a^b f(x)\, dx$.

1. First evaluate the indefinite integral $\int f(x)\, dx$ and suppose the result is $g(x)$.
2. Next find $g(b)$ and $g(a)$.
3. Finally, the value of the definite integral is obtained by subtracting $g(a)$ from $g(b)$.

Thus, $\int_a^b f(x)\, dx = [g(x)]_a^b = g(b) - g(a)$

| Example 1 Evaluate

(i) $\int_0^1 \frac{1}{3+4x} dx$ (ii) $\int_0^{\pi/2} \sin^4 x\, dx$

Sol. (i) Here, $I = \int_0^1 \frac{1}{3+4x} dx = \left[\frac{\ln(3+4x)}{4}\right]_0^1$

$$= \frac{1}{4}[\ln 7 - \ln 3] = \frac{1}{4}\ln\left(\frac{7}{3}\right)$$

(ii) Let $I = \int_0^{\pi/2} \sin^4 x\, dx$

$$= \frac{1}{4}\int_0^{\pi/2} (2\sin^2 x)^2 dx = \frac{1}{4}\int_0^{\pi/2} (1-\cos 2x)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3 - 4\cos 2x + \cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\left[3x - \frac{4}{2}\sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{2} - 2\sin \pi + \frac{1}{4}\sin 2\pi\right) - 0\right]$$

$$= \frac{1}{8}\left(\frac{3\pi}{2} - 0 + 0\right) = \frac{3\pi}{16}$$

| Example 2 The value of $\int_{-1}^1 \left[\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)\right] dx$ is

(a) $\pi/2$ (b) $\pi/4$ (c) $-\pi/2$ (d) None of these

Sol. Let $I = \int_{-1}^1 \left[\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)\right] dx$

Here, $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\therefore$ $I = \int_{-1}^1 -\frac{1}{1+x^2} dx = -\int_{-1}^1 \frac{1}{1+x^2} dx$

$$= -(\tan^{-1} x)_{-1}^1 = -[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= -\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Hence, (c) is the correct answer.

Remark

Note that $\int_{-1}^1 \left(\frac{d}{dx}\tan^{-1}\frac{1}{x}\right) dx = \left(\tan^{-1}\frac{1}{x}\right)_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

is Incorrect, because $\tan^{-1}\left(\frac{1}{x}\right)$ is not a anti-derivative (primitive) of $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ on the interval $[-1, 1]$.

Example 3 If $I_n = \int_1^e (\log x)^n dx$, then $I_n + nI_{n-1}$ is equal to

(a) $\frac{1}{e}$ (b) e (c) $e-1$ (d) None of these

Sol. We have, $I_n = \int_1^e (\log x)^n dx = \int_1^e \underbrace{(\log x)^n}_{\text{I}} \cdot \underbrace{1}_{\text{II}} dx$

$\therefore \quad I_n = [x\cdot(\log x)^n]_1^e - \int_1^e n\cdot(\log x)^{n-1}\cdot\frac{1}{x}\cdot x dx$

$= (e-0) - n\int_1^e (\log_e x)^{n-1} dx = e - n\cdot I_{n-1}$

$\therefore I_n + n\cdot I_{n-1} = e$

Hence, (b) is the correct answer.

Example 4 All the values of 'a' for which $\int_1^2 \{a^2 + (4-4a)x + 4x^3\} dx \le 12$ are given by

(a) $a = 3$ (b) $a \le 4$

(c) $0 \le a < 3$ (d) None of these

Sol. We have, $\int_1^2 \{a^2 + (4-4a)x + 4x^3\} dx \le 12$

$\Rightarrow \quad [a^2x + (2-2a)x^2 + x^4]_1^2 \le 12$

$\Rightarrow a^2(2-1) + (2-2a)(4-1) + (2^4 - 1^4) \le 12$

$\Rightarrow \quad a^2 + 3(2-2a) + 15 \le 12$

$\Rightarrow \quad a^2 - 6a + 9 \le 0$

$\Rightarrow \quad (a-3)^2 \le 0$

$\therefore \quad a = 3$

Hence, (a) is the correct answer.

Geometrical Interpretation of Definite Integral

If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$ and $x = b$ i.e. $\int_a^b f(x)\, dx$

In general, $\int_a^b f(x)\, dx$ represents the algebraic sum of the areas of the figures bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$ and $x = b$. The areas above X-axis are taken with plus sign and the areas below X-axis are taken with minus sign,

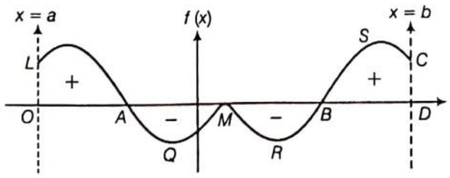


Figure 2.1

i.e. $\int_a^b f(x)\, dx = \text{Area}(OLA) - \text{Area}(AQM) - \text{Area}(MRB) + \text{Area}(BSCD)$

Remark

$\int_a^b f(x)\, dx$, represents algebraic sum of areas means that area of function $y = f(x)$ is asked between a to b.

$\Rightarrow$ Area bounded $= \int_a^b |f(x)|\, dx$ and not been represented by $\int_a^b f(x)\, dx$. e.g. If someone asks for the area of $y = x^3$ between -1 to 1, then $y = x^3$ could be plotted as

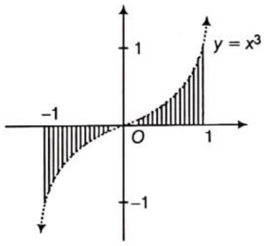


Figure 2.2

$\therefore \quad \text{Area} = \int_{-1}^0 -x^3\, dx + \int_0^1 x^3\, dx = \frac{1}{2}$

or using above definition, area $= \int_{-1}^1 |x^3|\, dx = 2\int_0^1 x^3\, dx$

$= 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$

But, if we integrate x^3 between -1 to 1.

$\Rightarrow \quad \int_{-1}^1 x^3\, dx = 0$ which does not represent the area.

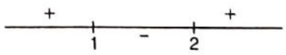
Thus, students are adviced to make difference between area and definite integral.

Example 5 Evaluate $\int_0^3 |(x-1)(x-2)|\, dx$.

Sol. Let $\quad I = \int_0^3 |(x-1)(x-2)|\, dx$

We know,

$$|(x-1)(x-2)| = \begin{cases} (x-1)(x-2), & x < 1 \text{ or } x > 2 \\ -(x-1)(x-2), & 1 < x < 2 \end{cases}$$



Using number line rule,

$I = \int_0^3 |(x-1)(x-2)|\, dx$

$$= \int_0^1 (x-1)(x-2)\,dx - \int_1^2 (x-1)(x-2)\,dx + \int_2^3 (x-1)(x-2)\,dx$$

$$= \int_0^1 (x^2-3x+2)\,dx - \int_1^2 (x^2-3x+2)\,dx + \int_2^3 (x^2-3x+2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2 + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$$

$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - \frac{12}{2} + 4 - \frac{1}{3} + \frac{3}{2} - 2\right) + \left(\frac{27}{3} - \frac{27}{2} + 6 - \frac{8}{3} + \frac{12}{2} - 4\right) = \frac{11}{6}$$

Evaluation of Definite Integrals by Substitution

Sometimes, the indefinite integral may need substitution, say $x = \phi(t)$. Then, in that case don't forget to change the limits of integration a and b corresponding to the new variable t. The substitution $x = \phi(t)$ is not valid, if it is not continuous in the interval $[a, b]$.

| Example 6 Show that

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab};\ a, b > 0.$$

Sol. Let $I = \int_{x=0}^{x=\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$= \int_{x=0}^{x=\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

(divide numerator and denominator by $\cos^2 x$)

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = \int_{t=0}^{t=\infty} \frac{dt}{a^2 + b^2 t^2}$$

We find the new limits of integration $t = \tan x \Rightarrow t = 0$ when $x = 0$ and $t = \infty$ when $x = \pi/2$.

$$\Rightarrow \quad I = \frac{1}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{1}{b^2}\cdot\frac{1}{a/b}\left[\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$

$$= \frac{1}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2ab}$$

| Example 7 Evaluate $\int_{-2}^{2} \frac{dx}{4+x^2}$ directly as well as by the substitution $x = 1/t$. Examine as to why the answer don't valid?

Sol. Let $I = \int_{-2}^{2} \frac{dx}{4+x^2}$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

On the other hand; if $x = 1/t$, then

$$I = \int_{-2}^{2} \frac{dx}{4+x^2} = -\int_{-1/2}^{1/2} \frac{dt}{t^2(4 + 1/t^2)} = -\int_{-1/2}^{1/2} \frac{dt}{4t^2+1}$$

$$= -\left[\frac{1}{2}\tan^{-1}(2t)\right]_{-1/2}^{1/2}$$

$$= -\frac{1}{2}\tan^{-1}(1) - \left(-\frac{1}{2}\tan^{-1}(-1)\right)$$

$$= -\frac{\pi}{8} - \frac{\pi}{8} = -\frac{\pi}{4}$$

$\therefore \ I = -\frac{\pi}{4}$, when $x = \frac{1}{t}$

In above two results, $I = -\pi/4$ is wrong. Since, the integrand $\frac{1}{4+x^2} > 0$ and therefore the definite integral of this function cannot be negative.

Since, $x = 1/t$ is discontinuous at $t = 0$, then substitution is not valid. $(\because I = \pi/4)$

Remark

It is important that the substitution must be continuous in the interval of integration.

| Example 8 Evaluate $\int_0^{1/2} x\cdot\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$.

Sol. Let $I = \int_0^{1/2} x\cdot\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $\sin^{-1}x = \theta$, then $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Also when $x = 0$, then $\theta = 0$ and when $x = \frac{1}{2}$,

then $\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \sin\theta\cdot\frac{\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\, d\theta = \int_0^{\pi/6} \theta\cdot\sin\theta\, d\theta$$

$$= (-\theta\cdot\cos\theta)_0^{\pi/6} + \int_0^{\pi/6}\cos\theta\, d\theta,$$

using integeration by parts.

$$= (-\theta\cos\theta)_0^{\pi/6} + (\sin\theta)_0^{\pi/6}$$

$$= -\frac{\pi}{6}\cos\frac{\pi}{6} + 0 + \sin\frac{\pi}{6} - 0 = \frac{-\sqrt{3}\pi}{12} + \frac{1}{2}$$

| Example 9 For any $n > 1$, evaluate the integral $\int_0^{\infty} \frac{1}{(x+\sqrt{x^2+1})^n} dx$.

Sol. Let $I = \int_0^{\infty} \frac{1}{(x+\sqrt{1+x^2})^n} dx$

Put $x + \sqrt{1+x^2} = t \quad \Rightarrow \sqrt{1+x^2} = t - x$

$\Rightarrow 1 + x^2 = (t-x)^2 \quad \Rightarrow x = \frac{t^2-1}{2t}$ or $x = \frac{1}{2}\left(t - \frac{1}{t}\right)$

$\therefore \quad dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt$

$$\therefore \quad I = \int_{t=1}^{\infty} \frac{1}{t^n} \cdot \frac{1}{2}\left(1+\frac{1}{t^2}\right) dt = \frac{1}{2}\int_1^{\infty} (t^{-n} + t^{-n-2}) dt$$

$$= \frac{1}{2}\left[\frac{t^{1-n}}{1-n} + \frac{t^{-n-1}}{-(n+1)}\right]_1^{\infty} = \frac{1}{2}\left[0 - \left(\frac{1}{1-n} - \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-2n}{1-n^2}\right] = \frac{n}{n^2-1}$$

| Example 10 The value of

$$\int_0^{e-1} \frac{e^{\frac{x^2+2x-1}{2}}}{(x+1)} dx + \int_1^e x \log x \cdot e^{\frac{x^2-2}{2}} dx \text{ is equal to}$$

(a) $(\sqrt{e})^{(e^2+1)}$ (b) $(\sqrt{e})^{e^2-1}$ (c) 0 (d) $(\sqrt{e})^{e^2-2}$

Sol. Let $I = \int_0^{e-1} \frac{e^{\frac{x^2+2x-1}{2}}}{(x+1)} dx + \int_1^e x \log x \cdot e^{\frac{x^2-2}{2}} dx$

Put $x + 1 = t$ in first integral

$$\therefore \quad I = \int_1^e \frac{e^{\frac{t^2-2}{2}}}{t} dt + \int_1^e x \log x \cdot e^{\frac{x^2-2}{2}} dx$$

$$= \int_1^e e^{\frac{t^2-2}{2}} \left\{\frac{1}{t} + t \cdot \log t\right\} dt = \left(\log t \cdot e^{\frac{t^2-2}{2}}\right)_1^e$$

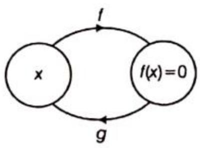
$$= (\sqrt{e})^{e^2-2}$$

Hence, (d) is the correct answer.

| Example 11 Let $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f. Then, the value of $g'(0)$ is

(a) 1 (b) 17 (c) $\sqrt{17}$ (d) None of these

Sol. Here, $f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$



Now, $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

When $y = 0$ i.e. $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$, then $x = 2$

Therefore, $g'(0) = \sqrt{1+16} = \sqrt{17}$

Hence, (c) is the correct answer.

| Example 12 Let $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t$, then $\lim_{n\to\infty} \sum_{n=1}^{n} \frac{a_n}{n}$ is equal to

(a) 1/2 (b) 1
(c) 4/3 (d) 3/2

Sol. We have, $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t \, dt$

Let $1 - \sin t = u \Rightarrow -\cos t \, dt = du$

$$\therefore \quad a_n = 2\int_0^1 u^n(1-u) du = 2\left(\int_0^1 u^n du - \int_0^1 u^{n+1} du\right)$$

$$= 2\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Therefore, $\frac{a_n}{n} = 2\left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right)$

$$\therefore \lim_{n\to\infty} \sum_{n=1}^{n} \frac{a_n}{n} = 2 \left[\sum\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum\left(\frac{1}{n} - \frac{1}{n+2}\right)\right]$$

$$= 2\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 2(1) - \left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \dots\right] = 2 - \frac{3}{2} = \frac{1}{2}$$

Hence, (a) is the correct answer.

| Example 13 The value of $x > 1$ satisfying the equation $\int_1^x t \ln t \, dt = \frac{1}{4}$ is

(a) $\sqrt{e}$ (b) e (c) e^2 (d) $e - 1$

Sol. Let $I = \int_1^x t \ln t \, dt = \left[\ln t \cdot \frac{t^2}{2}\right]_1^x$

$$- \frac{1}{2}\int_1^x \frac{1}{t} \cdot t^2 \, dt = \frac{x^2}{2}\ln x - \frac{1}{2}\left[\frac{t^2}{2}\right]_1^x$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{4}[x^2 - 1] = \frac{1}{4}$$

$$\therefore \frac{x^2 \ln x}{2} - \frac{1}{4}x^2 = 0 \Rightarrow [2\ln x - 1] = 0 \quad \text{(as } x > 1\text{)}$$

$$\Rightarrow \quad \ln x = \frac{1}{2} \Rightarrow x = \sqrt{e}$$

Hence, (a) is the correct answer.

Example 14 If $\lim_{a\to\infty}\frac{1}{a}\int_0^\infty \frac{x^2+ax+1}{1+x^4}\cdot\tan^{-1}\left(\frac{1}{x}\right)dx$ is equal to $\frac{\pi^2}{k}$, where $k\in N$, then k equals to

(a) 4 (b) 8 (c) 16 (d) 32

Sol. Let $I=\int_0^\infty \frac{x^2+ax+1}{1+x^4}\cdot\tan^{-1}\left(\frac{1}{x}\right)dx$

Put $x=1/t$ and adding, we get

[using $\tan^{-1}(1/x)+\cot^{-1}x=\pi/2$]

$$I=\frac{\pi}{4}\int_0^\infty \frac{(x^2+1)+ax}{1+x^4}dx$$

$$=\frac{\pi}{4}\left[\int_0^\infty \frac{(x^2+1)}{1+x^4}dx+a\int_0^\infty \frac{x\,dx}{1+x^4}\right]$$

$$=\frac{\pi}{4}\left[\frac{\pi}{2\sqrt{2}}+\frac{a\pi}{4}\right]=\left[\frac{\pi^2}{8\sqrt{2}}+\frac{\pi^2 a}{16}\right]$$

$$\therefore\quad I=\lim_{a\to\infty}\frac{1}{a}\left[\frac{\pi^2}{8\sqrt{2}}+\frac{\pi^2 a}{16}\right]=\lim_{a\to\infty}\left[\frac{\pi^2}{(8\sqrt{2})a}+\frac{\pi^2}{16}\right]=\frac{\pi^2}{16}$$

$\Rightarrow\quad k=16$

Hence, (c) is the correct answer.

Example 15 If the value of definite integral $\int_1^a x\cdot a^{-[\log_a x]}dx$, where $a>1$ and [x] denotes the greatest integer, is $\frac{e-1}{2}$, then the value of 'a' equals to

(a) $\sqrt{e}$ (b) e (c) $\sqrt{e+1}$ (d) $e-1$

Sol. Let $I=\int_1^a x\cdot a^{-[\log_a x]}dx$

Put $\log_a x=t \Rightarrow a^t=x$

$$\therefore\quad I=\ln a\cdot\int_0^1 (a^t\cdot a^{-[t]}\cdot a^t)\,dt=\ln a\cdot\int_0^1 (a^{t-[t]}\cdot a^t)dt$$

$$=\ln a\cdot\int_0^1 (a^{\{t\}}\cdot a^t)dt=\ln a\cdot\int_0^1 a^{2t}dt$$

$$=\left[\frac{\ln a}{2}\cdot\frac{a^{2t}}{\ln a}\right]_0^1=\frac{1}{2}(a^2-1)\qquad[\text{as }\{t\}=t,\text{ if }t\in(0,1)]$$

$$\therefore\quad \frac{1}{2}(a^2-1)=\frac{e-1}{2}\Rightarrow a=\sqrt{e}$$

Aliter $x\in(1,a)$

$\Rightarrow\quad \log_a x\in(0,1)\Rightarrow[\log_a x]=0$

$$\therefore\quad I=\int_1^a x\,dx=\frac{1}{2}(a^2-1)=\frac{e-1}{2}\Rightarrow a=\sqrt{e}$$

Hence, (a) is the correct answer.

Exercise for Session 1

1. $\int_0^{\pi/4}\cos^2 x\,dx$

2. $\int_0^{\pi/2}\frac{dx}{1+\cos x}$

3. $\int_0^{\pi/2}\sqrt{1+\cos x}\,dx$

4. $\int_0^{\pi/6}\sin 2x\cdot\cos x\,dx$

5. $\int_1^2\frac{dx}{\sqrt{x}-\sqrt{x-1}}$

6. $\int_0^1 \log x\,dx$

7. $\int_0^{\pi/4}\frac{(\sin x+\cos x)}{9+16\sin 2x}dx$

8. $\int_a^b\frac{1}{\sqrt{(x-a)(b-x)}}dx, b>a$

9. $\int_a^b\sqrt{\frac{x-a}{b-x}}dx$

10. $\int_0^{\pi/4}\sqrt{\tan x}\,dx$

11. $\int_0^\pi \cos 2x.\log(\sin x)dx$

12. $\int_0^{\pi/4}e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx$

13. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right)+x^2 f(x)=0$ for all non-zero x, then $\int_{\sin\theta}^{\text{cosec}\,\theta}f(x)dx$ is equal to

14. The value of $\int_0^1\left(\prod_{r=1}^{n}(n+r)\right)\left(\sum_{k=1}^{n}\frac{1}{x+k}\right)dx$ equals to

(a) n (b) $n!$ (c) $(n+1)!$ (d) $n\cdot n!$

15. The true set of values of 'a' for which the inequality $\int_x^0(3^{-2x}-2\cdot3^{-x})\,dx\geq 0$ is true, is

(a) [0, 1] (b) $[-\infty,-1]$ (c) $[0,\infty]$ (d) $[-\infty,-1]\cup[1,\infty]$

Session 2

Properties of Definite Integral

Properties of Definite Integrals

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. The integration is independent of the change of variable.

Proof Let $\phi(x)$ be a primitive of $f(x)$, then

$$\frac{d}{dx}[\phi(x)] = f(x) \Rightarrow \frac{d}{dt}[\phi(t)] = f(t)$$

Therefore, $\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$ ...(i)

and $\int_a^b f(t)\,dt = [\phi(t)]_a^b = \phi(b) - \phi(a)$ ...(ii)

From Eqs. (i) and (ii), we have

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e. if the limits of definite integral are interchanged, then its value changes by minus sign only.

Proof Let $\phi(x)$ be a primitive of $f(x)$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

and $-\int_b^a f(x)\,dx = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$

$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property III. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (King's property)

Proof On RHS put $(a - x) = t$, so that $dx = -dt$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

$\therefore \int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$

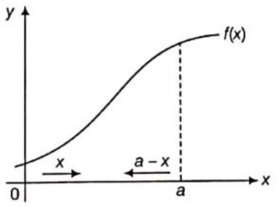
$\therefore \int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$

Remark

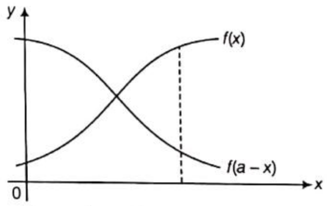
This property is useful to evaluate a definite integral without first finding the corresponding indefinite integrals which may be difficult or sometimes impossible to find.

Geometrically $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

This property says that when integrating from 0 to a, we will get the same result whether we use the function $f(x)$ or $f(a-x)$. The justification for this property will become clear from the figures below :



As x progresses from 0 to a, the variable $a - x$ progresses from a to 0. Thus, whether we use x or $a - x$, the entire interval $[0, a]$ is still covered.



The function $f(a-x)$ can be obtained from the function $f(x)$ by first flipping $f(x)$ along the y-axis and then shifting it right by a units. Notice that in the interval $[0, a]$, $f(x)$ and $f(a-x)$ describe precisely the same area.

There are two ways to look at the justification of this property, as described in the figures on the left and right respectively.

| Example 16 Show that

(i) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(ii) $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$

(iii) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} \sqrt{2}\, f(\cos 2x)\cdot\cos x\,dx$
$= \int_0^{\pi/2} f(\sin 2x)\cdot\cos x\,dx$

(iv) $\int_0^{\pi} x\, f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ **[IIT JEE 1982]**

Sol. (i) We know,

$$\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\cos x)\,dx$$

$$\therefore \int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$$

(ii) $\displaystyle\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\cot x)\,dx$$

$$\therefore \int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$$

(iii) We know, $I = \displaystyle\int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ ...(i)

$$= \int_0^{\pi/2} f\left[\sin 2\left(\frac{\pi}{2}-x\right)\right]\cdot\sin\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\sin(\pi-2x))\cos x\,dx$$

$$I = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

Put $x + \dfrac{\pi}{4} = \left(\dfrac{\pi}{2} - \theta\right)$ $\left(\text{i.e. } x = \dfrac{\pi}{4} - \theta\right)$

$$\Rightarrow \quad 2I = -\sqrt{2}\int_{\pi/4}^{-\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta \text{ (since it is even)}$$

$$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

(iv) Let $I = \displaystyle\int_0^{\pi} x\,f(\sin x)\,dx$ ...(i)

Replacing x by $(\pi - x)$, we get

$$I = \int_0^{\pi} (\pi - x)\,f(\sin(\pi - x))\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\,f(\sin x)\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \pi\,f(\sin x)\,dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\therefore \int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

| Example 17 If f and g are continuous functions satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that $\displaystyle\int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$.

Sol. Let $I = \displaystyle\int_0^a f(x)\,g(x)\,dx = \int_0^a f(a-x)\,g(a-x)\,dx$

$$= \int_0^a f(x)\cdot[2 - g(x)]\,dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$\because$ $f(x) = f(a-x)$ and $g(a-x) + g(x) = 2$ (given)

$$\therefore \int_0^a f(x)\cdot g(x)\,dx = 2\int_0^a f(x)\,dx - \int_0^a f(x)\cdot g(x)\,dx$$

or $2\displaystyle\int_0^a f(x)\cdot g(x)\,dx = 2\int_0^a f(x)\,dx$

$$\Rightarrow \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

| Example 18 Evaluate

(i) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ (ii) $\displaystyle\int_0^{\pi/2} \log(\tan x)\,dx$

(iii) $\displaystyle\int_0^{\pi/4} \log(1+\tan x)\,dx$ (iv) $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx$

Sol. (i) Let $I = \displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ ...(i)

Then, $I = \displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\cos(\pi/2 - x)} + \sqrt{\sin(\pi/2 - x)}}\,dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} 1\,dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2} - 0 \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

(ii) Let $I = \displaystyle\int_0^{\pi/2} \log(\tan x)\,dx$ ...(i)

Then, $I = \displaystyle\int_0^{\pi/2} \log\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log(\cot x)\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log(\tan x)\,dx + \int_0^{\pi/2} \log(\cot x)\,dx$$

$$= \int_0^{\pi/2} (\log\tan x + \log\cot x)\,dx$$

$$= \int_0^{\pi/2} \log(\tan x\cdot\cot x)\,dx = \int_0^{\pi/2} \log(1)\,dx$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

(iii) Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ...(i)

$= \int_0^{\pi/4} \log[1+\tan(\pi/4 - x)]\,dx$

$= \int_0^{\pi/4} \log\left(1+\frac{\tan \pi/4 - \tan x}{1+\tan(\pi/4)\cdot\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log(2)\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$

$\Rightarrow I = (\log 2)(x)_0^{\pi/4} - I$ [using Eq. (i)]

$\Rightarrow 2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$

(iv) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x\cdot\sin x}\,dx$ ...(ii)

Adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx \Rightarrow$

$2I = 0 \Rightarrow I = 0$

| Example 19 The value of $\int_0^a \log(\cot a + \tan x)\,dx$, where $a \in (0, \pi/2)$ is equal to

(a) $a\log(\sin a)$ (b) $-a\log(\sin a)$
(c) $-a\log(\cos a)$ (d) None of these

Sol. Let $I = \int_0^a \log(\cot a + \tan x)\,dx$

$= \int_0^a \log\left(\frac{\cos a}{\sin a} + \frac{\sin x}{\cos x}\right)dx$

$= \int_0^a \log\left(\frac{\cos(a-x)}{\sin a \cos x}\right)dx$

$= \int_0^a \log[\cos(a-x)]\,dx - \int_0^a \log(\sin a)\,dx - \int_0^a \log(\cos x)\,dx$

$= \int_0^a \log\cos(x)\,dx - \int_0^a \log(\sin a)\,dx - \int_0^a \log(\cos x)\,dx$

$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \text{ to first integral}\right]$

$= -\log(\sin a)\int_0^a dx = -a\log(\sin a)$

Hence, (b) is the correct answer.

Property IV. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ (King's property)

Proof Put $x = a+b-t \Rightarrow dx = -dt$

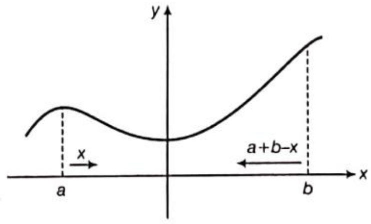
Also, when $x = a$, then $t = b$, and when $x = b$

$\therefore \int_a^b f(x)\,dx = \int_b^a f(a+b-t)(-dt) = -\int_b^a f(a+b-t)\,dt$

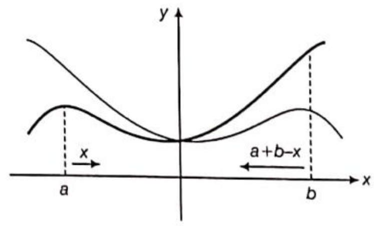
$= \int_a^b f(a+b-t)\,dt = \int_a^b f(a+b-x)\,dx$

$\therefore \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Geometrically $\int_0^a f(x)dx = \int^{a^b} f(a+b-x)\,dx$



As the variable x varies from a to b, the variable $a+b-x$ varies from b to a. Thus, whether we use x or $a+b-x$, the entire interval $[a, b]$ is covered in both the cases and the areas will be the same.



The graph of $f(a+b-x)$ can be obtained from the graph of $f(x)$ by first flipping the graph of $f(x)$ along the y-axis and then shifting it $(a+b)$ units towards the right; the areas described by $f(x)$ and $f(a+b-x)$ in the interval $[a, b]$ are precisely the same.

| Example 20 Evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$.

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\cdot dx}{\sqrt{\cos x}+\sqrt{\sin x}}$ ... (i)

then, $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx$

($\because a+b = \pi/2$)

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ...(ii)

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} 1\,dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

| Example 21 Prove that

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}.$$

Sol. Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ ...(i)

then, $I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}\,dx$

$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$ ...(ii)

Adding Eqs. (i) and (ii), we get

$$2I = \int_a^b \frac{f(a+b-x)+f(x)}{f(a+b-x)+f(x)}\,dx$$

$$\Rightarrow \quad 2I = \int_a^b 1\,dx = (b-a) \Rightarrow I = \frac{b-a}{2}$$

| Example 22 Solve

$$I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}\,dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}}.$$

Sol. We have, $I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}\,dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}}$ ...(i)

$\therefore \quad I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(\cos 2t - z)}\,dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}}$ ..(ii)

$$\left[\text{using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\cos^4 t}^{-\sin^4 t} dz \Rightarrow 2I = (z)_{\cos^4 t}^{-\sin^4 t}$$

$$\therefore \quad I = -\frac{1}{2}(\sin^4 t + \cos^4 t) = -\frac{1}{2}(1 - 2\sin^2 t \cos^2 t)$$

$$= -\frac{1}{2}\left(1 - \frac{1}{2}\sin^2 2t\right) = -\frac{1}{2} + \frac{1}{4}\sin^2 2t$$

Directions (Ex 23-25) *Let the function f satisfies*

$$f(x)\cdot f'(-x) = f(-x)\cdot f'(x) \text{ for all } x \text{ and } f(0) = 3.$$

| Example 23 The value of $f(x)\cdot f(-x)$ for all x is

(a) 4 (b) 9

(c) 12 (d) 16

Sol. Given, $f(x)\cdot f'(-x) = f(-x)\cdot f'(x)$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{f'(-x)}{f(-x)}$$

Integrating both sides, we get

$$\ln f(x) = -f(-x) + C$$

$$\ln[f(x)\cdot f(-x)] = C$$

$$f(x)\cdot f(-x) = C$$

But $f(0) = 3$

$\Rightarrow \quad f^2(0) = C \therefore C = 9$

$\therefore \quad f(x)\cdot f(-x) = 9$

Aliter $f(x)\cdot f'(x) - f(-x)\cdot f'(x) = 0$

$$\Rightarrow \quad \frac{d}{dx}[f(-x)\cdot f(x)] = 0$$

Integrating both sides, we get $f(x)\cdot f(-x) =$ Constant

Hence, (b) is the correct answer.

| Example 24 $\int_{-51}^{51} \frac{dx}{3+f(x)}$ has the value equal to

(a) 17 (b) 34

(c) 102 (d) 0

Sol. Let $I = \int_{-51}^{51} \frac{dx}{3+f(x)} = \int_{-51}^{51} \frac{dx}{3+f(-x)}$

$$\left[\text{using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$2I = \int_{-51}^{51} \frac{6+f(x)+f(-x)}{[3+f(x)][3+f(-x)]}\,dx$$

$$= \int_{-51}^{51} \frac{6+f(x)+f(-x)}{9+3[f(x)+f(-x)]+f(x)\cdot f(-x)}\,dx$$

$$= \int_{-51}^{51} \frac{6+f(x)+f(-x)}{18+3[f(x)+f(-x)]}\,dx$$

$$= \frac{1}{3}\int_{-51}^{51} dx = \frac{2\cdot 51}{3}$$

$$\Rightarrow \quad I = \frac{51}{3} = 17$$

Hence, (a) is the correct answer.

| Example 25 Number of roots of $f(x) = 0$ in $[-2, 2]$ is

(a) 0 (b) 1

(c) 2 (d) 4

Sol. Let $x = \alpha$ be the root of $f(x) = 0$.

$$\therefore \quad f(\alpha) = 0$$

$$f(x)\cdot f(-x) = 9$$

Put $x = \alpha$, then $0 = 9$ (impossible)

Therefore, $f(x)$ has no root but $f(0) = 3$.

$\therefore f(x) > 0, \forall x \in R$ as f is continuous possible function $f(x) = 3e^{-x}$.

Hence, (a) is the correct answer.

Exercise for Session 2

1. The value of $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$ is equal to

(a) $\frac{\pi}{2}\log 2$ (b) $-\frac{\pi}{4}\log 2$ (c) $\frac{\pi}{8}\log 2$ (d) None of these

2. For any integer n, the value of $\int_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\cdot dx$ is equal to

(a) 0 (b) 1 (c) -1 (d) None of these

3. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ is equal to

(a) 1/2 (b) 1/3 (c) 1/4 (d) None of these

4. The value of $\int_0^2 \frac{dx}{(17+8x-4x^2)(e^{6(1-x)}+1)}$ is equal to

(a) $-\frac{1}{8\sqrt{21}}\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right|$ (b) $-\frac{1}{8\sqrt{21}}\log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|$

(c) $-\frac{1}{8\sqrt{21}}\left\{\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right|-\log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|\right\}$ (d) None of these

5. If f is an odd function, then the value of $\int_{-a}^{a}\frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}\,dx$ is equal to

(a) 0 (b) $f(\cos x)+f(\sin x)$ (c) 1 (d) None of these

6. If $[x]$ stands for the greatest integar function, then $\int_4^{10}\frac{[x^2]\,dx}{[x^2-28x+196]+[x^2]}$ is

(a) 1 (b) 2 (c) 3 (d) 4

7. The value of $\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\cdot\sin\alpha}(0<\alpha<\pi)$ is

(a) $\frac{\pi}{\sin\alpha}$ (b) $\frac{\pi\alpha}{\sin\alpha}$ (c) $\frac{\alpha}{\sin\alpha}$ (d) $\frac{\sin\alpha}{\alpha}$

8. If f, g, h be continuous functions on $[0, a]$ such that $f(a-x)=f(x)$, $g(a-x)=-g(x)$ and $3h(x)=5+4h(a-x)$, then the value of $\int_0^a f(x)\cdot g(x)\cdot h(x)dx$ is

(a) 0 (b) 1 (c) a (d) $2a$

9. If $2f(x)+f(-x)=\frac{1}{x}\sin\left(x-\frac{1}{x}\right)$, then the value of $\int_{1/e}^{e} f(x)dx$ is

(a) 0 (b) e (c) $1/e$ (d) $e+1/e$

10. Prove that $\int_0^{\pi}xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi}f(\sin x)dx$.

11. Evaluate $\int_0^{\pi}\frac{x^2\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx}{(2x-\pi)}$.

12. Number of positive continuous function $f(x)$ defined on $[0, 1]$ for which $\int_0^1 f(x)dx=1$, $\int_0^1 xf(x)dx=2$ and $\int_0^1 x^2 f(x)dx=4$.

13. Let $I_1=\int_0^1\frac{e^x}{1+x}dx$ and $I_2=\int_0^1\frac{x^2}{e^{x^3}(2-x^3)}dx$. Then $\frac{I_1}{I_2}$ is equal to

(a) $\frac{3}{e}$ (b) $\frac{3}{e}$ (c) $3e$ (d) $\frac{1}{3e}$

14. If $f(x)=\frac{e^x}{1+e^x}$, $I_1=\int_{f(-a)}^{f(x)}x\cdot g\{x(1-x)\}dx$ and $I_2=\int_{f(-a)}^{f(a)}g\{x(1-x)\}dx$, then the value of $\frac{I_2}{I_1}$ is

(a) 1 (b) -3 (c) -1 (d) 2

Session 3

Applications of Piecewise Function Property

Applications of Piecewise Function Property

Property V (a). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $c \leftrightarrow R$

Proof Let $\phi(x)$ be primitive of $f(x)$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

and $\int_a^c f(x)dx + \int_c^b f(x)dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)]$

$$= \phi(b) - \phi(a) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Generalisation Property V(a) can be generalised into the following form

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)dx$$

where, $\qquad a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$

Property V (b).

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

Proof As we know,

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx$$

Put $x = a - t \Rightarrow dx = -dt$ in the second integral also, when $x = a/2$, then $t = a/2$ and when $x = a$, then $t = 0$.

$$\therefore \quad \int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^0 f(a-t)(-dt)$$

$$= \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-t)\,dt$$

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

Property V (c).

$$\int_a^b f(x)\,dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$$

Proof Let us consider the function $f(x)$ on $[a, b]$ when $f(a+x) = f(b-x)$, then f is even symmetric about the mid-point $x = \frac{a+b}{2}$ on the interval $[a, b]$ when $f(a+x) = -f(b-x)$, then f is odd symmetric about the mid-point $x = \frac{a+b}{2}$ of the interval $[a, b]$.

$$\therefore \text{Using } \int_{-a}^a f(x)\,dx = \begin{cases} 0, & \text{if } f \text{ is an odd function} \\ 2f(x)dx, & \text{if } f \text{ is an even function} \end{cases}$$

$$\Rightarrow \int_a^b f(x)dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$$

| Example 26 Given function, $f(x) = \begin{cases} x^2, & \text{for } 0 \le x < 1 \\ \sqrt{x}, & \text{for } 1 \le x \le 2 \end{cases}$.

Evaluate $\int_0^2 f(x)\,dx$.

Sol. Here, $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$$\therefore \quad \int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 \sqrt{x}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^{3/2}}{3/2}\right]_1^2$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2}{3}x\sqrt{x}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \frac{2}{3}[2\sqrt{2} - 1]$$

$$= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} = \frac{4\sqrt{2}}{3} - \frac{1}{3} = \frac{1}{3}(4\sqrt{2} - 1)$$

| Example 27 Evaluate the integral $I = \int_0^2 |1 - x|\,dx$.

Sol. By definition, $|a - b| = \begin{cases} b - a, & \text{if } a < b \\ a - b, & \text{if } a > b \end{cases}$

Then, $\qquad |1 - x| = \begin{cases} (1 - x), & 0 \le x \le 1 \\ (x - 1), & 1 \le x \le 2 \end{cases}$

$$\therefore \int_0^2 |1-x|\,dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$$

$$= \left\{\left(1-\frac{1}{2}\right) - (0-0)\right\} + \left\{\left(\frac{4}{2}-2\right) - \left(\frac{1}{2}-1\right)\right\}$$

$$= \frac{1}{2} + \left\{0 + \frac{1}{2}\right\} = 1$$

| Example 28 Evaluate

(i) $\int_0^{\pi} |\cos x|\,dx$ (ii) $\int_0^2 |x^2+2x-3|\,dx$

Sol. (i) $\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$

$$= \int_0^{\pi/2} (\cos x)\,dx - \int_{\pi/2}^{\pi} (\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= (1-0) - (0-1) = 2$$

(ii) $\int_0^2 |x^2+2x-3|\,dx$

$$= \int_0^1 |x^2+2x-3|\,dx + \int_1^2 |x^2+2x-3|\,dx \quad \text{...(i)}$$

We have, $x^2+2x-3 = (x+3)(x-1)$

$\therefore \quad x^2+2x-3>0 \quad \text{for} \quad x<-3 \quad \text{or} \quad x>1$

and $x^2+2x-3<0 \quad \text{for} \quad -3<x<1$

So, $|x^2+2x-3|$

$$= \begin{cases} (x^2+2x-3), & \text{for} \quad x<-3 \text{ or } x>1 \\ -(x^2+2x-3), & \text{for} \quad -3<x<1 \end{cases}$$

$\therefore$ Eq. (i) becomes

$$I = \int_0^1 -(x^2+2x-3)\,dx + \int_1^2 (x^2+2x-3)\,dx$$

$$= -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$$= 4$$

| Example 29 Evaluate $\int_{-1}^1 (x-[x])\,dx$, where [.] denotes the greatest integral part of x.

Sol. Let $I = \int_{-1}^1 (x-[x])\,dx = \int_{-1}^1 x\cdot dx - \int_{-1}^1 [x]\,dx$

$$= \left(\frac{x^2}{2}\right)_{-1}^1 - \left(\int_{-1}^0 [x]\,dx + \int_0^1 [x]\,dx\right)$$

$$= \frac{1}{2}(1-1) - \left(\int_{-1}^0 -1\,dx + \int_0^1 0\,dx\right)$$

$$= 0 + (x)_{-1}^{0} - 0 = 1$$

| Example 30 Evaluate $\int_0^2 \{x\}\,dx$, where $\{x\}$ denotes the fractional part of x.

Sol. $\int_0^2 \{x\}\,dx = \int_0^2 (x-[x])\,dx = \int_0^2 x\,dx - \int_0^2 [x]\,dx$

$$= \left(\frac{x^2}{2}\right)_0^2 - \left(\int_0^1 [x]\,dx + \int_1^2 [x]\,dx\right)$$

$$= \frac{1}{2}(4-0) - \left(\int_0^1 0\,dx + \int_1^2 1\,dx\right)$$

$$= 2 - (x)_1^2 = 2 - (1) = 1$$

Remark

In above example, for greatest integer less than or equal to x, it is compulsory to break it at integral limits.

| Example 31 Evaluate $\int_0^9 \{\sqrt{x}\}\,dx$, where $\{x\}$ denotes the fractional part of x.

Sol. $\int_0^9 \{\sqrt{x}\}\,dx = \int_0^9 \left(\sqrt{x} - [\sqrt{x}]\right)dx$

$$= \int_0^9 (x^{1/2})\,dx - \int_0^9 [\sqrt{x}]\,dx$$

$$= \frac{2}{3}(x^{3/2})_0^9 - \int_0^9 [\sqrt{x}]\,dx$$

$$= \frac{2}{3}[27] - \int_0^9 [\sqrt{x}]\,dx$$

As $\int_0^9 [\sqrt{x}]\,dx \Rightarrow 0 \le x \le 9$ and $0 \le \sqrt{x} \le 3$

Thus, it should be divided into three parts

$$0 \le \sqrt{x} \le 1,\ 1 \le \sqrt{x} \le 2,\ 2 \le \sqrt{x} \le 3$$

i.e. $I = 2(9) - \int_0^9 [\sqrt{x}]\,dx$

$$= 18 - \left[\int_0^1 [\sqrt{x}]\,dx + \int_1^4 [\sqrt{x}]\,dx + \int_4^9 [\sqrt{x}]\,dx\right]$$

$$= 18 - \left(\int_0^1 0\,dx + \int_1^4 1\,dx + \int_4^9 2\,dx\right)$$

$$= 18 - \left(0 + (x)_1^4 + (2x)_4^9\right) = 18 - (3+10) = 5$$

| Example 32 If for a real number y, $[y]$ is the greatest integer less than or equal to y, then find the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$.

Sol. We know, $\quad -1 \le \sin x \le 1$ as $x \in [\pi/2, 3\pi/2]$

$\Rightarrow \quad -2 \le 2\sin x \le 2$

$\therefore$ $2\sin x$ must be divided (or broken) at $x = 5\pi/6, \pi, 7\pi/6$.

As $2\sin x = +1, 0, -1$ at these points.

$$\therefore \int_{\pi/2}^{3\pi/2} [2\sin x]\,dx = \int_{\pi/2}^{5\pi/6} [2\sin x]\,dx + \int_{5\pi/6}^{\pi} [2\sin x]\,dx + \int_{\pi}^{7\pi/6} [2\sin x]\,dx + \int_{7\pi/6}^{3\pi/2} [2\sin x]\,dx$$

$$= \int_{\pi/2}^{5\pi/6} 1\,dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6} (-1)\,dx + \int_{7\pi/6}^{3\pi/2} (-2)\,dx$$

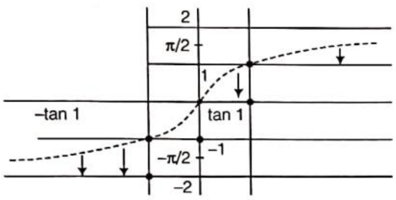
$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + 0 - \left(\frac{7\pi}{6} - \pi\right) - 2\left(\frac{3\pi}{2} - \frac{7\pi}{6}\right) = -\frac{\pi}{2}$$

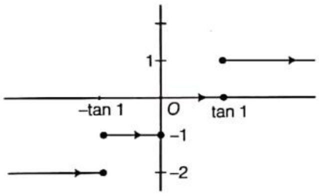
Example 33 The value of $\int_0^{100} [\tan^{-1} x]\, dx$ is equal to (where [.] denotes the greatest integer function)

(a) $\tan 1 - 100$ (b) $\pi/2 - \tan 1$
(c) $100 - \tan 1$ (d) None of these

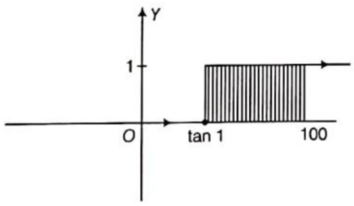
Sol. Let $I = \int_0^{100} [\tan^{-1} x]\, dx$

where $[\tan^{-1} x]$ is shown as





$\therefore \int_0^{100} [\tan^{-1} x]\, dx$ is shown as



$$= 1 \times (100 - \tan 1)$$

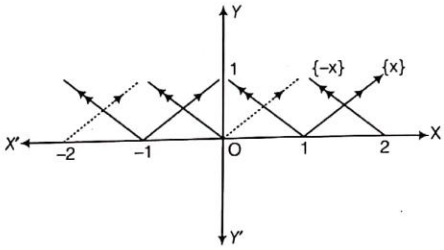
Hence, (c) is the correct answer.

Example 34 The value of $\int_{-2}^{2} \min\{x - [x], -x - [-x]\}\, dx$ is equal to (where [.] denotes the greatest integer function)

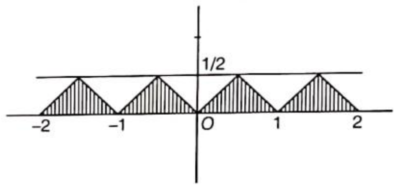
(a) 1/2 (b) 1
(c) 3/2 (d) 2

Sol. Let $f(x) = \min(x - [x], -x - [-x]) = \min(\{x\}, \{-x\})$

Graphically, $\{x\}$ and $\{-x\}$ could be plotted as;



From the above graph, we need $\int_{-2}^{2} \min(\{x\}, \{-x\})$ shown as



$$\therefore \quad \int_{-2}^{2} f(x)\, dx = 4\int_0^1 f(x)\, dx = 4 \times \frac{1}{2} \times 1 \times \frac{1}{2} = 1$$

Hence, (b) is the correct answer.

Example 35 The value of $\int_1^2 (x^{[x^2]} + [x^2]^x)\, dx$ is equal to where [.] denotes the greatest integer function

(a) $\frac{5}{4} + \sqrt{3} + (2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(b) $\frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(c) $\frac{5}{4} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(d) None of the above

Sol. Let $I = \int_1^2 (x^{[x^2]} + [x^2]^x)\, dx$

$$= \int_1^{\sqrt{2}} (x + 1)\, dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^x)\, dx + \int_{\sqrt{3}}^{2} (x^3 + 3^x)\, dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^{\sqrt{2}} + \left[\frac{x^3}{3} + \frac{2^x}{\log 2}\right]_{\sqrt{2}}^{\sqrt{3}} + \left[\frac{x^4}{4} + \frac{3^x}{\log 3}\right]_{\sqrt{3}}^{2}$$

$$= \frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(3^2 - 3^{\sqrt{3}})$$

Hence, (b) is the correct answer.

Example 36 The value of $\int_0^{2\pi} [|\sin x| + |\cos x|]\, dx$ is equal to

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

Sol. Let $f(x) = [|\sin x| + |\cos x|]$

As, $|\sin x| \ge \sin^2 x$ and $|\cos x| \ge \cos^2 x$

$\therefore$ $|\sin x| + |\cos x| \ge 1$

and $|\sin x| + |\cos x| \le \sqrt{1^2 + 1^2}$

$\Rightarrow$ $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

Thus, $[|\sin x| + |\cos x|] = 1$

$\therefore$ $\int_0^{2\pi} [|\sin x| + |\cos x|]\, dx = \int_0^{2\pi} 1\, dx = 2\pi$

Hence, (d) is the correct answer.

| Example 37 The value of the definite integral $\int_0^{\pi/2} \sin|2x - \alpha|\, dx$, where $\alpha \in [0, \pi]$, is

(a) 1 (b) $\cos\alpha$ (c) $\frac{1+\cos\alpha}{2}$ (d) $\frac{1-\cos\alpha}{2}$

Sol. Let $I = \int_{-\alpha}^{(\pi-\alpha)} \sin|t|\, dt$, where $2x - \alpha = t \Rightarrow dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_{-\alpha}^{0} -\sin t\, dt + \frac{1}{2}\int_0^{\pi-\alpha} \sin t\, dt$$

$$= \left[\frac{1}{2}\cos t\right]_{-\alpha}^{0} - \left[\frac{1}{2}\cos t\right]_0^{\pi-\alpha}$$

$$= \frac{1}{2}[1 - \cos\alpha] - \frac{1}{2}[-\cos\alpha - 1]$$

$$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$$

$$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$$

Hence, (a) is the correct answer.

| Example 38 Let f be a continuous functions satisfying

$$f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ e^x - 1 & \text{for } x > 1 \end{cases}$$

and $f(0) = 0$, then $f(x)$ can be defined as

(a) $f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 1 - e^x, & \text{if } x > 0 \end{cases}$ (b) $f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ e^x - 1, & \text{if } x > 0 \end{cases}$

(c) $f(x) = \begin{cases} x, & \text{if } x < 0 \\ e^x, & \text{if } x > 0 \end{cases}$ (d) $f(x) = \begin{cases} x, & \text{if } x \le 0 \\ e^x - 1, & \text{if } x > 0 \end{cases}$

Sol. $f'(\ln x) = \begin{cases} 1, & \text{for } 0 < x \le 1 \\ x, & \text{for } x > 1 \end{cases}$

Put $\ln x = t \Rightarrow x = e^t$

For $x > 1$; $f'(t) = e^t$ for $t > 0$

Integrating $f(t) = e^t + C$; $f(0) = e^0 + C \Rightarrow C = -1$

[given, $f(0) = 0$]

$\therefore$ $f(t) = e^t - 1$, for $t > 0$ (corresponding to $x > 1$)

Therefore, $f(x) = e^x - 1$, for $x > 0$ ...(i)

Again, for $0 < x \le 1$,

$f'(\ln x) = 1$ $(\because x = e^t)$

$f'(t) = 1$, for $t \le 0$

$f(t) = t + C$

$f(0) = 0 + C \Rightarrow C = 0 \Rightarrow f(t) = t$, for $t \le 0$

$\Rightarrow$ $f(x) = x$, for $x \le 0$

Hence, (d) is the correct answer.

| Example 39 The integral $\int_{\pi/4}^{5\pi/4} (|\cos t| \sin t + |\sin t| \cos t)\, dt$ has the value equal to

(a) 0 (b) 1/2

(c) $1/\sqrt{2}$ (d) 1

Sol. Let

$$I = \int_{\pi/4}^{\pi/2} 2\sin t \cos t\, dt + \underbrace{\int_{\pi/2}^{\pi} [(-\sin t \cos t) + (\sin t \cos t)]\, dt}_{\text{zero}} + \int_{\pi}^{5\pi/4} (-2\sin t \cos t)\, dt$$

$$= \int_{\pi/4}^{\pi/2} \sin 2t\, dt - \int_{\pi}^{5\pi/4} \sin 2t\, dt$$

These two integrals cancels

$\Rightarrow$ Zero.

Hence, (a) is the correct answer.

| Example 40 The value of $\int_0^2 f(x)\, dx$, where

$$f(x) = \begin{cases} 0, \text{ when } x = \frac{n}{n+1}, & n = 1, 2, 3 \ldots \\ 1, \text{elsewhere} \end{cases}$$ 3 is equal to

(a) 1 (b) 2

(c) 3 (d) None of these

Sol. Here, $\int_0^2 f(x)\, dx = \int_0^{1/2} 1\, dx + \int_{1/2}^{2/3} 1\, dx + \int_{2/3}^{3/4} 1\, dx + \ldots + \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} 1\, dx + \ldots + \int_1^2 1\, dx$

$$= \left(\frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + \ldots + \left(\frac{n}{n+1} - \frac{n-1}{n}\right) + \ldots + 1$$

$$= \frac{n}{n+1} + \ldots + 1, \text{ as } n \to \infty$$

We take, limit $n \to \infty$

We have, $\int_0^2 f(x)\, dx = 1 + 1 = 2$

Hence, (b) is the correct answer.

Exercise for Session 3

1. The value of $\int_{-1}^{3} \{|x-2|+[x]\}\,dx$, where [.] denotes the greatest integer function, is equal to

(a) 5 (b) 6 (c) 7 (d) None of these

2. The value of $\int_{-1}^{3} (|x|+|x-1|)\,dx$ is equal to

(a) 9 (b) 6 (c) 3 (d) None of these

3. Let $f(x)=x-[x]$, for every real number x (where, $[x]$ is integral part of x). Then, the value of $\int_{-1}^{1} f(x)\,dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) None of these

4. The value of $\int_0^2 [x+[x+[x]]]\,dx$ (where, [.] denotes the greatest integer function) is equal to

(a) 2 (b) 3 (c) −3 (d) None of these

5. The value of $\int_0^{[x]} \frac{2^x}{2^{[x]}}\,dx$ is equal to (where, [.] denotes the greatest integer function)

(a) $\frac{[x]}{\log 2}$ (b) $\frac{[x]}{2\log 2}$ (c) $\frac{[x]}{4\log 2}$ (d) None of these

6. The value of $\int_0^4 \{x\}\,dx$ (where, {.} denotes fractional part of x) is equal to

(a) $\frac{4}{3}$ (b) $\frac{5}{3}$ (c) $\frac{7}{3}$ (d) None of these

7. The value of $\int_1^4 \{x\}^{[x]}\,dx$ (where, [.] and {.} denote the greatest integer and fractional part of x) is equal to

(a) $\frac{11}{12}$ (b) $\frac{13}{12}$

(c) $\frac{7}{12}$ (d) $\frac{19}{12}$

8. The value of $\int_0^x [t+1]^3\,dt$ (where, [.] denotes the greatest integer function of x) is equal to

(a) $\left(\frac{[x]([x]+1)}{2}\right)^2+([x]+1)^3\{x\}$ (b) $\left(\frac{[x]([x]+1)}{2}\right)^3+([x]+1)^3\{x\}$

(c) $\left(\frac{[x]([x]+1)}{2}\right)^3+([x]+1)^2\{x\}$ (d) None of these

9. The value of $\int_0^{10\pi}[\tan^{-1}x]\,dx$ (where, [·] denotes the greatest integer function of x) is equal to

(a) tan 1 (b) 10π

(c) $10\pi-\tan 1$ (d) None of these

10. If $f(x)=\min\{|x-1|,|x|,|x+1|\}$, then the value of $\int_{-1}^{1} f(x)\,dx$ is equal to

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{1}{8}$

11. The value of $\int_0^{\infty}[2e^{-x}]\,dx$ (where, [·] denotes the greatest integer function of x) is equal to

(a) 1 (b) $\log_e 2$

(c) 0 (d) $\frac{1}{e}$

12. The value of $\int_1^{10\pi}([\sec^{-1}x]+[\cot^{-1}x])dx$ (where, [·] denotes the greatest integer function) is equal to

(a) $(\sec 1)-10\pi$
(b) $10\pi-\sec 1$
(c) $\pi-\sec 1$
(d) None of these

13. The value of $\int_{-\pi/2}^{\pi/2}[\cot^{-1}x]dx$ (where, [·] denotes greatest integer function) is equal to

(a) $\pi+\cot 1$
(b) $\pi+\cot 2$
(c) $\pi+\cot 1+\cot 2$
(d) $\cot 1+\cot 2$

14. The value of $\int_0^{\pi/4}(\tan^n(x-[x])+\tan^{n-2}(x-[x]))dx$ (where, [·] denotes greatest integer function) is equal to

(a) $\frac{1}{n}$
(b) $\frac{1}{n-1}$
(c) $\frac{1}{n(n-1)}$
(d) $\frac{1}{n(n+1)}$

15. The value of $\int_0^2[x^2-x+1]dx$ (where, [·] denotes the greatest integer function) is equal to

(a) $\frac{5+\sqrt{5}}{2}$
b) $\frac{1+\sqrt{5}}{2}$
(c) $\frac{1-\sqrt{5}}{2}$
(d) $\frac{5-\sqrt{5}}{2}$

16. Evaluate $\int_0^a[x^n]dx$, (where, [·] denotes the greatest integer function).

17. Prove that $\int_0^x[x]dx=x[x]-\frac{1}{2}[x]([x]+1)$, where [·] denotes the greatest integer function.

18. If $f(n)=\dfrac{\int_0^n[x]dx}{\int_0^n\{x\}dx}$ (where, [·] and {} denotes greatest integer and fractional part of x and $n\in N$). Then, the value of $f(4)$ is ...

19. If $f(n)=\int_0^x[\cos t]dt$, where $x\in\left(2n\pi, 2n\pi+\frac{\pi}{2}\right)$; $n\in N$ and [·] denotes the greatest integer function. Then, the value of $\left|f\left(\frac{1}{\pi}\right)\right|$ is ...

20. If $\int_0^x[x]dx=\int_0^{[x]}xdx$, $x\notin$ integer (where, [·] and {} denotes the greatest integer and fractional parts respectively, then the value of $4\{x\}$ is equal to ...

Session 4

Applications of Even-Odd Property and Half the Integral Limit Property

Applications of Even-Odd Property and Half the Integral Limit Property

Property VI.

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

Proof We know,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \text{ if } a < c < b$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad \text{...(i)}$$

Now, $\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-t)(-dt)$, where $t = -x$

$$= -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt$$

$$= \int_0^a f(-x)\,dx \text{ (using properties I and II)}$$

$$= \begin{cases} \int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ -\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is odd} \end{cases} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

| Example 41 Evaluate $\int_{-1}^{1} (x^3 + 5x + \sin x)dx$.

Sol. Let, $f(x) = x^3 + 5x + \sin x$

$\therefore \quad f(-x) = -x^3 - 5x - \sin x = -f(x)$

So, $f(x)$ is an odd function).

Hence, $\int_{-1}^{1} (x^3 + 5x + \sin x)dx = 0$

$$\left[\text{using } \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, f(x) \text{ is even} \\ 0, f(x) \text{ is odd} \end{cases}\right]$$

| Example 42 Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$.

Sol. Let $f(x) = x^3 \sin^4 x$, then

$$f(-x) = (-x)^3 \sin^4(-x) = -x^3 [\sin(-x)]^4$$
$$= -x^3(-\sin x)^4 = -x^3 \sin^4 x = -f(x)$$

So, $f(x)$ is an odd function.

Hence, $\int_{-\pi/4}^{\pi/4} f(x)\,dx = 0$

i.e. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx = 0$

| Example 43 Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$.

Sol. Let $f(x) = \sin^2 x$, then

$$f(-x) = \sin^2(-x) = [\sin(-x)]^2 = (-\sin x)^2$$
$$= \sin^2 x = f(x)$$

So, $f(x)$ is an even function, hence

$$\int_{-\pi/2}^{\pi/2} (\sin^2 x)\,dx = 2\int_0^{\pi/2} (\sin^2 x)\,dx$$
$$= 2\int_0^{\pi/2} \frac{1-\cos 2x}{2}\,dx$$
$$= \left(x - \frac{\sin 2x}{2}\right)_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{2}$$

| Example 44 The value of $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) -1 (d) 0

Sol. Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$

Now, $f(-x) = \log\left(\frac{2+x}{2-x}\right) = \log\left(\frac{2-x}{2+x}\right)^{-1}$

$$= -\log\left(\frac{2-x}{2+x}\right)$$

$$\therefore \quad f(-x) = -\log\left(\frac{2-x}{2+x}\right) = -f(x)$$

i.e. $f(x)$ is an odd function.

So, $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0$

$$\left[\because \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \end{cases}\right]$$

Hence, (d) is the correct answer.

| Example 45 The value of

$$\int_0^{\pi} \frac{x\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx \text{ is equal to}$$

(a) $\frac{8}{\pi}$ (b) $\frac{\pi}{8}$ (c) $\frac{8}{\pi^2}$ (d) $\frac{\pi^2}{8}$

Sol. Let $I = \int_0^{\pi} \frac{x\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx$ ...(i)

$$I = \int_0^{\pi} \frac{(\pi-x)\cdot\sin 2(\pi-x)\cdot\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}\,dx$$

$$I = \int_0^{\pi} \frac{(\pi-x)\cdot\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{-(2x-\pi)}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(2x-\pi)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx$$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi} 2\sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right) dx$$

(put $\cos x = t$, then $-\sin x\,dx = dt$)

$$= -\int_1^{-1} t\sin\left(\frac{\pi}{2}t\right) dt$$

$$= 2\int_0^1 t\cdot\sin\left(\frac{\pi}{2}t\right) dt \quad \text{(using by parts)}$$

$$= 2\left\{\left(t\cdot\frac{\cos\left(\frac{\pi}{2}t\right)}{-\frac{\pi}{2}}\right)_0^1 - \int_0^1 1\cdot\frac{\cos\left(\frac{\pi}{2}t\right)}{-\frac{\pi}{2}}\,dt\right\}$$

$$= 2\left\{0 + \frac{2}{\pi}\cdot\left(\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}}\right)_0^1\right\} = \frac{8}{\pi^2}$$

Hence, (c) is the correct answer.

| Example 46 If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$, then value of $\int_{-\pi/2}^{\pi/2} (x^2+1)[f(x)+f''(x)]\,dx$ is equal to

(a) 1 (b) −1 (c) 2 (d) None of these

Sol. As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$

$\Rightarrow f(-x) = -f(x)$

$\Rightarrow f(x)$ is odd. $\Rightarrow f'(x)$ is even.

$\Rightarrow f''(x)$ is odd.

Thus, $f(x) + f''(x)$ is odd function, let

$$\phi(x) = (x^2+1)\cdot\{f(x)+f''(x)\}$$

$$\Rightarrow \quad \phi(-x) = -\phi(x)$$

i.e. $\phi(x)$ is odd.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \phi(x)\,dx = 0$$

Hence, (d) is the correct answer.

| Example 47 The value of

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx \text{ is equal to}$$

(a) $4\sqrt{2}$ (b) $4(\sqrt{2}-1)$ (c) $4(\sqrt{2}+1)$ (d) None of these

Sol. Let $I = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx$

$$= 2\int_0^1 \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx$$

$$\left[\text{using } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$$\therefore \quad I = 2\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot\sin^{-1}(2\sin\theta\cos\theta)\cdot\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/4} \sin\theta\cdot 2\theta\,d\theta + 2\int_{\pi/4}^{\pi/2} (\pi - 2\theta)\sin\theta\,d\theta$$

$$\left[\text{using } \sin^{-1}(\sin 2\theta) = \begin{cases} 2\theta, & 0 < \theta < \pi/4 \\ \pi - 2\theta, & \pi/4 < \theta < \pi/2 \end{cases}\right]$$

$$= 4\int_0^{\pi/4}\theta\cdot\sin\theta\,d\theta + 2\pi\int_{\pi/4}^{\pi/2}\sin\theta\,d\theta - 4\int_{\pi/4}^{\pi/2}\theta\cdot\sin\theta\,d\theta$$

$$= 4\{\theta(-\cos\theta)\}_0^{\pi/4} - 4\int_0^{\pi/4} 1\cdot(-\cos\theta)d\theta + 2\pi(-\cos\theta)_{\pi/4}^{\pi/2}$$

$$- 4\{\theta(-\cos\theta)\}_{\pi/4}^{\pi/2} + 4\int_{\pi/4}^{\pi/2}(-\cos\theta)\,d\theta$$

$$= -\frac{\pi}{\sqrt{2}} + 2\sqrt{2} + \sqrt{2}\pi - \frac{\pi}{\sqrt{2}} - 4 + 2\sqrt{2} = 4\sqrt{2} - 4$$

$$= 4(\sqrt{2}-1)$$

Hence, (b) is the correct answer.

| Example 48 Suppose the function $g_n(x)=x^{2n+1}+a_nx+b_n (n\in N)$ satisfies the equation $\int_{-1}^{1}(px+q)g_n(x)dx=0$ for all linear functions $(px+q)$, then

(a) $a_n=b_n=0$ (b) $b_n=0; a_n=-\frac{3}{2n+3}$

(c) $a_n=0; b_n=-\frac{3}{2n+3}$ (d) $a_n=\frac{3}{2n+3}; b_n=-\frac{3}{2n+3}$

Sol. We have, $\int_{-1}^{1}(px+q)(x^{2n+1}+a_nx+b_n)dx=0$

Equating the odd component to be zero and integrating, we get

$$\frac{2p}{2n+3}+\frac{2a_np}{3}+2b_nq=0 \text{ for all } p, q$$

Therefore, $b_n=0$ and $a_n=-\frac{3}{2n+3}$

Hence, (b) is the correct answer.

Property VII (a).

$$\int_0^{2a} f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x)\\ 0, & \text{if } f(2a-x)=-f(x)\end{cases}$$

Proof We know,

$$\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_a^{2a} f(x)\,dx \quad \text{...(i)}$$

Consider the integral $\int_a^{2a} f(x)\,dx$; putting $x=2a-t$, so that $dx=-dt$

Also, when $x=a$, then $t=a$ and when $x=2a$, then $t=0$.

$$\therefore \int_a^{2a} f(x)\,dx=\int_a^0 f(2a-t)(-dt)=-\int_a^0 f(2a-t)\,dt$$

$$=\int_0^a f(2a-t)\,dt=\int_0^a f(2a-x)\,dx$$

$$=\begin{cases}\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x)\\ -\int_0^a f(x)\,dx, & \text{if } f(2a-x)=-f(x)\end{cases} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\int_0^{2a} f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x)\\ 0, & \text{if } f(2a-x)=-f(x)\end{cases}$$

| Example 49 Evaluate $\int_0^{\pi}\frac{x}{1+\cos^2 x}dx$.

Sol. Let $I=\int_0^{\pi}\frac{x}{1+\cos^2 x}dx$

$$=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2(\pi-x)}=\int_0^{\pi}\frac{\pi\,dx}{1+\cos^2 x}-\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$$

$$\therefore \quad I=\pi\int_0^{\pi}\frac{dx}{1+\cos^2 x}-I$$

$$\Rightarrow \quad 2I=\pi\int_0^{\pi}\frac{dx}{1+\cos^2 x}=2\pi\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$

$$\left[\text{using} \int_0^{2a} f(x)\,dx=\begin{cases}0, & f(2a-x)=-f(x)\\ 2\int_0^a f(x)\,dx, & f(2a-x)=f(x)\end{cases}\right]$$

$$\Rightarrow \quad I=\pi\int_0^{\pi/2}\frac{\sec^2 x}{\sec^2 x+1}dx$$

(dividing numerator and denominator by $\cos^2 x$)

$$I=\pi\int_0^{\pi/2}\frac{\sec^2 x}{2+\tan^2 x}dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

Also, when $x=0$, then $t=0$ and when $x=\pi/2$, then $t=\infty$

Hence, $$I=\pi\int_0^{\infty}\frac{dt}{2+t^2}=\frac{\pi}{\sqrt{2}}\left(\tan^{-1}\frac{t}{\sqrt{2}}\right)_0^{\infty}$$

$$=\frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{2\sqrt{2}}$$

| Example 50 Prove that $\int_0^{\pi/2}\log(\sin x)\,dx=\int_0^{\pi/2}\log(\cos x)\,dx=-\frac{\pi}{2}\log 2$.

Sol. Let $I=\int_0^{\pi/2}\log(\sin x)\,dx$ ...(i)

Then, $I=\int_0^{\pi/2}\log\sin(\pi/2-x)\,dx$

$=\int_0^{\pi/2}\log(\cos x)\,dx$ ...(ii)

Adding eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\log\sin x\,dx+\int_0^{\pi/2}\log\cos x\,dx$$

$$=\int_0^{\pi/2}(\log\sin x+\log\cos x)\,dx$$

$$=\int_0^{\pi/2}\log(\sin x\cos x)\,dx=\int_0^{\pi/2}\log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$$

$$=\int_0^{\pi/2}\log(\sin 2x)\,dx-\int_0^{\pi/2}(\log 2)\,dx$$

$$=\int_0^{\pi/2}\log\sin 2x\,dx-(\log 2)(x)_0^{\pi/2}$$

$$\Rightarrow \quad 2I=\int_0^{\pi/2}\log(\sin 2x)\,dx-\frac{\pi}{2}\log 2 \quad \text{...(iii)}$$

Let $I_1=\int_0^{\pi/2}\log(\sin 2x)\,dx$

$$I_1 = \int_0^{\pi} \log \sin t \frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt \text{ (putting } 2x = t)$$

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log(\sin t)\,dt$$

$$\left[\text{using } \int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = f(x), \end{cases}\right.$$

$$= \int_0^{\pi/2} \log(\sin x)\,dx$$

$\therefore$ Eq. (iii) becomes $2I = I - \frac{\pi}{2}\log 2$

Hence, $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2.$

Remark

Students are advised to learn

$\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx = -\frac{\pi}{2}\log 2.$

| Example 51 If $f(x) = -\int_0^x \log(\cos t)\,dt$, then the value of $f(x) - 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) + 2f\left(\frac{\pi}{4}-\frac{x}{2}\right)$ is equal to

(a) $-x\log 2$ (b) $\frac{x}{2}\log 2$

(c) $\frac{x}{3}\log 2$ (d) None of these

Sol. Here, $f\left(\frac{\pi}{4}+\frac{x}{2}\right) = -\int_0^{\pi/4 + x/2} \log(\cos t)\,dt$

$$= -\int_0^{\pi/4} \log(\cos t)\,dt - \int_{\pi/4}^{\pi/4+x/2} \log(\cos t)\,dt \quad \text{...(i)}$$

$$f\left(\frac{\pi}{4}-\frac{x}{2}\right) = -\int_0^{\pi/4 - x/2} \log(\cos t)\,dt$$

$$= -\int_0^{\pi/4} \log(\cos t)\,dt - \int_{\pi/4}^{\pi/4-x/2} \log(\cos t)\,dt \quad \text{...(ii)}$$

$$\therefore 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) - 2f\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$= 2\int_{\pi/4}^{\pi/4 - x/2} \log(\cos t)\,dt - 2\int_{\pi/4}^{\pi/4+x/2} \log(\cos t)\,dt$$

Put $t = \frac{\pi}{4} - z$ in first integral and $t = \frac{\pi}{4} + z$ in second integral, we get

$$= -2\int_0^{x/2} \log\cos\left(\frac{\pi}{4} - z\right)dz - 2\int_0^{x/2} \log\cos\left(z + \frac{\pi}{4}\right)dz$$

$$= -2\int_0^{x/2} \log\left(\frac{1}{2}(\cos^2 z - \sin^2 z)\right)dz$$

$$= 2\int_0^{x/2} (\log 2)\,dx - 2\int_0^{x/2} \log(\cos 2z)\,dz$$

$$= x\log 2 - 2\int_0^{x/2} \log(\cos 2z)\,dz$$

$$\therefore 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) - 2f\left(\frac{\pi}{4}-\frac{x}{2}\right) = x(\log 2) - 2\int_0^{x/2} \log(\cos 2z)\,dz$$

$$\Rightarrow 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) - 2f\left(\frac{\pi}{4}-\frac{x}{2}\right) = x(\log 2) + f(x)$$

or $$f(x) - 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) + 2f\left(\frac{\pi}{4}-\frac{x}{2}\right) = -x\log 2$$

Hence, (a) is the correct answer.

| Example 52 If $\int_0^{\pi} \left(\frac{x}{1+\sin x}\right)^2 dx = A$, then the value for $\int_0^{\pi} \frac{2x^2 \cdot \cos^2 x/2}{(1+\sin x)^2}\,dx$ is equal to

(a) $A + 2\pi - \pi^2$ (b) $A - 2\pi + \pi^2$

(c) $2\pi - A - \pi^2$ (d) None of these

Sol. Let $$B = \int_0^{\pi} \frac{2x^2 \cos^2 x/2}{(1+\sin x)^2}\,dx$$

$$\therefore \quad B - A = \int_0^{\pi} \frac{x^2(2\cos^2 x/2 - 1)}{(1+\sin x)^2}\,dx$$

$$= \int_0^{\pi} \frac{x^2 \cdot \cos x}{(1+\sin x)^2}\,dx$$

Using by parts,

$$B - A = \left\{-\frac{x^2}{1+\sin x}\right\}_0^{\pi} + 2\int_0^{\pi} \frac{x}{(1+\sin x)}\,dx$$

$$B - A = -\pi^2 + 2K \quad \text{...(i)}$$

where, $$K = \int_0^{\pi} \frac{x\,dx}{1+\sin x} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\sin x}$$

$$\therefore \quad K = \pi\int_0^{\pi} \frac{dx}{1+\sin x} - K,$$

$$\left[\text{using } \int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = f(x), \end{cases}\right.$$

$$\Rightarrow 2K = 2\pi\int_0^{\pi/2} \frac{dx}{1+\sin x} = 2\pi\int_0^{\pi/2} \frac{dx}{1+\cos x}$$

$$= 2\pi\int_0^{\pi/2} \frac{1}{2}\sec^2\frac{x}{2}\,dx = 2\pi\left\{\tan\frac{x}{2}\right\}_0^{\pi/2} = 2\pi$$

$\therefore \quad K = \pi$

Thus, $$B - A = -\pi^2 + 2K$$

$$\Rightarrow \quad B = A - \pi^2 + 2\pi$$

Hence, (a) is the correct answer.

Example 53 The value of

$\int_{-a}^{a} (\cos^{-1} x - \sin^{-1} \sqrt{1-x^2})\, dx \ (a>0)$ (where, $\int_0^a \cos^{-1} x\, dx = A$) is

(a) $\pi a - A$ (b) $\pi a + 2A$

(c) $\pi a - 2A$ (d) $\pi a + A$

Sol. Let $I = \int_{-a}^{a} (\cos^{-1} x - \sin^{-1} \sqrt{1-x^2})\, dx$ (as $a>0$)

$$= \int_{-a}^{0} (\cos^{-1} x - \sin^{-1} \sqrt{1-x^2})\, dx + \int_0^a (\cos^{-1} x - \sin^{-1} \sqrt{1-x^2})\, dx$$

$$= \int_{-a}^{0} \cos^{-1} x\, dx + A - 2\int_0^a \sin^{-1} \sqrt{1-x^2}\, dx$$

$$= -\int_a^0 (\pi - \cos^{-1} x)\, dx + A - 2A$$

$$= \pi a - \int_0^a \cos^{-1} x\, dx - A$$

$$= \pi a - A - A = \pi a - 2A$$

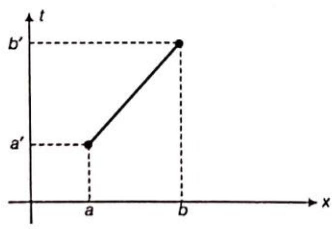
Hence, (c) is the correct answer.

Property VII (b)

$$\int_a^b f(x)\, dx = (b-a)\int_0^1 f[(b-a)x + a]\, dx$$

Sometimes, it is convenient to change the limits of integration into some other limits. For example, suppose we have to add two definite integrals I_1 and I_2; the limits of integration of these integrals are different. if we could somehow change the limits of I_2 into those of I_1 or vice-versa, or infact change the limits of both I_1 and I_2 into a third (common) set of limits, the addition could be accomplished easily.

Suppose that $I = \int_a^b f(x)\, dx$. We need to change the limits (a to b) to (a' to b'). As x varies from a to b, we need a new variable t (in terms of x) which varies from a' to b'.



As x varies from a to b, t varies from a to b, t varies from a' to b'.

Thus, $$\frac{t-a'}{x-a} = \frac{b'-a'}{b-a}$$

As described in the figure above, the new variable t is given by,

$$t = a' + \left(\frac{b'-a'}{b-a}\right)(x-a).$$

Thus, $$dt = \frac{b'-a'}{b-a} dx$$

$\Rightarrow$ $$I = \int_a^b f(x)\, dx$$

$$= \int_{a'}^{b'} f\left(a + \left(\frac{b-a}{b'-a'}\right)(t-a')\right)\left(\frac{b-a}{b'-a'}\right) dt$$

The modified integral has the limits (a' to b'). A particular case of this property is modifying the arbitrary integration limits (a to b) to (0 to 1) i.e. $a'=0$ and $b'=1$. For this case,

$$I = \int_a^b f(x)\, dx = (b-a)\int_0^1 f(a+(b-a)t)\, dt$$

Example 54 Evaluate

$$\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx.$$

Sol. Here, we know $\int e^{x^2} dx$ cannot be evaluated by indefinite integral.

Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx = (-5+4)\int_0^1 e^{[(-5+4)x-4+5]^2} dx$

$\therefore$ $$I_1 = -\int_0^1 e^{(x-1)^2} dx \quad \text{...(i)}$$

Again, let $I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$

$$= \left(\frac{2}{3} - \frac{1}{3}\right)\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{2}\right)x + \frac{1}{3} - \frac{2}{3}\right]^2} dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2} dx = \frac{1}{3}(-I_1) \quad \text{...(ii)}$$

where, $I = I_1 + 3I_2 = I_1 + 3\left(-\frac{I_1}{3}\right) = I_1 - I_1 = 0$

$\therefore$ $$\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx = 0$$

Property VII (c) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\, dt$ is an even function.

Proof We have, $\phi(-x) = \int_a^{-x} f(t)\, dt$

$\Rightarrow$ $$\phi(-x) = \int_a^{-a} f(t)\, dt + \int_{-a}^{-x} f(t)\, dt$$

$\Rightarrow$ $$\phi(-x) = 0 + \int_{-a}^{-x} f(t)\, dt$$

$\left[\because f(t) \text{ is an odd function, then } \int_{-a}^{a} f(t)\, dt = 0\right]$

$\Rightarrow \quad \phi(-x) = -\int_a^x f(-y)\,dy$, where $t = -y$

$\Rightarrow \quad \phi(-x) = \int_a^x f(y)\,dy$

[$\because f$ is an odd function, then $f(-y) = -f(y)$]

$\Rightarrow \quad \phi(-x) = \int_a^x f(t)\,dt$

$\Rightarrow \quad \phi(-x) = \phi(x)$

Hence, $\phi(x) = \int_a^x f(t)\,dt$ is an even function, if $f(t)$ is odd.

| Example 55 If $f(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right) dt$, then discuss whether even or odd?

Sol. Let $\quad \phi(t) = \log\left(\frac{1-t}{1+t}\right)$

$\therefore \quad \phi(-t) = \log\left(\frac{1+t}{1-t}\right) = -\log\left(\frac{1-t}{1+t}\right) = -\phi(t)$

$\Rightarrow \quad \phi(-t) = -\phi(t)$, i.e. $\phi(t)$ is odd function

$\therefore \quad \phi(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right) dt$ is an even function.

Property VII (d) If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

Proof We have, $\phi(-x) = \int_0^{-x} f(t)\,dt = -\int_0^x f(-y)\,dy$, where $t = -y$

$\Rightarrow \quad \phi(-x) = -\int_0^x f(y)\,dy$

[$\because f$ is even function $\Rightarrow f(-y) = f(y)$]

$\Rightarrow \quad \phi(-x) = -\int_0^x f(t)\,dt$

$\Rightarrow \quad \phi(-x) = -\phi(x)$

Hence, $\phi(x)$ is an odd function.

Remark

If $f(t)$ is an even function, then for non-zero 'a', $\int_a^x f(t)\,dt$ is not necessarily an odd function. It will be an odd function, if $\int_0^a f(t)\,dt = 0$, because, if $\phi(x) = \int_a^x f(t)\,dt$, it is an odd function.

$\Rightarrow \quad \phi(-x) = -\phi(x)$

$\Rightarrow \quad \int_a^{-x} f(t)\,dt = -\int_a^x f(t)\,dt$

$\Rightarrow \quad \int_a^0 f(t)\,dt + \int_0^{-x} f(t)\,dt = -\int_a^0 f(t)\,dt - \int_0^x f(t)\,dt$

$\Rightarrow \quad \int_a^0 f(t)\,dt - \int_0^x f(-y)\,dy = -\int_0^a f(t)\,dt - \int_0^x f(t)\,dt$

[where, $y = -t$ in second integral of LHS $\Rightarrow f(-y) = f(y)$]

$\Rightarrow \quad 2\int_a^0 f(t)\,dt = \int_0^x f(y)\,dy - \int_0^x f(t)\,dt$

$\Rightarrow \quad 2\int_a^0 f(t)\,dt = 0$

$\Rightarrow \quad -\int_0^a f(t)\,dt = 0 \Rightarrow \int_0^a f(t)\,dt = 0$

or $\quad \phi(x) = \int_a^x f(t)\,dt$ is an odd function, [when $f(t)$ is even.]

Only, if $\int_0^a f(t)\,dt = 0$

Exercise for Session 4

1. Let $f: R \to R$ and $g: R \to R$ be continous functions, then the value of $\int_{-\pi/2}^{\pi/2} [f(x)+f(-x)][g(x)-g(-x)]\,dx$ is equal to

(a) -1 (b) 0 (c) 1 (d) None of these

2. The value of $\int_{-1}^{1} (x|x|)\,dx$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) None of these

3. The value of $\int_{-1}^{1} \left(\frac{x^2+\sin x}{1+x^2}\right) dx$ is equal to

(a) $2-\pi$ (b) $\pi-2$ (c) $2-\frac{\pi}{2}$ (d) None of these

4. If $f(x)$ is an odd function, then the value of $\int_{-a}^{a} \frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}\,dx$ is equal to

(a) 0 (b) $f(\cos x)+f(\sin x)$ (c) 1 (d) None of these

5. The value of $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)}{1+e^x}\,dx$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{\sqrt{3}}$ (c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi}{3\sqrt{3}}$

6. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$, where $a>0$, is

[IITJEE 2001]

(a) π (b) $a\pi$ (c) 2π (d) $\frac{\pi}{2}$

7. The integral $\int_{-1/2}^{1/2} \left\{[x]+\log_e\left(\frac{1+x}{1-x}\right)\right\} dx$ is equal to (where, [·] denotes greatest integer function)

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2\log\left(\frac{1}{2}\right)$

8. The value of $\int_{-\pi/2}^{\pi/2} \frac{1}{e^{\sin x}+1}\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

9. If [·] denotes greatest integer function, then the value of $\int_{-\pi/2}^{\pi/2} \left(\left[\frac{\pi}{2}\right]+ \quad \right) dx$ is

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $-\frac{\pi}{2}$

10. The equation $\int_{-\pi/4}^{\pi/4} \left\{a|\sin x|+\frac{b\sin x}{1+\cos^2 x}+c\right\} dx = 0$, where a, b, c are constants gives a relation between

(a) a, b and c (b) a and c (c) a and b (d) b and c

11. The value of $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx$, where [·] denotes greatest integer function, is

(a) 1 (b) 0 (c) $4-\sin 4$ (d) None of these

12. Let $f(x)$ be a continuous function such that $\int_{m}^{n+1} f(x)dx = n^3, n \in Z$. Then, the value of $\int_{-3}^{3} f(x)dx$ is

(a) 9 (b) -27 (c) -9 (d) 27

13. Let $f(x) = \dfrac{e^x + 1}{e^x - 1}$ and $\int_0^1 x^3 \cdot \dfrac{e^x + 1}{e^x - 1} dx = \alpha$. Then, $\int_{-1}^1 t^3 f(t)\, dt$ is equal to

(a) 0 (b) α (c) 2α (d) None of these

14. Let $f : R \to R$ be a continuous function given by $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. if $\int_0^2 f(x)\, dx = \alpha$, then $\int_{-2}^2 f(x)\, dx$ is equal to

(a) 2α (b) α (c) 0 (d) None of these

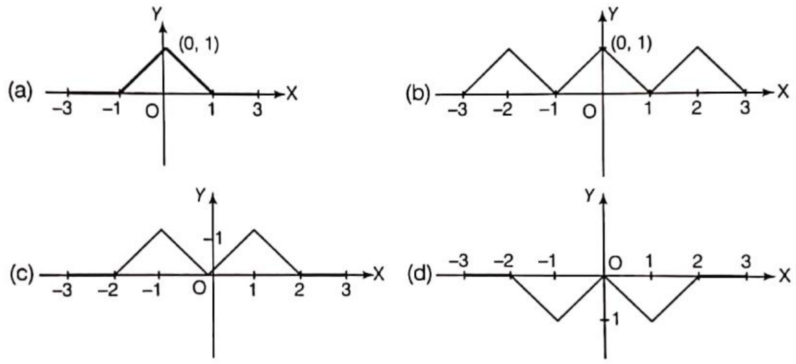
15. The value of $\int_{-2}^2 |[x]|\, dx$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

▪ **Directions** (Q. Nos. 16 to 17)

Let $f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ and $g(x) = f(x + 1) + f(x - 1)$ for all $x \in R$.

16. The graph for $g(x)$ is given by



17. The value of $\int_{-3}^3 g(x)dx$ is

(a) 2 (b) 3 (c) 4 (d) 5

▪ **Direction** (Q. Nos. 18 to 20)

Let $I_n = \int_{-\pi}^{\pi} \dfrac{\sin x}{(1 + \pi^x)\sin x} dx;\ n = 0, 1, 2, \ldots$

18. The value of $I_{n+2} - I_n$ is equal to

(a) $n\pi$ (b) π (c) $-\pi$ (d) 01

19. The value of $\sum_{m=1}^{10} I_{2m+1}$ is equal to

(a) 0 (b) 5π
(c) 10π (d) None of these

20. The value of $\sum_{m=1}^{10} I_{2m}$ is equal to

(a) 0 (b) 5π
(c) 10π (d) None of these

Session 5

Applications of Periodic Functions and Newton- Leibnitz's Formula

Applications of Periodic Functions and Newton-Leibnitz's Formula

If $f(x)$ is a periodic function with period T, then the area under $f(x)$ for n periods would be n times the area under $f(x)$ for one period, i.e.

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

Now, consider the periodic function $f(x) = \sin x$ as an example. The period of $\sin x$ is 2π.

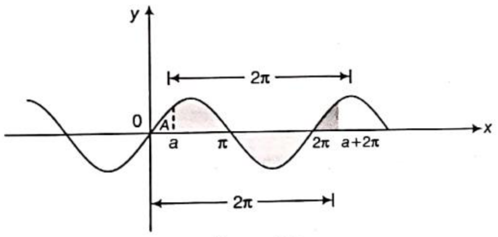


Figure 2.6

Suppose we intend to calculate $\int_a^{a+2\pi} \sin x\,dx$ as depicted above. Notice that the darkly shaded area in the interval $[2\pi, a+2\pi]$ can precisely cover the area marked as

Thus, $$\int_a^{a+2\pi} \sin x\,dx = \int_0^{2\pi} \sin x\,dx$$

This will hold true for every periodic function, i.e.

$$\int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx$$

(where T is the period of $f(x)$)

This also implies that

$$\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)dx = n\int_0^{T} f(x)\,dx$$

and $$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$$

and $$\int_a^{b+nT} f(x)\,dx = \int_a^{b} f(x)dx + n\int_0^{T} f(x)\,dx$$

| Example 56 Evaluate $\int_0^{4\pi} |\cos x|\,dx$.

Sol. Note that $|\cos x|$ is a periodic with period π.

Let $$I = 4\int_0^{\pi} |\cos x|\,dx$$

$$\left[\text{using property, } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\right]$$

$$= 4\left\{\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx\right\}$$

$$= 4\left\{\left(\sin x\right)_0^{\pi/2} - \left(\sin x\right)_{\pi/2}^{\pi}\right\} = 4\{1+1\} = 8$$

| Example 57 Prove that $\int_0^{25} e^{x-[x]}\,dx = 25(e-1)$.

Sol. Since, $x-[x]$ is a periodic function with period one.

Therefore, $e^{x-[x]}$ has period one

$\therefore$ $$I = \int_0^{25\times 1} e^{x-[x]}\,dx = 25\int_0^{1} e^{x-[x]}\,dx$$

$$\left[\text{using property, } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\right]$$

$$= 25\int_0^{1} e^{x-0}\,dx = 25\,(e^x)_0^1 = 25\,(e^1 - e^0)$$

$$= 25\,(e-1)$$

| Example 58 The value of $\int_0^{2n\pi} [\sin x + \cos x]\,dx$ is equal to

(a) $-n\pi$ (b) $n\pi$ (c) $-2n\pi$ (d) None of these

(where [.] denotes the greatest integer function)

Sol. Let $I = \int_0^{2n\pi} [\sin x + \cos x]\,dx$

We know, $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ ...(i)

$$\therefore \quad \sin x + \cos x = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le 3\pi/4 \\ -1, & 3\pi/4 < x \le \pi \\ -2, & \pi < x < 3\pi/2 \\ -1, & 3\pi/2 \le x < 7\pi/4 \\ 0, & 7\pi/4 \le x \le 2\pi \end{cases}$$

$$\therefore \int_0^{2\pi} [\sin x + \cos x]dx = \int_0^{\pi/2} (1)\,dx + \int_{\pi/2}^{3\pi/4} (0)\,dx$$

$$+\int_{3\pi/4}^{\pi}(-1)dx+\int_{\pi}^{3\pi/2}(-2)\,dx+\int_{3\pi/2}^{7\pi/4}(-1)\,dx+\int_{7\pi/4}^{2\pi}(0)\,dx$$

$$=\left(\frac{\pi}{2}\right)+(0)-\left(\pi-\frac{3\pi}{4}\right)-2\left(\frac{3\pi}{2}-\pi\right)-1\left(\frac{7\pi}{4}-\frac{3\pi}{2}\right)+0$$

$$=-\pi$$

Since, $\sin x+\cos x$ has the period 2π.

So, $I=\int_0^{2n\pi}[\sin x+\cos x]\,dx=n\int_0^{2\pi}[\sin x+\cos x]\,dx$

$$=-n\pi$$

Hence, (a) is the correct answer.

| Example 59 The value of $\int_{-5}^{5} f(x)\,dx$; where $f(x)=minimum\ (\{x+1\},\{x-1\}),\forall\ x\in R$ and $\{.\}$ denotes fractional part of x, is equal to

(a) 3 (b) 4 (c) 5 (d) 6

Sol. We know, $\{x+1\}=\{x-1\}=\{x\}$

Thus, $f(x)=\text{minimum}\ (\{x+1\},\{x-1\})=\{x\}$

$$\Rightarrow \int_{-5}^{5} f(x)\,dx=\int_{-5}^{5}\{x\}\,dx=(5-(-5))\int_0^1\{x\}\,dx$$

[as $\{x\}$ is periodic with period '1']

$$=10\int_0^1(x-[x])\,dx=10\int_0^1 x\,dx$$

$$=10\left(\frac{x^2}{2}\right)_0^1=5$$

Hence, (c) is the correct answer.

| Example 60 Show $\int_0^{n\pi+V}|\sin x|\,dx=(2n+1)-\cos V$, where n is a positive integer and $0\le V<\pi$.

[IIT JEE 1994, 2004]

Sol. $\int_0^{n\pi+V}|\sin x|\,dx=\int_0^V|\sin x|\,dx+\int_V^{n\pi+V}|\sin x|\,dx$

$$=\int_0^V\sin x\,dx+n\int_0^{\pi}|\sin x|\,dx$$

$\left[\text{using property, } \int_a^{a+nT}f(x)dx=n\int_0^T f(x)dx \text{ i.e. } \int_V^{n\pi V}|\sin x|dx=n\int_V^{\pi}|\sin x|dx\right]$

$$=\left(-\cos x\right)_0^V+n\int_0^{\pi}\sin x\,dx$$

$$=(-\cos V+1)+n\left(-\cos x\right)_0^{\pi}$$

$$=-(\cos V)+1+n\,(1+1)=(2n+1)-\cos V$$

$\therefore \int_0^{n\pi+V}|\sin x|\,dx=(2n+1)-\cos V$, where n is a positive integer and $0\le V<\pi$.

| Example 61 The value of $\int_{-2\pi}^{5\pi}\cot^{-1}(\tan x)\,dx$ is equal to

(a) $\frac{7\pi}{2}$ (b) $\frac{7\pi^2}{2}$

(c) $\frac{3\pi}{2}$ (d) π^2

Sol. Let $I=\int_{-2\pi}^{5\pi}\cot^{-1}(\tan x)\,dx$

$$\Rightarrow I=7\int_0^{\pi}\cot^{-1}\left(\cot\left(\frac{\pi}{2}-x\right)\right)dx \quad \text{...(i)}$$

$$\left[\because \int_{nT}^{mT}f(x)\,dx=(m-n)\int_0^T f(x)\,dx\right]$$

As we know, $\cot^{-1}(\cot x)=\begin{cases} x &, 0<x<\pi/2\\ \pi+x, & \pi/2<x<\pi\end{cases}$

$$\therefore \quad I=7\left\{\int_0^{\pi/2}\left(\frac{\pi}{2}-x\right)dx+\int_{\pi/2}^{\pi}\left(\pi+\frac{\pi}{2}-x\right)dx\right\}$$

$$=7\left\{\left(\frac{\pi}{2}x-\frac{x^2}{2}\right)_0^{\pi/2}+\left(\frac{3\pi}{2}x-\frac{x^2}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$=7\left\{\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right)+\left(\frac{3\pi^2}{2}-\frac{\pi^2}{2}-\frac{3\pi^2}{4}+\frac{\pi^2}{8}\right)\right\}=\frac{7\pi^2}{2}$$

Hence, (b) is the correct answer.

| Example 62 Let $g(x)$ be a continuous and differentiable function such that $\int_0^2\left\{\int_{\sqrt{2}}^{\sqrt{5/2}}[2x^2-3]\,dx\right\}\cdot g(x)\,dx=0$, then $g(x)=0$ when $x\in(0,2)$ has (where, [.] denotes the greatest integer function)

(a) exactly one real root (b) atleast one real root

(c) no real root (d) None of these

Sol. As, $1<2x^2-3<2,\ \forall\ x\in(\sqrt{2},\sqrt{5/2})$

$$\Rightarrow \int_{\sqrt{2}}^{\sqrt{5/2}}[2x^2-3]\,dx>0,\ \forall\ x\in(0,2)$$

$\Rightarrow g(x)=0$ should have atleast one root in $(0,2)$.

$[\because g'(x)\ne 0]$

Hence, (b) is the correct answer.

| Example 63 The value of x satisfying $\int_0^{2[x+14]}\left\{\frac{x}{2}\right\}dx=\int_0^{\{x\}}[x+14]\,dx$ is equal to

(where, [.] and {.} denotes the greatest integer and fractional part of x)

(a) $[-14,-13)$ (b) $(0,1)$

(c) $(-15,-14]$ (d) None of these

Sol. Given, $\int_0^{2[x+14]} \left\{ \frac{x}{2} \right\} dx = \int_0^{\{x\}} [x+14]\, dx$

$$\Rightarrow \int_0^{28+2[x]} \left\{ \frac{x}{2} \right\} dx = \int_0^{\{x\}} (14+[x])\, dx$$

$$\Rightarrow \int_0^{28} \left\{ \frac{x}{2} \right\} dx + \int_{28}^{28+2[x]} \left\{ \frac{x}{2} \right\} dx = (14+[x])\{x\}$$

$$\Rightarrow 14\int_0^{2} \left\{ \frac{x}{2} \right\} dx + \int_0^{2[x]} \left\{ \frac{x}{2} \right\} dx = (14+[x])\{x\}$$

$$\left[\begin{array}{l} \text{using } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx \text{ and} \\ \int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx; \text{ where } T \text{ is period of } f(x) \end{array}\right]$$

$$\Rightarrow 14+[x] = (14+[x])\{x\}$$

$$\Rightarrow (14+[x])(1-\{x\}) = 0$$

$$\Rightarrow [x] = -14$$

$$\Rightarrow x \in [-14, -13)$$

Hence, (a) is the correct answer.

Property IX. Leibnitz's Rule for the Differentiation under the Integral Sign

(i) If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and are differentiable at a point $x \in (a, b)$ and $f(x, t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right]$$
$$= \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt + \left\{\frac{d\,\psi(x)}{dx}\right\} f^{(x,\,\psi(x))} - \left\{\frac{d\,\phi(x)}{dx}\right\} f^{(x,\,f(x))}$$

(ii) If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = \frac{d}{dx}\{\psi(x)\}\, f(\psi(x)) - \frac{d}{dx}\{\phi(x)\}\, f(\phi(x))$$

| Example 64 Find the derivative of the following with respect to x.

(i) $\int_0^x \cos t\, dt$ (ii) $\int_0^{x^2} \cos t^2 dt$

Sol. (i) Let $f(x) = \int_0^x \cos t\, dt$

$$\therefore \frac{d}{dx}(f(x)) = \cos(x)\left\{\frac{d}{dx}(x)\right\} - \cos 0 \cdot \left\{\frac{d}{dx}(0)\right\} = \cos x$$

$$\left[\text{using Leibnitiz's rule, } \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt = \frac{d}{dx}\{\psi(x)\} f(\psi(x)) - \frac{d}{dx}\{\phi(x)\} + f(\phi(x))\right]$$

$$\Rightarrow \frac{d}{dx}\left\{\int_0^x \cos t\, dt\right\} = \cos x$$

(ii) Let $f(x) = \int_0^{x^2} \cos t^2\, dt$

$$\therefore \frac{d}{dx}(f(x)) = \cos(x^2)^2 \cdot \left\{\frac{d}{dx}(x^2)\right\} - \cos(0)^2 \left\{\frac{d}{dx}(0)\right\}$$
$$= 2x \cdot \cos x^4$$

$$\Rightarrow \frac{d}{dx}\left(\int_0^{x^2} (\cos t^2)\, dt\right) = 2x \cos x^4$$

| Example 65 Evaluate $\frac{d}{dx}\left(\int_{1/x}^{\sqrt{x}} \cos t^2\, dt\right)$.

Sol. Let $f(x) = \int_{1/x}^{\sqrt{x}} \cos t^2\, dt$

$$\therefore \frac{d}{dx}(f(x)) = \cos(\sqrt{x})^2 \cdot \left\{\frac{d}{dx}(\sqrt{x})\right\} - \cos\left(\frac{1}{x}\right)^2 \left\{\frac{d}{dx}\left(\frac{1}{x}\right)\right\}$$
$$= \frac{1}{2\sqrt{x}} \cos x + \frac{1}{x^2} \cdot \cos\left(\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\int_{1/x}^{\sqrt{x}} \cos t^2\, dt\right) = \frac{1}{2\sqrt{x}} \cos x + \frac{1}{x^2} \cos\left(\frac{1}{x^2}\right)$$

| Example 66 If $\frac{d}{dx}\left(\int_0^y e^{-t^2}\, dt + \int_0^{x^2} \sin^2 t\, dt\right) = 0$, find $\frac{dy}{dx}$.

Sol. We know, $\frac{d}{dx}\left(\int_0^y e^{-t^2}\, dt + \int_0^{x^2} \sin^2 t\, dt\right) = 0$

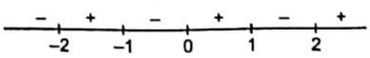
$$\Rightarrow e^{-y^2} \cdot \left\{\frac{d}{dx}(y)\right\} - e^{-0}\left\{\frac{d}{dx}(0)\right\} + \sin^2(x^2)\left\{\frac{d}{dx}(x^2)\right\} - \sin^2 0 \left\{\frac{d}{dx}(0)\right\} = 0$$

$$\Rightarrow e^{-y^2}\frac{dy}{dx} + 2x \sin^2 x^2 = 0 \Rightarrow \frac{dy}{dx} = -2x\, e^{y^2} \sin^2 x^2$$

| Example 67 Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$.

Sol. Let $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$

$$\therefore f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} \cdot 2x - 0$$
$$= \frac{(x-1)(x+1)(x-2)(x+2) \cdot 2x}{2 + e^{x^2}}$$



From the wavy curve, it is clear that $f'(x)$ changes its sign at $x = \pm 2 \pm 1\ 0$ and hence the points of maxima are $-1, 1$ (as sign changes from + ve to − ve) and of the minima are $-2, 0, 2$ (as sign changes from − ve to + ve).

Example 68 Find $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right)$.

Sol. $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right) = \frac{1}{\log x^3}\cdot\frac{d}{dx}(x^3) - \frac{1}{\log x^2}\cdot\frac{d}{dx}(x^2)$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$\therefore \quad \frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right) = \frac{1}{\log x}(x^2 - x)$$

Example 69 If $y = \int_0^x f(t)\sin\{K(x-t)\}\,dt$, then prove that $\frac{d^2y}{dx^2} + K^2y = Kf(x)$.

Sol. We have, $y = \int_0^x f(t)\sin\{K(x-t)\}\,dt$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}\{f(t)\sin K(x-t)\}\,dt + \frac{d}{dx}(x)\cdot\{f(x)\sin K(x-x)\} - \frac{d}{dx}(0)\{f(0)\sin K(x-0)\}$$

$$= K\int_0^x f(t)\cos K(x-t)\,dt + 0 - 0$$

$$\Rightarrow \frac{dy}{dx} = K\int_0^x f(t)\cos K(x-t)\,dt$$

Again, differentiating both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = K\left\{\int_0^x \frac{\partial}{\partial x}\{f(t)\cos K(x-t)\}\,dt + \left\{f(x)\cdot\cos K(x-x)\cdot\frac{d}{dx}(x)\right\} - \{f(0)\cos K(x-0)\cdot\frac{d}{dx}(0)\}\right\}$$

$$= K\left[-K\int_0^x f(t)\sin K(x-t)\,dt + f(x) - 0\right]$$

$$= -K^2\int_0^x f(t)\sin K(x-t)\,dt + Kf(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -K^2y + Kf(x) \Rightarrow \frac{d^2y}{dx^2} + K^2y = Kf(x)$$

Example 70 If $\int_{\pi/3}^x \sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t\,dt = 0$, then evaluate $\frac{dy}{dx}$.

Sol. Differentiating w.r.t. x, we have

$$\frac{d}{dx}\left(\int_{\pi/3}^x \sqrt{3-\sin^2 t}\,dt\right) + \frac{d}{dx}\int_0^y (\cos t)\,dt = 0$$

$$\Rightarrow \sqrt{3-\sin^2 x}\cdot\frac{d}{dx}(x) - \sqrt{3-\sin^2\frac{\pi}{3}}\cdot\frac{d}{dx}\left(\frac{\pi}{3}\right) + \cos y\cdot\frac{dy}{dx} - \cos(0)\frac{d}{dx}(0) = 0$$

$$\Rightarrow \quad \sqrt{3-\sin^2 x} + \cos y\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{3-\sin^2 x}}{\cos y}$$

Example 71 Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}, x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}\,dx = F(K) - F(1)$, then the possible value of K is

(a) 10 (b) 14 (c) 16 (d) 18

Sol. We have, $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$

$$\Rightarrow \quad \int \frac{e^{\sin x}}{x}\,dx = F(x) \quad \text{...(i)}$$

Now, $\int_1^4 \frac{2e^{\sin x^2}}{x}\,dx = \int_1^4 \frac{e^{\sin(x^2)}}{x^2}\,d(x^2) = \int_1^{16}\frac{e^{\sin t}}{t}\,dt \quad (\because x^2 = t)$

$$= [F(t)]_1^{16} = F(16) - F(1)$$

$$\Rightarrow \quad K = 16$$

Hence, (c) is the correct answer.

Example 72 The function $f(x) = \int_0^x \log_{|\sin t|}\left(\sin t + \frac{1}{2}\right)dt$, where $x \in (0, 2\pi)$, then $f(x)$ strictly increases in the interval

(a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(\frac{5\pi}{6}, 2\pi\right)$

(c) $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$ (d) $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$

Sol. Here, $f'(x) = \log_{|\sin x|}\left(\sin x + \frac{1}{2}\right) > 0$

$$\Rightarrow \quad \sin x + \frac{1}{2} < 1 \text{ and } \left(\sin x + \frac{1}{2}\right) > 0$$

$$\Rightarrow \quad 0 < \sin x + \frac{1}{2} < 1$$

$$\Rightarrow \quad -1/2 < \sin x < 1/2$$

$$\Rightarrow \quad x \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right) \text{ as } x \in (0, 2\pi)$$

Hence, (d) is the correct answer.

Example 73 Let $f:(0,\infty)\to R$ and $F(x)=\int_0^x t\,f(t)\,dt$. If $F(x^2)=x^4+x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to

(a) 216 (b) 219 (c) 221 (d) 223

Sol. Here, $F(x^2)=x^4+x^5=\int_0^{x^2} t\,f(t)\,dt$

Differentiating w.r.t. x, we get

$$x^2\,f(x^2)\cdot 2x=4x^3+5x^4$$

$$\Rightarrow \quad f(x^2)=2+\frac{5}{2}x \Rightarrow f(r^2)=2+\frac{5}{2}r$$

$$\therefore \quad \sum_{r=1}^{n} f(r^2)=2n+\frac{5}{2}n(n+1)=\frac{n(5n+13)}{4}$$

$$\Rightarrow \quad \sum_{r=1}^{12} f(r^2)=\frac{12(60+13)}{4}=219$$

Hence, (b) is the correct answer.

Example 74 A function $f(x)$ satisfies $f(x)=\sin x+\int_0^x f'(t)(2\sin t-\sin^2 t)\,dt$, then $f(x)$ is

(a) $\frac{x}{1-\sin x}$ (b) $\frac{\sin x}{1-\sin x}$ (c) $\frac{1-\cos x}{\cos x}$ (d) $\frac{\tan x}{1-\sin x}$

Sol. Differentiating both the sides w.r.t. x, we get

$$f'(x)=\cos x+f'(x)(2\sin x-\sin^2 x)$$

$$\Rightarrow (1+\sin^2 x-2\sin x)f'(x)=\cos x$$

$$\Rightarrow \quad f'(x)=\frac{\cos x}{1+\sin^2 x-2\sin x}=\frac{\cos x}{(1-\sin x)^2}$$

Integrating, we get $f(x)=\int\frac{\cos x}{(1-\sin x)^2}\,dx$

Put $1-\sin x=t$, $\quad f(x)=-\int\frac{dt}{t^2}=\frac{1}{t}=\frac{1}{1-\sin x}+C$

Also, $f(0)=0$, hence $C=-1$

$$f(x)=\frac{1}{1-\sin x}-1=\frac{1-1+\sin x}{1-\sin x}=\frac{\sin x}{1-\sin x}$$

Hence, (b) is the correct answer.

Example 75 If $F(x)=\int_1^x f(t)\,dt$, where $f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du$, then the value of $F''(2)$ equals to

(a) $\frac{7}{4\sqrt{17}}$ (b) $\frac{15}{\sqrt{17}}$ (c) $\sqrt{257}$ (d) $\frac{15\sqrt{17}}{68}$

Sol. Here, $f'(t)=\frac{\sqrt{1+t^8}\cdot 2t}{t^2}=\frac{2\sqrt{1+t^8}}{t}$ ...(i)

Now, $F(x)=\int_1^x f(t)\,dt \Rightarrow F'(x)=f(x)$

$$F''(x)=f'(x) \Rightarrow F''(2)=f'(2)$$

From Eq. (i), $f'(2)=\sqrt{256+1}=\sqrt{257}$

Hence, (c) is the correct answer.

Property X. Let a function $f(x,\alpha)$ be continuous for $a\le x\le b$ and $c\le\alpha\le d$, then for any $\alpha\in[c,d]$, if $I(\alpha)=\int_a^b f(x,\alpha)\,dx$, then $\frac{dI(\alpha)}{d\alpha}=\int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$

Example 76 Evaluate $I(b)=\int_0^1 \frac{x^b-1}{\ln x}\,dx; b\ge 0$.

Sol. We have, $I(b)=\int_0^1\frac{x^b-1}{\ln x}\,dx$

$$\Rightarrow \quad \frac{d}{db}(I(b))=\int_0^1\frac{\partial}{\partial b}\left(\frac{x^b-1}{\ln x}\right)dx+0-0$$

$$=\int_0^1\frac{x^b\ln x}{\ln x}\,dx=\int_0^1 (x^b)\,dx=\left(\frac{x^{b+1}}{b+1}\right)_0^1$$

$$=\frac{1}{b+1}[1-0]=\frac{1}{b+1}$$

$$\therefore \quad \frac{d}{dx}(I(b))=\frac{1}{b+1}$$

Integrating both the sides w.r.t. b, we get

$$I(b)=\log(b+1)+C \quad \text{...(i)}$$

Given, $I(b)=\int_0^1\frac{x^b-1}{\ln x}\,dx$

$\therefore \quad I(0)=0$ (when $b=0$) ...(ii)

Also, from Eq. (i), $I(0)=\log(1)+C$

$\therefore \quad I(0)=C$ ...(iii)

From Eqs. (ii) and (iii), $C=0$

$$\Rightarrow \quad I(b)=\log(b+1)$$

Example 77 Prove that $\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi(a^2+b^2)}{4a^3b^3}$.

Sol. Let $I=\int_0^{\pi/2}\frac{1\,dx}{a^2\sin^2 x+b^2\cos^2 x}$

Then, $I=\int_0^{\pi/2}\frac{\sec^2 x}{a^2\tan^2 x+b^2}\,dx=\int_0^{\infty}\frac{dt}{a^2t^2+b^2}$

(where, $t=\tan x$)

$$\Rightarrow \quad I=\frac{1}{ab}\left[\tan^{-1}\left(\frac{at}{b}\right)\right]_0^{\infty}=\frac{1}{ab}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{2ab}$$

Thus, $\int_0^{\pi/2}\frac{1}{a^2\sin^2 x+b^2\cos^2 x}\,dx=\frac{\pi}{2ab}=f(a,b)$ ...(i)

Differentiating both the sides w.r.t. 'a', we get

$$\int_0^{\pi/2} \frac{-2a\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = -\frac{\pi}{2a^2b}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3b} \quad \text{...(ii)}$$

Similarly, by differentiating Eq. (i) w.r.t. 'b', we get

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab^3} \quad \text{...(iii)}$$

Adding Eqs. (ii) and (iii), we get

$$\int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)\,dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3}$$

$$\Rightarrow \int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b^3}(a^2 + b^2)$$

Example 78 The value of $\int_0^{\pi/2} \frac{\log(1 + x\sin^2\theta)}{\sin^2\theta}\,d\theta;\ x \ge 0$ is equal to

(a) $\pi(\sqrt{1+x} - 1)$
(b) $\pi(\sqrt{1+x} - 2)$
(c) $\sqrt{\pi}(\sqrt{1+x} - 1)$
(d) None of these

Sol. Given, $f(x) = \int_0^{\pi/2} \frac{\log(1 + x\sin^2\theta)}{\sin^2\theta}\,d\theta;\ x \ge 0$

As above integral is function of x. Thus, differentiating both the sides w.r.t. 'x', we get

$$f'(x) = \int_0^{\pi/2} \frac{1}{1 + x\sin^2\theta}\cdot\frac{\sin^2\theta}{\sin^2\theta}\,d\theta + 0 - 0$$

(using Newton-Leibnitz's formula)

$$= \int_0^{\pi/2} \frac{d\theta}{1 + x\sin^2\theta} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta\,d\theta}{\cot^2\theta + (1 + x)}$$

$$= \left(-\frac{1}{\sqrt{1+x}}\cdot\tan^{-1}\frac{\cot\theta}{\sqrt{1+x}}\right)_0^{\pi/2} = \frac{\pi}{2\sqrt{1+x}}$$

$$\therefore \quad f'(x) = \frac{\pi}{2\sqrt{1+x}}$$

$$\Rightarrow \quad f(x) = \pi\sqrt{1+x} + C$$

where, $f(0) = 0 \Rightarrow C = -\pi$

$$\Rightarrow \quad f(x) = \pi\sqrt{1+x} - \pi = \pi(\sqrt{1+x} - 1)$$

Hence, (a) is the correct answer.

Example 79 Let $f(x)$ be a continuous functions for all x, such that $(f(x))^2 = \int_0^x f(t)\cdot\frac{2\sec^2 t}{4 + \tan t}\,dt$ and $f(0) = 0$, then

(a) $f\left(\frac{\pi}{4}\right) = \log\frac{5}{4}$
(b) $f\left(\frac{\pi}{4}\right) = \frac{3}{4}$
(c) $f\left(\frac{\pi}{2}\right) = 2$
(d) None of these

Sol. Here, $(f(x))^2 = \int_0^x f(t)\cdot\frac{2\sec^2 t}{4 + \tan t}\,dt,$

On differentiating both the sides w.r.t. x, we get

$$2f(x)\,.\,f'(x) = f(x)\cdot\frac{2\sec^2 x}{4 + \tan x}$$

$$\Rightarrow \quad f'(x) = \frac{\sec^2 x}{4 + \tan x}$$

Integrating both the sides, we get

$$f(x) = \int \frac{\sec^2 x}{4 + \tan x}\,dx = \log(4 + \tan x) + C$$

Since, $f(0) = 0$

$\Rightarrow \quad 0 = \log(4) + C$

$\Rightarrow \quad C = -\log 4$

$\therefore \quad f(x) = \log(4 + \tan x) - \log(4)$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \log(4 + 1) - \log(4) = \log\frac{5}{4}$$

Hence, (a) is the correct answer.

Exercise for Session 5

1. The value of $\int_{-1}^{10} \text{sgn}(x-[x])dx$ is equal to (where, $[\cdot]$ denotes the greatest integer function)

(a) 9 (b) 10 (c) 11 (d) 12

2. The value of $\int_0^{[x]}(x-[x])dx$ (where, $[\cdot]$ denotes the greatest integer function)

(a) $[x]$ (b) $\frac{[x]}{2}$ (c) $x[x]$ (d) None of these

3. The value of $\int_{-\pi/4}^{n\pi-\pi/4} |\sin x+\cos x|\,dx$ is equal to

(a) $2\sqrt{2}n$ (b) $\sqrt{2}n$ (c) $\frac{1}{2\sqrt{2}}n$ (d) None of these

4. Let $f(x)=x-[x]$ for every real number x (where, $[\cdot]$ denotes greatest integer function), then $\int_{-1}^{1} f(x)dx$ is equal to

(a) 1 (b) 2 (c) 0 (d) $\frac{1}{2}$

5. If $\int_0^x f(t)dt = x+\int_x^1 tf(t)\,dt$, then the value of $f(1)$ is

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) $-\frac{1}{2}$

6. The least value of the function $\phi(x)=\int_{5\pi/4}^{x}(3\sin t+4\cos t)dt$ on the interval $\left[\frac{5\pi}{4},\frac{4\pi}{3}\right]$ is

(a) $\sqrt{3}+\frac{3}{2}$ (b) $-2\sqrt{3}+\frac{3}{2}+\frac{1}{\sqrt{2}}$

(c) $\frac{3}{2}+\frac{1}{\sqrt{2}}$ (d) None of these

7. The points of extremum of $\phi(x)=\int_1^x e^{-t^2/2}(1-t^2)dt$ are

(a) $x=1,-1$ (b) $x=-1, 2$ (c) $x=2, 1$ (d) $x=-2, 1$

8. If $f(x)$ is periodic function, with period T, then

(a) $\int_a^b f(x)dx=\int_a^{b+T} f(x)dx$ (b) $\int_a^b f(x)dx=\int_{a+T}^{b+T} f(x)dx$

(c) $\int_a^b f(x)dx=\int_{a+T}^{b} f(x)dx$ (d) $\int_a^b f(x)dx=\int_{a+T}^{b+2T} f(x)dx$

9. Let $\frac{d}{dx}(F(x))=\frac{e^{\sin x}}{x}$, $x>0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx=F(k)-F(1)$, then one of the possible value of k is

(a) 4 (b) 8 (c) 16 (d) 32

10. Let $f:(0,\infty)\to R$ and $F(x)=\int_0^x f(t)dt$. If $F(x^2)=x^2(1+x)$, then $f(4)$ is equal to

(a) $\frac{5}{4}$ (b) 7 (c) 4 (d) 2

11. Let $T>0$ be a fixed real number. Suppose f is continuous function such that $f(x+T)=f(x)$ for all $x\in R$. If $I=\int_0^T f(x)dx$, then the value of $\int_3^{3+3T} f(2x)\,dx$ is

(a) $\frac{3}{2}I$ (b) $2I$ (c) $3I$ (d) $6I$

12. Let $f(x)=\int_1^x \sqrt{2-t^2}dt$, then the real roots of the equation $x^2-f'(x)=0$ are

(a) ±1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) $\pm\sqrt{2}$

13. Let $f(x)$ be an odd continuous function which is periodic with period 2. If $g(x)=\int_0^x f(t)\,dt$, then

(a) $g(x)$ is an odd function
(b) $g(n)=0$ for all $n\in N$
(c) $g(2n)=0$ for all $n\in N$
(d) $g(x)$ is non-periodic

14. Let $f(x)$ be a function defined by $f(x)=\int_1^x t(t^2-3t+2)dt,\ 1\le x\le 3$. Then, the range of $f(x)$ is

(a) $[0, 2]$
(b) $\left[-\frac{1}{4}, 4\right]$
(c) $\left[-\frac{1}{4}, 2\right]$
(d) None of these

15. The value of $\lim\limits_{x\to 0}\dfrac{2\int_0^{\cos x}\cos^{-1}(t)}{2x-\sin 2x}dt$ is

(a) 0
(b) $\frac{1}{2}$
(c) $-\frac{1}{2}$
(d) $\frac{2}{3}$

16. If $\int_0^x \dfrac{bt\cos 4t-a\sin 4t}{t^2}dt=\dfrac{a\sin 4x}{x}$ for all $x\ne 0$, then a and b are given by

(a) $a=\frac{1}{4}, b=1$
(b) $a=2, b=2$
(c) $a=-1, b=4$
(d) $a=2, b=4$

17. If $f(x)=\int_0^x \{f(t)\}^{-1}dt$ and $\int_0^1\{f(t)\}^{-1}=\sqrt{2}$, then

(a) $f(x)=\sqrt{2x}$
(b) $f(x)=\sqrt{2\log_e x}$
(c) $f(x)=\sqrt{3x-1}$
(d) None of these

18. Let f be a real valued function defined on the interval $(-1,1)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$ and let f^{-1} be the inverse of f. Then, $(f^{-1})'(2)$ is equal to *[IIT JEE 2010]*

(a) 1
(b) 1/3
(c) 1/2
(d) $1/e$

■ **Directions** (Q. Nos. 19 to 20) *Consider the function defined on* $[0, 1]\to R$

$$f(x)=\frac{\sin x-x\cos x}{x^2},\ \text{if } x\ne 0 \text{ and } f(0)=0.$$

[IIT JEE 2012]

19. $\int_0^1 f(x)dx$ is equal to

(a) $1-\sin(1)$
(b) $\sin(1)-1$
(c) $\sin(1)$
(d) $-\sin(1)$

20. $\lim\limits_{t\to 0}\dfrac{1}{t^2}\int_0^t f(x)\,dx$ is equal to

(a) 1/3
(b) 1/6
(c) 1/12
(d) 1/24

Session 6

Integration as Limit of a Sum, Applications of Inequality Gamma Function, Beta Function, Walli's Formula

Integration as Limit of a Sum

Applications of Inequality and Gamma Integrals

An alternative way of describing $\int_a^b f(x)\,dx$ is that the definite integral $\int_a^b f(x)\,dx$ is a limiting case of summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$, i.e. $\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=1}^{n-1} f(a+rh)$, where $h = \frac{b-a}{n}$. The converse is also true, *ie*, if we have an infinite series of the above form, it can be expressed as definite integral.

Method to Express the Infinite Series as Definite Integral

(i) Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$.

(ii) Then, the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)$.

(iii) Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by (dx) and $\lim_{n\to\infty} \sum$ by the sign of $\int$.

(iv) The lower and the upper limit of integration are limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Some particular cases of the above are

(a) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$

or $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

(b) $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{n} = 0$ (as $r = 1$)

and $\beta = \lim_{n\to\infty} \frac{r}{n} = p$ (as $r = pn$)

Some Important Results to Remember

(i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

(iv) In GP, sum of n terms, $S_n = \begin{cases} \frac{a(r^n-1)}{(r-1)}, & |r|>1 \\ an, & r=1 \\ \frac{a(1-r^n)}{(1-r)}, & |r|<1 \end{cases}$

(v) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha + (\overline{n-1})\beta]$
$= \frac{\sin n\beta/2}{\sin\beta/2} \cdot \sin[\alpha + (n-1)\beta/2]$

(vi) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (\overline{n-1})\beta]$
$= \frac{\sin n\beta/2}{\sin\beta/2} \cdot \cos[\alpha + (n-1)\beta/2]$

(vii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots \infty = \frac{\pi^2}{12}$

(viii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots \infty = \frac{\pi^2}{6}$

(ix) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$

(x) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots \infty = \frac{\pi^2}{24}$

(xi) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(xii) $\cos h\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sin h\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

Remark

The method of evaluate the integral, as limit of the sum of an infinite series is known as integration by first principles.

Example 80 Evaluate the following :

(i) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n-1}{n^2}\right)$

(ii) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)$

(iii) $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{2n^2}\right)$

(iv) $\lim_{n\to\infty}\frac{(1^P+2^P+\ldots+n^P)}{n^{P+1}}, P>0$

Sol. Let

(i) $S_n=\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n-1}{n^2}$

$$=\sum_{r=2}^{n}\frac{r-1}{n^2}=\sum_{r=2}^{n}\frac{1}{n}\left(\frac{r-1}{n}\right)$$

$$\Rightarrow\quad S=\lim_{n\to\infty}\sum_{r=2}^{n}\frac{1}{n}\left(\frac{r}{n}\right)\qquad\left(\text{as } n\to\infty\Rightarrow\frac{1}{n}\to 0\right)$$

$$=\int_0^1 x\,dx=\frac{1}{2}$$

(ii) Let $S_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=\sum_{r=1}^{n}\frac{1}{n+r}$

$$=\sum_{r=1}^{n}\frac{1}{n\left(1+\frac{r}{n}\right)}$$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$

$$=\int_0^1\frac{dx}{1+x}=\left[\log(1+x)\right]_0^1=\log 2$$

(iii) Let $S_n=\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{2n^2}$

$$=\sum_{r=1}^{n}\frac{n}{n^2+r^2}=\sum_{r=1}^{n}\frac{1}{n(1+r^2/n^2)}$$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r^2}{n^2}}$

$$=\int_0^1\frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$$

(iv) Let $S_n=\frac{1^P+2^P+\ldots+n^P}{n^{P+1}}=\sum_{r=1}^{n}\frac{r^P}{n^{P+1}}=\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)^P$

$$\therefore\quad S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^P$$

$$=\int_0^1 x^P\,dx=\left[\frac{x^{P+1}}{P+1}\right]_0^1=\frac{1}{P+1}$$

Example 81 Evaluate $S=\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}}$ as $n\to\infty$.

Sol. Let $S_n=\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}}=\sum_{r=0}^{n-1}\frac{1}{n\sqrt{4-(r/n)^2}}$

$$=\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}}$$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}}$

$$=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\left[\sin^{-1}\frac{x}{2}\right]_0^1$$

$$\Rightarrow\quad S=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}\qquad\therefore\ S=\frac{\pi}{6}$$

Example 82 Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{6n}\right).$$

Sol. Let $S_n=\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{6n}\right)$

$$=\sum_{r=1}^{4n}\frac{1}{2n+r}=\sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{2+(r/n)}$$

$$\Rightarrow\quad S=\lim_{n\to\infty}S_n=\int_0^4\frac{dx}{2+x}=[\ln|2+x|]_0^4$$

$$=\log 6-\log 2=\log 3$$

$\therefore\quad S=\ln 3$

Example 83 Evaluate $\int_1^4(ax^2+bx+c)\,dx$ from the first principle.

Sol. Let $x=1+rh$, where $h=\frac{4-1}{n}=\frac{3}{n}$

As $x\to 1, r\to 0$ and $x\to 4,\ r\to n$

$$\therefore\quad\int_1^4(ax^2+bx+c)\,dx=\lim_{n\to\infty}\sum_{r=0}^{n}h\,f(1+rh)$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}h\,[a(1+rh)^2+b(1+rh)+c]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{3}{n}\left[a\left(1+\frac{3r}{n}\right)^2+b\left(1+\frac{3r}{n}\right)+c\right]$$

$$=3\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n}\left[a\left(1+\frac{9r^2}{n^2}+\frac{6r}{n}\right)+b\left(1+\frac{3r}{n}\right)+c\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[a\left(n+\frac{9n(n+1)(2n+1)}{6n^2}+\frac{6n(n+1)}{2n}\right)+b\left(n+\frac{3n(n+1)}{2n}\right)+cn\right]$$

$$=3[a(1+3+3)+b(1+3/2)+c]=21a+\frac{15}{2}b+3c$$

Example 84 The value of

$$\lim_{n\to\infty}\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\ldots\sin\frac{(n-1)\pi}{n}\right)^{1/n}$$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None of these

Sol. Let $A=\left\{\lim_{n\to\infty}\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\ldots\sin\frac{2(n-1)\pi}{2n}\right\}^{1/n}$

$$\therefore \log A=\lim_{n\to\infty}\frac{1}{n}\log\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\ldots\sin\frac{2(n-1)\pi}{2n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2(n-1)}\log\sin\frac{r\pi}{2n}$$

$$=\int_0^2\log\sin\left(\frac{\pi x}{2}\right)dx$$

$$\left[\text{using }\lim_{n\to\infty}\Sigma\frac{1}{n}f\left(\frac{r}{n}\right)=\int_a^b f(x)\,dx\right]$$

$$=\int_0^{\pi}\log(\sin t)\cdot\frac{2}{\pi}dt \qquad \left[\text{putting }\frac{\pi x}{2}=t\right]$$

$$=\frac{2\cdot 2}{\pi}\int_0^{\pi/2}\log(\sin t)\,dt$$

$$\left[\text{using }\int_0^{2a}f(x)\,dx=2\int_0^a f(x)\,dx,\text{ if }f(2a-x)=f(x)\right]$$

$$=\frac{4}{\pi}\left\{-\frac{\pi}{2}\log 2\right\}\left[\text{using}\int_0^{\pi/2}\log(\sin x)dx=-\frac{\pi}{2}\log 2\right]$$

$$=-2\log 2$$

$$\therefore \quad \log A=\log(1/4)\Rightarrow A=1/4$$

Hence, (c) is the correct answer.

Example 85 The interval [0, 4] is divided into n equal sub-intervals by the points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ where $0=x_0<x_1<x_2<x_3<\ldots<x_n=4$.

If $\delta x=x_i-x_{i-1}$ for $i=1,2,3,\ldots,n$, then $\lim_{\delta x\to 0}\sum_{i=1}^{n}x_i\delta x$ is equal to

(a) 4 (b) 8 (c) $\frac{32}{3}$ (d) 16

Sol. $\lim_{\delta x\to 0}\delta x(x_1+x_2+x_3+\ldots+x_n)$

$$=\lim_{n\to\infty}\frac{4}{n}\left[\frac{4}{n}+\frac{8}{n}+\frac{12}{n}+\ldots\ldots+4\cdot\frac{n}{n}\right] \qquad \left(\because \delta x=\frac{4}{n}\right)$$

$$=\lim_{n\to\infty}\frac{16}{n^2}(1+2+3+\ldots+n)=\lim_{n\to\infty}\frac{16}{n^2}\cdot\frac{n(n+1)}{2}=8$$

Hence, (b) is the correct answer.

Example 86 The value of $\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$ is equal to

(a) $1+\sqrt{5}$ (b) $-1+\sqrt{5}$ (c) $-1+\sqrt{2}$ (d) $1+\sqrt{2}$

Sol. We have, $\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{\frac{r}{n}}{\sqrt{1+\frac{r^2}{n^2}}}=\int_0^2\frac{x}{\sqrt{1+x^2}}dx=(\sqrt{x^2+1})_0^2=\sqrt{5}-1$$

Hence, (b) is the correct answer.

Applications of Inequality

Sometimes you are asked to prove inequalities involving definite integrals or to estimate the upper and lower boundary values of definite integral, where the exact value of the definite integral is difficult to find. Under these circumstances, we use the following types

Type I. If $f(x)$ is defined on $[a,b]$, then

$$\left|\int_a^b f(x)\,dx\right|\le\int_a^b|f(x)|\,dx.$$

Equality sign holds, where $f(x)$ is entirely of the same sign on $[a,b]$.

Example 87 Estimate the absolute value of the integral $\int_{10}^{19}\frac{\sin x}{1+x^8}dx$.

Sol. To find, $I=\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|\le\int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ ...(i)

(using type I)

Since, $|\sin x|\le 1$ for $x\ge 10$

The inequality $\left|\frac{\sin x}{1+x^8}\right|\le\frac{1}{|1+x^8|}$ ...(ii)

Also, $10\le x\le 19\Rightarrow 1+x^8>10^8$

$\Rightarrow \quad \frac{1}{1+x^8}<\frac{1}{10^8}$ or $\frac{1}{|1+x^8|}<10^{-8}$ ...(iii)

From Eqs. (ii) and (iii), $\left|\frac{\sin x}{1+x^8}\right|<10^{-8}$ is fulfilled.

$$\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<\int_{10}^{19}10^{-8}\,dx$$

$$\therefore \quad \left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<(19-10)\cdot 10^{-8}<10^{-7}$$

($\because$ the true value of integral $\approx 10^{-8}$)

| Example 88 The minimum odd value of 'a' ($a > 1$) for which $\int_{10}^{19} \frac{\sin x}{1+x^a}\, dx < \frac{1}{9}$, is equal to

(a) 1 (b) 3 (c) 5 (d) 9

Sol. Let $I = \int_{10}^{19} \frac{\sin x \, dx}{1+x^a} < \int_{10}^{19} \frac{dx}{1+x^a}$

$$\left(\because \sin x < 1 \Rightarrow \frac{\sin x}{1+x^a} < \frac{1}{1+x^a}\right)$$

$$\therefore \quad I < \int_{10}^{19} \frac{dx}{1+x^a} < \int_{10}^{19} \frac{dx}{1+10^a}$$

$$(\because 10 < x < 19 \Rightarrow 10^a + 1 < 1 + x^a < 19^a + 1)$$

$$\Rightarrow \quad I < \frac{9}{1+10^a}$$

$\therefore \frac{9}{1+10^a} < \frac{1}{9} \Rightarrow 1 + 10^a > 81$ or $10^a > 80$ i.e. $a = 2, 3, 4, 5, \ldots$

$\therefore$ Minimum odd value of 'a' is 3.

Hence, (b) is the correct answer.

Type II.

$$\left|\int_a^b f(x)\, g(x)\, dx\right| \le \sqrt{\left(\int_a^b f^2(x)\, dx\right)\left(\int_a^b g^2(x)\, dx\right)},$$

where $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$.

| Example 89 Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)}\, dx$ cannot exceed $\sqrt{15/8}$.

Sol. $\int_0^1 \sqrt{(1+x)(1+x^3)}dx \le \sqrt{\left(\int_0^1 (1+x)dx\right)\left(\int_0^1 (1+x^3)\, dx\right)}$

$$\le \sqrt{\left(x + \frac{x^2}{2}\right)_0^1 \left(x + \frac{x^4}{4}\right)_0^1}$$

$$\le \sqrt{\frac{3}{2} \cdot \frac{5}{4}} \le \sqrt{\frac{15}{8}}$$

Type III. If $f(x) \,|\, g(x)$ on $[a, b]$, then $\int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx$. In particular, if $f(x) \ge 0$, then $\int_a^b f(x)\, dx \,|\, 0$.

| Example 90 If $f(x)$ is a continuous function such that $f(x) \,|\, 0, \forall\, x \in [2, 10]$ and $\int_4^8 f(x)\, dx = 0$, then find $f(6)$.

Sol. $f(x)$ is above the X-axis or on the X-axis for all $x \in [2, 10]$.

If $f(x)$ is greater than zero at any sub-interval of [4, 8], then $\int_4^8 f(x)\, dx$ must be greater than zero. But $\int_4^8 f(x)\, dx = 0$, which shows $f(x)$ can't have any value greater than zero in the sub-interval [4, 8].

$\Rightarrow f(x)$ is constant in the sub-interval [4, 8] and has to be zero at all points, $x \in [4, 8]$.

$$\Rightarrow \quad f(x) = 0, \quad \forall\, x \in [4, 8]$$

$$\Rightarrow \quad f(6) = 0$$

Type IV. For a given function $f(x)$ continuous on $[a, b]$, if you are able to find two continuous functions $f_1(x)$ and $f_2(x)$ on $[a, b]$ such that $f_1(x) \le f(x) \le f_2(x), \forall\, x \in [a, b]$, then

$$\int_a^b f_1(x)\, dx \le \int_a^b f(x)\, dx \le \int_a^b f_2(x)\, dx.$$

| Example 91 Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$.

Sol. Since, $4 - x^2 \,|\, 4 - x^2 - x^3 \,|\, 4 - 2x^2 > 1, \forall\, x \in [0, 1]$

$$\sqrt{4 - x^2} \,|\, \sqrt{4 - x^2 - x^3} \,|\, \sqrt{4 - 2x^2} > 1, \forall\, x \in [0, 1]$$

$$\Rightarrow \frac{1}{\sqrt{4 - x^2}} \le \frac{1}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{4 - 2x^2}}, \forall\, x \in [0, 1]$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4 - x^2}} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}}$$

$$\le \int_0^1 \frac{1}{\sqrt{4 - 2x^2}}\, dx, \forall\, x \in [0, 1]$$

$$\Rightarrow \quad \left(\sin^{-1} \frac{x}{2}\right)_0^1 \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{2}} \left(\sin^{-1} \frac{x}{\sqrt{2}}\right)_0^1$$

$$\Rightarrow \quad \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$$

Type V. If m and M be global minimum and global maximum of $f(x)$ respectively in $[a, b]$, then

$$m(b - a) \le \int_a^b f(x)\, dx \le M(b - a).$$

Proof We have, $m \le f(x) \le M$ for all $x \in [a, b]$

$$\Rightarrow \quad \int_a^b m\, dx \le \int_a^b f(x)\, dx \le \int_a^b M\, dx$$

$$\Rightarrow \quad m(b - a) \le \int_a^b f(x)\, dx \le M(b - a)$$

| Example 92 Prove that $4 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$.

Sol. Since, the function $f(x) = \sqrt{3 + x^3}$ increases monotonically on the interval [1, 3].

$\therefore \quad M$ = Maximum value of $\sqrt{3 + x^3} = \sqrt{3 + 3^3} = \sqrt{30}$

and $\quad m$ = Minimum value of $\sqrt{3 + x^3} = \sqrt{3 + 1^3} = 2$

$\therefore \quad m = 2, M = \sqrt{30}, b - a = 2$

Hence, $\quad 2 \cdot 2 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$

$\Rightarrow \quad 4 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$

Example 93 Prove that $1 \le \int_0^1 e^{x^2} dx \le e$.

Sol. For $0 \le x \le 1$, we have $e^0 \le e^{x^2} \le e^1$

$\therefore \quad e^0 (1 - 0) \le \int_0^1 e^{x^2}\, dx \le e^1 (1 - 0)$

$\Rightarrow \quad 1 \le \int_0^1 e^{x^2} dx \le e$

Gamma Function

If n is a positive number, then the improper integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$ is defined as Gamma function and is denoted by Γn.

i.e. $\quad \Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx$, where $x \in Q^+$.

Example 94 Evaluate

(i) $\Gamma 1$ (ii) $\Gamma 2$

Sol. (i) $\Gamma 1 = \int_0^\infty e^{-x} . x^{1-1}\, dx = \lim_{b \to \infty} \int_0^b e^{-x}\, dx$

$= \lim_{b \to \infty} \left[- e^{-x} \right]_0^b = \lim_{b \to \infty} (- e^{-b} + e^0)$

$= 0 + 1 = 1$

(ii) $\Gamma 2 = \int_0^\infty e^{-\infty}\, x^{2-1}\, dx = \lim_{b \to \infty} \int_0^b e^{-x} \cdot x\, dx$

$= \lim_{b \to \infty} \left[- x\, e^{-x} - e^{-x} \right]_0^b$

$= \lim_{b \to \infty} [(- b\, e^{-b} - e^{-b}) - (0 - 1)]$

$= \lim_{b \to \infty} \left[\frac{-b}{e^b} - \frac{1}{e^b} + 1 \right] = \lim_{b \to \infty} \frac{-(b+1)}{e^b} + 1$

$= \left(\lim_{b \to \infty} - \frac{1}{e^b} \right) + 1$ (using L'Hospital's rule)

$= 1$

Properties of Gamma Function

Gamma function has following properties :

(i) $\Gamma 1 = 1$, $\Gamma 0 = \infty$ and $\Gamma (n + 1) = n\, \Gamma n$

e.g. $\Gamma 5 = 4\, \Gamma 4 = 4 \times 3\, \Gamma 3 = 4 \times 3 \times 2\, \Gamma 2$

$= 4 \times 3 \times 2 \times 1 \Gamma 1 = 4 \times 3 \times 2 \times 1$

(ii) If $n \in N$, then $\Gamma (n + 1) = (n)\,!$

(iii) $\Gamma (1/2) = \sqrt{\pi}$

(iv) $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$

(v) $\Gamma n\, \Gamma (1 - n) = \dfrac{\pi}{\sin n\pi}, 0 < n < 1$

(vi) $\Gamma m\, \Gamma \left(m + \frac{1}{2} \right) = \dfrac{\sqrt{\pi}}{2^{2m-1}} \Gamma\, 2m$

(vii) $\Gamma \frac{1}{n} \Gamma \left(\frac{2}{n} \right) \dots \Gamma \left(\frac{n-1}{n} \right) = \dfrac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Example 95 Evaluate $\int_0^\infty e^{-x}\, x^3\, dx$.

Sol. By definition of Gamma function,

$\int_0^\infty e^{-x}\, x^3\, dx = \int_0^\infty e^{-x}\, x^{4-1}\, dx = \Gamma 4$

$\therefore \quad \int_0^\infty e^{-x}\, x^3\, dx = 6$

Example 96 Evaluate $\int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx$.

Sol. Let $I = \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx$

Put $\log \frac{1}{x} = t \Rightarrow dx = - e^{-t}\, dt$

$\therefore \quad I = \int_\infty^0 t^{n-1} (- e^{-t})\, dt = \int_0^\infty e^{-t}\, t^{n-1}\, dt$

$\therefore \quad \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx = \Gamma n$

Beta Function

The beta function is

$$B(m, n) = \int_0^1 x^{m-1}\, (1 - x)^{n-1}\, dx,$$

where $m, n > 0$

Properties of Beta Function

(i) $B(m, n) = B(n, m)$, where $m, n > 0$

(ii) $B(m, n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma (m + n)}$, where $m, n > 0$

(iii) $B(m, n) = \int_0^\infty \dfrac{x^{m-1}}{(1 + x)^{m+n}}\, dx$

| Example 97 Evaluate $\int_0^1 x^6 \sqrt{(1-x^2)}\,dx$.

Sol. Let, $I = \int_0^1 x^6 \sqrt{(1-x^2)}\,dx$

Put let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\therefore \quad I = \frac{1}{2}\int_0^1 t^{5/2}\sqrt{(1-t)}\,dt$

$$I = \frac{1}{2} B\left(\frac{7}{2}, \frac{3}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma 7/2\,\Gamma 3/2}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{4\cdot 3\cdot 2\cdot 1} = \frac{5\pi}{256}$$

Walli's Formula

An easy way to evaluate $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$, where $m, n \in I_+$.

We have, $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \int_0^{\pi/2} \sin^n x \cdot \cos^m x\,dx$

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2).(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\frac{\pi}{2},$$

when both m and $n \in$ even integer.

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)},$$

when either of m or $n \in$ odd integer.

Remark

If n be a positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \; n \text{ is even.}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \; n \text{ is odd.}$$

| Example 98 Evaluate $\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x\,dx$.

Sol. Using Gamma function, we have

$$\int_0^{\pi/4} \sin^4 x\cdot\cos^6 x\,dx = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\Gamma\left(\frac{4+6+2}{2}\right)} = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\Gamma 6}$$

$$= \frac{\left(\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)\right)\left(\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)\right)}{2\times 5!} = \frac{3\pi}{512}$$

| Example 99 The value of $\int_0^\infty e^{-a^2x^2}\,dx$ is equal to

(a) $\frac{\sqrt{\pi}}{2a}$ (b) $\frac{\pi}{2a}$

(c) $\frac{\pi}{\sqrt{2a}}$ (d) None of these

Sol. Let $I = \int_0^\infty e^{-a^2x^2}\,dx$

$$\therefore \quad I = \int_0^\infty e^{-t}\cdot\frac{1}{2a\sqrt{t}}\,dt = \frac{1}{2a}\int_0^\infty e^{-t}\cdot t^{-1/2}\,dt$$

(put $a^2x^2 = t \Rightarrow 2a^2x\,dx = dt$)

$$= \frac{1}{2a}\int_0^\infty t^{1/2-1}\cdot e^{-t}\cdot dt \quad \left(\text{using } \int_0^\infty e^{-x}\,x^{n-1}dx = \Gamma n\right)$$

$$= \frac{1}{2a}\Gamma\frac{1}{2} = \frac{1}{2a}\sqrt{\pi} \quad \left(\text{using } \Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

Hence, (a) is the correct answer.

Exercise for Session 6

1. The value of $f(x) = \int_0^{\pi/2} \frac{\log(1 + x\sin^2\theta)}{\sin^2\theta} d\theta;\ x \geq 0$ is equal to

(a) $\frac{1}{\pi}(\sqrt{1+x} - 1)$ (b) $\sqrt{\pi}(\sqrt{1+x} - 1)$ (c) $\pi(\sqrt{1+x} - 1)$ (d) None of these

2. The value of $\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n+r}\right)$ is equal to

(a) $1 - \log 2$ (b) $\log 4 - 1$ (c) $\log 2$ (d) None of these

3. The value of $\lim_{n \to \infty} \frac{1}{n}\{(n+1)(n+2)(n+3)\ldots(n+n)\}^{1/n}$ is equal to

(a) $4e$ (b) $\frac{e}{4}$ (c) $\frac{4}{e}$ (d) None of these

4. If $m, n \in N$, then the value of $\int_a^b (x-a)^m (b-x)^n\, dx$ is equal to

(a) $\frac{(b-a)^{m+n} \cdot m!\,n!}{(m+n)!}$ (b) $\frac{(b-a)^{m+n+1} \cdot m!\,n!}{(m+n+1)!}$

(c) $\frac{(b-a)^m \cdot m!}{m!}$ (d) None of these

5. The value of $\lim_{n \to \infty} \left(\frac{n!}{n^n}\right)^{\frac{2n^4+1}{5n^5+1}}$ is equal to

(a) e (b) $\frac{2}{e}$ (c) $\left(\frac{1}{e}\right)^{\frac{2}{5}}$ (d) None of these

6. The value of $\lim_{n \to \infty} n\left\{\frac{1}{3n^2+8n+4} + \frac{1}{3n^2+16n+16} + \ldots + n \text{ terms}\right\}$ is equal to

(a) $\frac{1}{2}\log\left(\frac{9}{5}\right)$ (b) $\frac{1}{3}\log\left(\frac{9}{5}\right)$

(c) $\frac{1}{4}\log\left(\frac{9}{5}\right)$ (d) None of these

7. The value of $\lim_{n \to \infty} \frac{1}{n^2}\left\{\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + \ldots + n\sin^3\frac{n\pi}{4n}\right\}$ is equal to

(a) $\frac{\sqrt{2}}{9\pi^2}(52 - 15\pi)$ (b) $\frac{2}{9\pi^2}(52 - 15\pi)$

(c) $\frac{1}{9\pi^2}(15\pi - 15)$ (d) None of these

8. The value of $f(k) = \int_0^{\pi/2} \log(\sin^2\theta + k^2\cos^2\theta)\, d\theta$ is equal to

(a) $\pi\log(1+K) - \pi\log 2$ (b) $\pi\log 2 - \log(1+K)$ (c) $\log(1+K) - \pi\log 2$ (d) None of these

9. If $m, n \in N$, then $I_{m,n} = \int_0^1 x^m (1-x)^n\, dx$ is equal to

(a) $\frac{m!\,n!}{(m+n+2)!}$ (b) $\frac{2m!\,n!}{(m+n+1)!}$ (c) $\frac{m!\,n!}{(m+n+1)!}$ (d) None of these

10. The value of $I(n) = \int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2\theta} d\theta$ is $(\forall\, n \in N)$

(a) $n\pi$ (b) $\frac{n\pi}{2}$ (c) $\frac{n\pi}{4}$ (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

• ***Ex. 1*** *If* $\int_0^\infty f(x)dx = \frac{\pi}{2}$ *and* $f(x)$ *is an even function, then* $\int_0^\infty f\left(x - \frac{1}{x}\right)dx$ *is equal to*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) None of these

Sol. Here, $\int_0^\infty f(x)dx = \frac{\pi}{2}$

Put $x = t - \frac{1}{t}$

$$\Rightarrow \quad dx = \left(1 + \frac{1}{t^2}\right)dt$$

$$\therefore \int_0^\infty f(x)dx = \int_1^\infty f\left(t - \frac{1}{t}\right)\cdot\left(1 + \frac{1}{t^2}\right)dt$$

$$= \int_1^\infty f\left(t - \frac{1}{t}\right)dt + \int_1^\infty f\left(t - \frac{1}{t}\right)\cdot\frac{1}{t^2}dt$$

$$= \int_1^\infty f\left(t - \frac{1}{t}\right)dt + \int_1^0 f\left(\frac{1}{y} - y\right)\cdot(-dy) \quad \left(\text{put } t = \frac{1}{y}\right)$$

$$= \int_1^\infty f\left(t - \frac{1}{t}\right)dt - \int_1^0 f\left(y - \frac{1}{y}\right)dy \quad \text{as } f(x) \text{ is even}$$

$$= \int_1^\infty f\left(t - \frac{1}{t}\right)dt + \int_0^1 f\left(t - \frac{1}{t}\right)dt = \int_0^\infty f\left(t - \frac{1}{t}\right)dt$$

$$\therefore \quad \int_0^\infty f\left(x - \frac{1}{x}\right)dx = \int_0^\infty f(x)dx = \frac{\pi}{2}$$

• ***Ex. 2*** *The value of* $\int_0^1 \left(\prod_{r=1}^{n}(x+r)\right)\left(\sum_{k=1}^{n}\frac{1}{x+k}\right)dx$ *equals to*

(a) n (b) $n!$ (c) $(n+1)!$ (d) $n \cdot n!$

Sol. The given integrand is perfect coefficient of $\prod_{r=1}^{n}(x+r)$

$$\therefore \quad I = \left[\prod_{r=1}^{n}(x+r)\right]_0^1 = (n+1)! - n! = n \cdot n!$$

Aliter $(x+1)(x+2)(x+3)\ldots(x+n) = e^t$

So that, when $x = 0$, then $t = \ln n!$ when $x = 1$, then $t = \ln(n+1)!$

$[\ln(x+1) + \ln(x+2) + \ldots\ldots + \ln(x+n)] = t$

$$\therefore \quad \left(\frac{1}{(x+1)} + \frac{1}{(x+2)} + \ldots + \frac{1}{(x+n)}\right)dx = dt$$

Therefore, $I = \int_{\ln n!}^{\ln(n+1)!} e^t dt = [e^t]_{\ln n!}^{\ln(n+1)!}$

$$= e^{\ln(n+1)!} - e^{\ln n!} = (n+1)! - n! = n \cdot n!$$

Hence, (d) is the correct answer.

• ***Ex. 3*** *The true set of values of 'a' for which the inequality* $\int_a^0 (3^{-2x} - 2\cdot 3^{-x})\,dx \geq 0$ *is true, is*

(a) [0, 1] (b) $(-\infty, -1]$ (c) $[0, \infty)$ (d) $(-\infty, -1] \cup [0, \infty)$

Sol. We have, $\int_a^0 3^{-x}(3^{-x} - 2)\,dx \geq 0$

Put $3^{-x} = t \Rightarrow 3^{-x}\ln 3\,dx = -dt$

$$\Rightarrow \quad \ln 3\int_1^{3^{-a}}(t-2)\,dt \geq 0 = \left[\frac{t^2}{2} - 2t\right]_1^{3^{-a}} \geq 0$$

$$\Rightarrow \quad \left(\frac{3^{-2a}}{2} - 2\cdot 3^{-a}\right) - \left(\frac{1}{2} - 2\right) \geq 0$$

$$\Rightarrow \quad 3^{-2a} - 4\times 3^{-a} + 3 > 0 \Rightarrow (3^{-a} - 3)(3^{-a} - 1) > 0$$

$$\Rightarrow \quad 3^{-a} > 3^1 \Rightarrow a < 1 \text{ or } 3^{-a} < 3^0 \Rightarrow a > 0$$

Thus, $a \in (-\infty, -1] \cup [0, \infty)$

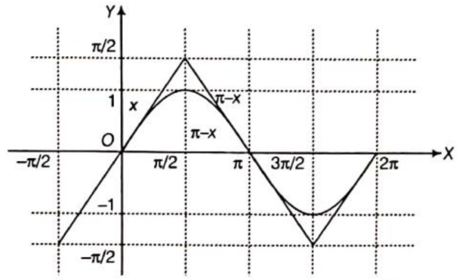
Hence, (d) is the correct answer.

• ***Ex. 4*** *The value of the definite integral* $\int_0^{2n\pi} \max(\sin x, \sin^{-1}(\sin x))dx$ *equals to (where,* $n \in I$*)*

(a) $\frac{n(\pi^2 - 4)}{2}$ (b) $\frac{n(\pi^2 - 4)}{4}$

(c) $\frac{n(\pi^2 - 8)}{4}$ (d) $\frac{n(\pi^2 - 2)}{4}$

Sol. Let $I = \int_0^{2n\pi} \max(\sin x, \sin^{-1}(\sin x))dx$



$$= n\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx + \int_{\pi}^{2\pi}(\sin x)\,dx\right]$$

$$= n\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{1}{2}\left(\pi^2 - \frac{\pi^2}{4}\right) - 2\right]$$

$$= n\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{3\pi^2}{8} - 2\right] = \frac{n(\pi^2 - 8)}{4}$$

Hence, (c) is the correct answer.

Ex. 5 $\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \cdot (1^1\cdot 2^2\cdot 3^3 .....n^n)^{\frac{1}{n^2}}$ *is equal to*

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$ (c) $\frac{1}{\sqrt[4]{e}}$ (d) $\sqrt[4]{e}$

Sol. Let $L = \lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \cdot (1^1\cdot 2^2\cdot 3^3 ...n^n)^{\frac{1}{n^2}}$

$$\therefore \ \ln L = \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n}(k\ln k - k\ln n + k\ln n)$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln\frac{k}{n} + \frac{\ln n}{n^2}\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} + \frac{\ln n}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \int_0^1 x\ln x\,dx + \frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$= \int_0^1 \underset{\text{II}}{x}\ \underset{\text{I}}{\ln x}\,dx = -\frac{1}{4} \Rightarrow L = e^{-\frac{1}{4}}$$

Hence, (c) is the correct answer.

Ex. 6 $\lim_{x\to\frac{\pi}{4}} \dfrac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ *is equal to* *[IIT JEE 2007]*

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$ (c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Sol. $\lim_{x\to\frac{\pi}{4}} \dfrac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \frac{\pi^2}{16}} = \lim_{x\to\frac{\pi}{4}} \dfrac{f(\sec^2 x)\cdot 2\sec^2 x\tan x - 0}{2x}$

(applying L'Hospital rule)

$$= \frac{f(2)\cdot 4}{\pi/2} = \frac{8f(2)}{\pi}$$

Hence, (a) is the correct answer.

Ex. 7 *Let f be a non-negative function defined on the interval* $[0,1]$. *If* $\int_0^x \sqrt{1-(f'(t))^2}\,dx = \int_0^x f(t)\,dt, 0\le x\le 1$ *and* $f(0)=0$, *then* *[IIT JEE 2009]*

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Sol. Given, $\int_0^x \sqrt{1-(f'(t))^2}dt = \int_0^x f(t)dt, 0\le x\le 1$

Applying Leibnitz theorem, we get

$$\sqrt{1-(f'(x))^2} = f(x)$$

$$\Rightarrow \quad 1-(f'(x))^2 = f^2(x)$$

$$\Rightarrow \quad (f'(x))^2 = 1 - f^2(x) \Rightarrow f'(x) = \pm\sqrt{1-f^2(x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{1-y^2}, \text{ where } y = f(x)$$

$$\Rightarrow \quad \frac{dy}{\sqrt{1-y^2}} = \pm dx$$

Integrating both the sides, we get $\sin^{-1}(y) = \pm x + C$

$\therefore \quad f(0) = 0 \Rightarrow C = 0$

$\therefore \quad y = \pm\sin x$

$\Rightarrow \quad y = \sin x = f(x)$, given $f(x)\ge 0$ for $x\in[0,1]$

Its known that, $\sin x < x, \forall x\in R^+$

$$\sin\left(\frac{1}{2}\right) < \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } \sin\left(\frac{1}{3}\right) < \frac{1}{3}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) < \frac{1}{3}$$

Hence, (c) is the correct answer.

Ex. 8 *The value of* $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ *[IIT JEE 2010]*

(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

Sol. $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\log(1+t)}{4+t^4}dt$

Using L'Hospital's rule,

$$\lim_{x\to 0}\frac{\frac{x\log(1+x)}{4+x^4}}{3x^2} = \lim_{x\to 0}\frac{\log(1+x)}{3x}\cdot\frac{1}{4+x^4} = \frac{1}{12}$$

$$\left[\text{using } \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

Hence, (b) is the correct answer.

Ex. 9 *The value(s) of* $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ *is (are)* *[IIT JEE 2010]*

(a) $\frac{22}{7}-\pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15}-\frac{3\pi}{2}$

Sol. Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$$= \int_0^1 \frac{(x^4-1)(1-x)^4 + (1-x)^4}{(1+x^2)}dx$$

$$= \int_0^1 (x^2-1)(1-x)^4 dx + \int_0^1 \frac{(1+x^2-2x)^2}{(1+x^2)}dx$$

$$= \int_0^1\left\{(x^2-1)(1-x)^4 + (1+x^2) - 4x + \frac{4x^2}{(1+x^2)}\right\}dx$$

$$= \int_0^1\left((x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1-x^2}\right)dx$$

$$=\int_0^1\left(x^6-4x^5+5x^4+4-\frac{4}{1+x^2}\right)dx$$

$$=\frac{1}{7}-\frac{4}{6}+\frac{5}{5}-\frac{4}{3}+4-4\left(\frac{\pi}{4}-0\right)=\frac{22}{7}-\pi$$

Hence, (a) is the correct answer.

- **Ex. 10** *If* $\int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx \left(\theta\neq\frac{n\pi}{2}, n\in I\right)$ *is equal to*

(a) $-\cos\theta\int_1^{\tan\theta} f(x\sin\theta)dx$

(b) $-\tan\theta\int_{\cos\theta}^{\sin\theta} f(x)dx$

(c) $\sin\theta\int_1^{\tan\theta} f(x\cos\theta)dx$

(d) $\frac{1}{\tan\theta}\int_{\sin\theta}^{\sin\theta\tan\theta} f(x)dx$

Sol. Let, $I=\int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx$. Put $x\tan\theta=z\sin\theta$

$\therefore \quad I=\int_{\tan\theta}^{1} f(z\sin\theta)\cos\theta\, dz$

$\Rightarrow \quad dx=\cos\theta\, dz$

$$=-\cos\theta\int_1^{\tan\theta} f(z\sin\theta)\,dz=-\cos\theta\int_1^{\tan\theta} f(x\sin\theta)\,dx$$

Hence, (a) is correct option.

JEE Type Solved Examples : More than One Correct Option Type Questions

- **Ex. 11** *Let $f(x)$ be an even function which is mapped from $(-\pi,\pi)$. Then, the value of* $\int_{-\pi}^{\pi}\left(\int_0^x f(t)\,dt+[f(x)]\right)dx$ *can be (where, $[\cdot]$ denotes greatest integer function)*

(a) 0 (b) π

(c) 2π (d) 4π

Sol. As, $f(x)$ is an even function, then $\int_0^x f(x)\,dx$ is an odd function.

Also, $[f(x)]=1,2$ [as $1<f(x)<3$]

$$\therefore \quad g(x)=\int_{-\pi}^{\pi}\left(\int_0^x f(t)\,dt\right)dx+\int_{-\pi}^{\pi}[f(x)]\,dx$$

$$=0+\int_{-\pi}^{\pi}[f(x)]\,dx, \quad [\because \text{since } [f(x)]=1 \text{ or } 2]$$

$$=\int_{-\pi}^{\pi}1\,dx$$

or $\int_{-\pi}^{\pi}2\,dx=2\pi$ or 4π

Hence, options (c) and (d) are correct.

- **Ex. 12** *Let,* $A_1=\int_n^{n+1}(\min\{|x-n|,|x-(n+1)|\})\,dx$,

$$A_2=\int_{n+1}^{n+2}(|x-n|-|x-(n+1)|)\,dx,$$

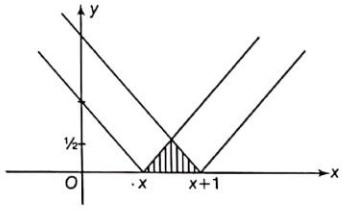
$$A_3=\int_{n+2}^{n+3}(|x-(n+4)|-|x-(n+3)|)\,dx$$

and $g(x)=A_1+A_2+A_3$, then

(a) $A_1+A_2+A_3=9$ (b) $A_1+A_2+A_3=\frac{9}{4}$

(c) $\sum_{n=1}^{100} g(x)=\frac{900}{4}$ (d) $\sum_{n=1}^{100} g(x)=300$

Sol. Here, $\min\{|x-n|,|x-(n+1)|\}$ can be shown as



$$\therefore \quad A_1=\int_n^{n+1}(\min\{|x-n|,|x-(n+1)|\})\,dx$$

$$=\frac{1}{2}\times 1\times\frac{1}{2}=\frac{1}{4}$$

Now, $A_2=\int_{n+1}^{n+2}(|x-n|-|x-(n+1)|)\,dx$

$$=\int_1^2(|t|-|t-1|)dt \quad [\text{put } x=n+t \Rightarrow dx=dt]$$

$$=\int_1^2(t-(t-1))dt=\int_1^2 1\,dt=(t)_1^2=1$$

and $A_3=\int_{n+2}^{n+3}(|x-(n+4)|-|x-(n+3)|)dx,$

$$=\int_2^3(|t-4|-|t-3|)dt \quad [\text{put } x=n+t\Rightarrow dx=dt]$$

$$=\int_2^3((4-t)-(3-t))dt=\int_2^3 1\,dt=1$$

Also, $g(x)=A_1+A_2+A_3=\frac{1}{4}+1+1=\frac{9}{4}$

$$\therefore \sum_{n=1}^{100} g(x)=g(1)+g(2)+g(3)+\ldots+g(100)$$

$$=\frac{9}{4}+\frac{9}{4}+\ldots+\frac{9}{4}=\frac{900}{4}$$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Q. Nos. 13 to 15) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows.*
  (a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
  (b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
  (c) Statement I is true, Statement II is false.
  (d) Statement I is false, Statement II is true.

• ***Ex. 13*** **Statement I** *If* $f(x)=\int_0^1 (x\,f(t)+1)\,dt$, *then* $\int_0^3 f(x)\,dx=12$

Statement II $f(x)=3x+1$

Sol. Let $\int_0^1 f(t)\,dt=k$, so $f(x)=xk+1$

Now, $\int_0^1 (kt+1)\,dt=k \Rightarrow \frac{k}{2}+1=k$, so $k=2$

$\therefore \quad f(x)=2x+1$

Also $\int_0^3 f(x)\,dx=12$

Hence, (c) is the correct answer.

• ***Ex. 14*** **Statement I** *The function* $f(x)=\int_0^x \sqrt{1+t^2}\,dt$ *is an odd function and* $g(x)=f'(x)$ *is an even function.*

Statement II *For a differentiable function* $f(x)$ *if* $f'(x)$ *is an even function, then* $f(x)$ *is an odd function.*

Sol. If $f(x)$ is of odd, then $f'(x)$ is even but converse is not true.

e.g. If $f'(x)=x\sin x$, then $f(x)=\sin x - x\cos x + C$

$$f(-x)=-\sin x + x\cos x + C$$

On adding, $f(x)+f(-x)=$ constant which need not to be zero.

For Statement I $f(x)=\int_0^x \sqrt{1+t^2}\,dt;\ g(x)=\sqrt{1+x^2}$

$$f(-x)=\int_0^{-x}\sqrt{1+t^2}\,dt;\ t=-y$$

$$f(-x)=-\int_0^x \sqrt{1+y^2}\,dy$$

$\therefore \quad f(x)+f(-x)=0$

$\Rightarrow f$ is odd and g is obviously even.

Hence, (c) is the correct answer.

• ***Ex. 15*** *Given,* $f(x)=\sin^3 x$ *and* $P(x)$ *is a quadratic polynomial with leading coefficient unity.*

Statement I $\int_0^{2\pi} P(x)\cdot f''(x)\,dx$ *vanishes.*

Statement II $\int_0^{2\pi} f(x)\,dx$ *vanishes.*

Sol. $P(x)=ax^2+bx+c;\ f(x)=\sin^3 x$

$$I=\int_0^{2\pi}\underbrace{P(x)}_{\text{I}}\cdot\underbrace{f''(x)}_{\text{II}}\,dx$$

Using I.B.P. $\underbrace{P(x)\cdot f'(x)|_0^{2\pi}}_{\text{zero}}-\int_0^{2\pi}\underbrace{P'(x)}_{\text{I}}\cdot\underbrace{f'(x)}_{\text{II}}\,dx$

$$=-\left[[P'(x)\cdot f(x)]_0^{2\pi}-\int_0^{2\pi}P''(x)\cdot f(x)\,dx\right]$$

$$=\int_0^{2\pi}P''(x)\cdot f(x)\,dx=2\int_0^{2\pi}\sin^3 x\,dx=0$$

Hence, (a) is the correct answer.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Q. Nos. 16 to 18)

Suppose we define the definite integral using the following formula $\int_a^b f(x)\,dx=\frac{b-a}{2}(f(a)+f(b))$, *for more accurate result for* $c\in(a,b)$, $F(c)=\frac{c-a}{2}(f(a)+f(c))+\frac{b-c}{2}(f(b)+f(c))$.

When $c=\frac{a+b}{2}$, then $\int_a^b f(x)\,dx=\frac{b-a}{4}(f(a)+f(b)+2f(c))$.

[IIT JEE 2007]

● ***Ex. 16*** $\int_0^{\pi/2} \sin x\,dx$ *is equal to*

(a) $\frac{\pi}{8}(1+\sqrt{2})$ (b) $\frac{\pi}{4}(1+\sqrt{2})$

(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

Sol. $\int_0^{\pi/2} \sin x\,dx = \frac{\frac{\pi}{2}-0}{4}\left(\sin(0)+\sin\left(\frac{\pi}{2}\right)+2\sin\left(\frac{0+\frac{\pi}{2}}{2}\right)\right)=\frac{\pi}{8}(1+\sqrt{2})$

Hence, (a) is the correct answer.

● ***Ex. 17*** *If $f(x)$ is a polynomial and if*

$$\lim_{t\to a}\frac{\int_a^t f(x)\,dx-\frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3}=0$$ *for all a, then the degree of $f(x)$ can atmost be*

(a) 1 (b) 2

(c) 3 (d) 4

Sol. Applying L' Hospital's rule,

$$\lim_{t\to a}\frac{f(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{3(t-a)^2}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{3(t-a)^2}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f'(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{12(t-a)}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f''(t)}{12}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f'(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{12(t-a)}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f''(t)}{12}=0\ f''(a)=0 \text{ for any } a.$$

$\Rightarrow$ $f(a)$ is atmost of degree 1.

Hence, (a) is the correct answer.

● ***Ex. 18*** *If $f''(x)<0, \forall\ x\in(a,b)$ and c is a point such that $a<c<b$ and $(c, f(c))$ is the point on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to*

(a) $\frac{f(b)-f(a)}{b-a}$ (b) $\frac{2(f(b)-f(a))}{b-a}$

(c) $\frac{2f(b)-f(a)}{2b-a}$ (d) 0

Sol. $F'(c)=(b-a)'(c)+f(a)-f(b)$

$F''(c)=f''(c)(b-a)<0$

$\Rightarrow$ $F'(c)=0$

$\Rightarrow$ $f'(c)=\frac{f(b)-f(a)}{b-a}$

Hence, (a) is the correct answer.

Passage II

(Q. Nos. 19 to 20)

$$\text{Let } f(\alpha,\beta)=\begin{vmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta\\ \sin\alpha & \cos\alpha & \sin\beta\\ -\cos\alpha & \sin\alpha & \cos\beta\end{vmatrix}.$$

● ***Ex. 19*** *The value of*

$$I=\int_0^{\pi/2} e^{\beta}\left(f(0,0)+f\left(\frac{\pi}{2},\beta\right)+f\left(\frac{3\pi}{2},\frac{\pi}{2}-\beta\right)\right)d\beta \text{ is}$$

(a) $e^{\pi/2}$ (b) 0

(c) $2(2e^{\pi/2}-1)$ (d) None of these

Sol. Here, $f(\alpha,\beta)=2\cos\beta$ i.e. independent of α

$$\therefore\quad I=\int_0^{\pi/2} e^{\beta}\left(f(0,0)+f\left(\frac{\pi}{2},\pi\right)+f\left(\frac{3\pi}{2},\frac{\pi}{2}-\beta\right)\right)d\beta$$

$$=\int_0^{\pi/2} e^{\beta}(2+2\cos\beta+2\sin\beta)\,d\beta$$

Hence, $=[e^{\beta}(2+2\sin\beta)]_0^{\pi/2}$

$=4e^{\pi/2}-2=2(2e^{\pi/2}-1)$

Hence, (c) is the correct answer.

● ***Ex. 20*** *If* $I=\int_{-\pi/2}^{\pi/2}\cos^2\beta\left(f(0,\beta)+f\left(0,\frac{\pi}{2}-\beta\right)\right)d\beta$, *then* $[I]$ *is*

(a) $e^{\pi/2}$ (b) 2

(c) $2(2e^{\pi/2}-1)$ (d) None of these

Sol. Again, $I=\int_{-\pi/2}^{\pi/2}\cos^2\beta\left(f(0,\beta)+f\left(0,\frac{\pi}{2}-\beta\right)\right)d\beta$

$$=\int_{-\pi/2}^{\pi/2}\cos^2\beta(\cos\beta+\sin\beta)\,d\beta$$

$$=\int_{-\pi/2}^{\pi/2}\cos^3\beta\,d\beta+\int_{-\pi/2}^{\pi/2}\cos^2\beta\sin\beta\,d\beta$$

$$=2\int_0^{\pi/2}\cos^3\beta\,d\beta+0$$

$\Rightarrow$ $I=\frac{\theta}{3}\Rightarrow [I]=2$

Hence, (b) is the correct answer.

JEE Type Solved Examples : Matching Type Questions

● **Ex. 21** *Match the following.*

Column I	Column II
(A) The function $f(x)=\dfrac{e^{x\cos x}-1-x}{\sin x^2}$ is not defined at $x=0$. The value of $f(0)$, so that f is continuous at $x=0$, is	(p) −1
(B) The value of the definite integral $\int_0^1 \dfrac{dx}{\sqrt{x}+\sqrt[3]{x}}$ is equal to $a+b\ln 2$, where a and b are integers, then $(a+b)$ is equal to	(q) 0
(C) Given, $e^n\int_0^n \dfrac{\sec^2\theta-\tan\theta}{e^\theta}\,d\theta=1$, then the value of $\tan n$ is equal to	(r) 1/2
(D) Let $a_n=\int_{\frac{1}{n+1}}^{\frac{1}{n}} \tan^{-1}(nx)\,dx$ and $b_n=\int_{\frac{1}{n+1}}^{\frac{1}{n}} \sin^{-1}(nx)\,dx$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}$ has the value equal to	(s) 1

Sol. (A) → (r), (B) → (p), (C) → (s), (D) → (r)

(A) $\lim\limits_{x\to 0}\dfrac{e^x\cos x-1-x}{x^2\cdot\dfrac{\sin x^2}{x^2}}=\lim\limits_{x\to 0}\dfrac{e^{x\cos x}-1-x}{x^2}$

(applying L'Hospital's twice)

$$=\lim_{x\to 0}\frac{e^{x\cos x}\cdot(-x\sin x+\cos x)-1}{2x}$$

$$=\lim_{x\to 0}\frac{e^{x\cos x}(-x\cos x-\sin x-\sin x)+e^{x\cos x}(-x\sin x+\cos x)}{2}$$

$$=\frac{1}{2}$$

(B) Put $x=u^6\Rightarrow dx=6u^5\,du$

$$\therefore\quad I=\int_0^1\frac{6u^5\,du}{u^3+u^2}=6\int_0^1\frac{u^3+1-1}{u+1}\,du=5-6\ln 2$$

$$\Rightarrow a+b=5-6=-1$$

(C) $e^n\int_0^n e^{-\theta}(\sec^2\theta-\tan\theta)\,d\theta=1$

Put $-\theta=t\Rightarrow d\theta=-dt$

$$-e^n\int_0^{-n}e^t(\sec^2 t+\tan t)\,dt=1$$

[use $\int e^x(f(x)+f'(x))=e^x f(x)$]

$$\Rightarrow -e^n[e^t\tan t]_0^{-n}=1\Rightarrow -e^n[-e^{-n}\tan n]=1\Rightarrow \tan n=+1$$

(D) $\lim\limits_{n\to\infty}\dfrac{\int_{1/n+1}^{1/n}\tan^{-1}(nx)\,dx}{\int_{1/n+1}^{1/n}\sin^{-1}(nx)\,dx}$

Put $nx=t\Rightarrow dx=\dfrac{1}{n}dt$

$$\therefore\quad \lim_{n\to\infty}\frac{\frac{1}{n}\int_{n/n+1}^{1}\tan^{-1}(t)\,dt}{\frac{1}{n}\int_{n/n+1}^{1}\sin^{-1}(t)\,dt}\qquad\left(\frac{0}{0}\text{ form}\right)$$

Use L'Hospital's rule, $\lim\limits_{n\to\infty}\dfrac{\tan^{-1}\left(\dfrac{n}{n+1}\right)}{\sin^{-1}\left(\dfrac{n}{n+1}\right)}=\dfrac{\frac{\pi}{4}}{\frac{\pi}{2}}=\dfrac{1}{2}$

● **Ex. 22** *Match the following* **[IIT JEE 2006]**

Column I	Column II
(A) $\int_{-1}^{1}\dfrac{dx}{1+x^2}$	(p) $\dfrac{1}{2}\log\left(\dfrac{2}{3}\right)$
(B) $\int_0^1\dfrac{dx}{\sqrt{1-x^2}}$	(q) $\dfrac{1}{2}\log\left(\dfrac{3}{2}\right)$
(C) $\int_2^3\dfrac{dx}{1-x^2}$	(r) $\dfrac{\pi}{3}$
(D) $\int_1^2\dfrac{dx}{x\sqrt{x^2-1}}$	(s) $\dfrac{\pi}{2}$

Sol. (A) → (s), (B) → (s), (C) → (q), (D) → (r)

(A) $\int_{-1}^{1}\dfrac{dx}{1+x^2}$

$\because\quad f(x)=\dfrac{1}{1+x^2}$ is an reven function.

$$\therefore\quad I=2\int_0^1\frac{dx}{1+x^2}=[2(\tan^{-1}x)]_0^1=\frac{\pi}{2}$$

(B) $\int_0^1\dfrac{dx}{\sqrt{1-x^2}}=[\sin^{-1}x]_0^1=\dfrac{\pi}{2}$

(C) $\int_2^3\dfrac{dx}{1-x^2}=-\int_2^3\dfrac{dx}{x^2-1}=-\left[\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|\right]_2^3$

$$=-\frac{1}{2}\ln\frac{2}{3}=\frac{1}{2}\ln\left(\frac{3}{2}\right)$$

(D) Let $I=\int_1^2\dfrac{dx}{x\sqrt{x^2-1}}$ [put $x=\sec\theta\Rightarrow dx=\sec\theta\tan\theta\,d\theta$]

$$=\int_0^{\pi/3}\frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta}$$

$$=\int_0^{\pi/3}1\,d\theta=\frac{\pi}{3}$$

JEE Type Solved Examples : Single Integer Answer Type Questions

- **Ex. 23** *Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)\,dt$. Then, the value of $f(\ln 5)$ is____.* **[IIT JEE 2009]**

Sol. (0) From given integral equation, $f(0) = 0$

Also, differentiating the given integral equation w.r.t. x, we get

$$f'(x) = f(x)$$

If $\quad f(x) \neq 0$, then $\dfrac{f'(x)}{f(x)} = 1 \Rightarrow f(x) = e^c e^x$

$\because \quad f(0) = 0 \quad \Rightarrow e^c = 0$, a contradiction

$\therefore \quad f(x) = 0, \forall\, x \in R \Rightarrow f(\ln 5) = 0$

- **Ex. 24** *For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by*

$$f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}.$$

Then, the value of $\dfrac{\pi^2}{10}\displaystyle\int_{-10}^{10} f(x) \cos \pi x\, dx$ is. **[IIT JEE 2010]**

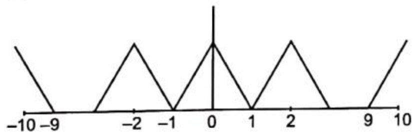
Sol. (4) We have, $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$

$f(x)$ and $\cos \pi x$ are both periodic with period 2 and are both even.

$$\therefore \int_{-10}^{10} f(x) \cos \pi x\, dx = 2\int_0^{10} f(x) \cos \pi x\, dx$$

$$= 10 \int_0^2 f(x) \cos \pi x\, dx$$

$$\int_0^1 f(x) \cos \pi x\, dx = \int_0^1 (1 - x) \cos \pi x\, dx = -\int_0^1 u \cos \pi u\, du$$



$$\int_1^2 f(x) \cos \pi x\, dx = \int_1^2 (x - 1) \cos \pi x\, dx = -\int_0^1 u \cos \pi u\, du$$

$$\therefore \quad \int_{-10}^{10} f(x) \cos \pi x\, dx = -20\int_0^1 u \cos \pi u\, du = \frac{40}{\pi^2}$$

$$\Rightarrow \frac{\pi^2}{10}\int_{-10}^{10} f(x) \cos \pi x\, dx = 4$$

- **Ex. 25** *Let $f(x)$ be a differentiable function satisfying $f(x) + f\left(x + \frac{1}{2}\right) = 1, \forall x \in R$ and $g(x) = \int_0^x f(t)\,dt$. If $g(1) = 1$, then the value of $\displaystyle\sum_{n=2}^{\infty}\left(\frac{8}{\sum_{k=1}^{n}(g(x + k^2) - g(x + k))}\right)$ is.*

Sol. (6) Here, $f(x) + f\left(x + \frac{1}{2}\right) = 1$

Replace x by $\left(x + \frac{1}{2}\right)$, we get $f\left(x + \frac{1}{2}\right) + f(x + 1) = 1$

On subtracting, $f(x) = f(x + 1)$ ...(i)

Also, $g(x + 1^2) = \int_0^{x+1^2} f(t)\,dt = \int_0^x f(t)\,dt + \int_n^{x+1^2} f(t)\,dt$

Since, $f(x + 1) = f(x)$

$$\therefore \quad g(x + 1^2) = \int_0^x f(t)\,dt + 1^2 \cdot \int_0^1 f(t)dt$$

$$g(x + 2^2) = \int_0^x f(t)\,dt + 2^2 \cdot \int_0^1 f(t)dt$$

$$g(x + k^2) = \int_0^x f(t)\,dt + k^2 \cdot \int_0^1 f(t)dt \quad \text{...(ii)}$$

and $\quad g(x + k) = \int_0^x f(t)\,dt + k \cdot \int_0^1 f(t)dt$ ...(iii)

Thus, $\displaystyle\sum_{k=1}^{n} (g(x + k^2) - g(x + k)) = \sum_{k=1}^{n} (k^2 - k) \cdot \int_0^1 f(t)\,dt$

$= \displaystyle\sum_{k=1}^{n} (k^2 - k) \cdot g(1)$, given $g(x) = \int_0^x f(t)\,dt$

$$= \left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right) \times 1$$

$$= \frac{n(n-1)(n+1)}{3}$$

$$\therefore \sum_{n=2}^{\infty} \frac{8}{\sum_{k=2}^{n}(g(x + k^2) - g(x + k))} = \sum_{n=2}^{\infty} \frac{8 \times 3}{(n-1)n(n+1)}$$

$$= 12\sum_{n=2}^{\infty}\left(\frac{(n+1) - (n-1)}{(n-1)\cdot n(n+1)} = \frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right)$$

$$= 12\left[\left(\frac{1}{1 \times 2} - \frac{1}{2 \times 3}\right) + \left(\frac{1}{2 \times 3} - \frac{1}{3 \times 4}\right) + \ldots + \left(\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right)\right]$$

$$= 12\left(\frac{1}{2} - \frac{1}{n(n+1)}\right)_{n \to \infty} = 6$$

Subjective Type Questions

• **Ex. 26** *Find the error in steps to evaluate the following integral.*

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{\sec^2 x + 2\tan^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

$$= \frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)]_0^{\pi} = 0$$

Sol. Here, the Newton-Leibnitz's formula for evaluating the definite integrals is not applicable because the anti-derivative,

$$f(x) = \frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)]$$

has a discontinuity at the point $x = \frac{\pi}{2}$ which lies in the interval $[0, \pi]$.

$$\text{LHL} = \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2}-h\right)\right] \text{ at } x = \frac{\pi}{2}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\cot h)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\infty) = \frac{\pi}{2\sqrt{3}} \quad \text{...(i)}$$

Also, $$\text{RHL} = \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2}+h\right)\right] \text{ at } x = \frac{\pi}{2}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(-\sqrt{3}\cot h)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(-\infty) = -\frac{\pi}{2\sqrt{3}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), LHL $\neq$ RHL at $x = \pi/2$

$\Rightarrow$ Anti-derivative $f(x)$ is discontinuous at $x = \pi/2$.

So, the correct solution for above integral;

$$I = \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x} \quad \text{...(iii)}$$

Using, $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = f(x) \end{cases}$$

We know, if $f(x) = \frac{\sec^2 x}{1+3\tan^2 x}$ $f(\pi - x) = f(x)$

$\therefore$ Eq. (iii) reduces to $I = 2\int_0^{\pi/2} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$

$$= 2\cdot\frac{1}{\sqrt{3}}\int_0^{\infty} \frac{dt}{1+t^2} \quad (\text{put } \sqrt{3}\tan x = t \Rightarrow \sqrt{3}\sec^2 x\,dx = dt)$$

$$= \frac{2}{\sqrt{3}}[\tan^{-1}(t)]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}(\tan^{-1}\infty - \tan^{-1}0) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{\sqrt{3}}$$

$$\therefore \quad \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \frac{\pi}{\sqrt{3}}$$

Remark

Students are advised to check continuity of anti-derivatives before substitution of integral limits.

• **Ex. 27** *If* $\int_a^b |\sin x|\,dx = 8$ *and* $\int_0^{a+b} |\cos x|\,dx = 9$, *then find the value of* $\int_a^b x\sin x\,dx$.

Sol. We know, $\int_a^b |\sin x|\,dx$ represents the area under the curve from $x = a$ to $x = b$.

We also know, area from $x = a$ to $x = a + \pi$ is 2.

$$\therefore \quad \int_a^b |\sin x|\,dx = 8$$

$$\Rightarrow \quad b - a = \frac{8\pi}{2} \quad \text{...(i)}$$

Similarly, $$\int_0^{a+b} |\cos x|\,dx = 9$$

$$\Rightarrow \quad a + b - 0 = \frac{9\pi}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $a = \pi/4, b = 17\pi/4$

Hence, $$\int_a^b x\sin x\,dx = \int_{\pi/4}^{17\pi/4} x\sin x\,dx$$

$$= [-x\cos x]_{\pi/4}^{17\pi/4} + \int_{\pi/4}^{17\pi/4} \cos x\,dx$$

$$= -\frac{17\pi}{4}\cos\frac{17\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} + [\sin x]_{\pi/4}^{17\pi/4}$$

$$= -\frac{4\pi}{\sqrt{2}}$$

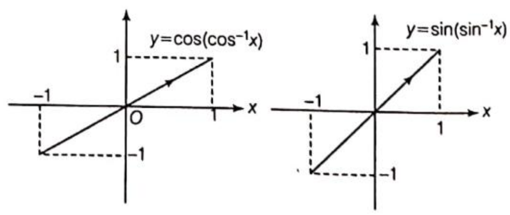
$$\therefore \quad \int_a^b x\sin x\,dx = -2\sqrt{2}\pi$$

• **Ex. 28** *Evaluate* $\int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \left|\frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\right| dx.$

Sol. We know, $\cos(\cos^{-1}x)$ and $\sin(\sin^{-1}x)$ could be plotted as

$\therefore$ $\sin(\sin^{-1}x)$ and $\cos(\cos^{-1}x)$ are identical functions.

$$\therefore \quad \int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\,dx = \int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} 1\,dx = (\beta - \alpha)$$



Ex. 29 *Evaluate* $\int_\alpha^\beta \sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx.$

Sol. Let $\quad I=\int_\alpha^\beta \sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx \quad$ ...(i)

Put $\quad x=\alpha\cos^2 t+\beta\sin^2 t$

$\therefore \quad x-\alpha=\alpha\cos^2 t+\beta\sin^2 t-\alpha=\beta[\sin^2(t)+\alpha(\cos^2 t-1)]$

$=\beta\sin^2 t-\alpha\sin^2 t \Rightarrow (x-\alpha)=(\beta-\alpha)\sin^2 t$

Similarly, $\beta-x=(\beta-\alpha)\cos^2 t$

$\therefore \quad dx=2(\beta-\alpha)\sin t\cos t\,dt$

where, $\quad x=\alpha\Rightarrow\sin^2 t=0 \Rightarrow t=0 \quad (\because \alpha<\beta)$

and $\quad x=\beta\Rightarrow\cos^2 t=0 \Rightarrow t=\pi/2$

$\therefore$ Eq. (i) reduces to

$$I=\int_0^{\pi/2}\sqrt{\frac{(\beta-\alpha)\sin^2 t}{(\beta-\alpha)\cos^2 t}}\cdot 2(\beta-\alpha)\sin t\cos t\,dt$$

$$=\int_0^{\pi/2}\frac{\sin t}{\cos t}\cdot 2(\beta-\alpha)\sin t\cos t\,dt$$

$$\left[\because \sqrt{\frac{\sin^2 t}{\cos^2 t}}=+\frac{\sin t}{\cos t}\text{ as } t\in[0,\pi/2]\right]$$

$$=(\beta-\alpha)\int_0^{\pi/2}2\sin^2 t\,dt=(\beta-\alpha)\int_0^{\pi/2}(1-\cos 2t)\,dt$$

$$=(\beta-\alpha)\left(t-\frac{\sin 2t}{2}\right)_0^{\pi/2}$$

$$=(\beta-\alpha)\left[\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(0-\frac{1}{2}\sin 0\right)\right]=(\beta-\alpha)\left[\frac{\pi}{2}\right]$$

$\therefore \quad I=\dfrac{\pi}{2}(\beta-\alpha)$

Aliter Let $\quad I=\int_\alpha^\beta\sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx \quad \left[\begin{array}{l}\text{put } \sqrt{\beta-x}=t\\ \text{or } \beta-x=t^2\\ \Rightarrow -dx=2t\,dt\end{array}\right]$

When $x=\alpha$, then $\beta-\alpha=t^2 \Rightarrow \sqrt{\beta-\alpha}=t$

When $x=\beta$, then $0=t^2 \Rightarrow 0=t$

$$\therefore \quad I=\int_{\sqrt{\beta-\alpha}}^0\frac{\sqrt{(\beta-t^2)-\alpha}}{t}(-2t)\,dt$$

$$=-2\int_{\sqrt{\beta-\alpha}}^0\sqrt{(\beta-\alpha)-t^2}\,dt$$

$$=2\int_0^{\sqrt{\beta-\alpha}}\sqrt{(\sqrt{\beta-\alpha})^2-t^2}\,dt$$

$$=2\left\{\frac{t}{2}\sqrt{(\beta-\alpha)-t^2}+\frac{(\beta-\alpha)}{2}\sin^{-1}\left(\frac{t}{\sqrt{\beta-\alpha}}\right)\right\}_0^{\sqrt{\beta-\alpha}}$$

$$=2\left[\left\{0+\frac{(\beta-\alpha)}{2}\sin^{-1}(1)\right\}-\left\{0+\frac{(\beta-\alpha)}{2}\sin^{-1}(0)\right\}\right]$$

$$=\frac{2(\beta-\alpha)}{2}\cdot\frac{\pi}{2}=\frac{\pi}{2}(\beta-\alpha)$$

Ex. 30 *Evaluate* $\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}}\dfrac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx.$

Sol. Let $I=\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}}\dfrac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx \quad$ ...(i)

Put $\quad x^2=a^2\cos^2 t+b^2\sin^2 t$

$\Rightarrow \quad 2x\,dx=[2a^2\cos t(-\sin t)+2b^2\sin t(\cos t)]\,dt$

$\Rightarrow \quad x\,dx=\dfrac{1}{2}(b^2-a^2)\sin 2t\,dt$

For $\quad x^2=\dfrac{a^2+b^2}{2}=a^2\cos^2 t+b^2\sin^2 t,$

$a^2+b^2=2(1-\sin^2 t)a^2+2b^2\sin^2 t$

or $(a^2+b^2)=2a^2+2(b^2-a^2)\sin^2 t$

$\Rightarrow \quad \sin^2 t=\dfrac{1}{2} \Rightarrow \cos 2t=0 \Rightarrow t=\dfrac{\pi}{4}$

For $\quad x^2=\dfrac{3a^2+b^2}{4}=a^2\cos^2 t+b^2\sin^2 t,$

$3a^2+b^2=4a^2+4(b^2-a^2)\sin^2 t$

$\Rightarrow \quad \sin^2 t=\dfrac{1}{4} \Rightarrow \cos 2t=\dfrac{1}{2} \Rightarrow t=\dfrac{\pi}{6}$

$\therefore$ Eq. (i) reduces to

$$I=\int_{\pi/6}^{\pi/4}\frac{1}{2}\cdot\frac{(b^2-a^2)\sin 2t}{\sqrt{(b^2-a^2)\sin^2 t(b^2-a^2)\cos^2 t}}\,dt$$

$$=\int_{\pi/6}^{\pi/4}dt=(t)_{\pi/6}^{\pi/4}=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\pi}{12}$$

Ex. 31 *Evaluate* $\int_0^{\pi/4}\dfrac{e^{\sec x}\left[\sin\left(x+\dfrac{\pi}{4}\right)\right]}{\cos x(1-\sin x)}\,dx.$

Sol. Let $I=\int_0^{\pi/4}\dfrac{e^{\sec x}\left[\sin\left(x+\dfrac{\pi}{4}\right)\right]}{\cos x(1-\sin x)}\,dx$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{e^{\sec x}(\sin x+\cos x)(1+\sin x)}{\cos x(1-\sin^2 x)}\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{e^{\sec x}(\sin x+\cos x)(1+\sin x)}{\cos^2 x\cdot\cos x}\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot(\sec x\tan x+\sec x)(\sec x+\tan x)\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot[\sec x\tan x(\sec x+\tan x)+\sec^2 x+\sec x\tan x]\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot[\sec x\tan x(\sec x+\tan x+1)+\sec^2 x]\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot\sec x\tan x(\sec x+\tan x+1)+\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot\sec^2 x\,dx$$

Applying integration by parts,

$$I = \frac{1}{\sqrt{2}}\{(\sec x + \tan x + 1)\, e^{\sec x}\}_0^{\pi/4}$$

$$-\frac{1}{\sqrt{2}}\int_0^{\pi/4}(\sec x \tan x + \sec^2 x)\cdot e^{\sec x} dx + \frac{1}{\sqrt{2}}\int_0^{\pi/4} e^{\sec x}\cdot \sec^2 x\, dx$$

$$= \frac{1}{\sqrt{2}}\{(\sqrt{2}+1+1)\, e^{\sqrt{2}} - (1+1)\, e\} - \frac{1}{\sqrt{2}}[e^{\sec x}]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}}(\sqrt{2}+2)\, e^{\sqrt{2}} - \sqrt{2}e - \frac{1}{\sqrt{2}}(e^{\sqrt{2}} - e) = \frac{1}{\sqrt{2}}(1+\sqrt{2})\, e^{\sqrt{2}} - \frac{e}{\sqrt{2}}$$

● ***Ex. 32*** *Evaluate* $\int_0^{\pi/4} \frac{x^2 (\sin 2x - \cos 2x)\, dx}{(1+\sin 2x)\cos^2 x}$.

Sol. Let $I = \int_0^{\pi/4} \frac{x^2(\sin 2x - \cos 2x)\, dx}{(1+\sin 2x)\cos^2 x}$,

$$= \frac{1}{8}\int_0^{\pi/2} \frac{t^2(\sin t - \cos t)}{(1+\sin t)(\cos^2 t/2)} dt \qquad \text{(put } 2x = t\text{)}$$

$$I = \frac{1}{4}\int_0^{\pi/2} \frac{t^2(\sin t - \cos t)}{(1+\sin t)(1+\cos t)} dt \qquad \text{...(i)}$$

$$\left[\text{using, } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \frac{1}{4}\int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - t\right)^2 \cdot (\cos t - \sin t)}{(1+\cos t)(1+\sin t)} dt \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \frac{1}{4}\int_0^{\pi/2} \frac{\left(\pi t - \frac{\pi^2}{4}\right)(\sin t - \cos t)}{(1+\cos t)(1+\sin t)} dt \qquad \text{...(iii)}$$

where, $\int_0^{\pi/2} \frac{\sin t - \cos t}{(1+\cos t)(1+\sin t)} dt = 0$

$$\left[\text{as } \int_0^{2a} f(x)\, dx = 0, \text{ if } f(2a-x) = -f(x)\right]$$

∴ Eq. (iii) becomes

$$I = \frac{1}{8}\int_0^{\pi/2} \frac{\pi t(\sin t - \cos t)\, dt}{(1+\cos t)(1+\sin t)} + 0$$

$$= \frac{\pi}{8}\int_0^{\pi/2} \frac{t\{(1+\sin t) - (1+\cos t)\}\, dt}{(1+\cos t)(1+\sin t)}$$

$$= \frac{\pi}{8}\int_0^{\pi/2} \frac{t\, dt}{1+\cos t} - \frac{\pi}{8}\int_0^{\pi/2} \frac{t\, dt}{1+\sin t}$$

$$= \frac{\pi}{8}\int_0^{\pi/2} \frac{t\, dt}{1+\cos t} - \frac{\pi}{8}\int_0^{\pi/2} \frac{(\pi/2 - t)\, dt}{1+\cos t}$$

$$= \frac{2\pi}{8}\int_0^{\pi/2} \frac{t\, dt}{1+\cos t} - \frac{\pi^2}{16}\int_0^{\pi/2} \frac{dt}{1+\cos t}$$

$$= \frac{\pi}{8}\int_0^{\pi/2} t(\sec^2 t/2)\, dt - \frac{\pi^2}{16}\int_0^{\pi/2} (\sec^2 t/2)\, dt$$

$$= \frac{\pi}{8}\left\{\left(2t\cdot \tan\frac{t}{2}\right)_0^{\pi/2} - \int_0^{\pi/2} 1\cdot 2\tan\frac{t}{2}\, dt\right\} - \frac{\pi^2}{16}\left(2\tan\frac{t}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{8}\left\{(\pi - 0) + 4\left(\log\cos\frac{t}{2}\right)_0^{\pi/2}\right\} - \frac{\pi^2}{8}\{1-0\}$$

$$= \frac{\pi}{8}\left\{\pi + 4\log\frac{1}{\sqrt{2}}\right\} - \frac{\pi^2}{8} = \frac{\pi^2}{8} + \pi\log\frac{1}{\sqrt{2}} - \frac{\pi^2}{8}$$

$$= \frac{\pi^2}{8} - \frac{1}{2}\pi\log 2 = \frac{\pi^2}{16} - \frac{\pi}{4}\log 2$$

● ***Ex. 33*** *Evaluate* $\int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2} dx$.

Sol. Let $I = \int_0^{\pi} \frac{x^2\cos x}{(1+\sin x)^2} dx = \int_0^{\pi} x^2\{(1+\sin x)^{-2}\cos x\}\, dx$

Applying integration by parts,

$$I = \left(x^2 \frac{(1+\sin x)^{-1}}{-1}\right)_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{(1+\sin x)^{-1}}{-1} dx$$

$$= (-\pi^2 + 0) + 2\int_0^{\pi} \frac{x}{1+\sin x} dx \qquad \text{...(i)}$$

$$\left[\text{using, } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = -\pi^2 + 2\int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = -2\pi^2 + 2\int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

$$= -2\pi^2 + 2\pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$= -2\pi^2 + 2\pi[\tan x - \sec x]_0^{\pi}$$

$$= -2\pi^2 + 2\pi[0 - (-1-1)] = -2\pi^2 + 4\pi$$

∴ $I = -\pi^2 + 2\pi$

● ***Ex. 34*** *Compute the following integrals.*

(i) $\int_0^{\infty} f(x^n + x^{-n}) \ln x \frac{dx}{x} = 0$

(ii) $\int_0^{\infty} f(x^n + x^{-n}) \ln x \frac{dx}{1+x^2} = 0$

Sol. (i) Let $t = \ln x \Rightarrow x = e^t$

$dx = e^t\, dt$ or $\frac{dx}{x} = dt$

Also, $x = 0 \Rightarrow t = -\infty$

and $x = \infty \Rightarrow t = \infty$

∴ $\int_0^{\infty} f(x^n + x^{-n}) \ln x \frac{dx}{x} = \int_{-\infty}^{\infty} f(e^{nt} + e^{-nt})\cdot t\, dt = 0$

(∵ integrand is an odd function of t)

(ii) $\int_0^{\infty} f(x^n + x^{-n}) \ln x \cdot \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} f(e^{nt} + e^{-nt})\cdot t\cdot \frac{e^t}{1+e^{2t}}\cdot dt$

Now, if $h(t) = f(e^{nt} + e^{-nt}) \cdot \dfrac{t e^t}{1 + e^{2t}}$

Then, $h(-t) = f(e^{-nt} + e^{nt}) \cdot (-t) \cdot \dfrac{1}{e^t + e^{-t}}$ $\left(\because \dfrac{e^t}{1 + e^{2t}} = \dfrac{1}{e^{-t} + e^t} \right)$

$h(-t) = -h(t)$

Thus, integrand is an odd function and hence

$$\int_0^\infty f(x^n + x^{-n}) \ln x \cdot \frac{dx}{1 + x^2} = 0$$

• Ex. 35 *Show that*

(a) $\int_0^\infty \sin x \, dx = 1$ (b) $\int_0^\infty \cos x \, dx = 0$

Sol. Let us first evaluate;

$$I = \int e^{-sx} \sin x \, dx \text{ and } J = \int e^{-sx} \cos x \, dx$$

Using integer by parts, we get

$$I = -e^{-sx} \cos x - sJ \quad \text{...(i)}$$

and $$J = e^{-sx} \sin x + sI \quad \text{...(ii)}$$

Subtracting Eqs. (i) and (ii), we get

$$I = -e^{-sx} \left| \frac{\cos x + s \sin x}{1 + s^2} \right|$$

$$\Rightarrow \quad J = e^{-sx} \left| \sin x - \frac{s^2}{s^2 + 1} \sin x - \frac{s}{1 + s^2} \cos x \right|$$

$$= e^{-sx} \left| \frac{\sin x - s \cos x}{1 + s^2} \right|$$

Thus, $\int_0^\infty e^{-sx} \sin x \, dx = \dfrac{1}{s^2 + 1}$

and $\int_0^\infty e^{-sx} \cos x \, dx = \dfrac{s}{s^2 + 1}$

Now, $\int_0^\infty \sin x \, dx = \lim\limits_{s \to 0} \int_0^\infty e^{-sx} \sin x \, dx = \lim\limits_{s \to 0} \dfrac{1}{s^2 + 1} = 1$

and $\int_0^\infty \cos x \, dx = \lim\limits_{s \to 0} \int_0^\infty e^{-sx} \cos x \, dx = \lim\limits_{s \to 0} \dfrac{s}{s^2 + 1} = 0$

• Ex. 36 *Find a function $g : R \to R$, continuous in $[0, \infty)$ and positive in $(0, \infty)$ satisfying $g(0) = 1$ and* $\dfrac{1}{2} \int_0^x g^2(t) \, dt = \dfrac{1}{x} \left(\int_0^x g(t) \, dt \right)^2$.

Sol. Let $$f(x) = \int_0^x g(t) \, dt$$

$$\Rightarrow \quad f'(x) = g(x)$$

$$\Rightarrow \quad \frac{1}{2} \int_0^x (f'(t))^2 \, dt = \frac{1}{2} \int_0^x g^2(t) \, dt \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{1}{2} \left[\int_0^x g(t) \right]^2 = \frac{1}{2} (f(x))^2$$

$$\Rightarrow \quad \frac{1}{2} \frac{\left[\int_0^x g(t) \right]^2}{x} = \frac{1}{2} \frac{(f(x))^2}{x}$$

$$\Rightarrow \quad \frac{1}{2} \cdot \frac{1}{2} \left[\int_0^x g^2(t) \right] dt = \frac{1}{2} \frac{(f(x))^2}{x}$$

$$\Rightarrow \quad \frac{1}{2} \int_0^x (f'(t))^2 \, dt = \frac{1}{2} \frac{(f(x))^2}{x} \quad \text{[using Eq. (i)]}$$

Differentiating both the sides w.r.t. x, we get

$$\frac{1}{2} [f'(x)]^2 = \frac{x 2 f(x) f'(x) - (f(x))^2}{x^2}$$

$$\Rightarrow \quad \frac{1}{2} [g(x)]^2 x^2 = 2x f(x) (g(x)) - (f(x))^2$$

$$\Rightarrow \quad \left(\frac{x g(x)}{f(x)} \right)^2 = 4 \left(\frac{x g(x)}{f(x)} \right) - 2$$

$$\Rightarrow \quad t^2 - 4t + 2 = 0, \text{ where } t = \frac{x g(x)}{f(x)}$$

$$\Rightarrow \quad t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \text{ or } \frac{x f'(x)}{f(x)} = 2 \pm \sqrt{2}$$

$$\Rightarrow \quad \ln f(x) = 2 \pm \sqrt{2} \ln x + \text{constant}$$

$$\Rightarrow \quad f(x) = cx^{2 + \sqrt{2}} \text{ or } cx^{2 - \sqrt{2}}$$

$$\Rightarrow \quad g(x) = f'(x) = c_1 x^{1 + \sqrt{2}} \text{ or } c_2 x^{1 - \sqrt{2}}$$

where, c_1 and c_2 are constants of integration.

But g is continuous on $[0, \infty)$, then $c_2 x^{1 - \sqrt{2}}$ is ruled out.

Hence, $g(x) = c_1 x^{1 + \sqrt{2}}$

Also, $g(0) = c^1 = 1 \Rightarrow g(x) = x^{1 + \sqrt{2}}$

• Ex. 37 *Let* $I_n = \int_0^{\pi/4} \tan^n x \, dx$ *($n > 1$ and is an integer). Show that*

(i) $I_n + I_{n-2} = \dfrac{1}{n - 1}$ (ii) $\dfrac{1}{2(n + 1)} < I_n < \dfrac{1}{2(n - 1)}$

Sol. (i) Given, $I_n = \int_0^{\pi/4} \tan^n x \, dx = \int_0^{\pi/4} \tan^{n-2} x \cdot \tan^2 x \, dx$

$$= \int_0^{\pi/4} \sec^2 x \cdot \tan^{n-2} x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \int_0^1 t^{n-2} \, dt - I_{n-2}, \text{ where } t = \tan x$$

$$I_n + I_{n-2} = \left(\frac{t^{n-1}}{n - 1} \right)_0^1 \quad \therefore \; I_n + I_{n-2} = \frac{1}{n - 1}$$

(ii) For $0 < x < \pi / 4$, we have $0 < \tan^n x < \tan^{n-2} x$

So that; $$0 < I_n < I_{n-2}$$

$$\therefore \quad I_n + I_{n+2} < 2I_n < I_n + I_{n-2}$$

$$\Rightarrow \quad \frac{1}{n + 1} < 2I_n < \frac{1}{n - 1}$$

$$\Rightarrow \quad \frac{1}{2(n + 1)} < I_n < \frac{1}{2(n - 1)}$$

• Ex. 38 *If $U_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx$, where n is positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$. Hence, deduce that $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} = \frac{1}{2} n\pi$.*

Sol. $U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{[1-\cos(n+2)x] - [1-\cos(n+1)x]}{1-\cos x}\,dx$

$$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}\,dx$$

$$= \int_0^{\pi} \frac{2\sin\left(n+\frac{3}{2}\right)x\cdot\sin\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx$$

$$U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}\,dx \quad \text{...(i)}$$

$$\Rightarrow U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n) = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx$$

$$U_{n+2} + U_n - 2U_{n+1} = \int_0^{\pi} \frac{2\cos(n+1)x\cdot\sin x/2}{\sin x/2}\,dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx = 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$$

$\Rightarrow U_{n+2} + U_n = 2U_{n+1}$

$\Rightarrow U_n, U_{n+1}, U_{n+2}$ are in AP.

Now, $U_0 = \int_0^{\pi} \frac{1-1}{1-\cos x}\,dx = 0$

$U_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x}\,dx = \pi$

$U_1 - U_0 = \pi$ (common difference)

$U_n = U_0 + n\pi = n\pi$

$U_n = n\pi$

Now, $I_n = \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}\,d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta}\,d\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx = \frac{1}{2}n\pi$$

• Ex. 39 *Prove that for any positive integer K,*

$$\frac{\sin 2Kx}{\sin x} = 2[\cos x + \cos 3x + \ldots + \cos(2K-1)x]$$

Hence, prove that $\int_0^{\pi/2} \sin 2Kx\cdot\cot x\,dx = \frac{\pi}{2}$.

Sol. To prove; $\sin 2Kx = 2\sin x[\cos x + \cos 3x + \cos 5x + \ldots + \cos(2K-1)x]$

Taking RHS, $[2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + \ldots + 2\sin x\cos(2K-1)x]$

$= (\sin 2x) + (\sin 4x - \sin 2x) + [\sin 6x - \sin 4x) + \ldots + (\sin 2Kx - \sin(2K-2)x]$

$= \sin 2Kx$

Now, $\int_0^{\pi/2} \sin 2Kx\cdot\cot x\,dx = \int_0^{\pi/2} \left(\frac{\sin 2Kx}{\sin x}\right)\cdot\cos x\,dx$

$= \int_0^{\pi/2} 2\cos x[\cos x + \cos 3x + \ldots + \cos(2K-1)x]\,dx$

$= \int_0^{\pi/2} (1+\cos 2x)\,dx + \int_0^{\pi/2} (\cos 4x + \cos 2x)\,dx + \ldots + \int_0^{\pi/2} [\cos 2Kx + \cos(2K-2)x]\,dx$

But we know that,

$$\int_0^{\pi/2} (\cos 2nx)\,dx = 0, \forall\, n \in I \text{ and } n \neq 0$$

$$\Rightarrow \int_0^{\pi/2} \sin 2Kx\cdot\cot x\,dx = \int_0^{\pi/2} 1\,dx + 0 = \frac{\pi}{2}$$

• Ex. 40 *Evaluate $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$.*

Sol. Let $I = \int_0^{\sqrt{3}} \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & \text{if } x > 1 \end{cases}$

$\therefore I = \int_0^1 \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx + \int_1^{\sqrt{3}} \frac{1}{1+x^2}\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$= \int_0^1 \frac{2\tan^{-1}x}{1+x^2}\,dx + \int_1^{\sqrt{3}} \frac{\pi - 2\tan^{-1}x}{1+x^2}\,dx$

$= 2\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx + \pi\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx - 2\int_1^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx$

$= 2\int_0^{\pi/4} t\,dt + \pi(\tan^{-1}x)_1^{\sqrt{3}} - 2\int_{\pi/4}^{\pi/3} t\,dt$ (put $\tan^{-1}x = t$)

$= 2\left(\frac{t^2}{2}\right)_0^{\pi/4} + \pi\{\tan^{-1}\sqrt{3} - \tan^{-1}1\} - 2\left(\frac{t^2}{2}\right)_{\pi/4}^{\pi/3}$

$= \frac{\pi^2}{16} + \pi\left\{\frac{\pi}{3} - \frac{\pi}{4}\right\} - \left\{\frac{\pi^2}{9} - \frac{\pi^2}{16}\right\} = \frac{7}{72}\pi^2$

• Ex. 41 *Prove that $\int_0^x e^{xt}\cdot e^{-t^2}\,dt = e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$.*

Sol. Let $I = \int_0^x e^{xt}\cdot e^{-t^2}\,dt$. Put $t = \frac{x+z}{2} \Rightarrow dt = \frac{1}{2}dz$

$\therefore I = \int_0^x e^{xt}\cdot e^{-t^2}\,dt = \frac{1}{2}\int_{-x}^{x} e^{\left(\frac{x+z}{2}\right)x}\cdot e^{-\left(\frac{x+z}{2}\right)^2}\,dz$

$= \frac{1}{2}\int_{-x}^{x} e^{\left(\frac{x+z}{2}\right)\left(x - \frac{x+z}{2}\right)}\,dz = \frac{1}{2}\int_{-x}^{x} e^{\left(\frac{x+z}{2}\right)\left(\frac{x-z}{2}\right)}\,dz$

$$=\frac{1}{2}\int_{-x}^{x} e^{\frac{x^2}{4}-\frac{z^2}{4}}\,dz=\frac{1}{2}e^{x^2/4}\cdot\int_{-x}^{x} e^{-z^2/4}\,dz$$

$$=\frac{1}{2}e^{x^2/4}\int_{-x}^{x} e^{-t^2/4}\,dt=\frac{1}{2}e^{x^2/4}\cdot 2\int_{0}^{x} e^{-t^2/4}\,dt$$

$$\left[\text{using, }\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx,\text{ when } f(-x)=f(x)\right]$$

$$=e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$$

$$\Rightarrow\quad \int_0^x e^{xt}\cdot x^{-t^2}\,dt=e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$$

• Ex. 42 *If $f(x)=e^x+\int_0^1 (e^x+te^{-x})\,f(t)\,dt$, find $f(x)$.*

Sol. We can write $f(x)=Ae^x+Be^{-x}$, where

$$A=1+\int_0^1 f(t)\,dt \quad\text{and}\quad B=\int_0^1 tf(t)\,dt$$

$$\therefore\quad A=1+\int_0^1 (Ae^t+Be^{-t})\,dt=1+(Ae^t-Be^{-t})_0^t$$

$$A=1+A(e^1-1)-B(e^{-1}-1)$$

$$\Rightarrow\quad (2-e)A+(e^{-1}-1)B=1 \qquad \text{...(i)}$$

$$B=\int_0^1 t(Ae^t+Be^{-t})\,dt$$

$$=A(te^t-e^t)_0^t+B(-te^{-t}-e^{-t})_0^1$$

$$B=A+B(1-2e^{-1})$$

$$\Rightarrow\quad A-2e^{-1}B=0 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$A=\frac{2(e-1)}{4e-2e^2},\ B=\frac{e-1}{4-2e}$$

Hence, $$f(x)=\frac{2(e-1)}{4e-2e^2}\cdot e^x+\frac{e-1}{4-2e}\cdot e^{-x}$$

• Ex. 43 *If $|a|<1$, show that*

$$\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\,dx=\pi\sin^{-1}a.$$

Sol. Given, $|a|<1$

Let $$f(a)=\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\,dx$$

$$\therefore\quad f'(a)=\int_0^\pi \frac{\cos x}{\cos x\cdot(1+a\cos x)}\,dx=\int_0^\pi \frac{dx}{1+a\cos x}$$

(differentiating w.r.t. 'a' using Leibnitz rule)

$$\left(\begin{array}{l}\text{put }\tan\frac{x}{2}=t\Rightarrow\frac{dx}{dt}=\frac{2}{1+t^2},\text{ when } x=0,t=0\\ \text{and when } x=\pi, t\to\infty\end{array}\right)$$

$$\Rightarrow\quad f'(a)=\int_0^\infty \frac{\frac{2dt}{1+t^2}}{1+a\left(\frac{1-t^2}{1+t^2}\right)}$$

$$=\int_0^\infty \frac{2dt}{(1+t^2)+a(1-t^2)}=\int_0^\infty \frac{2dt}{(1-a)t^2+(1+a)}$$

$$=\frac{2}{1-a}\int_0^\infty \frac{dt}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2}$$

$$=\frac{2}{1-a}\cdot\frac{1}{\sqrt{(1+a)/(1-a)}}\cdot\left(\tan^{-1}t\sqrt{\frac{1-a}{1+a}}\right)_0^\infty=\frac{2}{\sqrt{1-a^2}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{1-a^2}}$$

(integrating both the sides w.r.t. 'a')

$$\Rightarrow\quad f(a)=\pi\sin^{-1}a+c$$

But $$f(0)=\int_0^\pi \frac{\log(1+0\cos x)\,dx}{\cos x}=0$$

$$\Rightarrow\quad c=0$$

$$\therefore\quad f(a)=\pi\sin^{-1}a$$

• Ex. 44 *Evaluate* $\int_0^{\pi/2}\operatorname{cosec}\theta\tan^{-1}(c\sin\theta)\,d\theta$.

Sol. Let $$f(c)=\int_0^{\pi/2}\operatorname{cosec}\theta\cdot\tan^{-1}(c\sin\theta)\,d\theta$$

$$\therefore\quad f'(c)=\int_0^{\pi/2}\operatorname{cosec}\theta\cdot\frac{(\sin\theta)}{1+c^2\sin^2\theta}\,d\theta+0-0$$

$$=\int_0^{\pi/2}\frac{d\theta}{1+c^2\sin^2\theta}$$

$$=\int_0^{\pi/2}\frac{\operatorname{cosec}^2\theta\,d\theta}{c^2+\operatorname{cosec}^2\theta}=\int_0^{\pi/2}\frac{\operatorname{cosec}^2\theta\,d\theta}{(c^2+1)+\cot^2\theta}$$

$$=-\frac{1}{\sqrt{c^2+1}}\left(\tan^{-1}\frac{\cot\theta}{\sqrt{c^2+1}}\right)_0^{\pi/2}$$

$$f'(c)=\frac{\pi}{2\sqrt{c^2+1}}$$

Integrating both the sides, we get

$$f(c)=\int\frac{\pi\,dc}{2\sqrt{c^2+1}}=\frac{\pi}{2}\log(c+\sqrt{c^2+1})+A$$

where, $f(0)=A$

But $$f(0)=\int_0^{\pi/2}\operatorname{cosec}\theta\tan^{-1}(0)\,d\theta=0$$

$$\Rightarrow\quad f(c)=\frac{\pi}{2}\log(c+\sqrt{c^2+1})$$

• Ex. 45 *Evaluate* $\int_0^{\pi/2}\sec\theta\cdot\tan^{-1}(a\cos\theta)\,d\theta$.

Sol. Given definite integral is a function of 'a'. Let its value be $I(a)$.

Then, $$I(a)=\int_0^{\pi/2}\sec\theta\cdot\tan^{-1}(a\cos\theta)\,d\theta$$

$$\Rightarrow\quad I'(a)=\int_0^{\pi/2}\sec\theta\cdot\frac{1}{1+a^2\cos^2\theta}\cdot\cos\theta\,d\theta+0-0$$

(using Leibnitz rule)

$$=\int_0^{\pi/2}\frac{1}{1+a^2\cos^2\theta}\,d\theta=\int_0^{\pi/2}\frac{\sec^2\theta}{\sec^2\theta+a^2}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2 \theta}{1 + \tan^2 \theta + a^2} d\theta \qquad \text{(put } \tan\theta = t)$$

$$= \int_0^{\infty} \frac{dt}{t^2 + (a^2+1)} = \frac{1}{\sqrt{a^2+1}} \left(\tan^{-1} \frac{t}{\sqrt{a^2+1}} \right)_0^{\infty}$$

$$= \frac{1}{\sqrt{a^2+1}} \left(\frac{\pi}{2} - 0 \right) \Rightarrow I'(a) = \frac{\pi}{2\sqrt{a^2+1}}$$

Integrating both the sides w.r.t. 'a', we get

$$I(a) = \frac{\pi}{2} \log | a + \sqrt{a^2+1} | + C$$

Since, $I(0) = 0 + C$

$\Rightarrow$ $I(a) = \frac{\pi}{2} \log | a + \sqrt{1+a^2} |$

• Ex. 46 *Let f be a continuous function on $[a, b]$. Prove that there exists a number $x \in [a, b]$ such that $\int_a^x f(t)\, dt = \int_x^b f(t)\, dt$.*

Sol. Let $g(x) = \int_a^x f(t)\,dt - \int_x^b f(t)\,dt,\ x \in [a, b]$

We have, $g(a) = -\int_a^b f(t)\,dt$ and $g(b) = \int_a^b f(t)\,dt$

$\Rightarrow$ $g(a) \cdot g(b) = -\left(\int_a^b f(t)\,dt \right)^2 \le 0$

Clearly, $g(x)$ is continuous in $[a, b]$ and $g(a) \cdot g(b) \le 0$.

It implies that $g(x)$ will become zero at least once in $[a,b]$.

Hence, $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for atleast one value of $x \in [a, b]$.

• Ex. 47 *If $f(x) = x + \int_0^1 (xy^2 + x^2 y)(f(y))\,dy$, find $f(x)$.*

Sol. Given, $f(x) = x + x\int_0^1 y^2 f(y)\,dy + x^2 \int_0^1 y f(y)\,dy$

$$= x\left(1 + \int_0^1 y^2 f(y)\,dy\right) + x^2\left(\int_0^1 y f(y)\,dy \right)$$

$\Rightarrow$ $f(x)$ is a quadratic expression.

$\Rightarrow$ $f(x) = ax + bx^2$ or $f(y) = ay + by^2$ ...(i)

where, $a = 1 + \int_0^1 y^2 f(y)\,dy = 1 + \int_0^1 y^2 (ay + by^2)\,dy$

$$= 1 + \left(\frac{ay^4}{4} + \frac{by^5}{5} \right)_0^1 \ a = 1 + \left(\frac{a}{4} + \frac{b}{5} \right)$$

$\Rightarrow$ $20a = 20 + 5a + 4b$ or $15a - 4b = 20$ ...(ii)

and $b = \int_0^1 y f(y)\,dy = \int_0^1 y(ay + by^2)\,dy$

$$= \left(\frac{ay^3}{3} + \frac{by^4}{4} \right)_0^1 \Rightarrow b = \frac{a}{3} + \frac{b}{4}$$

$\Rightarrow$ $12b = 4a + 3b$ or $9b - 4a = 0$ ...(iii)

From Eqs. (ii) and (iii),

$$a = \frac{180}{119} \text{ and } b = \frac{80}{119}$$

$\therefore$ Eq. (i) reduces to

$$f(x) = \frac{80x^2 + 180x}{119}$$

• Ex. 48 *Prove that*

$$\int_0^x \left\{ \int_0^u f(t)\,dt \right\} du = \int_0^x (x-u) f(u)\,du.$$

Sol. Here, applying integration by parts to

$$\underbrace{\int_0^x 1}_{\text{II}} \cdot \underbrace{\left\{ \int_0^u f(t)\,dt \right\} du}_{\text{I}}$$

i.e. taking '1' as second function and $\int_0^u f(t)\,dt$ as first function, we have

$$\int_0^x 1 \cdot \left\{ \int_0^u f(t)\,dt \right\} du = \left\{ \int_0^u f(t)\,dt \right\}_0^x \cdot (u)_0^x - \int_0^x f(u) \cdot u\,du$$

$$= \left(u \int_0^u f(t)\,dt \right)_0^x - \int_0^x u f(u)\,du$$

$$= x \int_0^x f(t)\,dt - \int_0^x u f(u)\,du$$

$$= \int_0^x (x-u) f(u)\,du$$

• Ex. 49 *Evaluate $\int_0^{3\pi/2} (\log | \sin x |)(\cos(2nx))\,dx, n \in N$.*

Sol. Let $I(n) = \int_0^{3\pi/2} (\log|\sin x|)(\cos 2nx)\,dx$

$$I(n) = \left(\overset{\text{I}}{\log|\sin x|} \cdot \overset{\text{II}}{\frac{\sin 2nx}{2n}} \right)_0^{3\pi/2} - \int_0^{3\pi/2} \cot x \cdot \frac{\sin 2nx}{2n} dx$$

(using integration by parts)

$$I(n) = 0 - \frac{1}{2n} \int_0^{3\pi/2} \frac{\sin 2nx \cdot \cos x}{\sin x} dx = 0 - \frac{1}{2} I_1(n) \qquad \text{...(i)}$$

Let $I_1(n) = \int_0^{3\pi/2} \frac{\sin(2nx) \cos x}{\sin x} dx$

$$I_1(n+1) = \int_0^{3\pi/2} \frac{\sin(2n+2)x \cdot \cos x}{\sin x} dx$$

$$\therefore I_1(n+1) - I_1(n) = \int_0^{3\pi/2} \frac{(\sin(2n+2)x - \sin 2nx) \cdot \cos x}{\sin x} dx$$

$$= \int_0^{3\pi/2} \frac{2\cos(2n+1)x \cdot \sin x \cdot \cos x}{\sin x} dx$$

$$= \int_0^{3\pi/2} (\cos(2n+2)x + \cos 2nx)\,dx$$

$$= \left(\frac{\sin(2n+2)x}{(2n+2)} + \frac{\sin(2n)x}{2n} \right)_0^{3\pi/2} = 0 + 0$$

$\Rightarrow$ $I_1(n+1) = I_1(n) = \ldots = I_1(1)$

$$\therefore I_1(1) = \int_0^{3\pi/2} \frac{\sin 2x \cdot \cos x}{\sin x} dx = \int_0^{3\pi/2} \frac{2\sin x \cos x \cdot \cos x}{\sin x} dx$$

$$= \int_0^{3\pi/2} 2\cos^2 x\, dx = \int_0^{3\pi/2} (1+\cos 2x)\, dx$$

$$= \left(x + \frac{\sin 2x}{2} \right)_0^{3\pi/2}$$

$$\Rightarrow \quad I_1(1) = \frac{3\pi}{2} = I_1(n) \Rightarrow I(n) = -\frac{1}{2n} I_1(n) \qquad \text{[using Eq. (i)]}$$

$$\therefore \quad I(n) = -\frac{3\pi}{4n}$$

• Ex. 50 *Evaluate* $\int_0^{\infty} e^{-x} \sin^n x\, dx$, *if n is an even integer.*

Sol. Here, $I_n = \int_0^{\infty} e^{-x} \sin^n x\, dx$

$$= (-\sin^n x\, e^{-x})_0^{\infty} + n \int_0^{\infty} \sin^{n-1} x \cos x\, e^{-x}\, dx$$

$$= n \int_0^{\infty} (\sin^{n-1} x \cdot \cos x)(e^{-x})\, dx \; [\text{where } (-\sin^n x\, e^{-x})_0^{\infty} = 0]$$

$$\text{I} \qquad \text{II}$$

$$\Rightarrow \quad I_n = n[(-\sin^{n-1} x \cdot \cos x\, e^{-x})_0^{\infty}$$

$$+ (n-1)\int_0^{\infty} \sin^{n-2} x \cos^2 x\, e^{-x} dx - \int_0^{\infty} \sin^n x\, e^{-x}\, dx]$$

$$I_n = n(n-1)\int_0^{\infty} e^{-x} \sin^{n-2} x\, dx - n^2 \int_0^{\infty} e^{-x} \sin^n x\, dx$$

$$= n(n-1) I_{n-2} - n^2 I_n$$

$$\Rightarrow \quad (n^2+1) I_n = n(n-1) I_{n-2}$$

$$\therefore \quad I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$$

Replacing n by $(n-2), (n-4), \ldots, 2$, we get

$$I_{n-2} = \frac{(n-2)(n-3)}{(n-2)^2+1} I_{n-4} = \frac{(n-1)(n-3)}{(n-2)^2+1} \cdot \frac{(n-4)(n-5)}{(n-4)^2+1} I_{n-6}$$

... and so on.

$$I_2 = \frac{2(2-1)}{2^2+1} I_0$$

$$\therefore \quad I_n = \frac{n(n-1)}{n^2+1} \cdot \frac{(n-2)(n-3)}{(n-2)^2+1} \cdot \frac{(n-4)(n-5)}{(n-3)^2+1} \cdots \frac{2(2-1)}{2^2+1} I_0$$

$$= \frac{n!}{\prod_{r=1}^{n/2} \{1+(2r)^2\}} \cdot I_0 = \frac{n!}{\prod_{r=1}^{n/2} 1+4r^2} \quad \left(\because I_0 = \int_0^{\infty} e^{-x}\, dx = 1 \right)$$

• Ex. 51 *Evaluate* $\int_0^1 (tx+1-x)^n\, dx, n \in N$ *and t is independent of x. Hence, show*

$$\int_0^1 x^k (1-x)^{n-k}\, dx = \frac{1}{{}^nC_k (n+1)}.$$

Sol. Here, $I = \int_0^1 (tx+1-x)^n\, dx = \int_0^1 ((t-1)x+1)^n\, dx$

$$= \left\{ \frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)} \right\}_0^1$$

$$= \frac{1}{n+1} \{t^n + t^{n-1} + t^{n-2} + \ldots + t + 1\} \qquad \text{...(i)}$$

Again, $I = \int_0^1 (tx+1-x)^n\, dx = \int_0^1 \{(1-x)+tx\}^n\, dx$

$$= \int_0^1 \{{}^nC_0 (1-x)^n + {}^nC_1 (1-x)^{n-1}(tx) + {}^nC_2 (1-x)^{n-2}$$

$$(tx)^2 + \ldots + {}^nC_r (1-x)^{n-r}(tx)^r + \ldots + {}^nC_n (tx)^n\}\, dx$$

$$= \int_0^1 \left\{ \sum_{r=0}^{n} {}^nC_r (1-x)^{n-r} (tx)^r \right\} dx$$

$$= \sum_{r=0}^{n} {}^nC_r t^r \int_0^1 \{(1-x)^{n-r} x^r\}\, dx \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\sum_{r=0}^{n} {}^nC_r t^r \int_0^1 (1-x)^{n-r} x^r\, dx = \frac{1}{n+1} \{t^n + t^{n-1} + t^{n-2} + \ldots + t + 1\}$$

Equating coefficient of t^k on both the sides, we get

$${}^nC_k \int_0^1 (1-x)^{n-k} x^k\, dx = \frac{1}{n+1}$$

$$\Rightarrow \quad \int_0^1 (1-x)^{n-k} x^k\, dx = \frac{1}{{}^nC_k (n+1)}$$

• Ex. 52 *Given a real valued function $f(x)$ which is monotonic and differentiable, prove that for any real numbers a and b;*

$$\int_a^b \{f^2(x) - f^2(a)\} dx = \int_{f(a)}^{f(b)} 2x\{b - f^{-1}(x)\} dx$$

Sol. As, $f(x)$ is differentiable and monotonic.

$\therefore f^{-1}(x)$ exists.

Let $\quad f^{-1}(x) = t \Rightarrow x = f(t)$ or $dx = f'(t)\, dt$

As, $\quad x = f(a) \Rightarrow f^{-1}\{f(a)\} = t \Rightarrow t = a$

and $\quad x = f(b) \Rightarrow f^{-1}\{f(b)\} = t \Rightarrow t = b$

$$\therefore \quad \int_{f(a)}^{f(b)} 2x\{b - f^{-1}(x)\}\, dx = \int_a^b 2f(t)(b-t) f'(t)\, dt$$

$$= \int_a^b (b-t)\{2f(t) f'(t)\}\, dt$$

$$\text{I} \qquad \text{II}$$

$$= ((b-t)\{f(t)\}^2)_a^b + \int_a^b \{f(t)\}^2\, dt$$

$$= -(b-a)\{f(a)\}^2 + \int_a^b \{f(t)\}^2\, dt$$

$$= \int_a^b (\{f(t)\}^2 - \{f(a)\}^2)\, dt$$

$$= \int_a^b \{f^2(x) - f^2(a)\}\, dx$$

• Ex. 53 *Evaluate*

$$\int_0^1 \frac{\sin\theta (\cos^2\theta - \cos^2 \pi/5)(\cos^2\theta - \cos^2 2\pi/5)}{\sin 5\theta} d\theta.$$

Sol. We know,

$$z^{10} - 1 = (z^2-1)\left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]\left[z^2 - 2\left(\cos\frac{2\pi}{5}\right)z + 1\right]$$

$$\times \left(z^2 - 2\cos\frac{4\pi}{5} z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{5} z + 1\right)$$

or $z^5-\frac{1}{z^5}=\left(z-\frac{1}{z}\right)\left(z-2\cos\frac{\pi}{5}+\frac{1}{z}\right)\left(z-2\cos\frac{2\pi}{5}+\frac{1}{z}\right)$

$$\times\left(z-2\cos\frac{4\pi}{5}+\frac{1}{z}\right)\left(z-2\cos\frac{6\pi}{5}+\frac{1}{z}\right)$$

Put $z=\cos\theta+i\sin\theta$

$\Rightarrow 2i\sin 5\theta=(2i\sin\theta)(2\cos\theta-2\cos\pi/5)(2\cos\theta-2\cos 2\pi/5)$
$\times(2\cos\theta-2\cos 4\pi/5)(2\cos\theta-2\cos 6\pi/5)$

$\Rightarrow \sin 5\theta=16\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)$

$\Rightarrow \frac{\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)}{\sin 5\theta}=\frac{1}{16}$

$\therefore \int_0^1 \frac{\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)}{\sin 5\theta}d\theta$

$$=\int_0^1\frac{1}{16}d\theta=\frac{1}{16}$$

• **Ex. 54** *Show that* $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^nC_K}{n^K(K+3)}=e-2$.

Sol. $\lim_{n\to\infty}\sum_{K=0}^{n}\frac{{}^nC_K}{n^K(K+3)}=\lim_{n\to\infty}\sum_{K=0}^{n}\frac{1}{K+3}{}^nC_K\cdot\frac{1}{n^K}$

$=\lim_{n\to\infty}\sum_{K=0}^{n}{}^nC_K\cdot\frac{1}{n^K}\int_0^1 x^{K+2}dx\left(\because \frac{1}{K+3}=\int_0^1 x^{K+2}dx\right)$

$=\int_0^1\left(x^2\lim_{n\to\infty}\sum_{K=0}^{n}{}^nC_K\cdot\left(\frac{x}{n}\right)^K\right)dx$

$=\int_0^1 x^2\left\{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right\}dx$

$=\int_0^1 x^2\cdot e^x dx \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x\right]$

$=(x^2\cdot e^x)_0^1-\int_0^1 2x\cdot e^x dx=e-2\left\{(xe^x)_0^1-\int_0^1 e^x dx\right\}$

$=e-2\{e-e+1\}=e-2$

• **Ex. 55** *Let* $I=\int_0^{\pi/2}\frac{\cos x}{a\cos x+b\sin x}dx$ *and* $J=\int_0^{\pi/2}\frac{\sin x}{a\cos x+b\sin x}dx$, *where* $a>0$ and $b>0$. *Compute the values of I and J.*

Sol. $\because \quad aI+bJ=\frac{\pi}{2}$ ...(i)

and $bI-aJ=\int_0^{\pi/2}\frac{b\cos x-a\sin x}{a\cos x+b\sin x}dx$

$\therefore \quad bI-aJ=\ln[a\cos x+b\sin x]_0^{\pi/2}$

$\Rightarrow \quad bI-aJ=\ln\left(\frac{b}{a}\right)$ ...(ii)

From Eqs. (i) and (ii),

$a^2I+abJ=\frac{a\pi}{2}$

$b^2I-abJ=b\ln(b/a)$

$$I=\frac{1}{a^2+b^2}\left[\frac{a\pi}{2}+b\ln\left(\frac{b}{a}\right)\right]$$

Again, $abI+b^2J=\frac{b\pi}{2}$

and $abI-a^2J=a\ln(b/a)$

On subtracting, we get $J=\frac{1}{a^2+b^2}\left[\frac{b\pi}{2}-a\ln\left(\frac{b}{a}\right)\right]$

Aliter Convert $a\cos x+b\sin x$ into a single cosine say $\cos(x+\phi)$ and put $(x+\phi)=t$.

• **Ex. 56** *Evaluate* $\int_0^\infty\frac{\ln x\,dx}{x^2+2x+4}$.

Sol. $I=\int_0^\infty\frac{\ln x\,dx}{x^2+2x+4}$ (put $x=2t\Rightarrow dx=2dt$ to make coefficient of x^2 and constant term same)

$$=2\int_0^\infty\frac{\ln 2+\ln t}{4(t^2+t+1)}dt=\frac{\ln 2}{2}\underbrace{\int_0^\infty\frac{dt}{t^2+t+1}}_{I_1}+\frac{1}{2}\underbrace{\int_0^\infty\frac{\ln t\,dt}{t^2+t+1}}_{I_2=\text{zero}}$$

$I_2=\int_0^\infty\frac{\ln t\,dt}{t^2+t+1}\left(\text{put } t=\frac{1}{y}\Rightarrow dt=-\frac{1}{y^2}dy\right)$

$I_2=\int_\infty^0\frac{\ln y\cdot(+1)}{\left(\frac{1}{y^2}+\frac{1}{y}+1\right)y^2}dy$

$I_2=\int_\infty^0\frac{\ln y\,dy}{y^2+y+1}=-\int_\infty^0\frac{\ln y\,dy}{y^2+y+1}=-I_2$

Now, $I_1=\int_0^\infty\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{[t+(1/2)]^2}{\sqrt{3}}\right]_0^\infty$

$=\frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right]=\frac{2\pi}{3\sqrt{3}} \qquad \therefore I=\frac{\ln 2}{2}\cdot\frac{2\pi}{3\sqrt{3}}=\frac{\pi\ln 2}{3\sqrt{3}}$

Note For $a>0$, $I=\int_0^\infty\frac{\ln x\,dx}{ax^2+bx+a}=0$

[Hint By putting $x=1/t$, we get $I=-I$, so $I=0$]

• **Ex. 57** *Find a function f, continuous for all x (and not zero everywhere) such that* $f^2(x)=\int_0^x\frac{f(t)\sin t}{2+\cos t}dt$.

Sol. We have, $f^2(x)=\int_0^x\frac{f(t)\sin t}{2+\cos t}dt$

(Note that $f^2(x)$ being an integral function of a continuous function is continuous and differentiable)

$$2f(x)f'(x)=\frac{f(x)\cdot\sin x}{2+\cos x}$$

Integrating, we get $2f(x)=C-\ln(2+\cos x)$

$\because \quad x=0\Rightarrow f(0)=0\Rightarrow C=\ln 3$

$\therefore \quad f(x)=\frac{1}{2}\ln\frac{3}{2+\cos x}$

● **Ex. 58** *Evaluate* $\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$, *where a is a parameter.*

Sol. Let $I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$

$$\therefore \quad \frac{dI}{da} = \int_0^{\infty} \frac{1 \cdot x}{(1+a^2x^2)x} dx = \int_0^{\infty} \frac{dx}{1+a^2x^2}$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2 + \frac{1}{a^2}} = \frac{1}{a^2} [a \tan^{-1} x]_0^{\infty} = \frac{\pi}{2a}$$

$$\int dI = \frac{\pi}{2} \int \frac{da}{a} \Rightarrow I = \frac{\pi}{2} \ln a + C$$

When $a = 1$ and $I = 0$ then $C = 0$

Hence, $I = \frac{\pi}{2} \ln a$

● **Ex. 59** *Evalute* $\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{(1-x^2)}} dx \; (a^2 < 1)$.

Sol. Let $I = \int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{(1-x^2)}} dx, |a| < 1$

$$\therefore \quad \frac{dI}{da} = \int_0^1 \frac{-2ax^2}{(1-a^2x^2)} \cdot \frac{dx}{x^2\sqrt{1-x^2}} \quad \text{(put } x = \sin\theta\text{)}$$

$$= \int_0^{\pi/2} \frac{2a\, d\theta}{a^2\sin^2\theta - 1} = \int_0^{\pi/2} \frac{2a \sec^2\theta \, d\theta}{a^2\tan^2\theta - (1+\tan^2\theta)}$$

$$= -\int_0^{\pi/2} \frac{2a\sec^2\theta \, d\theta}{(1-a^2)\tan^2\theta + 1} \quad \text{(put } \tan\theta = 1\text{)}$$

$$= \int_0^{\infty} \frac{2a\, dt}{(1-a^2)t^2+1} \text{ or } -\frac{2a}{(1-a^2)} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{1-a^2}}\right)^2}$$

$$= -\frac{2a}{\sqrt{1-a^2}} [\tan^{-1} t\sqrt{1-a^2}]_0^{\infty} = -\frac{\pi \cdot 2a}{2\sqrt{1-a^2}} = -\frac{\pi a}{\sqrt{1-a^2}}$$

or $I = \pi\sqrt{1-a^2} + C$

If $a = 0$ and $I = 0$, then $C = -\pi$

$$\therefore \quad I = \pi\left(\sqrt{1-a^2} - 1\right)$$

● **Ex. 60** *Evaluate* $\int_0^{\pi/2} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right) \frac{dx}{\sin x} \; (|a| < 1)$.

Sol. Let $I = \int_0^{\pi/2} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right) \frac{dx}{\sin x} \; (|a| < 1)$

$$\therefore \quad \frac{dI}{da} = \int_0^{\pi/2} \frac{2\sin x}{(1-a^2\sin^2 x)} \cdot \frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x}{1+\tan^2 x - a^2\tan^2 x} dx$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x}{(1-a^2)\tan^2 x + 1} dx \quad \text{(put } \tan x = t\text{)}$$

$$= \int_0^{\infty} \frac{2\, dt}{(1-a^2)t^2+1} = \frac{2}{(1-a^2)} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{1-a^2}}\right)^2}$$

$$\Rightarrow \quad = \frac{2}{\sqrt{1-a^2}} [\tan^{-1}(t\sqrt{1-a^2})]_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}}$$

$$\Rightarrow \quad dI = \frac{\pi\, da}{\sqrt{1-a^2}}$$

When $a = 0$ and $I = 0$, then $C = 0$

$$\therefore \quad I = \pi \sin^{-1}(a)$$

Hence, $I = \pi \sin^{-1} a + C$

Definite Integral Exercise 1 : Single Option Correct Type Questions

1. The value of $\int_0^4 \frac{(y^2-4y+5)\sin(y-2)dy}{(2y^2-8y+1)}$ is

(a) 0 (b) 2
(c) −2 (d) None of these

2. Let $f(x)=x^2+ax+b$ and the only solution of the equation $f(x)=\text{minimum } f(x)$ is $x=0$ and $f(x)=0$ has root α and β, then $\int_\alpha^\beta x^3\,dx$ is equal to

(a) $\frac{1}{4}(\beta^4+\alpha^4)$ (b) $\frac{1}{4}(a^2-b^2)$
(c) 0 (d) None of these

3. $\int_{\pi/2}^{x}\sqrt{(3-2\sin^2 t)}\,dt+\int_0^y \cos t\,dt=0$, then $\left(\frac{dy}{dx}\right)$ at $x=\pi$ and $y=\pi$ is

(a) $\sqrt{3}$ (b) $-\sqrt{2}$
(c) $-\sqrt{3}$ (d) None of these

4. $\int_{-4}^{4}\frac{\sin^{-1}(\sin x)+\cos^{-1}(\cos x)}{(1+x^2)\left(1+\left[\frac{x^2}{17}\right]\right)}dx=\log\left(\frac{(1+x^2)}{a}\right)$ $+b\pi\tan^{-1}\left(\frac{c-\pi}{1+c\pi}\right)$ (where, [·] denotes greatest integer function), then number of ways in which $a-(2b+c)$ distinct object can distributed among $\frac{a-5}{c}$ persons equally, is

(a) $\frac{9!}{(3!)^3}$ (b) $\frac{12!}{(4!)^3}$ (c) $\frac{15!}{(5!)^3}$ (d) $\frac{10!}{(6!)^3}\times 3!$

5. The value of the definite integral $\int_0^\infty \frac{dx}{(1+x^a)(1+x^2)}(a>0)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) Some function of a

6. The value of the definite integral $\int_0^{3\pi/4}[(1+x)\sin x+(1-x)\cos x]\,dx$ is

(a) $2\tan\frac{3\pi}{8}$ (b) $2\tan\frac{\pi}{4}$
(c) $2\tan\frac{\pi}{8}$ (d) 0

7. Let $C_n=\int_{1/n+1}^{1/n}\frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$, then $\lim_{n\to\infty} n^2\cdot C_n$ is equal to

(a) 1 (b) 0
(c) −1 (d) $\frac{1}{2}$

8. If x satisfies the equation $\left(\int_0^1\frac{dt}{t^2+2t\cos\alpha+1}\right)x^2-\left(\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}dt\right)x-2=0\,(0<\alpha<\pi)$, then the value of x is

(a) $\pm\sqrt{\frac{\alpha}{2\sin\alpha}}$ (b) $\pm\sqrt{\frac{2\sin\alpha}{\alpha}}$
(c) $\pm\sqrt{\frac{\alpha}{\sin\alpha}}$ (d) $\pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$

9. If $f(x)=e^{g(x)}$ and $g(x)=\int_2^x\frac{t\,dt}{1+t^4}$, then $f'(2)$ is equal to

(a) 2/17 (b) 0
(c) 1 (d) Cannot be determined

10. If a, b and c are real numbers, then the value of $\lim_{t\to 0}\ln\left(\frac{1}{t}\int_0^t(1+a\sin bx)^{c/x}\,dx\right)$ equals

(a) abc (b) $\frac{ab}{c}$
(c) $\frac{bc}{a}$ (d) $\frac{ca}{b}$

11. The value of $\lim_{n\to\infty}\sum_{r=1}^{r=4n}\frac{\sqrt{n}}{\sqrt{r}\,(3\sqrt{r}+4\sqrt{n})^2}$ is equal to

(a) $\frac{1}{35}$ (b) $\frac{1}{14}$
(c) $\frac{1}{10}$ (d) $\frac{1}{5}$

12. Let $f(x)=\int_{-1}^{x}e^{t^2}\,dt$ and $h(x)=f(1+g(x))$ where $g(x)$ is defined for all x, $g'(x)$ exists for all x, and $g(x)<0$ for $x>0$. If $h'(1)=e$ and $g'(1)=1$, then the possible values which $g(1)$ can take

(a) 0 (b) −1
(c) −2 (d) −4

13. Let $f(x)$ be a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and g be the function satisfying $f(x)+g(x)=x^2$.

The value of the integral $\int_0^1 f(x)\,g(x)\,dx$ is

(a) $e-\frac{1}{2}e^2-\frac{5}{2}$ (b) $e-e^2-3$
(c) $\frac{1}{2}(e-3)$ (d) $e-\frac{1}{2}e^2-\frac{3}{2}$

14. Let $f(x)=\int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}}$,
where $g(x)=\int_0^{\cos x}(1+\sin t^2)\,dt$. Also, $h(x)=e^{-|x|}$ and $f(x)=x^2\sin\frac{1}{x}$, if $x\neq 0$ and $f(0)=0$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) $l'(0)$ (b) $h'(0^-)$
(c) $h'(0^+)$ (d) $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$

15. For $f(x)=x^4+|x|$, let $I_1=\int_0^{\pi} f(\cos x)\,dx$ and $I_2=\int_0^{\pi/2} f(\sin x)\,dx$, then $\frac{I_1}{I_2}$ has the value equal to

(a) 1 (b) 1/2
(c) 2 (d) 4

16. Let f be a positive function. Let
$I_1=\int_{1-k}^{k}(x)f(x(1-x))\,dx$; $I_2=\int_{1-k}^{k} f(x(1-x))\,dx$,
where $2k-1>0$. Then, $\frac{I_2}{I_1}$ is

(a) k (b) 1/2
(c) 1 (d) 2

17. Suppose that the quadratic function $f(x)=ax^2+bx+c$ is non-negative on the interval $[-1,1]$. Then, the area under the graph of f over the interval $[-1,1]$ and the X-axis is given by the formula

(a) $A=f(-1)+f(1)$
(b) $A=f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)$
(c) $A=\frac{1}{2}[f(-1)+2f(0)+f(1)]$
(d) $A=\frac{1}{3}[f(-1)+4f(0)+f(1)]$

18. Let $I(a)=\int_0^{\pi}\left(\frac{x}{a}+a\sin x\right)^2 dx$, where '$a$' is positive real. The value of 'a' for which $I(a)$ attains its minimum value, is

(a) $\sqrt{\pi\sqrt{\frac{2}{3}}}$ (b) $\sqrt{\pi\sqrt{\frac{3}{2}}}$ (c) $\sqrt{\frac{\pi}{16}}$ (d) $\sqrt{\frac{\pi}{13}}$

19. The set of values of 'a' which satisfy the equation $\int_0^2 (t-\log_2 a)\,dt=\log_2\left(\frac{4}{a^2}\right)$, is

(a) $a\in R$ (b) $a\in R^+$
(c) $a<2$ (d) $a>2$

20. $\lim_{x\to\infty}\left(x^3\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^t}\,dt\right)$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) 1 (d) 0

21. The value of $\sqrt{\pi\left(\int_0^{2008} x|\sin \pi x|\,dx\right)}$ is equal to

(a) $\sqrt{2008}$ (b) $\pi\sqrt{2008}$ (c) 1004 (d) 2008

22. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2x^2}$, $x>0$ is equal to

(a) $x\tan^{-1}(x)$ (b) $\tan^{-1}(x)$
(c) $\frac{\tan^{-1}(x)}{x}$ (d) $\frac{\tan^{-1}(x)}{x^2}$

23. Let $a>0$ and $f(x)$ is monotonic increasing such that $f(0)=0$ and $f(a)=b$, then $\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx$ is equal to

(a) $a+b$ (b) $ab+b$ (c) $ab+a$ (d) ab

24. If $\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx=k\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}$, then '$k$' is equal to

(a) π (b) 2π (c) 2 (d) 1

25. $\int_0^{\infty} f\left(x+\frac{1}{x}\right)\cdot\frac{\ln x}{x}\,dx$ is equal to

(a) 0 (b) 1
(c) $\frac{1}{2}$ (d) Cannot be evaluated

26. $\lim_{\lambda\to 0}\left(\int_0^1 (1+x)^{\lambda}\,dx\right)^{1/\lambda}$ is equal to

(a) $2\ln 2$ (b) $\frac{4}{e}$ (c) $\ln\frac{4}{e}$ (d) 4

27. If $g(x)$ is the inverse of $f(x)$ and $f(x)$ has domain $x\in[1,5]$, where $f(1)=2$ and $f(5)=10$, then the values of $\int_1^5 f(x)\,dx+\int_2^{10} g(y)\,dy$ is equal to

(a) 48 (b) 64 (c) 71 (d) 52

28. The value of the definite integral $\int_0^{\pi/2}\sin x\sin 2x\sin 3x\,dx$ is equal to

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $\frac{1}{6}$

29. If $f(x)=\int_0^x (f(t))^2\,dt$, $f:R\to R$ be differentiable function and $f(g(x))$ is differentiable function at $x=a$, then

(a) $g(x)$ must be differentiable at $x=a$
(b) $g(x)$ may be non-differentiable at $x=a$
(c) $g(x)$ may be discontinuous at $x=a$
(d) None of the above

30. The number of integral solutions of the equation $4\int_0^{\infty}\frac{\ln t\,dt}{x^2+t^2}-\pi\ln 2=0$; $x>0$, is

(a) 0 (b) 1 (c) 2 (d) 3

31. $\int_0^{16n^2/\pi}\cos\frac{\pi}{2}\left[\frac{x\pi}{n}\right]dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

32. If $\int_{-2}^{-1} (ax^2 - 5)\, dx = 0$ and $5 + \int_{1}^{2} (bx + c)\, dx = 0$, then

(a) $ax^2 - bx + c = 0$ has atleast one root in (1, 2)
(b) $ax^2 - bx + c = 0$ has atleast one root in (−2, −1)
(c) $ax^2 + bx + c = 0$ has atleast one root in (−2, −1)
(d) None of the above

33. The value of $\int_0^6 \left(\sqrt{x + \sqrt{12x - 36}} + \sqrt{x - \sqrt{12x - 36}}\right) dx$ is equal to

(a) $6\sqrt{3}$ (b) $4\sqrt{3}$
(c) $12\sqrt{3}$ (d) None of these

34. If $I_n = \int_{-n}^{n} (\{x + 1\}\{x^2 + 2\} + \{x^2 + 3\}\{x^3 + 4\})\, dx$, (where, {} denotes the fractional part), then I_1 is equal to

(a) $-\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{1}{3}$ (d) None of these

35. Let $I_n = \int_0^{\pi/2} \frac{\sin(2n - 1)x}{\sin x}\, dx$, $J_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}$, $n \in N$, then

(a) $J_{n+1} - J_n = I_n$ (b) $J_{n+1} - J_n = I_{n+1}$
(c) $J_{n+1} + J_n = J_n$ (d) $J_{n+1} - J_{n+1} = J_n$

Definite Integral Exercise 2 :
More than One Option Correct Type Questions

36. If $f(x) = [\sin^{-1}(\sin 2x)]$ (where, [] denotes the greatest integer function), then

(a) $\int_0^{\pi/2} f(x)\, dx = \frac{\pi}{2} - \sin^{-1}(\sin 1)$
(b) $f(x)$ is periodic with period π
(c) $\lim_{x \to \frac{\pi}{2}} f(x) = -1$
(d) None of the above

37. Which of the following definite integral(s) vanishes?

(a) $\int_0^{\pi/2} \ln(\cot x)\, dx$ (b) $\int_0^{2\pi} \sin^3 x\, dx$
(c) $\int_{1/e}^{e} \frac{dx}{x(\ln x)^{1/3}}$ (d) $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx$

38. The equation $10x^4 - 3x^2 - 1 = 0$ has

(a) atleast one root in (−1, 0)
(b) atleast one root in (0, 1)
(c) atleast two roots in (−1, 1)
(d) no root in (−1, 1)

39. Suppose $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)\, dx$;
$I_2 = \int_0^{\pi/2} \cos(2\pi \sin^2 x)\, dx$ and $I_3 = \int_0^{\pi/2} \cos(\pi \sin x)\, dx$, then

(a) $I_1 = 0$ (b) $I_2 + I_3 = 0$
(c) $I_1 + I_2 + I_3 = 0$ (d) $I_2 = I_3$

40. Let $f(x) = \int_{-1}^{1} (1 - |t|) \cos(xt)\, dt$, then which of the following holds true?

(a) $f(0)$ is not defined
(b) $\lim_{x \to 0} f(x)$ exists and is equal to 2
(c) $\lim_{x \to 0} f(x)$ exists and is equal to 1
(d) $f(x)$ is continuous at $x = 0$

41. The function f is continuous and has the property $f(f(x)) = 1 - x$ for all $x \in [0, 1]$ and $J = \int_0^1 f(x)\, dx$, then

(a) $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) = 1$
(b) the value of J equals to 1/2
(c) $f\left(\frac{1}{3}\right) \cdot f\left(\frac{2}{3}\right) = 1$
(d) $\int_0^{\pi/2} \frac{\sin x\, dx}{(\sin x + \cos x)^3}$ has the same value as J

42. Let $f(x)$ is a real valued function defined by $f(x) = x^2 + x^2 \int_{-1}^{1} t f(t)\, dt + x^3 \int_{-1}^{1} f(t)\, dt$, then which of the following hold(s) good?

(a) $\int_{-1}^{1} t f(t)\, dt = \frac{10}{11}$ (b) $f(1) + f(-1) = \frac{30}{11}$
(c) $\int_{-1}^{1} t f(t)\, dt > \int_{-1}^{1} f(t)\, dt$ (d) $f(1) - f(-1) = \frac{20}{11}$

43. Let $f(x)$ and $g(x)$ be differentiable functions such that $f(x) + \int_0^x g(t)\, dt = \sin x(\cos x - \sin x)$ and $(f'(x))^2 + (g(x))^2 = 1$, then $f(x)$ and $g(x)$ respectively, can be

(a) $\frac{1}{2}\sin 2x$, $\sin 2x$ (b) $\frac{\cos 2x}{2}$, $\cos 2x$
(c) $\frac{1}{2}\sin 2x$, $-\sin 2x$ (d) $-\sin^2 x$, $\cos 2x$

44. Let $f(x) = \int_{-x}^{x} (t \sin at + bt + c)\, dt$, where a, b, c are non-zero real numbers, then $\lim_{x \to 0} \frac{f(x)}{x}$ is

(a) independent of a
(b) independent of a and b, and has the value equals to c
(c) independent a, b and c
(d) dependent only on c

45. Let $L = \lim_{n \to \infty} \int_a^{\infty} \frac{n\, dx}{1 + n^2 x^2}$, where $a \in R$, then L can be

(a) π (b) $\pi/2$ (c) 0 (d) 1

Definite Integral Exercise 3 : Passage Based Questions

Passage I

(Q. Nos. 46 to 48)

Suppose $\lim_{x \to 0} \dfrac{\int_0^x \dfrac{t^2\, dt}{(a+t^r)^{1/p}}}{bx - \sin x} = l$, where

$p \in N, p \ge 2, a > 0, r > 0$ and $b \ne 0$.

46. If l exists and is non-zero, then

(a) $b > 1$ (b) $0 < b < 1$
(c) $b < 0$ (d) $b = 1$

47. If $p = 3$ and $l = 1$, then the value of 'a' is equal to

(a) 8 (b) 3
(c) 6 (d) 3/2

48. If $p = 2$ and $a = 9$ and l exists, then the value of l is equal to

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{7}{9}$

Passage II

(Q. Nos. 49 to 51)

Suppose $f(x)$ *and* $g(x)$ *are two continuous functions defined for* $0 \le x \le 1$. *Given,* $f(x) = \int_0^1 e^{x+1} \cdot f(t)\, dt$ *and*

$g(x) = \int_0^1 e^{x+t} \cdot g(t)\, dt + x$.

49. The value of $f(1)$ equals

(a) 0 (b) 1
(c) e^{-1} (d) e

50. The value of $g(0) - f(0)$ equals

(a) $\frac{2}{3-e^2}$ (b) $\frac{3}{e^2-2}$ (c) $\frac{1}{e^2-1}$ (d) 0

51. The value of $\frac{g(0)}{g(2)}$ equals

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{e^2}$ (d) $\frac{2}{e^2}$

Passage III

(Q. Nos. 52 to 54)

We are given the curves $y = \int_{-\infty}^x f(t)dt$ *through the point* $\left(0, \frac{1}{2}\right)$ *any* $y = f(x)$ *where* $f(x) > 0$ *and* $f(x)$ *is differentiable,* $\forall x \in R$ *through* $(0, 1)$. *Tangents drawn to both the curves at the points with equal abscissae intersect on the same point on the X-axis.*

52. The number of solutions $f(x) = 2ex$ is equal to

(a) 0 (b) 1
(c) 2 (d) None of these

53. $\lim_{x \to \infty} (f(x))^{f(-x)}$ is

(a) 3 (b) 6
(c) 1 (d) None of these

54. The function $f(x)$ is

(a) increasing for all x (b) non-monotonic
(c) decreasing for all x (d) None of these

Passage IV

(Q. Nos. 55 to 57)

$f(x) = \int_0^x (4t^4 - at^3)\, dt$ *and* $g(x)$ *is quadratic satisfying* $g(0) + 6 = g'(0) - c = g''(c) + 2b = 0$ $y = h(x)$ and $y = g(x)$ *intersect in 4 distinct points with abscissae* x_i; $i = 1, 2, 3, 4$ *such that* $\sum \frac{i}{x_i} = 8, a, b, c \in R^+$ *and* $h(x) = f'(x)$.

55. Abscissae of point of intersection are in

(a) AP (b) GP
(c) HP (d) None of these

56. 'a' is equal to

(a) 6 (b) 8 (c) 20 (d) 12

57. 'c' is equal to

(a) 25 (b) 25/2 (c) 25/4 (d) 25/8

Passage V

(Q. Nos. 58 to 60)

Let $y = \int_{u(x)}^{v(x)} f(t)\, dt$, *let us define* $\frac{dy}{dx}$ *as* $\frac{dy}{dx} = v'(x) f^2(v(x)) - u'(x) f^2(u(x))$ *and the equation of the tangent at* (a, b) *and* $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$.

58. If $y = \int_x^{x^2} t^2\, dt$, then the equation of tangent at $x = 1$ is

(a) $x + y = 1$ (b) $y = x - 1$
(c) $y = x$ (d) $y = x + 1$

59. If $y = \int_{x^2}^{x^4} (\ln t)dt$, then $\lim_{x \to 0^+} \frac{dy}{dx}$ is equal to

(a) 0 (b) 1 (c) 2 (d) -1

60. If $f(x) = \int_1^x e^{t^2/2} (1 - t^2)\, dt$, then $\frac{d}{dx} f(x)$ at $x = 1$ is

(a) 0 (b) 1
(c) 2 (d) -1

Passage VI (Q. Nos. 61 to 62)

Consider $f:(0, \infty)\rightarrow\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *defined as* $f(x)=\tan^{-1}\left(\frac{\log_e x}{(\log_e x)^2+1}\right)$.

61. The about function can be classified as
(a) Injective but nor surjective
(b) Surjective but not bijective
(c) Neither injective nor surjective
(d) Both injective and surjective

62. The value of $\int_0^{\infty}$ [tan] dx is equal to (where, [·] denotes the greatest integer function)
(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) -1 (d) 1

Definite Integral Exercise 4 : Matching Type Questions

63. Let $\lim_{T\to\infty}\frac{1}{T}\int_0^T(\sin x+\sin ax)^2\,dx=L$, then

Column I	Column II
(A) For $a=0$, the value of L is	(p) 0
(B) For $a=1$, the value of L is	(q) 1/2
(C) For $a=-1$, the value of L is	(r) 1
(D) For $a\in R-\{-1,0,1\}$, the value of L is	(s) 2

64. Let $f(\theta=\int_0^1(x+\sin\theta)^2\,dx$ and $g(\theta)=\int_0^1(x+\cos\theta)^2\,dx$ where $\theta\in[0,2\pi]$. The quantity $f(\theta)-g(\theta)$, $\forall\,\theta$ in the interval given in column I, is

Column I	Column II
(A) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	(p) negative
(B) $\left(\frac{3\pi}{4}, \pi\right]$	(q) positive
(C) $\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right)$	(r) non-negative
(D) $\left(0, \frac{\pi}{4}\right)\cup\left(\frac{7\pi}{4}, 2\pi\right)$	(s) non-positive

65. Match the following.

Column I	Column II
(A) $\int_0^1(1+2008x^{2008})\,e^{x^{2008}}\,dx$ equals	(p) e^{-1}
(B) The value of the definite integral $\int_0^1 e^{-x^2}\,dx+\int_1^{1/e}\sqrt{-\ln x}\,dx$ is equal to	(q) $e^{-1/4}$
(C) $\lim_{n\to\infty}\left(\frac{1^1\cdot2^2\cdot3^3\ldots(n-1)^{n-1}\cdot n^n}{n^{1+2+3+\ldots+n}}\right)^{\frac{1}{n^2}}$ equals	(r) $e^{1/2}$
	(s) e

66. Match the following.

Column I	Column II
(A) If $f(x)=\int_0^{g(x)}\frac{dt}{\sqrt{1+t^3}}$, where $g(x)=\int_0^{\cos x}(1+\sin t^2)\,dt$, then value of $f'\left(\frac{\pi}{2}\right)$ is	(p) -2
(B) If $f(x)$ is a non-zero differentiable function such that $\int_0^x f(t)\,dt=\{f(x)\}^2$, $\forall x\in R$, then $f(2)$ is equal to	(q) 2
(C) If $\int_a^b(2+x-x^2)\,dx$ is maximum, then $a+b$ is equal to	(r) 1
(D) If $\lim_{x\to0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=0$, then $3a+b$ has the value	(s) -1

Definite Integral Exercise 5 : Single Integer Answer Type Questions

67. If $f(x)=\sum_{n=1}\frac{\sin nx}{4^n}$ and $\int_0^{\pi} f(x)\,dx=\log\left(\frac{m}{n}\right)$, then the value of $(m+n)$ is ...

68. The value of $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x\,dx}{1+2[\sin^{-1}(\sin x)]}$ (where, [·] denotes greatest integer function) is ...

69. If $f(x)=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin^2 n\theta}{\sin^2\theta}d\theta\,(n\in N)$, then the value of $\frac{f(15)+f(13)}{f(15)-f(a)}$ ...

70. Let $f(x)=\int_{-2}^{x} e^{(1+t)^2}dt$ and $g(x)=(h/x)$, where $h(x)$ is defined for all $x\in h$. If $g^{1/2}=e^4$ and $h'(2)=1$. Then, absolute value of sun for all possible value of $h(2)$, is ...

71. If $=\int_0^{\pi/2}\sin x\cdot\log(\sin x)dx=\log\left(\frac{K}{e}\right)$. Then, the value of K is ...

Definite Integrals Exercise 6 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

72. The value of $\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ is equal to **[Only One Correct Option 2016]**

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$

(c) $\pi^2-e^{-\pi/2}$ (d) $\pi^2+e^{\pi/2}$

73. The total number of distincts $x\in[0,1]$ for which $\int_0^x\frac{t^2}{1+t^4}dt=2x-1$ is **[Integer Type 2016]**

74. Let $f(x)=7\tan^8 x+7\tan^6 x-3\tan^4 x-3\tan^2 x$ for all $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then, the correct expression(s) is/are **[More than One Correct Option 2015]**

(a) $\int_0^{\pi/4}x f(x)\,dx=\frac{1}{12}$ (b) $\int_0^{\pi/4}f(x)\,dx=0$

(c) $\int_0^{\pi/4}x f(x)\,dx=\frac{1}{6}$ (d) $\int_0^{\pi/4}f(x)\,dx=1$

75. Let $f'(x)=\frac{192x^3}{2+\sin^4\pi x}$ for all $x\in R$ with $f\left(\frac{1}{2}\right)=0$. If $m\le\int_{1/2}^{1}f(x)\,dx\le M$, then the possible values of m and M are **[More than One Correct Option 2015]**

(a) $m=13, M=24$ (b) $m=\frac{1}{4}, M=\frac{1}{2}$

(c) $m=-11, M=0$ (d) $m=1, M=12$

76. The option(s) with the values of a and L that satisfy the equation $\frac{\int_0^{4\pi}e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi}e^t(\sin^6 at+\cos^4 at)dt}=L$, is/are **[More than One Correct Option 2015]**

(a) $a=2, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a=2, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a=4, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a=4, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

- **Directions** (Q. Nos. 77 to 78) Let $F:R\to R$ be a thrice differentiable function. Suppose that $F(1)=0, F(3)=-4$ and $F'(x)<0$ for all $x\in(1,3)$. Let $f(x)=xF(x)$ for all $x\in R$.

77. The correct statement(s) is/are **[Passage Based Questions 2015]**

(a) $f'(1)<0$

(b) $f(2)<0$

(c) $f'(x)\ne 0$ for any $x\in(1,3)$

(d) $f'(x)=0$ for some $x\in(1,3)$

78. If $\int_1^3 x^2F'(x)\,dx=-12$ and $\int_1^3 x^3F''(x)\,dx=40$, then the correct expression(s) is/are

(a) $9f'(3)+f'(1)-32=0$ (b) $\int_1^3 f(x)\,dx=12$

(c) $9f'(3)-f'(1)+32=0$ (d) $\int_1^3 f(x)\,dx=-12$

79. Let $f : R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is **[Integer Answer Type 2015]**

80. If $\alpha = \int_0^1 (e^{9x + 3\tan^{-1} x}) \left(\frac{12 + 9x^2}{1 + x^2} \right) dx$, where $\tan^{-1} x$ takes only principal values, then value of $\left(\log_e |1 + \alpha| - \frac{3\pi}{4} \right)$ is **[Integer Answer Type 2015]**

81. The integral $\int_{\pi/4}^{\pi/2} (2 \operatorname{cosec} x)^{17} dx$ is equal to **[Only One Correct Option 2014]**

(a) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$
(b) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$
(c) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$
(d) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

82. Let $f : [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$.

Let $F(x) = \int_0^{x^2} f(\sqrt{t}) dt$, for $x \in [0, 2]$. If $F'(x) = f'(x), \forall x \in (0, 2)$, then $F(2)$ equals **[Only One Correct Option 2014]**

(a) $e^2 - 1$
(b) $e^4 - 1$
(c) $e - 1$
(d) e^4

83. Match the conditions/expressions in Column I with statement in Column II. **[Matching Type 2014]**

	Column I		Column II
A.	$\int_{-1}^{1} \frac{dx}{1+x^2}$	p.	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
B.	$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	q.	$2\log\left(\frac{2}{3}\right)$
C.	$\int_2^3 \frac{dx}{1-x^2}$	r.	$\frac{\pi}{3}$
D.	$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	s.	$\frac{\pi}{2}$

84. Match List I with List II and select the correct answer using codes given below the lists. **[Matching Type 2014]**

	List I		List II
A.	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\, dx = 1$, is	p.	8
B.	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	q.	2
C.	$\int_{-2}^{2} \frac{3x^2}{1+e^x} dx$ equals	r.	4
D.	$\frac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	s.	0

Codes

	A	B	C	D
(a)	r	q	s	p
(b)	q	r	s	p
(c)	r	q	p	s
(d)	q	r	p	s

85. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2}(1 - x^2)^5 \right\} dx$ is **[Integer Answer Type 2014]**

86. The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \log \frac{\pi - x}{\pi + x} \right) \cos x\, dx$ is **[Only One Correct Option 2012]**

(a) 0
(b) $\frac{\pi^2}{2} - 4$
(c) $\frac{\pi^2}{2} + 4$
(d) $\frac{\pi^2}{2}$

87. The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$ is **[Integer Answer Type 2011]**

(a) $\frac{1}{4}\log\frac{3}{2}$
(b) $\frac{1}{2}\log\frac{3}{2}$
(c) $\log\frac{3}{2}$
(d) $\frac{1}{6}\log\frac{3}{2}$

88. Let $f : [1, \infty] \to [2, \infty]$ be differentiable function such that $f(1) = 2$. If $6\int_1^x f(x) = dt = 3x f(x) - x^3, \forall x \ge 1$ then the value of $f(2)$ is .... **[Integer Answer Type 2011]**

89. The value(s) of $\int_0^1 \frac{x^4 (1-x)^4}{1+x^2} dx$ is (are) **[Only One Correct Option 2010]**

(a) $\frac{22}{7} - \pi$
(b) $\frac{2}{105}$
(c) 0
(d) $\frac{71}{15} - \frac{3\pi}{2}$

90. For $a \in R$ (the set of all real numbers), $a \ne -1$,

$$\lim_{n \to \infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1} [(na+1) + (na+2) + \ldots + (na+n)]} = \frac{1}{60}.$$

Then, a is equal to **[More than One Correct Option 2010]**

(a) 5
(b) 7
(c) $\frac{-15}{2}$
(d) $\frac{-17}{2}$

- **Directions** (Q Nos. 91 to 92) Let $f(x)=(1-x)^2\sin^2 x+x^2$, $\forall x\in R$ and $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt\ \forall x\in(1,\infty)$.

[Passage Based Questions 2010]

91. Consider the statements

P: There exists some $x\in R$ such that, $f(x)+2x=2(1+x^2)$.

Q: There exists some $x\in R$ such that $2f(x)+1=2x(1+x)$.

Then,

(a) both P and Q are true (b) P is true and Q is false
(c) P is false and Q is true (d) both P and Q are false

92. Which of the following is true?

(a) g is increasing on $(1,\infty)$
(b) g is decreasing on $(1,\infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2,\infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2,\infty)$

93. For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x)=\begin{cases} x-[x[, & \text{if } f(x) \text{ is odd}\\ 1+[x[-x, & \text{if } f(x) \text{ is even}\end{cases}$$

Then, the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx$ is...... **[Integer Answer Type 2010]**

94. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt$, $0\le x\le 1$ and $f(0)=0$, then **[Only One Option Correct 2009]**

(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$ (b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ (d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

95. If $I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$, $n=0,1,2,\ldots$, then

[More than One Correct 2009]

(a) $I_n=I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1}=10\pi$

(c) $\sum_{m=1}^{10} I_{2m}=0$ (d) $I_n=I_{n+1}$

96. Let $S_n=\sum_{k=0}^{n}\frac{n}{n^2+kn+k^2}$ and $T_n=\sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$, for $n=1,2,3,\ldots$, then **[More than One Correct Option 2008]**

(a) $S_n<\frac{\pi}{3\sqrt{3}}$ (b) $S_n>\frac{\pi}{3\sqrt{3}}$

(c) $T_n<\frac{\pi}{3\sqrt{3}}$ (d) $T_n>\frac{\pi}{3\sqrt{3}}$

(ii) JEE Main & AIEEE

97. The integral $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to **[2017 JEE Main]**

(a) -1 (b) -2
(c) 2 (d) 4

98. Let $I_n=\int\tan^n x\,dx,(n>1)$. $I_4+I_6=a\tan^5 x+bx^5+C$, where C is a constant of integration, then the ordered pair (a, b) is equal to **[2017 JEE Main]**

(a) $\left(-\frac{1}{5},0\right)$ (b) $\left(-\frac{1}{5},1\right)$ (c) $\left(\frac{1}{5},0\right)$ (d) $\left(\frac{1}{5},-1\right)$

99. $\lim_{n\to\infty}\left[\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right]^{1/n}$ is equal to **[2016 JEE Main]**

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$ (c) $\frac{9}{e^2}$ (d) $3\log 3-2$

100. The integral $\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to

[2015 JEE Main]

(a) 2 (b) 4 (c) 1 (d) 6

101. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ is equal to **[2014 JEE Main]**

(a) $\pi-4$ (b) $\frac{2\pi}{3}-4-4\sqrt{3}$

(c) $4\sqrt{3}-4$ (d) $4\sqrt{3}-4-\pi/3$

102. Statement I The value of the integral **[2013 JEE Main]** $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$.

Statement II $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

(a) Statement I is true; Statement II is true; Statement II is a true explanation for Statement I
(b) Statement I is true; Statement II is true; Statement II is not a true explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

103. The intercepts on X-axis made by tangents to the curve, $y=\int_0^x|t|\,dt$, $x\in R$, which are parallel to the line $y=2x$, are equal to **[2013 JEE Main]**

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

104. If $g(x)=\int_0^x\cos 4t\,dt$, then $g(x+\pi)$ equals **[2012 AIEEE]**

(a) $\frac{g(x)}{g(\pi)}$ (b) $g(x)+g(\pi)$ (c) $g(x)-g(\pi)$ (d) $g(x)\cdot g(\pi)$

105. The value of $\int_0^1\frac{8\log(1+x)}{1+x^2}dx$ is **[2011 AIEEE]**

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{2}\log 2$ (c) $\log 2$ (d) $\pi\log 2$

106. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then, f has **[2011 AIEEE]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

107. Let $p(x)$ be a function defined on R such that $\lim_{x \to \infty} \frac{f(3x)}{f(x)} = 1, p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then, $\int_0^1 p(x)\,dx$ equals **[2010 AIEEE]**

(a) $\sqrt{41}$ (b) 21
(c) 41 (d) 42

108. $\int_0^{\pi} [\cot x]\,dx$, [ ] denotes the greatest integer function, is equal to **[2009 AIEEE]**

(a) $\frac{\pi}{2}$ (b) 1
(c) -1 (d) $-\frac{\pi}{2}$

109. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$.

Then, which one of the following is true? **[2008 AIEEE]**

(a) $I > \frac{2}{3}$ and $J > 2$ (b) $I < \frac{2}{3}$ and $J < 2$
(c) $I < \frac{2}{3}$ and $J > 2$ (d) $I > \frac{2}{3}$ and $J < 2$

Answers

Exercise for Session 1

1. $\frac{\pi + 2}{8}$ 2. 1 3. 2 4. $\frac{2}{3} - \frac{\sqrt{3}}{4}$ 5. $\frac{4\sqrt{2}}{3}$
6. -1 7. $\frac{1}{20} \log_e 3$ 8. π 9. $\frac{\pi}{2}(b-a)$
10. $\frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \log(\sqrt{2} - 1)$ 11. $\frac{-\pi}{2}$ 12. $1 + \left(\frac{\pi}{4} - \sqrt{2}\right) e^{1/\sqrt{2}}$
13. 10 ! 14. $n \,.\, n!$ 15. $[-\infty - 1] \cup [1, \infty]$

Exercise for Session 2

1. $\frac{\pi}{8} \log 2$ 2. 0 3. $\frac{1}{2}$
4. $-\frac{1}{8\sqrt{21}} \left\{\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right| - \log\left|\frac{2+\sqrt{21}}{2-\sqrt{21}}\right|\right\}$ 5. 0 6. 3
7. $\frac{\pi\alpha}{\sin\alpha}$ 8. 0 9. 0 11. $\frac{16}{\pi}$ 12. 0 13. $3e$
14. 2

Exercise for Session 3

1. 7 2. 9 3. 1 4. 3 5. $\frac{[x]}{\log 2}$ 7. $\frac{13}{12}$
8. $\left(\frac{[x][x]+1}{2}\right)^2$ 9. $10\pi - \tan 1$ 10. $\frac{1}{2}$ 11. $\log_e^2$
12. $10\pi - \sec 1$ 13. $\pi + \cot 1 + \cot 2$ 14. $\frac{1}{\eta - 1}$ 15. ?
16. $\sum_{r=1}^{k} (r-1) \cdot (r^{1/n} - (r-1)^{1/n}) + k(9 - k^{1/n})$
18. 3 19. 1 20. 2

Exercise for Session 4

1. 0 2. 0 3. $2 - \frac{\pi}{2}$ 4. 0 5. $\frac{\pi}{2\sqrt{3}}$ 6. $\frac{\pi}{2}$
7. $-\frac{1}{2}$ 8. $\frac{\pi}{2}$ 9. 0 10. a and c 11. 0 12. -27
13. 2α 14. 0 15. 4 16. (c) 17. 2 18. 0
19. 10π 20. 0

Exercise for Session 5

1. 11 2. $\frac{[x]}{2}$ 3. $2\sqrt{2}n$ 4. 1 5. $\frac{1}{2}$
6. $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$ 7. $x = 1, -1$
8. $\int_a^b f(x)\,dx = \int_{f(x)dx}^{b+T} f(x)\,dx$ 9. 16 10. 4 11. $3I$
12. ± 1 13. $g(2n) = 0$ 14. $\left[-\frac{1}{4}, 2\right]$ 15. $-\frac{1}{2}$
16. $a = \frac{1}{4}, b = 1$ 17. $f(x) = \sqrt{2x}$ 18. $\frac{1}{3}$
19. (1) − sin (1) 20. $\frac{1}{6}$

Exercise for Session 6

1. (c)	2. (a)	3. (c)	4. (b)	5. (c)	6. (c)
7. (a)	8. (a)	9. (c)	10. (a)		

Chapter Exercises

1. (a)	2. (c)	3. (a)	4. (a)	5. (a)	6. (a)
7. (d)	8. (d)	9. (a)	10. (a)	11. (c)	12. (c)
13. (d)	14. (c)	15. (c)	16. (d)	17. (d)	18. (a)
19. (b)	20. (a)	21. (d)	22. (c)	23. (d)	24. (a)
25. (a)	26. (b)	27. (a)	28. (d)	29. (a)	30. (c)
31. (a)	32. (b)	33. (a)	34. (b)	35. (b)	36. (a,b,c)
37. (c,d)	38. (a)	39. (a,b,c)		40. (c,d)	41. (a,b,d)
42. (b,d)	43. (c)	44. (d)	45. (a,b,c)	46. (d)	47. (a)
48. (b)	49. (a)	50. (a)	51. (b)	52. (b)	53. (c)
54. (a)	55. (a)	56. (c)	57. (a)	58. (b)	59. (a)
60. (a)	61. (c)	62. (c)			

63. (A) → (q), (B) → (s), (C) → (p), (D) → (r)
64. (A) → (q), (B) → (r), (C) → (s), (D) → (p)
65. (A) → (s), (B) → (p), (C) → (q)
66. (A) → (s), (B) → (r), (C) → (r), (D) → (q)

67. (8)	68. (0)	69. (3)	70. (2)	71. (2)	72. (a)
73. (1)	74. (a,b)	75. (b)	76. (a,c)	77. (a,b,c)	78. (c,d)
79. (0)	80. (9)	81. (a)	82. (b)		
83. A→s, B→s, C→p, D→r,			84. (d)	85. (2)	86. (b)
87. (a)	88. (8/3)	89. (a)	90. (b,d)	91. (c)	92. (b)
93. (4)	94. (c)	95. (a,b,c)	96. (d)	97. (c)	98. (c)
99. (b)	100. (c)	101. (d)	102. (d)	103. (a)	104. (c)
105. (d)	106. (c)	107. (b)	108. (d)	109. (b)	

Solutions

1. Let $I = \int_0^4 \frac{(y^2 - 4y + 5)\sin(y-2)dy}{(2y^2 - 8y + 1)}$

Put $y - 2 = z$

$\Rightarrow$ $dy = dz$

$\therefore$ $I = \int_{z=-2}^{2} \frac{(z^2+1)\sin z}{(2z^2-7)} dz$

$\Rightarrow$ $I = 0$, as $\int_{-a}^{a} f(x)\,dx = 0$, when $f(-x) = -f(x)$

and $f(z) = \frac{(z^2+1)\sin z}{(2z^2-7)} dz$ is an odd function.

2. Hence, $f(x) = x^2 + ax - b$

The solution of $f(x) =$ minium $f(x)$ is $x = \frac{-a}{2}$.

Given, $f(x)_{\text{minimum}}$ is at $x = 0$.

$\Rightarrow$ $\frac{-a}{2} = 0$ or $a = 0$

$\therefore$ $f(x) = x^2 - b = 0$ has roots α and β.

$\Rightarrow$ $\alpha = \sqrt{b}$ and $\beta = -\sqrt{b}$

$\Rightarrow$ $\int_\alpha^\beta x^3 dx = \int_{\sqrt{b}}^{-\sqrt{b}} x^3 dx = 0$

$$\left[\int_{-a}^{a} f(x)dx = 0, \text{ when } f(-x) = f(x)\right]$$

3. Here, $\int_{\pi/2}^{x} \sqrt{(3 - 2\sin^2 t)}\,dt + \int_0^y \cos t\,dt = 0$

Differentiating both the sides, we get

$$(\sqrt{3 - 2\sin^2 x}) \cdot 1 + (\cos y)\left(\frac{dy}{dx}\right) = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-\sqrt{3 - 2\sin^2 x}}{\cos y}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(\pi,\pi)} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

4. Here, $-4 \le x \le 4 \Rightarrow 0 \le \frac{x^2}{17} < 1$

$\therefore$ $\left[\frac{x^2}{17}\right] = 0$ and $\sin^{-1}(\sin x)$ is an odd function.

Let, $I = \int_{-4}^{4} \frac{\sin^{-1}(\sin x)}{(1+x^2)\times 1} dx + \int_{-4}^{4} \frac{\cos^{-1}(\cos x)}{(1+x^2)\times 1} dx$

$$= 0 + 2\int_0^\pi \frac{\cos^{-1}(\cos x)}{(1+x^2)} dx$$

$$I = 2\left[\int_0^\pi \frac{x}{1+x^2} dx + \int_\pi^4 \frac{2\pi}{(1+x^2)} dx\right]$$

$$= \log(1+\pi^2) + 2\pi(\tan^{-1} x)_\pi^4 - [\log(1+16) - \log(1+\pi^2)]$$

$$= \log(1+\pi^2) + 2\pi(\tan^{-1} 4 - \tan^{-1}\pi) - \log\frac{17}{1+\pi^2}$$

$$= \log\frac{(1+\pi^2)\cdot(1+\pi^2)}{17} + 2\pi\tan^{-1}\left[\tan^{-1}\left(\frac{4-\pi}{1+4\pi}\right)\right]$$

$$= \log\left(\frac{(1+\pi^2)^2}{17}\right) + 2\pi\tan^{-1}\left(\frac{4-\pi}{1+4\pi}\right)$$

On comparing with $\log\left(\frac{(1+\pi^2)^2}{a}\right) + b\pi\tan^{-1}\left(\frac{c-\pi}{1+c\pi}\right)$, we get

$a = 17$, $b = 2$ and $c = 4$

$\therefore$ $a - (2b + c) = 17 - 8 = 9$ and $\frac{a-5}{c} = 3$

Thus, the number of ways to distribute 9 distinct bijective into 3 persons equally is $\frac{9!}{(3!)^3}$.

5. Put $x = \tan\theta$

$\therefore$ $I = \int_0^{\pi/2} \frac{d\theta}{1 + (\tan\theta)^a} = \int_0^{\pi/2} \frac{(\cos\theta)^a}{(\sin\theta)^a + (\cos\theta)^a} d\theta \Rightarrow I = \frac{\pi}{4}$

6. $I = \int_0^{3\pi/4} (\sin x + \cos x)\,dx + \int_0^{3\pi/4} \underset{\text{I}}{x}\underbrace{(\sin x - \cos x)}_{\text{II}}\,dx$

$$= \int_0^{3\pi/4} (\sin x + \cos x)\,dx$$

$$+ [x(-\cos x - \sin x)]_0^{3\pi/4} + \int_0^{3\pi/4} (\sin x + \cos x)\,dx$$

$$= 2\int_0^{3\pi/4} (\sin x + \cos x)\,dx + 0 = 2[-\cos x + \sin x]_0^{3\pi/4}$$

$$= 2\left[+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1\right] = 2(\sqrt{2} + 1) = 2\tan\frac{3\pi}{8}$$

7. We have, $C_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\tan^{-1}(nx)}{\sin^{-1}(nx)} dx$

$$C_n = \frac{1}{n}\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1}(t)}{\sin^{-1}(t)} dt \qquad \text{(put } nx = t\text{)}$$

Now, $L = \lim_{n\to\infty} n^2 \cdot C_n = \lim_{n\to\infty} n \int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t} dt \, (\infty \times 0)$

$$L = \frac{\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t} dt}{\frac{1}{n}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

Applying Leibnitz rule,

$$L = \lim_{n\to\infty} \frac{0 - \frac{\tan^{-1}\frac{n}{n+1}}{\sin^{-1}\frac{n}{n+1}}\left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}} = \frac{\pi}{4}\cdot\frac{2}{\pi} = \frac{1}{2}$$

8. $\int_{-3}^{3} \frac{t^2 \sin 2t}{t^2 + 1} dt = 0$ as the integrand is an odd function.

Also, $\int_0^1 \frac{dt}{t^2 + 2t\cos\alpha + 1}$

$$= \frac{1}{\sin\alpha}\left|\tan^{-1}\frac{t + \cos\alpha}{\sin\alpha}\right|_0^1 = \frac{\alpha}{2\sin\alpha}$$

Thus, the given equation reduces to

$$x^2 \cdot \frac{\alpha}{2\sin\alpha} - 2 = 0 \Rightarrow x = \pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$$

9. $f'(x) = e^{g(x)} \cdot g'(x)$ and $g'(x) = \dfrac{x}{1+x^4}$

$\therefore \quad f'(x) = e^{g(x)} \cdot \dfrac{x}{1+x^4} \quad e^{g(2)} = e^0 = 1$

Hence, $\quad f'(2) = e^{g(2)} \cdot g'(2) = e^0 \cdot \dfrac{2}{17} = \dfrac{2}{17}$

10. $\displaystyle \lim_{t\to 0} \ln\left(\frac{1}{t}\int_0^t (1 + a\sin bx)^{c/x}\, dx\right)$

$$= \ln \lim_{t\to 0} \frac{\int_0^t (1 + a\sin b\, x)^{c/x}\, dx}{t}$$

$$= \ln \lim_{t\to 0} \frac{(1 + a\sin bt\,)^{c/t}}{1} = \ln e^{\lim_{t\to 0} \frac{a\sin bt}{t/c}} = \ln e^{abc} = abc$$

11. $T_r = \dfrac{1}{\sqrt{\dfrac{r}{n}} \cdot n\left(3\sqrt{\dfrac{r}{n}} + 4\right)^2}$

$$S = \frac{1}{n}\sum_{1}^{4n} \frac{1}{\left(3\sqrt{\frac{r}{n}} + 4\right)^2 \cdot \sqrt{\frac{r}{n}}} = \int_0^4 \frac{dx}{\sqrt{x}\,(3\sqrt{x} + 4)^2}$$

Put $3\sqrt{x} + 4 = t \Rightarrow \dfrac{3}{2}\cdot\dfrac{1}{\sqrt{x}}\, dx = dt$

$$= \frac{2}{3}\int_4^{10} \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^{4} = \frac{2}{3}\left[\frac{1}{4} - \frac{1}{10}\right] = \frac{2}{3}\cdot\frac{6}{40} = \frac{1}{10}$$

12. Given, $f(x) = \int_{-1}^{x} e^{t^2}\, dt; h(x) = f(1 + g(x)); g(x) < 0$ for $x > 0$

Now, $\quad h(x) = \int_{-1}^{1+g(x)} e^{t^2}\, dt = f(1 + g(x))$ (given)

Differentiating, we get $\; h'(x) = e^{(1+g(x))^2} \cdot g'(x)$

Now, $\quad h'(1) = e$ (given)

$\therefore \quad e^{(1+g(1))^2} \cdot g'(1) = e$

$\Rightarrow \quad (1 + g(1))^2 = 1$

$\Rightarrow \quad 1 + g(1) = \pm 1$

$\Rightarrow \quad g(1) = 0$ (not possible)

or $\quad g(1) = -2$

13. Given, $f'(x) = f(x) \Rightarrow f(x) = Ce^x$

Since, $\; f(0) = 1 \quad \therefore \quad 1 = f(0) = C$

$\therefore\; f(x) = e^x$ and hence $g(x) = x^2 - e^x$

Thus, $\displaystyle \int_0^1 f(x)\, g(x)\, dx = \int_0^1 (x^2 e^x - e^{2x})\, dx$

$$= [x^2 e^x]_0^1 - 2\int_0^1 xe^x\, dx - \left[\frac{e^{2x}}{2}\right]_0^1$$

$$= (e - 0) - 2\,[(xe^x)_0^1 - (e^x)_0^1\,] - \frac{1}{2}(e^2 - 1)$$

$$= (e - 0) - 2\,[(e - 0) - (e - 1)] - \frac{1}{2}(e^2 - 1)$$

$$= e - \frac{1}{2}e^2 - \frac{3}{2}$$

14. Here, $\; f'(x) = \dfrac{1}{\sqrt{1 + g^2(x)}}\, g'(x);$

Now, $\; f'\left(\dfrac{\pi}{2}\right) = \dfrac{g'(\pi/2)}{\sqrt{1 + g^2(\pi/2)}}; \quad g\left(\dfrac{\pi}{2}\right) = 0 = g'\left(\dfrac{\pi}{2}\right)$

But $\quad g'(x) = [1 + \sin(\cos^2 x)](-\sin x)$

$$g'\left(\frac{\pi}{2}\right) = 1\,(-1) = -1$$

Hence, $\quad f'\left(\dfrac{\pi}{2}\right) = -1$ as $h'(0^+) = -1$

15. Clearly, f is an even function, hence

$$I_1 = \int_0^\pi f[\cos(\pi - x)]\, dx = \int_0^\pi f(-\cos x)\, dx$$

$$= \int_0^\pi f(\cos x)\, dx$$

$\therefore \quad I_1 = 2\displaystyle\int_0^{\pi/2} f(\cos x)\, dx \cdot 2\int_0^{\pi/2} f(\sin x)\, dx = 2I_2$

$\Rightarrow \dfrac{I_1}{I_2} = 2$

Aliter Let $u = \cos x \Rightarrow du = -\sin x\, dx$

$\therefore \quad I_1 = \displaystyle\int_{-1}^{1} \frac{f(u)}{\sqrt{1 - u^2}}\, du$

$\Rightarrow \quad I_1 = 2\displaystyle\int_0^1 \frac{f(u)}{\sqrt{1 - u^2}}\, du \quad$ ...(i)

Similarly, with $\sin t = t$,

$$I_2 = \int_0^1 \frac{f(t)}{\sqrt{1 - t^2}}\, dt \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\dfrac{I_1}{I_2} = 2$

16. Given, $I_1 = \displaystyle\int_{1-k}^{k} x f(x(1 - x))\, dx$ and

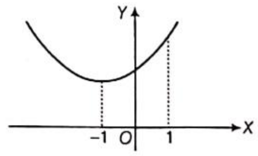
$$I_2 = \int_{1-k}^{k} f(x(1 - x))\, dx$$

Using King's property $I_1 = \displaystyle\int_{1-k}^{k} (1 - x)\, f(x(1 - x))\, dx$

$$2I_1 = \int_{1-k}^{k} f(x(1 - x))\, dx = I_2$$

$\therefore \quad \dfrac{I_2}{I_1} = 2$

17. $A = \displaystyle\int_{-1}^{1} (ax^2 + bx + c)\, dx = 2\int_0^1 (ax^2 + c)\, dx$



$$= 2\left[\frac{a}{3} + c\right] = \frac{1}{3}\,[2a + 6c]$$

$\therefore \quad A = \dfrac{1}{3}\,[f(-1) + 4f(0) + f(1)]$

18. $I(a)=\int_0^{\pi}\left(\frac{x^2}{a^2}+a^2\sin^2 x+2x\sin x\right)dx \qquad \left(\because \int_0^{\pi} x\sin x\,dx=\pi\right)$

$$\therefore \quad I(a)=\frac{\pi^3}{3a^2}+\frac{\pi a^2}{2}+2\pi=\pi\left[\frac{\pi^2}{3a^2}+\frac{a^2}{2}\right]+2\pi$$

$$=\pi\left[\left(\frac{\pi}{\sqrt{3}a}-\frac{a}{\sqrt{2}}\right)^2+\frac{2\pi}{\sqrt{6}}\right]+2\pi$$

$I(a)$ is minimum when $\frac{\pi}{\sqrt{3}a}=\frac{a}{\sqrt{2}} \Rightarrow a^2=\pi\sqrt{\frac{2}{3}}$

$$\Rightarrow \quad a=\sqrt{\pi\sqrt{\frac{2}{3}}}$$

Also, $\quad [I(a)]_{\min}=2\pi+\pi^2\sqrt{\frac{2}{3}}$

19. $$\left[\frac{t^2}{2}-\log_2 a\cdot t\right]_0^2=2-\log_2(a^2) \qquad (\because a>0)$$

$$\Rightarrow \quad (2-2\log_2 a)=2-2\log_2 a$$

$$\Rightarrow \quad 2\log_2 a=2\log_2 a \Rightarrow a\in R^+$$

20. Consider $I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^t}\,dt$ ...(i)

$$\Rightarrow \quad I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^{-t}}\,dt \qquad \text{(using King' property)}$$

$$I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)\,e^t}{1+e^t}\,dt \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_{-1/x}^{1/x}\ln(1+t^2)\,dt=2\int_0^{1/x}\ln(1+t^2)\,dt$$

$$\Rightarrow \quad I=\int_0^{1/x}\ln(1+t^2)\,dt$$

Hence, $\quad l=\lim_{x\to\infty}x^3\int_0^{1/x}\ln(1+t^2)\,dt$

$$=\lim_{x\to\infty}\frac{\int_0^{1/x}\ln(1+t^2)\,dt}{x^{-3}} \qquad \left(\frac{0}{0}\text{ form}\right)$$

Using L' Hospital's rule,

$$l=\lim_{x\to 0}\frac{x^4\ln\left(1+\frac{1}{x^2}\right)\cdot\left(-\frac{1}{x^2}\right)}{-3}=\frac{1}{3}\lim_{x\to\infty}x^2\ln\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{3}\lim_{x\to\infty}\ln\left(1+\frac{1}{x^2}\right)^{x^2} \qquad (1^{\infty}\text{ form})$$

$$=\lim_{x\to\infty}\frac{1}{3}x^2\left(1+\frac{1}{x^2}-1\right)=\frac{1}{3}$$

21. Put $\pi x=t \Rightarrow dx=\frac{dt}{\pi}$

$$\therefore \quad I=\frac{1}{\pi}\cdot\frac{\pi}{\pi}\int_0^{2008\pi}t\,|\sin t|\,dt=\frac{1}{\pi}\int_0^{2008\pi}t\,|\sin t|\,dt \qquad \text{...(i)}$$

$$\Rightarrow \quad I=\frac{1}{\pi}\int_0^{2008\pi}(2008\pi-t)\,|\sin t|\,dt \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I=\frac{2008\pi}{\pi}\int_0^{2008\pi}|\sin t|\,dt=(2008)^2\cdot\int_0^{\pi}|\sin t|\,dt$$

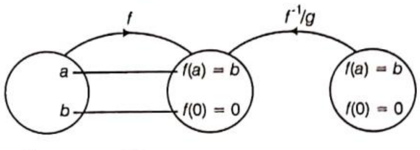
$$\Rightarrow \quad I=(2008)^2 \qquad \Rightarrow \sqrt{I}=2008$$

22. $$T_n=\frac{n}{n^2+k^2x^2}=\frac{1}{n[1+(k/n)^2x^2]};$$

$$S=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\underbrace{(k/n)^2}_{t}x^2}=\int_0^1\frac{dt}{1+t^2x^2}$$

$$=\frac{1}{x^2}\int_0^1\frac{dt}{t^2+(1/x^2)}=\left[\frac{1}{x}\tan^{-1}(tx)\right]_0^1=\frac{\tan^{-1}(x)}{x}$$

23. Let $y=f(x) \Rightarrow x=g(y)$ and $dy=f'(x)\,dx$



$$I=\int_0^a f(x)\,dx+\int_0^b g(y)\,dy;\, y=f(x)$$

$$\Rightarrow \quad x=f^{-1}(y)=g(y)=\int_0^a f(x)dx+\int_0^a xf'(x)\,dx$$

$$=\int_0^a (f(x)+xf'(x))\,dx=[xf(x)]_0^a=a\,f(a)=ab$$

24. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx$ ...(i)

$$\Rightarrow I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1-x^4}\right)dx \quad \text{(using King's property)}$$

$$\Rightarrow \quad I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\left(\pi-\cos^{-1}\frac{2x}{1-x^4}\right)dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\therefore \quad 2I=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\,dx \Rightarrow 2I=2\pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}\,dx$$

$$\therefore \quad k=\pi$$

25. Put $\ln x=t$ or $x=e^t \Rightarrow dx=e^t\,dt$

$$\therefore \quad I=\int_{-\infty}^{\infty}f(e^t+e^{-t})\frac{t}{e^t}e^t\,dt=\int_{-\infty}^{\infty}f(e^t+e^{-t})\,t\,dt=0$$

(as the function is odd)

Aliter I Put $x=\tan\theta$

$$\int_0^{\pi/2}f\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\ln\tan\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta$$

$$=\int_0^{\pi/2}f\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\ln\tan\theta}{\sin\theta\cos\theta}\,d\theta$$

Aliter II Put $x=1/t \Rightarrow I=-1 \Rightarrow 2I=0 \Rightarrow I=0$

26. $$\lim_{\lambda\to 0}\left(\int_0^1(1+x)^{\lambda}\,dx\right)^{1/\lambda}=\lim_{\lambda\to 0}\left(\left|\frac{(1+x)^{\lambda+1}}{\lambda+1}\right|_0^1\right)^{1/\lambda}$$

$$=\lim_{\lambda\to 0}\left(\frac{2^{\lambda+1}-1}{\lambda+1}\right)^{1/\lambda} \qquad (1^{\infty}\text{ form})$$

$$=e^{\lim_{\lambda\to 0}\frac{1}{\lambda}\left(\frac{2^{\lambda+1}-1-\lambda-1}{\lambda+1}\right)}=e^{\lim_{\lambda\to 0}\left(\frac{2^{\lambda+1}-2-\lambda}{\lambda(\lambda+1)}\right)}$$

$$=e^{\lim_{\lambda\to 0}\left(\frac{2(2^{\lambda}-1)}{\lambda}-1\right)}=e^{2\ln 2-1}=e^{\ln\left(\frac{4}{e}\right)}=\frac{4}{e}$$

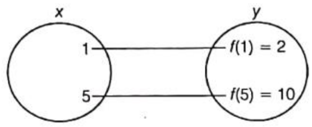
27. Let $y = f(x) \Rightarrow x = f^{-1}(y) = g(y)$

$$dy = f'(x)\,dx$$

$\therefore$ $$I = \int_1^5 f(x)\,dx + \int_1^5 xf'(x)\,dx$$

When $y = 2$, then $x = 1$

and when $y = 1$, then $x = 5$



$\therefore$ $$I = \int_1^5 (f(x) + x\,f'(x))dx = |x\,f(x)|_1^5$$

$$= 5f(5) - f(1) = 5 \cdot 10 - 2 = 48$$

28. $I = \int_0^{\pi/2} \sin x \sin 2x \sin 3x\,dx$ ...(i)

$$= \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) \sin 2\left(\frac{\pi}{2} - x\right) \sin 3\left(\frac{\pi}{2} - x\right) dx$$

$\Rightarrow$ $I = \int_0^{\pi/2} -\cos x \sin 2x \cos 3x\,dx$ ...(ii)

$\therefore$ $$2I = \int_0^{\pi/2} -\sin 2x\,(\cos x \cos 3x - \sin 3x \sin x)\,dx$$

$$= \int_0^{\pi/2} -\sin 2x \cos(4x)\,dx$$

$$= -\int_0^{\pi/2} \sin 2x\,(2\cos^2 2x - 1)\,dx$$

Put $\cos 2x = t \Rightarrow -\sin 2x \times 2\,dx = dt$

$\therefore$ $$2I = \int_1^{-1} \frac{1}{2}(2t^2 - 1)\,dt = \frac{1}{2}\left[\frac{2t^3}{3} - t\right]_1^{-1}$$

$$= \frac{1}{2}\left[\frac{2}{3}(-1)^3 - (-1) - \left(\frac{2}{3}(1)^3 - 1\right)\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3} + 1 - \frac{2}{3} + 1\right] = \frac{1}{2}\left[\frac{2}{3}\right] \Rightarrow I = \frac{1}{6}$$

29. Here, $f'(x) = (f(x))^2 > 0$; $\frac{d}{dx} f(g(x))|_{x=a}$

$$= f'(g(x)) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

It implies that $g(x)$ must be differentiable at $x = a$.

30. On putting $t = \frac{x^2}{z}$ and then solving,

$$\int_0^{\infty} \frac{\ln x}{x^2 + t^2}\,dt = \frac{2\pi \ln x}{x}$$

$\Rightarrow$ $$\frac{\ln x}{x} = \frac{\ln 2}{x}$$

$\Rightarrow$ $x = 2$ and 4 i.e. two solutions.

31. Put $\frac{x\pi}{n} = t$

$\therefore$ $$\int_0^{\frac{16n^2}{\pi}} \cos\frac{\pi}{2}\left[\frac{\pi x}{n}\right] dx = \frac{n}{\pi}\int_0^{16n} \cos\frac{\pi}{2}[t]\,dt$$

$$= \frac{4n^2}{\pi}\int_0^4 \cos\frac{\pi}{2}[t]\,dt = 0$$

32. $\int_{-2}^{-1} (ax^2 - 5)\,dx + \int_1^2 (bx + c)\,dx + 5$

$$= \int_{-2}^{-1} (ax^2 - 5 - bx + c + 5)\,dx = 0$$

Hence, $ax^2 - bx + c = 0$ has atleast one root in $(-2, -1)$.

33. $I = \int_3^6 ((\sqrt{x-3} + \sqrt{3}) + (\sqrt{3} - \sqrt{x-3}))\,dx = 6\sqrt{3}$

34. $I_1 = \int_{-1}^1 (\{x\} + \{x^3\})\{x^2\}\,dx$

$$= -2\int_0^1 \{x^2\}\,dx = \left[-2 \times \frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

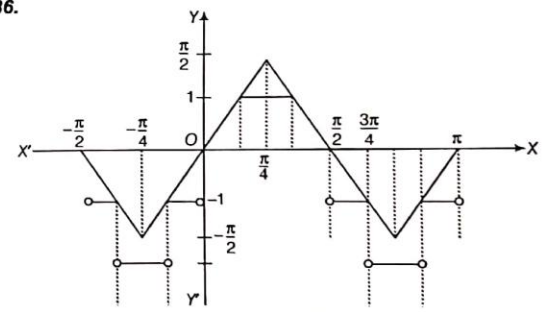
35. $J_n - J_{n-1} = \int_0^{\pi/2} \frac{\sin^2 nx - \sin^2 (n-1)x}{\sin^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x \cdot \sin x}{\sin^2 x}\,dx = I_n$$

i.e. $J_n - J_{n-1} = I_n$

$\Rightarrow$ $J_{n+1} - J_n = I_{n+1}$

36.



37. (a) $I = \int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = \int_0^{\pi/2} \ln(\tan x)\,dx$

$$I = -\int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = -I \Rightarrow I = 0$$

(b) $I = \int_0^{2\pi} \sin^3 x\,dx = -\int_0^{2\pi} \sin^3 x\,dx \Rightarrow I = 0$

(c) At $x = \frac{1}{t}$, $I = \int_e^{1/e} \frac{-(1/t^2)\,dt}{-1/t(\ln t)^{1/3}} = -\int_{1/e}^{e} \frac{dt}{t(\ln t)^{1/3}}$

$$I = -1 \text{ or } I = 0$$

(d) $\sqrt{\frac{1 + \cos 2x}{2}} > 0 \Rightarrow \int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\,dx > 0$

38. $I = \int_0^1 (10x^4 - 3x^2 - 1)\,dx = [2x^5 - x^3 - x]_0^1 = 0$

Since, $f(x)$ is even, hence must have a root in $(-1, 0)$.

39. We have, $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)\,dx$

$$I_1 = \int_0^{\pi/2} \cos(\pi \cos^2 x)\,dx$$

On adding, $2I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) + \cos(\pi \cos^2 x)\,dx$

$$= \int_0^{\pi/2} 2\cos\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\cos 2x\right) dx = 0$$

$\Rightarrow$ $I_1 = 0$ ...(i)

Now, $I_2 = \int_0^{\pi/2} \cos\{\pi(1-\cos 2x)\}\,dx$

$= -\int_0^{\pi/2} \cos(\pi \cos 2x)\,dx$

$= -\frac{1}{2}\int_0^{\pi} \cos(\pi \cos t)\,dt$ (put $2x = t$)

$= -\frac{2}{2}\int_0^{\pi/2} \cos(\pi \cos t)\,dt = I_3$

$\Rightarrow I_2 + I_3 = 0$

$I_3 = -\int_0^{\pi/2} \cos(\pi \sin t)\,dt$

$\therefore I_2 + I_3 = 0$ ...(ii)

Hence, $I_1 + I_2 + I_3 = 0$

40. $f(x) = 2\int_0^1 \underbrace{(1-t)}_{\text{I}}\,\underbrace{\cos(xt)}_{\text{II}}\,dt$

$$= 2\left[\left|(1-t)\frac{\sin xt}{x}\right|_0^1 + \frac{1}{x}\int_0^1 \sin xt\,dt\right] = 2\left[\left|0 - \frac{1}{x^2}\cos xt\right|_0^1\right]$$

$f(x) = 2\left[\frac{1-\cos x}{x^2}\right]$ $(x \neq 0)$

If $x = 0$, then $f(x) = \int_{-1}^1 (1-|t|)\,dt = 2\int_0^1 (1-t)\,dt = 1$

$\therefore$ option (c) is correct.

Hence, $f(x) = \begin{cases} 2\left(\frac{1-\cos x}{x^2}\right), & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

$\therefore$ f is continuous at $x = 0$.

$\therefore$ option (d) is correct.

41. Given, $f(f(x)) = -x + 1$

Replacing x by $f(x)$, we get

$f(f(f(x))) = -f(x) + 1$

$f(1-x) = -f(x) + 1$

$f(x) + f(1-x) = 1$ ...(i)

Now, $J = \int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ (using King's property)

$\Rightarrow 2J = \int_0^1 (f(x) + f(1-x))\,dx$

$\Rightarrow 2J = \int_0^1 dx = 1 \Rightarrow J = \frac{1}{2}$

Put $x = \frac{1}{4}$ in Eq. (i),

$F\left(\frac{1}{4}\right) + F\left(1 - \frac{1}{4}\right) = 1 \Rightarrow F\left(\frac{1}{4}\right) + F\left(\frac{3}{4}\right) = 1$

Now $I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)^3}\,dx$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right)^3}\,dx$$

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)^3}\,dx$

$$\therefore 2I = \int_0^{\pi/2} \frac{1}{(\sin x + \cos x)^2}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)^2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin^2\left(\frac{\pi}{4} + x\right)}$$

$$= \frac{1}{2}\int_0^{\pi/2} \operatorname{cosec}^2\left(\frac{\pi}{4} + x\right)dx = -\frac{1}{2}\left[\cot\left(\frac{\pi}{4} + x\right)\right]_0^{\pi/2}$$

$= -\frac{1}{2}[-1-1] = 1 \quad \Rightarrow I = \frac{1}{2}$

42. We have, $f(x) = x^2 + ax^2 + bx^3$

where, $a = \int_{-1}^1 t f(t)\,dt$ and $b = \int_{-1}^1 f(t)\,dt$

Now, $a = \int_{-1}^1 t[(a+1)t^2 + bt^3]\,dt$

$\Rightarrow a = 2b\int_0^1 t^4 dt = \frac{2b}{5}$ ...(i)

Again, $b = \int_{-1}^1 f(t)\,dt = \int_{-1}^1 ((a+1)t^2 + bt^3)\,dt$

$\Rightarrow b = 2\int_0^1 (a+1)t^2\,dt$

$\Rightarrow b = \frac{2(a+1)}{3}$ ...(ii)

From Eqs. (i) and (ii), $\frac{5a}{2} = \frac{2(a+1)}{3}$

$\Rightarrow \left(\frac{5}{2} - \frac{2}{3}\right)a = \frac{2}{3} \Rightarrow \frac{11}{6}a = \frac{2}{3}$

$\Rightarrow a = \frac{4}{11}$ and $b = \frac{10}{11}$

Hence, $\int_{-1}^1 t f(t)\,dt = \frac{4}{11}$ and $\int_{-1}^1 f(t)\,dt = \frac{10}{11}$

$\therefore$ $f(x) = (a+1)x^2 + bx^3$

$f(1) = (a+1) + b$

$f(-1) = (a+1) - b$

$\Rightarrow f(1) + f(-1) = 2(a+1) = \frac{30}{11}$

and $f(1) - f(-1) = 2b = \frac{20}{11}$

43. Given, $(f'(x))^2 + (g(x))^2 = 1$

$f(x) + \int_0^x g(t)\,dt = \sin x(\cos x - \sin x)$

Differentiating both the sides, we get

$f'(x) + g(x) = \cos 2x - \sin 2x$ ...(i)

Squaring both the sides of Eq. (i), we get

$(f'(x))^2 + (g(x))^2 = 2f'(x)\cdot g(x) = 1 - \sin 4x$

$\Rightarrow 1 + 2f'(x)\cdot g(x) = 1 - \sin 4x$

$\therefore 2f'(x)\,g(x) = -\sin 4x$

Now, substituting $g(x) = -\frac{\sin 4x}{2f'(x)}$ in Eq. (i), we get

$$f'(x) - \frac{\sin 4x}{2f'(x)} = \cos 2x - \sin 2x$$

Put $\quad f'(x) = t$

$\Rightarrow 2t^2 - 2(\cos 2x - \sin 2x)\, t - \sin 4x = 0$

$\Rightarrow t = \dfrac{2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8 \sin 4x}}{4}$

$\therefore \quad 4t = 2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8 \sin 4x}$

$\Rightarrow 2t = (\cos 2x - \sin 2x) \pm \sqrt{1 + \sin 4x}$

Taking + ve sign, $2t = \cos 2x - \sin 2x + \cos 2x + \sin 2x$

$\Rightarrow \quad t = \cos 2x$

Taking − ve sign, $t = -\sin 2x$

Since, $\quad f'(x) = \cos 2x$ or $f'(x) = -\sin 2x$

$$f(x) = \frac{1}{2}\sin 2x + C_1 \text{ or } f(x) = \frac{\cos 2x}{2} + C_2$$

$$f(0) = 0$$

$\Rightarrow \quad C_1 = 0$ and $C_2 = -1/2$

$\therefore \quad f(x) = \dfrac{1}{2}\sin 2x$ or $f(x) = \dfrac{\cos 2x - 1}{2}$

If $f'(x) = \cos 2x$, then $g(x) = -\sin 2x$

If $f'(x) = -\sin 2x$, then $g(x) = \cos 2x$

i.e. $f(x) = \dfrac{1}{2}\sin 2x$ and $g(x) = -\sin 2x$

$\Rightarrow f(x) = \dfrac{\cos 2x - 1}{2}$ and $g(x) = \cos 2x$

44. Consider $f(x) = \int_{-x}^{x} (\underbrace{t\sin at}_{\text{even}} + \underbrace{bt}_{\text{odd}} + \underbrace{c}_{\text{even}})dt$

$$= 2\int_0^x (t \sin at + c)\, dt$$

$$= 2\left[\left|-t\frac{\cos at}{a}\right|_0^x + \int_0^x \frac{\cos at}{a} dt + |ct|_0^x\right] \text{(using I.B.P.)}$$

$$= 2\left[\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax + cx\right]$$

$$\therefore \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} 2\left[-\frac{\cos ax}{a} + \frac{\sin ax}{a \cdot ax} + c\right]$$

$$= 2\left[-\frac{1}{a} + \frac{1}{a} + c\right] = 2c$$

45. Consider $I = \int_a^{\infty} \dfrac{n\,dx}{n^2\left(x^2 + \dfrac{1}{n^2}\right)} = \dfrac{1}{n}\cdot n(\tan^{-1} nx)_a^{\infty} = \left(\dfrac{\pi}{2} - \tan^{-1} an\right)$

$$\therefore \quad L = \lim_{n\to\infty}\left(\frac{\pi}{2} - \tan^{-1} an\right) = \begin{cases} \pi, & \text{if } a < 0 \\ \pi/2, & \text{if } a = 0 \\ 0, & \text{if } a > 0 \end{cases}$$

46. $\lim_{x\to 0} \dfrac{\int_0^x \dfrac{t^2 dt}{(a + t^r)^{1/p}}}{bx - \sin x} = \lim_{x\to 0} \dfrac{\dfrac{x^2}{(a + x^r)^{1/p}}}{b - \cos x}$ Using L'Hospital's rule

For existence of limit, $\lim_{x\to 0}$ denominator $= 0$

$\therefore \quad b - 1 = 0 \Rightarrow b = 1$

47. $l = \lim_{x\to 0} \dfrac{x^2}{(a + x^r)^{1/p}} \cdot \dfrac{x^2}{(1 - \cos x)} \cdot \dfrac{1}{x^2}$

$$= 2\lim_{x\to 0} \frac{1}{(a + x^r)^{1/p}} = \frac{2}{a^{1/p}}$$

If $p = 3$ and $l = 1$, then $1 = \dfrac{2}{a^{1/3}} \Rightarrow a = 8$

48. If, $p = 2$ and $a = 9$, then $l = \dfrac{2}{9^{1/2}} = \dfrac{2}{3}$

49. Here, $\quad f(x) = e^x \underbrace{\int_0^1 e^t \cdot f(t)\, dt}_{A \text{ (say)}}$

$f(x) = Ae^x \quad$ ...(i)

$\Rightarrow \quad f(t) = Ae^t$

where, $\quad A = \int_0^1 e^t \cdot f(t)\, dt$

$\Rightarrow \quad A = \int_0^1 e^t \cdot Ae^t\, dt;\ A = A\int_0^1 e^{2t}\, dt$

Now, $A\left[\int_0^1 e^{2t} dt - 1\right] = 0 \Rightarrow A = 0$, as $\int_0^1 e^{2t}\, dt \neq 0$

Hence, $\quad f(x) = 0 \Rightarrow f(1) = 0$

50. Again, $\quad g(x) = e^x \int_0^1 e^t g(t)\, dt + x$

$\Rightarrow \quad g(x) = Be^x + x \quad$ ...(ii)

$\Rightarrow \quad g(t) = Be^t + t$

where, $\quad B = \int_0^1 e^t g(t)\, dt \Rightarrow B = \int_0^1 e^t (Be^t + t)\, dt$

$\Rightarrow \quad B = B\int_0^1 e^{2t}\, dt + \int_0^1 e^t \cdot t\, dt$

But $\quad \int_0^1 e^{2t}\, dt = \dfrac{1}{2}(e^2 - 1)$ and $\int_0^1 te^t\, dt = 1$

$\therefore \quad B = \dfrac{B}{2}(e^2 - 1) + 1$

$\Rightarrow \quad 2B = B(e^2 - 1) + 2$

$\Rightarrow \quad 3B = Be^2 + 2 \quad \Rightarrow \quad B = \dfrac{2}{3 - e^2}$

From Eq. (ii), $g(x) = \left(\dfrac{2}{3 - e^2}\right)e^x + x \Rightarrow g(0) = \dfrac{2}{3 - e^2}$

Also, $\quad f(0) = 0$

$\therefore \quad g(0) - f(0) = \dfrac{2}{3 - e^2} - 0 = \dfrac{2}{3 - e^2}$

51. $g(2) = \dfrac{2e^2}{3 - e^2} + 2 = \dfrac{6}{3 - e^2}$

$\therefore \quad \dfrac{g(0)}{g(2)} = \dfrac{2}{3 - e^2} \cdot \dfrac{3 - e^2}{6} = \dfrac{1}{3}$

Solutions (Q. Nos. 52 to 54)

We have the equations of the tangents to the curve $y = \int_{-\infty}^{x} f(t)dt$ and $y = f(x)$ at arbitrary points on them are

$Y - \int_{-x}^{x} f(t)\, dt = f(x)(X - x) \quad$ ...(i)

and $\quad Y - f(x) = f'(x)(X - x) \quad$ ...(ii)

As Eqs. (i) and (ii) intersect at the same point on the X-axis

Putting $Y = 0$ and equating x-coordinates, we have

$$x - \frac{f(x)}{f'(x} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)}$$

$$\Rightarrow \qquad \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \qquad \int_{-\infty}^{x} f(t)\,dt = cf(x) \qquad \text{...(iii)}$$

As, $f(0) = 1 \Rightarrow \int_{-\infty}^{0} f(t)\,dt = c \times 1 \Rightarrow c = \frac{1}{2}$

$\Rightarrow \int_{-\infty}^{x} f(t)\,dt = \frac{1}{2} f(x)$; differentiating both the sides and integrating and using boundary conditions, we get $f(x) = e^{2x}$; $y = 2ex$ is tangent to $y = e^{2x}$.

$\therefore$ Number of solutions = 1.

Cearly, $f(x)$ is increasing for all x.

$$\therefore \qquad \lim_{x \to \infty} (e^{2x})^{e^{-2x}} = 1 \qquad (\infty^0 \text{ form})$$

Solutions (Q. Nos. 55 to 57)

We have, $g(x) = g(0) + xg'(0) + \frac{x^2}{2} g''(0) = -bx^2 + cx - 6$

$h(x) = g(x) = 4x^4 - ax^3 + bx^2 - cx + 6 = 0$ has 4 distinct real roots. Using Descarte's rule of signs.

Given biquadratic equation has 4 distinct positive roots.

Let the roots be x_1, x_2, x_3 and x_4.

Now, $$\frac{\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \frac{4}{x_4}}{4} \geq \sqrt[4]{\frac{24}{x_1 x_2 x_3 x_4}}$$

$$\Rightarrow \quad 2 \geq 2 \Rightarrow \frac{1}{x_1} = \frac{2}{x_2} = \frac{3}{x_3} = \frac{4}{x_4} = k$$

$$\Rightarrow \qquad \frac{1}{x_1} \cdot \frac{2}{x_2} \cdot \frac{3}{x_3} \cdot \frac{4}{x_4} = k^4$$

$$\Rightarrow \qquad \frac{24}{3/2} = k^4 \Rightarrow k = 2$$

$\therefore$ Roots are $\frac{1}{2}, 1, \frac{3}{2}$ and 2. Also, $a = 20$ and $c = 25$

58. At $x = 1, y = 0, \frac{dy}{dx} = 2x \cdot (x^4)^2 - (x^2)^2 = 1$

$\therefore$ Equation of the tangent is $y = x - 1$.

59. $\frac{dy}{dx} = 4x^3(\ln x^4)^2 - 3x^2 (\ln x^3)^2$

$= 64x^3 (\ln x)^2 - 27x^2 (\ln x)^2$

$\therefore \lim_{x \to 0^+} \frac{dy}{dx} = 64 \lim_{x \to 0^+} x^3 (\ln x)^2 - 27 \lim_{x \to 0^+} x^2 (\ln x)^2 = 0$

60. We have, $\quad f(x) = \int_1^x e^{t^2/2} (1 - t^2)\,dt$

$\therefore \quad f'(x) = [e^{x^2/2} (1 - x^2)]^2$

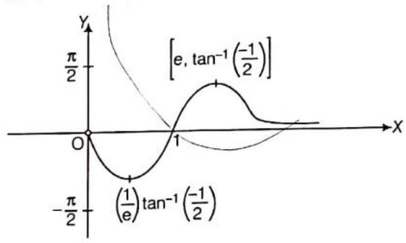
Now, $\quad f'(1) = e^{1/2} \cdot 0 = 0$

61. Here, $f(x) = \tan^{-1}\left(\frac{\log_e x}{\log_e x + 1}\right), 0 < x < \infty$

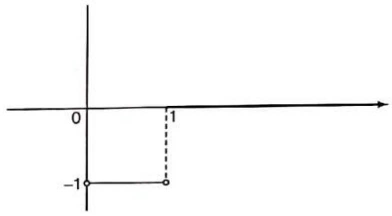
attains minimum at $x = \frac{1}{e}$ and maximum at $x = e$.

Also, $f(x)$ has $y = 0$ as asymptotes.

$\therefore$ $f(x)$ can be shown as



Clearly, $f(x)$ is neither injective nor subjective, also graph for $[f(x)]$ can be shown, as



62. $\int_0^{\infty} [f(x)]dx = \int_0^1 -1 \cdot dx + \int_1^{\infty} 0 \cdot dx = -(x)\Big|_0^1 = -1$

63. (A) For $a = 0$, $I = \int_0^T \sin^2 x\,dx = \int_0^T \frac{1 - \cos 2x}{2}\,dx$

$$= \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^T = \frac{T}{2} - \frac{1}{4}\sin 2T$$

$$\therefore \qquad L = \frac{1}{2} - \lim_{T \to \infty} \frac{1}{4}\frac{\sin 2T}{T} = \frac{1}{2}$$

(B) For $a = 1$, $\int_0^T 4\sin^2 x\,dx \Rightarrow L = 2$

(C) For $a = -1$, $\int_0^T 0\,dx = 0 \Rightarrow L = 0$

(D) For $a \neq 0, -1, 1$,

$$I = \int_0^T (\sin^2 x + \sin^2 ax + 2\sin x \cdot \sin ax)\,dx$$

$$= \int_0^T \left(\frac{1 - \cos 2x}{2} + \frac{1 - \cos 2ax}{2} + \cos(a-1)x - \cos(a+1)x\right)dx$$

$$= \left[x - \frac{1}{4}\sin 2x - \frac{1}{4a}\sin 2ax \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1}\right]_0^T$$

$$\therefore L = \lim_{T \to \infty} \frac{T}{T} - \lim_{T \to \infty} \frac{1}{T}$$

$$\left[\frac{1}{4}\sin 2x - \frac{1}{4a}\sin 2ax + \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1}\right]_0^T$$

$\Rightarrow L = 1$

64. $\because \quad f(\theta) = \left[\frac{(x + \sin\theta)^3}{3}\right]_0^1 = \frac{(1 + \sin\theta)^3 - \sin^3\theta}{3}$

and $g(\theta) = \left[\frac{(x + \cos\theta)^3}{3}\right]_0^1 = \frac{(1 + \cos\theta)^3 - \cos^3\theta}{3}$

$\therefore\ f(\theta) = \frac{1 + 3\sin\theta + 3\sin^2\theta}{3}$ and $g(\theta) = \frac{1 + 3\cos\theta + 3\cos^2\theta}{3}$

Now, $f(\theta) - g(\theta) = (\sin\theta - \cos\theta) + (\sin^2\theta - \cos^2\theta)$

and $f(\theta) - g(\theta) = (\sin\theta - \cos\theta)(1 + \sin\theta + \cos\theta)$

Now verify all matchings.

65. (A) Let $a = 2008$, then

$$I = \int_0^1 (1 + ax^a)\, e^{x^a}\, dx$$

$$I = \int_0^1 (e^{x^a} + ax^a e^{x^a})\, dx \qquad (\text{note} : ax^a = ax \cdot x^{a-1})$$

$$\therefore\ I = \int_0^1 (e^{x^a} + e^{x^a} \cdot x \cdot ax^{a-1})\, dx$$

$$= \int_0^1 (f(x) + x\, f'(x))\, dx, \text{ where } f(x) = e^{x^a}$$

Hence, $I = [xe^{x^a}]_0^1 = e$

(B) $I = I_1 + I_2$

Consider $I_2 = \int_1^{1/e} \sqrt{-\ln x}\, dx$

Put $\sqrt{-\ln x} = t \Rightarrow -\ln x = t^2 \Rightarrow x = e^{-t^2}$

$\Rightarrow\quad dx = -2t\, e^{-t^2}\, dt$

$$\therefore\ I_2 = \int_0^1 \underbrace{t}_{\text{I}} \cdot \underbrace{(-2te^{-t^2})}_{\text{II}}\, dt = [te^{-t^2}]_0^1 - \int_0^1 e^{-t^2} dt = \frac{1}{e} - \int_0^1 e^{-t^2} dt$$

Hence, $I_2 = \int_0^1 e^{-x}\, dx + \frac{1}{e} - \int_0^1 e^{-t^2}\, dt = \frac{1}{e} = e^{-1}$

Note that, if $f(x) = e^{-x^2}$, then $f^{-1}(x) = \sqrt{-\ln x}$

(C) $L = \lim_{n \to \infty} \left[\left(\frac{1}{n}\right)^1 \cdot \left(\frac{2}{n}\right)^2 \cdot \left(\frac{3}{n}\right)^3 \cdots \left(\frac{n}{n}\right)^n\right]^{\frac{1}{n^2}}$

$$\Rightarrow\quad \ln L = \lim_{n \to \infty} \frac{1}{n^2}$$

$$\left[1 \cdot \ln\left(\frac{1}{n}\right) + 2 \cdot \ln\left(\frac{2}{n}\right) + 3 \cdot \ln\left(\frac{3}{n}\right) + \ldots + n \cdot \ln\left(\frac{n}{n}\right)\right]$$

General term of $\ln L = \frac{r}{n^2} \ln \frac{r}{n}$

$$\text{Sum} = \frac{1}{n} \cdot \sum_{r=1}^{n} \frac{r}{n} \ln\left(\frac{r}{n}\right)$$

$$\ln L = \int_0^1 x \ln x\, dx = \left[\frac{x^2 \cdot \ln x}{2}\right]_0^1 - \frac{1}{2}\int_0^1 x^2\, \frac{1}{x}\, dx$$

$$= \left[0 - \frac{1}{2}\frac{x^2}{2}\right]_0^1 = -\frac{1}{4}$$

$\therefore\ L = e^{-1/4}$

66. (A) We have, $f'(x) = \frac{g'(x)}{\sqrt{1 + g^3(x)}}$

and $g'(x) = (1 + \sin(\cos^2 x))(-\sin x)$

Hence, $f'(x) = \frac{(1 + \sin(\cos^2 x))(-\sin x)}{\sqrt{1 + g^3(x)}}$

$$f'\left(\frac{\pi}{2}\right) = \frac{1 + 0}{\sqrt{1 + g^3\left(\frac{\pi}{2}\right)}} = -1,\ g\left(\frac{\pi}{2}\right) = 0$$

$\therefore\quad f'\left(\frac{\pi}{2}\right) = -1$

(B) We have, $\int_0^x f(x)\, dx = \{f(x)\}^2$

Differentiating both the sides, we get

$$f(x) = 2f(x) \cdot f'(x) \Rightarrow f'(x) = \frac{1}{2}$$

Integrating both the sides, $f(x) = \frac{1}{2}x + C$

where, $f(0) = 0 \Rightarrow C = 0$

$\Rightarrow\quad f(x) = \frac{x}{2} \Rightarrow f(2) = 1$

(C) Maximum when $a = -1$, $b = 2 \Rightarrow a + b = 1$

(D) If $\lim_{x \to 0} \frac{\sin 2x}{x^3} + a + \frac{b}{x^2} = 0$, then

$$\lim_{x \to 0} \frac{\sin 2x + ax^3 + bx}{x^3} = 0$$

For limit to exist $2 + b = 0 \Rightarrow b = -2$

$$\therefore\quad \lim_{x \to 0} \frac{\sin 2x + ax^3 - 2x}{x^3} = 0$$

Using left hand rule and solving, we get $a = \frac{4}{3}$

$\therefore\quad 3a + b = 2$

67. Let $B = \sum_{n=1} \frac{\sin nx}{4^n} = \frac{\sin x}{4} + \frac{\sin 2x}{4^2} + \ldots$

and $A = 1 + \cos\frac{x}{4} + \cos\frac{2x}{4^2} + \ldots$

$$\therefore\ A + iB = 1 + \frac{e^{ix}}{4} + \frac{e^{i2x}}{4^2} + \ldots = \frac{1}{1 - \frac{e^{ix}}{4}}$$

Thus, B imaginary part of $\frac{1}{1 - \frac{e^{ix}}{4}} = \frac{4}{4 - \cos x - i\sin x}$

$$\therefore\quad B = \frac{4\sin x}{17 - \theta\cos x} \Rightarrow f(x) = \frac{4\sin x}{17 - \theta\cos x}$$

and $\int_0^\pi f(x)dx = \int_0^\pi \frac{4\sin x}{17 - \theta\cos x} = \frac{1}{2}\log\left(\frac{25}{9}\right)$

$$= \log\left(\frac{5}{3}\right) = \log\left(\frac{m}{n}\right)$$

$\therefore\quad m + n = 8$

68. Here, $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x dx}{1 + 2[\sin^{-1}(\sin x)]}$

$$= \int_{-\pi/2}^{-1} \frac{\cos x}{-3} dx + \int_{-1}^{0} \frac{\cos x dx}{-1} + \int_0^1 \cos x dx + \int_1^{\pi/2} \frac{\cos x}{3} dx$$

$$= \frac{1}{3}\int_{-1}^{-\pi/2} \cos x dx + \int_0^{-1} \cos x dx + \int_0^1 \cos x dx + \frac{1}{3}\int_1^{\pi/2} \cos x dx$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos t(-dt) + \int_0^1 \cos(t) \cdot (-dt) + \int_0^1 \cos x dx + \frac{1}{3}\int_1^{\pi/2} \cos x dx$$

$= 0$

69. Here, $f(x+1)-f(x)=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin^2(n+1)\theta-\sin^2 n\theta}{\sin^2\theta}d\theta$

$$=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin(2n+1)\theta\cdot\sin\theta}{\sin^2\theta}d\theta$$

$$=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin(2n+1)\theta}{\sin^2\theta}d\theta$$

$$=\frac{1}{\pi}\left(\int_0^{\pi/2}\frac{\sin 2n\theta\cdot\cos\theta}{\sin\theta}d\theta+\int_0^{\pi/2}\cos 2n\theta\, d\theta\right)$$

Using, $\cos\theta+\cos 3\theta+\cos 5\theta+\ldots+\cos(2n-1)\theta$

$$=\frac{\sin\frac{n\cdot(2\theta)}{2}}{\sin\left(\frac{2\theta}{2}\right)}\cdot\cos(\theta+(n-1)\theta)$$

i.e. $\cos\theta+\cos 3\theta+\cos 5\theta+\ldots+\cos(2n-1)\theta=\frac{1}{2}\left(\frac{\sin 2n\theta}{\sin\theta}\right)$

$\therefore f(n+1)-f(n)=\frac{1}{\pi}$

$$\left[2\int_0^{\pi/2}\cos\theta+\cos 3\theta+\ldots+\cos(2n-1)\theta\cdot\cos\theta\, d\theta+0\right]$$

$$=\frac{1}{\pi}\int_0^{\pi/2}\{(2\cos\theta\cos\theta)+(2\cos 3\theta\cos\theta)+\ldots(2\cos(2n-1)\cdot\cos\theta)\}d\theta$$

$$=\frac{1}{\pi}\int_0^{\pi/2}1\cdot d\theta,\ \text{as}\int_0^{\pi/2}\cos 2n\theta\, d\theta=0$$

$\therefore \quad f(n+1)-f(n)=\frac{1}{2}$

If $\quad n=1,\ f(2)-f(1)=\frac{1}{2}\ \left(\text{as } f(1)=\frac{1}{2}\right)\Rightarrow f(2)=\frac{2}{2}$

If $\quad n=2,\ f(3)-f(2)=\frac{1}{2}$

$\Rightarrow f(3)=\frac{3}{2}$ and so on $\quad\therefore f(n)=\frac{n}{2}$

Hence, $\frac{f(15)+f(3)}{f(15)-f(9)}=\frac{\frac{15}{2}+\frac{3}{2}}{\frac{15}{2}-\frac{9}{2}}=3$

70. Here, $g(x)=\int_{-2}^{h(x)}e^{(1+t)^2}dt$

$\Rightarrow \quad g'(x)=h'(x)\cdot e^{(1+h(x))}$

$\Rightarrow \quad g'(x)=h'(x)\cdot e$

$\Rightarrow \quad g'(2)=h'(2)\cdot e^{(1+h(x))^2}$

$\Rightarrow e^4=1\cdot e^{(1+h(2))^2}$, given $g'(2)=e^4$ and $h'(2)=1$

$\therefore \quad (1+h(2))^2=4$

$\Rightarrow \quad 1+h(2)=2,-2$

$\therefore \quad h(2)=-3,1$

$\therefore$ Absolute sum for all possible values of $h(2)=|-3+1|=2$

71. Let $I=\int_0^{\pi/2}\sin x\cdot\log(\sin x)dx=\frac{1}{2}\int_0^{\pi/2}\sin x\cdot\log(\sin^2 x)dx$

$$=\frac{1}{2}\int_0^{\pi/2}\sin x\cdot\log(1-\cos^2 x)dx$$

Put $\cos x=t\Rightarrow -\sin x\, dx=dt$

$\therefore \quad I=\frac{1}{2}\int_0^1\log(1-t^2)dt$

$$=\frac{1}{2}\int_0^1\left(-t^2-\frac{(-t^2)^2}{2}+\frac{(-t^2)^3}{2}\ldots\right)dt$$

$$=\frac{1}{2}\left[\frac{-t^3}{3}-\frac{t^5}{10}+\frac{t^7}{21}\ldots\right]_0^1=\frac{1}{2}\left[\frac{-1}{3}-\frac{1}{10}+\frac{1}{21}\ldots\right]$$

$$=-\frac{1}{2}\left[\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\ldots\right]=-\left[\frac{1}{2\times 3}+\frac{1}{20}+\frac{1}{42}+\ldots\right]$$

$$=-\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\ldots\right]$$

$$=\log_e 2-1=\log_e\left(\frac{2}{e}\right)$$

$\therefore \quad K=2$

72. Let $\quad I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ ...(i)

$$\left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)\,dx\right]$$

$\Rightarrow \quad I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos(-x)}{1+e^{-x}}dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\left[\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\right]dx$$

$$=\int_{-\pi/2}^{\pi/2}x^2\cos x\cdot(1)\,dx$$

$$\left[\because \int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx,\text{ when } f(-x)=f(x)\right]$$

$\Rightarrow \quad 2I=2\int_0^{\pi/2}x^2\cos x\, dx$

Using integration by parts, we get

$$2I=2\left[x^2(\sin x)-(2x)(-\cos x)+(2)(-\sin x)\right]_0^{\pi/2}$$

$\Rightarrow \quad 2I=2\left[\frac{\pi^2}{4}-2\right]$

$\therefore \quad I=\frac{\pi^2}{4}-2$

73. Let $f(x)=\int_0^x\frac{t^2}{1+t^4}dt$

$\Rightarrow f'(x)=\frac{x^2}{1+x^4}>0$, for all $x\in[0,1]$

$\therefore f(x)$ is increasing.

At $x=0$, $f(0)=0$ and at $x=1$,

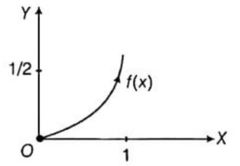
$$f(1)=\int_0^1\frac{t^2}{1+t^4}dt$$

Because, $\quad 0<\frac{t^2}{1+t^4}<\frac{1}{2}$

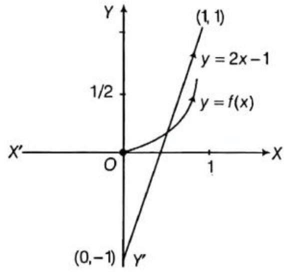
$\Rightarrow \quad \int_0^1 0\cdot dt<\int_0^1\frac{t^2}{1+t^4}dt<\int_0^1\frac{1}{2}\cdot dt$

$\Rightarrow \quad 0<f(1)<\frac{1}{2}$

Thus, $f(x)$ can be plotted as



$\therefore$ $y = f(x)$ and $y = 2x - 1$ can be shown as



From the graph, the total number of distinct solutions for $x \in (0, 1] = 1$. [as they intersect only at one point]

74. Here, $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$

for all $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore$ $f(x) = 7\tan^6 x \sec^2 x - 3\tan^2 x \sec^2 x$

$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

Now, $\displaystyle\int_0^{\pi/4} x f(x)dx = \int_0^{\pi/4} \underset{\text{I}}{x} \underset{\text{II}}{(7\tan^6 x - 3\tan^2 x)\sec^2 x}\, dx$

$= [x(\tan^7 x - \tan^3 x)]_0^{\pi/4}$

$\displaystyle - \int_0^{\pi/4} 1\,(\tan^7 x - \tan^3 x)dx$

$\displaystyle = 0 - \int_0^{\pi/4} \tan^3 x(\tan^4 x - 1)dx$

$\displaystyle = -\int_0^{\pi/4} \tan^3 x\,(\tan^2 x - 1)\sec^2 x\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore$ $\displaystyle\int_0^{\pi/4} x f(x)dx = -\int_0^1 t^3(t^2 - 1)dt$

$\displaystyle = \int_0^1 (t^3 - t^5)dt = \left[\frac{t^4}{4} - \frac{t^5}{5}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

Also, $\displaystyle\int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\sec^2 x\, dx$

$\displaystyle = \int_0^1 (7t^6 - 3t^2)dt = [t^7 - t^3]_0^1 = 0$

75. Here, $f'(x) = \dfrac{192\,x^3}{2 + \sin^4 \pi x}$

$\therefore$ $\dfrac{192x^3}{3} \le f'(x) \le \dfrac{192x^3}{2}$

On integrating between the limits $\frac{1}{2}$ to x, we get

$$\int_{1/2}^{x} \frac{192x^3}{3}dx \le \int_{1/2}^{x} f'(x)dx \le \int_{1/2}^{x} \frac{192x^3}{2}dx$$

$\Rightarrow$ $\dfrac{192}{12}\left(x^4 - \dfrac{1}{16}\right) \le f(x) - f(0) \le 24x^4 - \dfrac{3}{2}$

$\Rightarrow$ $16x^4 - 1 \le f(x) \le 24x^4 - \dfrac{3}{2}$

Again, integrating between the limits $\frac{1}{2}$ to 1, we get

$$\int_{1/2}^{1} (16x^4 - 1)dx \le \int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1}\left(24x^4 - \frac{3}{2}\right)dx$$

$\Rightarrow$ $\displaystyle\left[\frac{16x^5}{5} - x\right]_{1/2}^{1} \le \int_{1/2}^{1} f(x)dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{1/2}^{1}$

$\Rightarrow$ $\displaystyle\left(\frac{11}{5} + \frac{2}{5}\right) \le \int_{1/2}^{1} f(x)dx \le \left(\frac{33}{10} + \frac{6}{10}\right)$

$\Rightarrow$ $\displaystyle 2.6 \le \int_{1/2}^{1} f(x)dx \le 3.9$

76. Let $\displaystyle I_1 = \int_0^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$

$\displaystyle = \int_0^{\pi} e^t(\sin^6 at + \cos^6 at)dt + \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^6 at)dt$

$\displaystyle + \int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^6 at)dt + \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$

$\therefore$ $I_1 = I_2 + I_3 + I_4 + I_5$ ...(i)

Now, $\displaystyle I_3 = \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = \pi + t \Rightarrow dt = dt$

$\therefore$ $\displaystyle I_3 = \int_0^{\pi} e^{\pi + t} \cdot (\sin^6 at + \cos^6 at)dt$

$= e^t \cdot I_2$ ...(ii)

Now, $\displaystyle I_4 = \int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = 2\pi + t \Rightarrow dt = dt$

$\therefore$ $\displaystyle I_4 = \int_0^{\pi} e^{t + 2\pi}(\sin^6 at + \cos^6 at)dt$

$= e^{2\pi} \cdot I_2$ ...(iii)

and $\displaystyle I_5 = \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = 3\pi + t$

$\therefore$ $\displaystyle I_5 = \int_0^{\pi} e^{3\pi + t}(\sin^6 at + \cos^6 at)dt$

$= e^{3\pi} \cdot I_2$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), we get

$I_1 = I_2 + e^{\pi} \cdot I_2 + e^{2\pi} \cdot I_2 + e^{3\pi} \cdot I_2 = (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\, I_2$

$\therefore$ $L = \dfrac{\displaystyle\int_0^{4\pi} e^t(\sin^6 at + \cos^6 at)dt}{\displaystyle\int_0^{\pi} e^t(\sin^6 at + \cos^6 at)dt}$

$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) = \dfrac{1\cdot(e^{4\pi} - 1)}{e^{\pi} - 1}$ for $a \in R$

77. According to the given data, $F(x) < 0, \forall x \in (1, 3)$

We have, $f(x) = xF(x)$

$\Rightarrow$ $f'(x) = F(x) + xF'(x)$ ...(i)

$\Rightarrow$ $f'(1) = F(1) + F'(1) < 0$

[given $F(1) = 0$ and $F'(x) < 0$]

Also, $f(2) = 2F(2) < 0$ [using $F(x) < 0, \forall x \in (1, 3)$]

Now, $f'(x) = F(x) + xF'(x) < 0$

[using $F(x) < 0, \forall x \in (1, 3)$]

$\Rightarrow$ $f'(x) < 0$

78. Given, $\int_1^3 x^2F'(x)dx=-12$

$\Rightarrow \quad [x^2F(x)]_1^3-\int_1^3 2x\cdot F(x)dx=-12$

$\Rightarrow \quad 9F(3)-F(1)-2\int_1^3 f(x)dx=-12$ [$\because xF(x)=f(x)$, given]

$\Rightarrow \quad -36-0-2\int_1^3 f(x)dx=-12$

$\therefore \quad \int_1^3 f(x)dx=-12$

and $\int_1^3 x^3F''(x)dx=40$

$\Rightarrow \quad [x^3F'(x)]_1^3-\int_1^3 3x^2F'(x)dx=40$

$\Rightarrow \quad [x^2(xF'(x))]_1^3-3\times(-12)=40$

$\Rightarrow \quad \{x^2\cdot[f'(x)-F(x)]\}_1^3=4$

$\Rightarrow 9[f'(3)-F(3)]-[f'(1)-F(1)]=4$

$\Rightarrow \quad 9[f'(3)+4]-[f'(1)-0]=4$

$\Rightarrow \quad 9f'(3)-f'(1)=-32$

79. Here, $f(x)=\begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$

$$\therefore \quad I=\int_{-1}^{2}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$=\int_{-1}^{0}\frac{x f(x^2)}{2+f(x+1)}dx+\int_{0}^{1}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$+\int_{1}^{\sqrt2}\frac{x f(x^2)}{2+f(x+1)}dx+\int_{\sqrt2}^{\sqrt3}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$+\int_{\sqrt3}^{2}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$=\int_{-1}^{0}0\,dx+\int_0^1 0\,dx+\int_1^{\sqrt2}\frac{x\cdot 1}{2+0}dx+\int_{\sqrt2}^{\sqrt3}0\,dx+\int_{\sqrt3}^{2}0\,dx$$

$\because -1<x<0\Rightarrow 0<x^2<1\Rightarrow [x^2]=0,$

$0<x<1\Rightarrow 0<x^2<1\Rightarrow [x^2]=0,$

$1<x<\sqrt2\Rightarrow\begin{cases}1<x^2<2 & \Rightarrow [x^2]=1\\ 2<x+1<1+\sqrt2 & \Rightarrow f(x+1)=0,\end{cases}$

$\sqrt2<x<\sqrt3\Rightarrow 2<x^2<3\Rightarrow f(x^2)=0,$

and $\sqrt3<x<2\Rightarrow 3<x^2<4\Rightarrow f(x^2)=0$

$$\Rightarrow \quad I=\int_1^{\sqrt2}\frac{x}{2}dx=\left[\frac{x^2}{4}\right]_1^{\sqrt2}=\frac14(2-1)=\frac14$$

$\therefore \quad 4I=1\Rightarrow 4I-1=0$

80. Here, $\alpha=\int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Put $9x+3\tan^{-1}x=t$

$\Rightarrow \quad \left(9+\frac{3}{1+x^2}\right)dx=dt$

$\therefore \quad \alpha=\int_0^{9+3\pi/4}e^t\,dt=[e^t]_0^{9+3\pi/4}=e^{9+3\pi/4}-1$

$\Rightarrow \quad \log_e|1+\alpha|=9+\frac{3\pi}{4}$

$\Rightarrow \quad \log_e|\alpha+1|-\frac{3\pi}{4}=9$

81. Plan This type of question can be done using appropriate substitution.

Given, $I=\int_{\pi/4}^{\pi/2}(2\operatorname{cosec}x)^{17}\,dx$

$$=\int_{\pi/4}^{\pi/2}\frac{2^{17}(\operatorname{cosec}x)^{16}\operatorname{cosec}x(\operatorname{cosec}x+\cot x)}{(\operatorname{cosec}x+\cot x)}dx$$

Let $\operatorname{cosec}x+\cot x=t$

$\Rightarrow (-\operatorname{cosec}x\cdot\cot x-\operatorname{cosec}^2x)\,dx=dt$

and $\operatorname{cosec}x-\cot x=1/t$

$\Rightarrow \quad 2\operatorname{cosec}x=t+\frac1t$

$$\therefore \quad I=-\int_{\sqrt2+1}^{1}2^{17}\left(\frac{t+\frac1t}{2}\right)^{16}\frac{dt}{t}$$

Let $t=e^u\Rightarrow dt=e^u du$. When $t=1, e^u=1\Rightarrow u=0$

and when $t=\sqrt2+1, e^u=\sqrt2+1$

$\Rightarrow \quad u=\ln(\sqrt2+1)$

$$\Rightarrow \quad I=-\int_{\ln(\sqrt2+1)}^{0}2(e^u+e^{-u})^{16}\frac{e^u du}{e^u}$$

$$=2\int_0^{\ln(\sqrt2+1)}(e^u+e^{-u})^{16}\,du$$

82. Plan Newton-Leibnitz's formula

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)}f(t)\,dt\right]=f\{\psi(x)\}\left\{\frac{d}{dx}\psi(x)\right\}-f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$$

Given, $F(x)=\int_0^{x^2}f(\sqrt t)\,dt$

$\therefore \quad F'(x)=2x\,f(x)$

Also, $F'(x)=f'(x)$

$\Rightarrow \quad 2x\,f(x)=f'(x)$

$\Rightarrow \quad \frac{f'(x)}{f(x)}=2x$

$\Rightarrow \quad \int\frac{f'(x)}{f(x)}dx=\int 2x\,dx$

$\Rightarrow \quad \ln f(x)=x^2+c\Rightarrow f(x)=e^{x^2+c}$

$\Rightarrow \quad f(x)=K\,e^{x^2}$ $\quad [K=e^c]$

Now, $f(0)=1$

$\therefore \quad 1=K$

Hence, $f(x)=e^{x^2}$

$F(2)=\int_0^4 e^t dt=[e^t]_0^4=e^4-1$

83. (A) Let $I=\int_{-1}^{1}\frac{dx}{1+x^2}$

Put $x=\tan\theta$

$\Rightarrow dx=\sec^2\theta\,d\theta$

$\therefore \quad I=2\int_0^{\pi/4}d\theta=\frac{\pi}{2}$

(B) Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore \quad I = \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}$

(C) $\int_2^3 \frac{dx}{1-x^2} = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right]_2^3$

$= \frac{1}{2}\left[\log\left(\frac{4}{-2}\right) - \log\left(\frac{3}{-1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2}{3}\right)\right]$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

84. (A) **Plan**

(p) A polynomial satisfying the given conditions is taken.

(q) The other conditions are also applied and the number of polynomial is taken out.

Let $f(x) = ax^2 + bx + c$

$f(0) = 0 \Rightarrow c = 0$

Now, $\int_0^1 f(x)\, dx = 1 \Rightarrow \left(\frac{ax^3}{3} + \frac{bx^2}{2}\right)_0^1 = 1$

$\Rightarrow \quad \frac{\alpha}{3} + \frac{\beta}{2} = 1 \Rightarrow 2a + 3b = 6$

As a, b are non-negative integers.

So, $a = 0, b = 2$ or $a = 3, b = 0$

$\therefore \quad f(x) = 2x$ or $f(x) = 3x^2$

(B) **Plan** Such type of questions are converted into only sine or cosine expression and then the number of points of maxima in given interval are obtained.

$f(x) = \sin(x^2) + \cos(x^2)$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos(x^2) + \frac{1}{\sqrt{2}}\sin(x^2)\right]$

$= \sqrt{2}\left[\cos x^2 \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\sin(x^2)\right] = \sqrt{2}\cos\left(x^2 - \frac{\pi}{4}\right)$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi \Rightarrow x^2 = 2n\pi + \frac{\pi}{4}$

$\Rightarrow \quad x = \pm\sqrt{\frac{\pi}{4}}$, for $n = 0$

$x = \pm\sqrt{\frac{9\pi}{4}}$, for $n = 1$

So, $f(x)$ attains maximum at 4 points in $[-\sqrt{13}, \sqrt{13}]$.

(C) **Plan**

(p) $\int_{-a}^a f(x)\, dx = \int_{-a}^a f(-x)\, dx$

(q) $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(-x) = f(x)$, i.e. f is an even function.

$I = \int_{-2}^2 \frac{3x^2}{1+e^x}\, dx$ and $I = \int_{-2}^2 \frac{3x^2}{1+e^{-x}}\, dx$

$\Rightarrow \quad 2I = \int_{-2}^2 \left(\frac{3x^2}{1+e^x} + \frac{3x^2(e^x)}{e^x+1}\right) dx$

$2I = \int_{-2}^2 3x^2\, dx \Rightarrow 2I = 2\int_0^2 3x^2\, dx$

$I = [x^3]_0^2 = 8$

(D) **Plan** $\int_{-a}^a f(x)\, dx = 0$

If $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function.

Let $f(x) = \cos 2x \log\left(\frac{1+x}{1-x}\right)$

$f(-x) = \cos 2x \log\left(\frac{1-x}{1+x}\right) = -f(x)$

Hence, $f(x)$ is an odd function.

So, $\int_{-1/2}^{1/2} f(x)\, dx = 0$

(A) → (q); (B) → (r); (C) → (p); (D) → (s)

85. Plan Integration by parts

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx}[f(x)]\int g(x)\, dx\right) dx$$

Given, $I = \int_0^1 \underset{\text{I}}{4x^3} \underset{\text{II}}{\frac{d^2}{dx^2}(1-x^2)^5}\, dx$

$= \left[4x^3 \frac{d}{dx}(1-x^2)^5\right]_0^1 - \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5\, dx$

$= \left[4x^3 \times 5(1-x^2)^4\,(-2x)\right]_0^1$

$- 12\left\{[x^2(1-x^2)^5]_0^1 - \int_0^1 2x(1-x^2)^5\, dx\right\}$

$= 0 - 0 - 12(0-0) + 12\int_0^1 2x(1-x^2)^5\, dx$

$= 12 \times \left[-\frac{(1-x^2)^6}{6}\right]_0^1 = 12\left[0 + \frac{1}{6}\right] = 2$

86. $I = \int_{-\pi/2}^{\pi/2}\left[x^2 + \log\left(\frac{\pi-x}{\pi+x}\right)\right]\cos x\, dx$

As, $\int_{-a}^a f(x)\, dx = 0$, when $f(-x) = -f(x)$

$\therefore \quad I = \int_{-\pi/2}^{\pi/2} x^2\cos x\, dx + 0 = 2\int_0^{\pi/2}(x^2\cos x)\, dx$

$= 2\{(x^2\sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x\cdot\sin x\, dx\}$

$= 2\left[\frac{\pi^2}{4} - 2\{(-x\cdot\cos x)_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)\, dx\}\right]$

$= 2\left[\frac{\pi^2}{4} - 2(\sin x)_0^{\pi/2}\right] = 2\left[\frac{\pi^2}{4} - 2\right] = \left(\frac{\pi^2}{2} - 4\right)$

87. Put $x^2 = t \Rightarrow x\, dx = dt/2$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin t \cdot \frac{dt}{2}}{\sin t + \sin(\log 6 - t)}$ ...(i)

Using, $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin(\log 6 - (\log 2 + \log 3 - t))}\, dt$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin(t)}\, dt$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t}\, dt$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t + \sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t}\,dt$$

$\Rightarrow \quad 2I = \frac{1}{2}(t)_{\log 2}^{\log 3} = \frac{1}{2}(\log 3 - \log 2)$

$\therefore \quad I = \frac{1}{4}\log\left(\frac{3}{2}\right)$

88. Given, $f(1) = \frac{1}{3}$ and $6\int_1^x f(t)dt = 3x\, f(x) - x^3, \forall\, x \ge 1$

Using Newton-Leibnitz formula.
Differentiating both sides

$\Rightarrow \quad 6f(x)\cdot 1 - 0 = 3f(x) + 3xf'(x) - 3x^2$

$\Rightarrow \; 3xf'(x) - 3f(x) = 3x^2 \;\Rightarrow\; f'(x) - \frac{1}{x}f(x) = x$

$\Rightarrow \quad \frac{xf'(x) - f'(x)}{x^2} = 1 \quad \Rightarrow \quad \frac{d}{dx}\left\{\frac{x}{x}\right\} = 1$

On integrating both sides, we get

$\Rightarrow \quad \frac{f(x)}{x} = x + c \qquad \left[\because f(1) = \frac{1}{3}\right]$

$\frac{1}{3} = 1 + c \;\Rightarrow\; c = \frac{2}{3}$ and $f(x) = x^2 - \frac{2}{3}x$

$\therefore \quad f(2) = 4 - \frac{4}{3} = \frac{8}{3}$

Note Here, $f(1) = 2$, does not satisfy given function.

$\therefore \quad f(1) = \frac{1}{3}$

For that $f(x) = x^2 - \frac{2}{3}x$ and $f(2) = 4 - \frac{4}{3} = \frac{8}{3}$

89. Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$$= \int_0^1 \frac{(x^4-1)(1-x)^4 + (1-x)^4}{(1+x^2)}dx$$

$$= \int_0^1 (x^2-1)(1-x)^4\,dx + \int_0^1 \frac{(1+x^2-2x)^2}{(1+x^2)}dx$$

$$= \int_0^1 \left\{(x^2-1)(1-x)^4 + (1+x^2) - 4x + \frac{4x^2}{(1+x^2)}\right\}dx$$

$$= \int_0^1 \left((x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4} - 0\right) = \frac{22}{7} - \pi$$

90. Converting infinite series into definite integral

i.e. $\quad \lim_{n\to\infty} \frac{h(n)}{n}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=g(n)}^{h(n)} f\left(\frac{r}{n}\right) = \int f(x)dx$$

$\lim_{n\to\infty} \frac{g(n)}{n}$, where, $\frac{r}{n}$ is replaced with x.

Σ is replaced with integral.

Here, $\lim_{n\to\infty} \frac{1^a + 2^a + \ldots + n^a}{(n+1)^{a-1}\{(na+1) + (na+2) + \ldots + (na+n)\}} = \frac{1}{60}$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\cdot\left[n^2a + \frac{n(n+1)}{2}\right]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\cdot(2na + n + 1)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left(2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a\right)\cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{a-1}\cdot\left(2a + 1 + \frac{1}{n}\right)} = \frac{1}{60}$$

$\Rightarrow \quad 2\int_0^1 (x^a)dx \cdot \frac{1}{1\cdot(2a+1)} = \frac{1}{60}$

$\Rightarrow \quad \frac{2\cdot[x^{a+1}]_0^1}{(2a+1)\cdot(a+1)} = \frac{1}{60}$

$\therefore \quad \frac{2}{(2a+1)(a+1)} = \frac{1}{60}$

$\Rightarrow \quad (2a+1)(a+1) = 120$

$\Rightarrow \quad 2a^2 + 3a + 1 - 120 = 0$

$\Rightarrow \quad 2a^2 + 3a - 119 = 0$

$\Rightarrow \quad (2a+17)(a-7) = 0$

$\Rightarrow \quad a = 7, \frac{-17}{2}$

91. Here, $f(x) + 2x = (1-x)^2 \cdot \sin^2 x + x^2 + 2x$ ...(i)

where, $P: f(x) + 2x = 2(1+x)^2$ ...(ii)

$\therefore \quad 2(1+x^2) = (1-x)^2\sin^2 x + x^2 + 2x$

$\Rightarrow \quad (1-x)^2\sin^2 x = x^2 - 2x + 2$

$\Rightarrow \quad (1-x)^2\sin^2 x = (1-x)^2 + 1 \;\Rightarrow\; (1-x)^2\cos^2 x = -1$

which is never possible.

$\therefore$ P is false.

Again, let $Q: h(x) = 2f(x) + 1 - 2x(1+x)$

where, $\quad h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2f(1) + 1 - 4 = -3$, as $h(0)\,h(1) < 0$

$\Rightarrow$ $h(x)$ must have a solution.

$\therefore$ Q is true.

92. Here, $f(x) = (1-x)^2 \cdot \sin^2 x + x^2 \ge 0, \forall\, x$.

and $\quad g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log t\right) f(t)\,dt$

$\Rightarrow \quad g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\}\cdot \underbrace{f(x)}_{+\text{ve}}$ ...(i)

For $g'(x)$ to be increasing or decreasing,

let $\quad \phi(x) = \frac{2(x-1)}{(x+1)} - \log x$

$$\phi'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{x(x+1)^2}$$

$\phi'(x) < 0$, for $x > 1 \Rightarrow \phi(x) < \phi(1) \Rightarrow \phi(x) < 0$ ...(ii)

From Eqs. (i) and (ii), we get

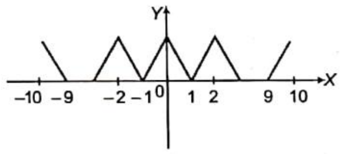
$$g'(x) < 0 \text{ for } x \in (1, \infty)$$

$\therefore$ $g(x)$ is decreasing for $x \in (1, \infty)$.

93. Given, $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd.} \\ 1 + [x] - x, & \text{if } [x] \text{ is even.} \end{cases}$

$f(x)$ and $\cos \pi x$ both are periodic with period 2 and both are even.

$$\therefore \quad \int_{-10}^{10} f(x) \cos \pi x \, dx = 2 \int_{0}^{10} f(x) \cos \pi x \, dx$$



$$= 10 \int_0^2 f(x) \cos \pi x \, dx$$

Now, $\int_0^1 f(x) \cos \pi x \, dx = \int_0^1 (1-x) \cos \pi x \, dx = -\int_0^1 u \cos \pi u \, du$

and $\int_1^2 f(x) \cos \pi x \, dx = \int_1^2 (x-1) \cos \pi x \, dx = -\int_0^1 u \cos \pi u \, du$

$$\therefore \quad \int_{-10}^{10} f(x) \cos \pi x \, dx = -20 \int_0^1 u \cos \pi u \, du = \frac{40}{\pi^2}$$

$$\Rightarrow \quad \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx = 4$$

94. Given $\int_0^x \sqrt{1 - \{f'(t)\}^2} \, dt = \int_0^x f(t) dt, 0 \le x \le 1$

Differentiating both sides w.r.t. x by using Leibnitz's rule, we get

$$\sqrt{1 - \{f'(x)\}^2} = f(x) \Rightarrow f'(x) = \pm \sqrt{1 - \{f(x)\}^2}$$

$$\Rightarrow \int \frac{f'(x)}{\sqrt{1 - \{f(x)\}^2}} dx = \pm \int dx \Rightarrow \sin^{-1} \{f(x)\} = \pm x + c$$

Put $\quad x = 0 \Rightarrow \sin^{-1} \{f(0)\} = c$

$\Rightarrow \quad c = \sin^{-1}(0) = 0 \quad [\because f(0) = 0]$

$\therefore \quad f(x) = \pm \sin x$

but $\quad f(x) \ge 0, \forall x \in [0, 1]$

$\therefore \quad f(x) = \sin x$

As we know that,

$$\sin x < x, \ \forall \ x > 0$$

$$\therefore \quad \sin\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } \sin\left(\frac{1}{3}\right) < \frac{1}{3}$$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } f\left(\frac{1}{3}\right) < \frac{1}{3}$$

95. Given $\quad I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$ ...(i)

Using $\int_a^b f(x) \, dx = \int_a^b f(b + a - x) \, dx$, we get

$$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1 + \pi^x) \sin x} dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we have

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx = 2 \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

$\left[\because f(x) = \frac{\sin nx}{\sin x} \text{ is an even function}\right]$

$$\Rightarrow \quad I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

Now, $\quad I_{n+2} - I_n = \int_0^{\pi} \frac{\sin (n+2)x - \sin nx}{\sin x} dx$

$$= \int_0^{\pi} \frac{2 \cos (n+1) x \cdot \sin x}{\sin x} dx$$

$$= 2 \int_0^{\pi} \cos (n+1)x \, dx = 2 \left[\frac{\sin (n+1) x}{(n+1)}\right]_0^{\pi} = 0$$

$\therefore \quad I_{n+2} = I_n$ ...(iii)

Since, $\quad I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$

$\Rightarrow \quad I_1 = \pi$ and $I_2 = 0$

From Eq. (iii) $\quad I_1 = I_3 = I_5 = \dots = \pi$

and $\quad I_2 = I_4 = I_6 = \dots = 0$

$$\Rightarrow \quad \sum_{m=1}^{10} I_{2m+1} = 10\pi \text{ and } \sum_{m=1}^{10} I_{2m} = 0$$

$\therefore$ Correct options are (a), (b), (c).

96. Given, $S_n = \sum_{k=0}^{n} \frac{n}{n^2 + kn + k^2}$

$$= \sum_{k=0}^{n} \frac{1}{n} \cdot \left(\frac{1}{1 + \frac{k}{n} + \frac{k^2}{n^2}}\right) < \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n} \left(\frac{1}{1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2}\right)$$

$$= \int_0^1 \frac{1}{1 + x + x^2} dx = \left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}} \cdot \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}} \quad \text{i.e. } S_n < \frac{\pi}{3\sqrt{3}}$$

Similarly, $T_n > \frac{\pi}{3\sqrt{3}}$

97.
$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x} \quad \text{...(i)}$$

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 - \cos x} \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii)

$$2I = \int_{\pi/4}^{3\pi/4} \frac{2}{\sin^2 x} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x \, dx$$

$$I = -(\cot x)_{\pi/4}^{3\pi/4} = 2$$

98. $I_4 + I_6 = \int (\tan^4 x + \tan^6 x) \, dx = \int \tan^4 x \sec^2 x \, dx$

$$= \frac{1}{5} \tan^5 x + c \quad \Rightarrow \quad a = \frac{1}{5}, b = 0$$

99. (b) Let $l = \lim_{n\to\infty}\left[\frac{(n+1)\cdot(n+2)\ldots(3n)}{n^{2n}}\right]^{\frac{1}{n}}$

$$= \lim_{n\to\infty}\left[\frac{(n+1)\cdot(n+2)\ldots(n+2n)}{n^{2n}}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\left(\frac{n+2n}{n}\right)\right]^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log l = \lim_{n\to\infty}\frac{1}{n}\left[\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{2n}{n}\right)\right\}\right]$$

$$\Rightarrow \log l = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{2n}{n}\right)\right]$$

$$\Rightarrow \log l = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log l = \int_0^2 \log(1+x)\,dx$$

$$\Rightarrow \log l = \left[\log(1+x)\cdot x - \int\frac{1}{1+x}\cdot x\,dx\right]_0^2$$

$$\Rightarrow \log l = [\log(1+x)\cdot x]_0^2 - \int_0^2\frac{x+1-1}{1+x}\,dx$$

$$\Rightarrow \log l = 2\cdot\log 3 - \int_0^2\left(1-\frac{1}{1+x}\right)dx$$

$$\Rightarrow \log l = 2\cdot\log 3 - [x - \log|1+x|]_0^2$$

$$\Rightarrow \log l = 2\cdot\log 3 - [2-\log 3]$$

$$\Rightarrow \log l = 3\cdot\log 3 - 2 \Rightarrow \log l = \log 27 - 2$$

$$\therefore\quad l = e^{\log 27 - 2} = 27\cdot e^{-2} = \frac{27}{e^2}$$

100. Central Idea Apply the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and then add. Let

$$I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$$

$$= \int_2^4 \frac{2\log x}{2\log x + \log(6-x)^2}dx$$

$$= \int_2^4 \frac{2\log x\,dx}{2[\log x + \log(6-x)]}$$

$$\Rightarrow I = \int_2^4 \frac{\log x\,dx}{[\log x + \log(6-x)]} \quad\ldots(i)$$

$$\Rightarrow I = \int_2^4 \frac{\log(6-x)}{\log(6-x)+\log x}dx \quad\ldots(ii)$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)}dx$$

$$\Rightarrow 2I = \int_2^4 dx = [x]_2^4$$

$$\Rightarrow 2I = 2 \Rightarrow I = 1$$

101. Plan Use the formula, $|x-a| = \begin{cases} x-a, & x \geq a \\ -(x-a), & x < a \end{cases}$

to break given integral in two parts and then integrate separately.

$$\int_0^\pi \sqrt{\left(1-2\sin\frac{x}{2}\right)^2}dx = \int_0^\pi \left|1-2\sin\frac{x}{2}\right|dx$$

$$= \int_0^{\frac{\pi}{3}}\left(1-2\sin\frac{x}{2}\right)dx - \int_{\frac{\pi}{3}}^{\pi}\left(1-2\sin\frac{x}{2}\right)dx$$

$$= \left(x+4\cos\frac{x}{2}\right)_0^{\frac{\pi}{3}} - \left(x+4\cos\frac{x}{2}\right)_{\frac{\pi}{3}}^{\pi} = 4\sqrt{3}-4-\frac{\pi}{3}$$

102. Let $I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ ...(i)

$$\therefore\quad I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$$

$$\Rightarrow\quad I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}} \quad\ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3}dx \Rightarrow 2I = [x]_{\pi/6}^{\pi/3}dx$$

$$\Rightarrow\quad I = \frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$$

Statement I is false.

But $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ is a true statement by property of definite integrals.

103. Given, $y = \int_0^x |t|\,dt$

$\therefore\quad \frac{dy}{dx} = |x|\cdot 1 - 0 = |x|$ [by Leibnitz's rule]

$\because$ Tangent to the curve $y = \int_0^x |t|\,dt$, $x \in R$ are parallel to the line $y = 2x$

$\therefore$ Slope of both are equal $\Rightarrow x = \pm 2$

Points, $y = \int_0^{\pm 2}|t|\,dt = \pm 2$

Equation of tangent is

$$y-2 = 2(x-2) \text{ and } y+2 = 2(x+2)$$

For x intercept put $y = 0$, we get

$$0-2 = 2(x-2) \text{ and } 0+2 = 2(x+2)$$

$$\Rightarrow\quad x = \pm 1$$

104. Given integral $g(x) = \int_0^x \cos 4t\,dt$

To find $g(x+\pi)$ in terms of $g(x)$ and $g(\pi)$.

$$g(x) = \int_0^x \cos 4t\,dt$$

$$\Rightarrow\quad g(x+\pi) = \int_{t=0}^{t=x+\pi}\cos 4t\,dt$$

$$= \int_0^x \cos 4t\,dt + \int_x^{x+\pi}\cos 4t\,dt$$

$$= g(x) + I_1 \quad \text{(say)}$$

$$I_1 = \int_x^{x+\pi}\cos 4t\,dt = \int_0^\pi \cos 4t\,dt \quad \text{(definite integral property)}$$

$$= g(\pi)$$

$\Rightarrow \quad g(x + \pi) = g(x) + g(\pi)$

But the value of I_1 is zero.

$$\Rightarrow \quad I_1 = \left[\frac{\sin 4t}{4}\right]_0^\pi = \left(\frac{\sin 4\pi}{4} - \frac{\sin 0}{4}\right) = 0$$

$\Rightarrow \quad g(x + \pi) = g(x) - g(\pi)$

In my opinion, the examiner has made this question keeping $g(x) + g(\pi)$ as the only answer in his/her mind. However, he/she did not realise that the value of the integral I_1 is actually zero. Hence, it does not matter whether you add to or subtract from $g(x)$.

105. $$I = \int_0^1 \frac{8 \log (1 + x)}{(1 + x^2)} dx$$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

When $x = 0 \Rightarrow \tan \theta = 0$

$\therefore \quad \theta = 0$

When $x = 1 = \tan\theta$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\pi/4} \frac{8 \log [1 + \tan\theta]}{1 + \tan^2 \theta} \cdot \sec^2 \theta \, d\theta$$

$$I = 8 \int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta \qquad \text{...(i)}$$

Using $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$, we get

$$I = 8 \int_0^{\pi/4} \log \left\{1 + \tan \left(\frac{\pi}{4} - \theta\right)\right\} d\theta$$

$$= 8 \int_0^{\pi/4} \log \left\{1 + \frac{1 - \tan \theta}{1 + \tan \theta}\right\} d\theta$$

$$= 8 \int_0^{\pi/4} \log \left\{\frac{2}{1 + \tan \theta}\right\} d\theta \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = 8 \int_0^{\pi/4} \left[\log \{1 + \tan \theta\} + \log \left\{\frac{2}{1 + \tan \theta}\right\}\right] d\theta$$

$$\Rightarrow \quad I = 4 \int_0^{\pi/4} \log 2 \, d\theta = 4 \cdot \log 2 (\theta)_0^{\pi/4}$$

$$= 4 \log 2 \cdot \left(\frac{\pi}{4} - 0\right) = \pi \log 2$$

106. If $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable for every x and $f(t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t) dt\right] = f[\psi(x)] \cdot \frac{d}{dx} \psi(x) - f[\phi(x)] \frac{d}{dx} \phi(x)$$

Here, $f(x) = \int_0^x \sqrt{t} \sin t \, dt$, where $x \in \left(0, \frac{5\pi}{2}\right)$

$$f'(x) = \{\sqrt{x} \sin x - 0\} \qquad \text{...(i)}$$

(using Newton-Leibnitz formula)

$\therefore \quad f'(x) = \sqrt{x} \sin x = 0$

$\Rightarrow \quad \sin x = 0$

$\therefore \quad x = \pi, 2\pi$

$$f''(x) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

At $x = \pi$, $f''(\pi) = -\sqrt{\pi} < 0$

$\therefore$ Local maximum at $x = \pi$. At $x = 2\pi$, $f''(2\pi) = \sqrt{2\pi} > 0$

$\therefore$ Local minimum at $x = 2\pi$.

107. We have, $p'(x) = p'(1 - x)$, $\forall x \in [0, 1]$, $p(0) = 1$, $p(1) = 41$

$\Rightarrow \quad p(x) = -p(1 - x) + C$

$\Rightarrow \quad 1 = -41 + C$

$\Rightarrow \quad C = 42$

$\therefore \quad p(x) + p(1 - x) = 42$

Now, $I = \int_0^1 p(x) dx = \int_0^1 p(1 - x) dx$

$$\Rightarrow \quad 2I = \int_0^1 (p(x) + p(1 - x)) dx = \int_0^1 42 dx = 42$$

$\Rightarrow \quad I = 21$

108. Let $I = \int_0^\pi [\cot x] \, dx \qquad \text{...(i)}$

$$\Rightarrow I = \int_0^\pi [\cot (\pi - x)] \, dx = \int_0^\pi [-\cot x] \, dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii),

$$2I = \int_0^\pi [\cot x] \, dx + \int_0^\pi [-\cot x] \, dx = \int_0^\pi (-1) \, dx$$

$[\because [x] + [-x] = -1$, if $x \notin z$ and 0, if $x \in z]$

$$= [-x]_0^\pi = -\pi$$

$$\therefore \quad I = -\frac{\pi}{2}$$

109. Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx$,

because in $x \in (0,1), x > \sin x$.

$$I < \int_0^1 \sqrt{x} \, dx = \frac{2}{3} [x^{3/2}]_0^1 \Rightarrow I < \frac{2}{3}$$

and $$J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 x^{-\frac{1}{2}} dx = 2$$

$J < 2$