4

Principle of Mathematical Induction

4.01 Introduction

We get special mathematical results with the help of general results of mathematics. Like,

General statement: Sum of digits of any number is divisible by 3 then number is divisible by 3.

Special statement: 210, is divisible by 3.

$\because$ Sum of digits of 210 = 2 + 1 + 0 = 3 this is divisible by 3.

$\therefore$ Hence, number 210 is also divisible by 3.

In this example a special mathematical result is inducted with the help of a general mathematical result. To find the mathematical result of this type is mathematical induction

Let us analyse a mathematical situation:

$$1 = \frac{1\cdot 2}{2}$$

$$1+2 = \frac{2\cdot 3}{2}$$

$$1+2+3 = \frac{3\cdot 4}{2}$$

$$1+2+3+4 = \frac{4\cdot 5}{2}$$

with the above pattern we come to the conclusion that

$$1+2+\ldots+n = \frac{n\cdot(n+1)}{2} \text{ where } n \in N$$

or

the sum of first n natural numbers is $\dfrac{n(n+1)}{2}$.

Induction is a process of conversion of results of special case into general results. But the result is not always true because these are the results obtained by some special cases for example in $n^2 + n + 41$, n = 1, 2 . . . 39

we get prime number but for n = 40

we do not get prime number. Hence taking some results as base a general result can not be calculated. Hence to find general results, it is necessary to follow a finite process. This process is called as principle of mathematical induction which is as follows:

The Principle of Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n such that

(i) *The statement is true for* $n=1$, i.e. P(1) *is true, and*

(ii) *If the statement P(m) is true then P(m+1) is also true.*

Hence, given statement is true for a natural number n = m then it is true for n = m + 1.

First step of this principle the statement of fact. In this we show that given statement is true for n = 1 but if it is true for $n \geq i$ then in this step at place of n = 1 true for n = i.

Second step of principle is the induction step of principle. In this step we consider true for natural number n = m and proves that statement is true for n = m + 1. For example,

(i) $P(n) = 3^{2n} - 1$, where $n \in N$, divisible by 8.

(ii) P(n) = 2 + 4 + . . . + 2n = n(n+1) where $n \in N$.

llustrative Examples

Example 1. Prove by using principle of mathematical induction

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ where } n \in N \text{ -}$$

Solution : We have to prove that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ where } n \in N \quad (1)$$

putting $n = 1$ L.H.S. $= 1^2 = 1$ and R.H.S. $= \frac{1.2.3}{6} = 1$

$\therefore$ L.H.S. = R.H.S.

(1) is true for, $n = 1$

Let (1), is true for $n = m$ i.e.

$$P(m) : 1^2 + 2^2 + \ldots + m^2 = \frac{m(m+1)(2m+1)}{6} \quad (2)$$

We shall now prove that $n = m+1$ is also true. Now, we have

$$1^2 + 2^2 + \ldots + m^2 + (m+1)^2 = \frac{(m+1)\{(m+1)+1\}\{2(m+1)+1\}}{6} \quad (3)$$

L.H.S. of (3) $= 1^2 + 2^2 + \ldots + m^2 + (m+1)^2$

$$= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \quad \text{[using (2)]}$$

$$= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6}$$

$$= \frac{(m+1)(2m^2 + 7m + 6)}{6}$$

$$= \frac{(m+1)(m+2)(2m+3)}{6}$$

$$=\frac{(m+1)\{(m+1)+1\}\{2(m+1)+1\}}{6}$$

= R.H.S.

Thus (1), *is true for* $n = m + 1$. Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers N.

Example 2. Using Principle of Mathematical Induction prove that

$$1\cdot 2+2\cdot 3+....+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}, \text{ where } n \in N.$$

Solution : We have to prove that

$$P(n):\ 1\cdot 2+2\cdot 3+....+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}, \text{ where } n \in N. \qquad (1)$$

For $n = 1$, L.H.S. $P(1):\ =1\cdot 2=2$

and R.H.S. $=\frac{1\cdot 2\cdot 3}{3}=2$

$\therefore$ L.H.S. = R.H.S.

Assume that $P(m)$ is true for some positive integer m, i.e.,

$$P(m):\ 1\cdot 2+2\cdot 3+....+m(m+1)=\frac{m(m+1)(m+2)}{3} \qquad (2)$$

We shall now prove that P(m + 1) is also true. Now, we have

$$1\cdot 2+2\cdot 3+....+m(m+1)+(m+1)(m+2)=\frac{(m+1)\{(m+1)+1\}\{(m+1)+2\}}{3} \qquad (3)$$

L.H.S. of (3) $=1\cdot 2+2\cdot 3+....+m(m+1)+(m+1)(m+2)$

$$=\frac{m(m+1)(m+2)}{3}+(m+1)(m+2) \qquad \text{[Using (2)]}$$

$$=(m+1)(m+2)\left(\frac{m}{3}+1\right)$$

$$=\frac{(m+1)(m+2)(m+3)}{3}$$

$$=\frac{(m+1)\{(m+1)+1\}\{(m+1)+2\}}{3}$$

=R.H.S.

Thus (1), *is true for* $n = m + 1$. Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers **N**.

Example 3. Prove the following by using the principle of mathematical induction.

$$\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+....+\frac{1}{(3n-1)\cdot(3n+2)}=\frac{n}{6n+4}, n\in N$$

Solution : We have to prove that

$$\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\ldots+\frac{1}{(3n-1)\cdot(3n+2)}=\frac{n}{6n+4},\ n\in\mathrm{N} \tag{1}$$

putting $n=1$, L.H.S. $=\frac{1}{2\cdot5}=\frac{1}{10}$ and R.H.S. $=\frac{1}{6+4}=\frac{1}{10}$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ P(1) is true

Assume that P(m) is true for some positive integer m i.e.,

$$P(m):\ \frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\ldots+\frac{1}{(3m-1)\cdot(3m+2)}=\frac{m}{6m+4} \tag{2}$$

We shall now prove that P (m + 1) is also true. Now, we have

$$P(m+1):\ \frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\ldots+\frac{1}{(3m-1)(3m+2)}+\frac{1}{(3m+2)(3m+5)}=\frac{m+1}{6(m+1)+4} \tag{3}$$

L.H.S. of (3)

$$=\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\ldots+\frac{1}{(3m-1)\cdot(3m+2)}+\frac{1}{(3m+2)\cdot(3m+5)}$$

$$=\frac{m}{6m+4}+\frac{1}{(3m+2)\cdot(3m+5)} \qquad \text{[using (2)]}$$

$$=\frac{1}{(3m+2)}\left(\frac{m}{2}+\frac{1}{3m+5}\right)=\frac{1}{(3m+2)}\left\{\frac{3m^2+5m+2}{2(3m+5)}\right\}$$

$$=\frac{1}{(3m+2)}\frac{(3m+2)(m+1)}{2(3m+5)}=\frac{(m+1)}{2(3m+5)}=\frac{m+1}{6m+10}$$

$$=\frac{m+1}{6(m+1)+4}$$

= R.H.S.

Thus $P(m+1)$ is true, whenever $P(m)$ is true.

Example 4. Prove the following by using the principle of mathematical induction

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{n}\right)=n+1,\ \text{where } n\in N.$$

Solution : We have to prove that

$$P(n):\ \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{n}\right)=n+1,\ \text{where } n\in N. \tag{1}$$

putting n = 1 in (1)

L.H.S. $= 1+1 = 2$

R.H.S. $= 1+1 = 2$

Thus, P(1), is true for $n = 1$

Let P(1), is true for $n = m$ i.e.

$$P(m) : \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{m}\right) = m+1 \tag{2}$$

We shall now prove that $P(m+1)$ is also true. Now, we have

$$P(m+1) : \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{m}\right)\left(1+\frac{1}{m+1}\right) = (m+1)+1 \tag{3}$$

L.H.S. of (3)

$$\left\{\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{m}\right)\right\}\left(1+\frac{1}{m+1}\right) = (m+1)\left(1+\frac{1}{m+1}\right) \qquad \text{[using (2)]}$$

$$= (m+1)\frac{m+2}{m+1} = (m+1)+1 = \text{R.H.S.}$$

Thus $P(m+1)$ is true, whenever $P(m)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural nubmer **N**.

Example 5. Prove that $n^2 > 2n$, by using principle of mathematical induction where $n \geq 3$

Solution : We have to prove that

$$n^2 > 2n, \quad n \geq 3 \tag{1}$$

In $P(n)$ putting $n = 3$ as $n \geq 3$ (given)

L.H.S. $= 3^2 = 9$

R.H.S. $= 2 \times 3 = 6$

$\therefore$ L.H.S. > R.H.S.

Hence P(3) is true.

Assume that $P(m)$ is true for any positive integer m, i.e.

$$m^2 > 2m \tag{2}$$

We shall now prove that $P(m+1)$ is true whenever $P(m)$ is true.

$$(m+1)^2 > 2(m+1) \tag{3}$$

L.H.S. of (3) $= (m+1)^2 = m^2 + 2m + 1$

$$> 2m + 2m + 1 = 2(m+1) + 2m - 1 \qquad \text{[Using (2)]}$$

$$> 2(m+1)$$

$\therefore$ L.H.S. > R.H.S.

Therefore, $P(m+1)$ is true when $P(m)$ is true. Hence, by principle of mathematical induction, $P(n)$ is true for every positive integer n.

Example 6. Using principle of mathematical induction prove that

$x^n - y^n$ is divisible by $x - y$ where $n \in N$.

Solution : We have to prove that

$P(n)$: $x^n - y^n$ is divisible by $x - y$ where $n \in N$

$x^n - y^n = k(x-y)$, where $n \in N$ and k, is a multiple of $(x - y)$ (1)

putting $n = 1$ in (1) L.H.S. $= x^1 - y^1$

$= 1(x - y)$

$= k(x-y)$ [here $k = 1 \in \boldsymbol{Z}$]

$=$ R.H.S.

Thus (1) is true for $n = 1$

Assume that $P(m)$ is true for any positive integer m, i.e.,

$P(m)$: $x^m - y^m = k_1(x-y)$, where k_1, is a factor of power $(m-1)$ in x and y (2)

We shall now prove that P(m + 1) is true whenever $P(m)$ is true.

$P(m+1)$: $x^{m+1} - y^{m+1} = k_2(x-y)$, where k_2, is a factor of power m in x and y (3)

L.H.S. of (3) $= x^{m+1} - y^{m+1}$

$= x^{m+1} - x^m y + x^m y - y^{m+1} = x^m(x-y) + y(x^m - y^m)$

$= x^m(x-y) + y.k_1(x-y) = (x^m + k_1 y)(x-y)$ [by (2)]

$= k_2(x-y)$, where k_2, is a linear polynomial in x and y

$=$ R.H.S.

Therefore, P(m + 1) is true when $P(m)$ is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.

Example 7. Using principle of mathematical induction prove that

$$\sin\theta + \sin 3\theta + \ldots. + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}, n \in N.$$

Solution : We have to prove that

$$P(n) : \sin\theta + \sin 3\theta + \ldots. + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}, n \in \mathrm{N}. \quad (1)$$

putting $n = 1$ in P(1) we have L.H.S.$= \sin\theta$ and R.H.S. $= \dfrac{\sin^2\theta}{\sin\theta} = \sin\theta$

$\therefore$ L.H.S. = R.H.S.

thus P(n) is true for $n = 1$

Assume that $P(m)$ is true for any positive integer m, i.e.,

$$P(m) : \sin\theta + \sin 3\theta + \ldots. + \sin(2m-1)\theta = \frac{\sin^2 m\theta}{\sin\theta} \quad (2)$$

We shall now prove that P(m + 1) is true whenever $P(m)$ is true.

$$P\ (m+1):\ \sin\theta+\sin 3\theta+....+\sin(2m-1)\theta+\sin(2m+1)\theta=\frac{\sin^2(m+1)\theta}{\sin\theta} \tag{3}$$

L.H.S. of (3)

$$=\sin\theta+....+\sin(2m-1)\theta+\sin(2m+1)\theta$$

$$=\frac{\sin^2 m\theta}{\sin\theta}+\sin(2m+1)\theta=\frac{1}{\sin\theta}\left\{\sin^2 m\theta+\sin\theta.\sin(2m+1)\theta\right\} \quad \text{[by (2)]}$$

$$=\frac{1}{\sin\theta}\left[\sin^2 m\theta+\frac{1}{2}\left\{\cos 2m\theta-\cos(2m+2)\theta\right\}\right]$$

$$=\frac{1}{\sin\theta}\left[\sin^2 m\theta+\frac{1}{2}\left\{1-2\sin^2 m\theta-1+2\sin^2(m+1)\theta\right\}\right]$$

$$=\frac{\sin^2(m+1)\theta}{\sin\theta}=\text{L.H.S.}$$

Therefore, $P(m + 1)$ is true when $P(m)$ is true. Hence, by principle of mathematical induction, $P(n)$ is true for every positive integer n.

Example 8. Using principle of mathematical induction prove that $10^{2n-1}+1$ is divisible by 11 for all $n\in N$.

Solution : Let $P(n): 10^{2n-1}+1$ is divisible by 11

for $n = 1$

$P(1): 10^{2-1}+1=11$ which is divisible by 11

i.e. $P(n)$: is true for $n = 1$

Assume that $P(m)$ is true for any positive integer m, i.e.

$$P(m): 10^{2m-1}+1=11k,\ k\in N \tag{1}$$

We shall now prove that $P(m + 1)$ is true whenever $P(m)$ is true

$$10^{2(m+1)-1}+1=10^{2m+2-1}+1$$

$$=10^2 10^{2m-1}+10^2-99$$

$$=100\left(10^{2m-1}+1\right)-99$$

$$=100\times 11k-99 \quad \text{[by (1)]}$$

$$=11(100k-9)$$

thus $10^{2(m+1)-1}+1$ is divisibly by 11

Therefore, $P(m+1)$ is true when $P(m)$ is true. Hence, by principle of mathematical induction, $P(n)$ is true for every positive integer n

Exericse 4.1

1. If $P(n):(n+3)<2^{n+3}$ then find $P(4)$

2. If $P(n): 1^2+3^2+5^2+\ldots+(2n-1)^2=\dfrac{n(2n-1)(2n+1)}{3}$ then test the validity of $P(4)$.

3. Find $1+(1+3)+(1+3+5)+\ldots.\ n^{\text{th}}$ term.

4. Find $1\cdot 4\cdot 7+2\cdot 5\cdot 8+3\cdot 6\cdot 9+\ldots.\ n^{\text{th}}$ term.

Prove the following by using the principle of mathematical induction for all $n \in N$:

5. $1+3+\ldots+(2n-1)=n^2$

6. $1+4+\ldots.+(3n-2)=\dfrac{n(3n-1)}{2}$

7. $1\cdot 3+3\cdot 5+\ldots+(2n-1)(2n+1)=\dfrac{n\left(4n^2+6n-1\right)}{3}$

8. $1\cdot 3+2\cdot 4+\ldots+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$

9. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$

10. $\dfrac{1}{3.5}+\dfrac{1}{5.7}+\ldots\dfrac{2}{(2n+1)(2n+3)}=\dfrac{n}{3(2n+3)}$

11. $1^3+2^3+\ldots+n^3=\left\{\dfrac{n(n+1)}{2}\right\}^2$

12. $1+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\ldots+\dfrac{1}{1+2+\ldots+n}=\dfrac{2n}{n+1}$

13. $1+5+5^2+\ldots+5^{n-1}=\dfrac{5^n-1}{4}$

14. $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)\ldots\left(1+\dfrac{2n+1}{n^2}\right)=(n+1)^2$

15. $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\ldots+\dfrac{1}{2^n}=1-\dfrac{1}{2^n}$

16. $1\cdot 3+2\cdot 3^2+\ldots+n3^n=\dfrac{(2n-1)3^{n+1}+3}{4}$

17. $2^n > n$

18. $(1+x)^n \geq 1+nx,\ x>0$

19. $1+2+...+n<\frac{1}{8}(2n+1)^2$.

20. $x^{2n}-y^{2n}$ is divisible by $(x+y)$.

21. $2^{3n}-1$ is divisible by 7.

22. $10^n+3.4^{n+2}+5$ is a divisible by 9.

23. 41^n-14^n is a divisible by 27.

24. $(2n+7)<(n+3)^2$

25. $1^2+2^2+...+n^2>\frac{n^3}{3}$

Answers

Exercise 4.1

1. $7<2^7$ 2. True 3. $1+3+...+(2n-1)$ 4. $n(n+3)(n+6)$