Chapter 2

Inverse Trigonometric Functions

Miscellaneous Exercise

Q. 1

Find the value of the following:

$\text{Cos}^{-1}\left(\cos\frac{13\pi}{6}\right)$

Answer:

$\text{Cos}^{-1}\left(\cos\frac{13\pi}{6}\right)$

(Forcos-1(cos x) type of problem we have to always check whether the angle is in the principal range or not. This angle must be in the principal range. [0, π])

So here, $\frac{13\pi}{6} \notin [0, \pi]$

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$can be written as,

$\text{Cos}^{-1}\left(\cos\frac{13\pi}{6}\right)$

$= \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$= \cos^{-1}\left(\cos\frac{\pi}{6}\right)$ where $\frac{\pi}{6} \in [0, \pi]$

$= \frac{\pi}{6}$

Hence, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$

Q. 2

Find the value of the following:

$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Answer:

$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

(For $\tan^{-1}(\tan x)$ type of problem we have to always check whether the angle is in the principal range or not. This angle must be in the principal range. $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

So here, $\frac{7\pi}{6} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as,

$= \tan^{-1}\left[\tan\left(\pi + \frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left(\tan\frac{\pi}{6}\right)$ where, $\frac{\pi}{6} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ [since, $\tan(\pi+x) = \tan x$]

$= \frac{\pi}{6}$

Hence, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

Q. 3

Prove that

$2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Answer:

Taking LHS

Let

$\text{Sin}^{-1}\frac{3}{5} = x$

Then,

$\text{Sin } x = \frac{3}{5}$

Therefore,

$\tan x = \frac{3}{\sqrt{5^2-3^2}} = \frac{3}{\sqrt{25-9}}$

$\therefore \tan x = \frac{3}{4} = x = \tan^{-1}\frac{3}{4}$

$= \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$ ……. (1)

and

$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$

Now, we know

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= 2\tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{1-\frac{9}{16}}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right)$

$= \tan^{-1}\frac{24}{7}$

As, LHS = RHS

Hence Proved!

Q. 4

Prove that

$\text{Sin}^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

Answer:

$\text{Sin}^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

$\text{Sin}^{-1} \left(\frac{8}{17}\right) = x$

$\text{Sin } x = \frac{8}{17}$

$\text{Cos } x = \sqrt{1 - \left(\frac{8}{17}\right)^2}$

$= \sqrt{\frac{225}{289}}$

$= \frac{15}{17}$

$\therefore \tan x = \frac{8}{15} = x = \tan^{-1} \frac{8}{15}$

$= \sin^{-1} \frac{8}{17} = \tan^{-1} \frac{8}{15} \ldots (1)$

Let sin-1 $\frac{3}{5} = y$

Then, $\sin y = \frac{3}{5}$

$= \cos y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$

$= \tan y = \frac{3}{4} = y = \tan^{-1} \frac{3}{4}$

$= \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \ldots (2)$

Now,

L.H.S. = $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

= $\tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$ putting the value from equation (1) and (2)

= $\tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$ $\left[since, tan^{-1}\, x - tan^{-1} y = tan^{-1}\frac{x-y}{1-xy}\right]$

= $\tan^{-1}\frac{32+45}{60-24}$

= $\tan^{-1}\frac{77}{36}$

= R.H.S

Hence proved.

Q. 5

Prove that

$\text{Cos}^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Answer:

$\text{Cos}^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Let $\cos^{-1}\frac{4}{5} = x$

Then, $\cos x = \frac{4}{5}$

= $\sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$

= $\tan x = \frac{3}{4} = x = \tan^{-1}\frac{3}{4}$

= $\cos^{-1}\frac{4}{3} = \tan^{-1}\frac{3}{4}$ … (1)

Let, $\cos^{-1}\frac{12}{13} = y$

Then cos y = $\frac{12}{13}$

= sin y = $\sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{13}$

= tan y = $\frac{5}{12}$ = y = $\tan^{-1}\frac{5}{12}$

= $\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12}$ … (2)

Let, $\cos^{-1}\frac{33}{65}$ = z

Then, cos z = $\frac{33}{65}$

= sin z = $\sqrt{1-\left(\frac{33}{65}\right)^2} = \frac{56}{65}$

tan z = $\frac{56}{33}$ = y = $\tan-1\ \frac{56}{33}$

= cos-1 $\frac{33}{65}$ = $\tan-1\ \frac{56}{33}$ … (3)

Now,

L.H.S. = $\cos-1\ \frac{4}{5} + \cos^{-1}\frac{12}{13}$

= $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12}$ putting the value from the equation (1) and (2)

= $\tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}$ $\left[since, tan^{-1}x - tan^{-1}y = tan^{-1}\frac{x-y}{1-xy}\right]$

= $\tan^{-1}\frac{36+20}{48-15}$

= $\tan^{-1}\frac{56}{33}$ … by equation (3)

= R.H.S.

Hence, proved.

Q. 6

Prove that

$\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Answer:

$\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

We can also solve this problem by using the identity Sin (A + B) = sin A Cos B + cos A Sin B

Let, sin-1 $\frac{3}{5}$ = A and cos-1 $\frac{12}{13}$ = B

So,

sin A = 3/5 and cos B = 12/13Therefore, cos A = 4/5 and sin B = 5/13

As R.H.S. is sin-1 we will use sin (A+B)

= sin (A + B) = sin A Cos B + cos A Sin B

$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13} = \frac{36}{65} + \frac{20}{65}$

$= \frac{56}{65}$

Thus, A + B = $\sin^{-1}\frac{56}{65}$

= R.H.S.

Hence Proved.

Q. 7

Prove that

$\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Answer:

$\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

let, $\sin^{-1}\frac{5}{13} = x$ then, $\sin x = \frac{5}{13}$

$= \cos x = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}$

$= \tan x = \frac{5}{12} = x = \tan^{-1}\frac{5}{12}$

$= \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$ … (1)

Let, $\cos^{-1}\frac{3}{5} = y$. then, $\cos y = \frac{3}{5}$

$= \sin y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$

$= \tan y = \frac{4}{3} = y = \tan^{-1}\frac{4}{3}$

$= \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3}$ … (2)

Now,

R.H.S. $= \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$ putting the value from equation (1) and (2)

$= \tan^{-1}\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}} \left(since, tan^{-1}x - tan^{-1}y = tan^{-1}\frac{x-y}{1-xy}\right)$

$= \tan^{-1}\frac{15+48}{36-20}$

$= \tan^{-1}\frac{63}{16}$

= L.H.S.

Hence, proved.

Q. 8

Prove that

$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Answer

L.H.S. = $\tan-1\ \frac{1}{5} + \tan-1\ \frac{1}{7} + \tan-1\ \frac{1}{3} + \tan-1\ \frac{1}{8}$

$= \tan-1\ \frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}} + \tan-1\ \frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{38}} \left[since, tan^{-1}x - tan^{-1}y = tan^{-1}\frac{x-y}{1-xy}\right]$

$= \tan-1\ \frac{7+5}{35-1} + \tan-1\ \frac{8+3}{24-1}$

$= \tan^{-1}\frac{12}{34} + \tan-1\ \frac{11}{23}$

$= \tan^{-1}\frac{6}{17} + \tan-1\ \frac{11}{23}$

$= \tan^{-1}\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}} \left[since, tan^{-1}x - tan^{-1}y = tan^{-1}\frac{x-y}{1-xy}\right]$

$= \tan^{-1}\frac{138+187}{391-66}$

$= \tan^{-1}\frac{325}{325} = \tan^{-1} = \frac{\pi}{4}$

= R.H.S.

Hence, proved.

Q. 9

Prove that

$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, x ∈ [0, 1]

Answer:

$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Let x = tan2θ.Then, $\sqrt{x}$ = tanθ

= θ = $\tan^{-1}\sqrt{x}$

$\therefore \frac{1-x}{1+x} = \frac{1-tan^2}{1+tan^2}$ = cos2 θ

So now putting the value, we get,

R.H.S. = $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

= $\frac{1}{2}\cos^{-1}$ (cos 2θ) = $\frac{1}{2} \times 2$ θ = $\tan^{-1}\sqrt{x}$

= L.H.S.

Q. 10

Prove that

$\text{Cot}^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, x $\in \left(0, \frac{\pi}{4}\right)$

Answer:

$\text{Cot}^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Consider, $\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

On Rationalizing, we get,

$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1+\sin x}\right)^2}$

$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$

$= \frac{2\left(1+\sqrt{1-sin^2 x}\right)}{2\sin x} = \frac{1+\cos x}{\sin x} = \frac{2cos^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}} = \cot\frac{x}{2}$

Now,

L.H.S. = cot^{-1} $\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

= cot^{-1} $\left(cot\,\frac{x}{2}\right) = \frac{x}{2}$

= R.H.S.

Hence Proved

Q. 11

Prove that

$\tan-1\ \left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos-1$ x

Answer:

tan-1 $\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos-1$ x

Let x = cos2θ so that θ = $\frac{1}{2}\cos-1$ x

Now,

L.H.S. = $\tan-1\ \left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

= $\tan-1\ \left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$

= tan^{-1} $\left(\frac{\sqrt{2cos^2\theta}-\sqrt{2sin^2\theta}}{\sqrt{2cos^2\theta}+\sqrt{2sin^2\theta}}\right)$

= tan^{-1} $\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

= tan^{-1} $\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$

$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$

$= \tan^{-1} 1 - \tan^{-1}(\tan\theta) = \frac{\pi}{4} - \theta$

$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$

= R.H.S.

Hence Proved.

Q. 12

Prove that

$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\pi$

Answer:

$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Now, L.H.S. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$

$= \frac{9}{4}\left(\frac{\pi}{2} - sin^{-1}\frac{1}{3}\right)$

$= \frac{9}{4} \times \cos^{-1}\frac{1}{3}$ … eq. (1)

Now, Let

$\text{Cos}^{-1}\frac{1}{3} = x$. then $\cos x = \frac{1}{3} = \sin x = \sqrt{1-\left(\frac{1}{3}\right)^3} = \frac{2\sqrt{2}}{3}$

$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} = \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$

L.H.S. $= \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

= R.H.S.

Hence Proved

Q. 13

Solve the following equations:

2 $\tan^{-1}$ (cos x) = $\tan^{-1}$ (2 cosec x)

Answer:

2 $\tan^{-1}$ (cos x) = $\tan^{-1}$ (2 cosec x)

= $\tan^{-1}\left(\frac{2\cos x}{1-cos^2 x}\right)$ = $\tan^{-1}$ (2 cosec x)

= $\frac{2\cos x}{1-cos^2 x}$ = 2 cosec x

= $\frac{2\cos x}{sin^2 x} = \frac{2}{\sin x}$ = cos x = sin x

= tan x = 1

Hence, x = $\frac{\pi}{4}$

Q. 14

Solve the following equations:

$\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}$ x, (x $>$ 0)

Answer:

$\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}$ x

As we know, $\tan^{-1}$ (x) – $\tan^{-1}$ (y) = $\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

= $\tan^{-1}$ 1 – $\tan^{-1}$ x = $\frac{1}{2}\tan^{-1}$ x

We know, tan $\frac{\pi}{4}$ = 1

So, $\tan^{-1}$ (1) = $\frac{\pi}{4}$

= $\frac{\pi}{4} - \tan^{-1}$ (x) = $\frac{1}{2}\tan^{-1}$ (x)

$= \frac{\pi}{4} = \frac{3}{2}$ $\tan^{-1}$ x

$= \tan^{-1} x = \frac{\pi}{6}$

$= x = \tan \frac{\pi}{6}$

Hence,

$x = \frac{1}{\sqrt{3}}$

Q. 15

Solve the following equations:

Sin ($\tan^{-1}$ x), |x| < 1 is equal to

A. $\frac{x}{\sqrt{1+x^2}}$

B. $\frac{1}{\sqrt{1-x^2}}$

C. $\frac{1}{\sqrt{1+x^2}}$

D. $\frac{x}{\sqrt{1+x^2}}$

Answer:

Let tan-1 x = y, then tan y = x = sin y = $\frac{x}{\sqrt{1+x^2}}$

$\therefore y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$= \tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$= \sin(\tan^{-1} x) = \sin\left(sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)$

$= \frac{x}{\sqrt{1+x^2}}$

Q. 16

Solve the following equations:

$\text{Sin}^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$, then x is equal to

A. 0, $\frac{1}{2}$

B. 1, $\frac{1}{2}$

C. 0

D. $\frac{1}{2}$

Answer:

$\text{Sin}^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$

Now we will put Now, we will put x= sin y in the given equation, and we get,

$\text{Sin}^{-1}(1 - \sin y) - 2\sin^{-1} \sin y = \frac{\pi}{2}$

$= \sin^{-1}(1 - \sin y) - 2y = \frac{\pi}{2}\pi$

$= \sin^{-1}(1 - \sin y) = \frac{\pi}{2} + 2y$

$= 1 - \sin y = \sin\left(\frac{\pi}{2} + 2y\right)$

$\Rightarrow 1 - \sin y = \cos 2y$ as $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

$\Rightarrow 1 - \cos 2y = \sin y$

$\Rightarrow 2\sin 2y = \sin y$

$\Rightarrow \sin y\,(2\sin y - 1) = 0$

$\Rightarrow \sin y = 0$ or $1/2$

$\therefore x = 0$ or $1/2$

Now, if we put then we will see that,

L.H.S. = sin^{-1} $\left(\frac{1}{2}\right)$ = - 2 sin^{-1} $\frac{1}{2}$

= - sin^{-1} $\frac{1}{2}$

= $-\frac{\pi}{6} \neq \frac{\pi}{2} \neq$ R.H.S

Hence, x = $\frac{1}{2}$is not the solution of the given equation.

Thus, x = 0

Q. 17

Solve the following equations:

tan^{-1} $\left(\frac{x}{y}\right) - $ tan^{-1} $\frac{x-y}{x+y}$ is equal to

A. $\frac{\pi}{2}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{4}$

D. $\frac{-3\pi}{4}$

Answer:

tan-^{1} $\frac{x}{y}$ – tan^{-1} $\frac{x-y}{x+y}$

= tan^{-1} $\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1-\frac{x}{y}.\frac{x-y}{x+y}}\right]$

= tan^{-1} $\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$

= tan^{-1} $\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$