Questions and Problems

1. Fundamentals of Mechanics

1.1. A wind is blowing with a constant velocity v in the direction denoted by the arrow in the figure. Two airplanes start out from a point A and fly with a constant speed c. One flies against the wind to a point B and then returns to point A, while the other flies in the direction perpendicular to the wind to a point C and then returns to point A. The distances AB and AC are the same.

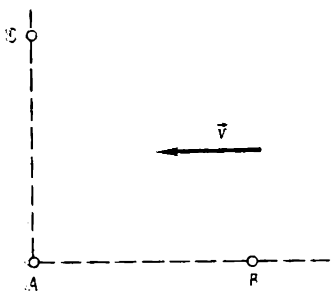


Fig. 1.1

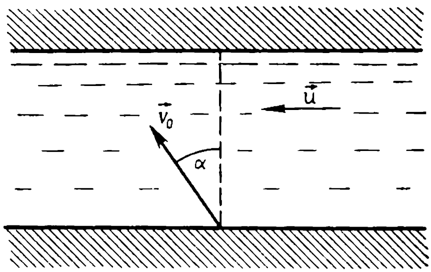


Fig. 1.2

Which plane will return to point A first and what will be the ratio of the flight times of the two planes?

1.2. A boat is moving across a river whose waters flow with a velocity u. The velocity of the boat with respect to the current, v_0, is directed at an angle α to the line perpendicular to the current. What will be the angle θ at which the boat moves with respect to this line? What will be the velocity v of the boat with respect to the river banks? What should be the angle at which the boat moves directly across the current with given u and v?

1.3. From a point A on a bank of a channel with still waters a person must get to a point B on the opposite bank. All the distances are shown in the figure. The person uses a boat to travel across the channel and then walks along the bank to point B. The velocity of the boat is v_1 and the velocity of the walking person is v_2. Prove that the fastest way for the person to get from A

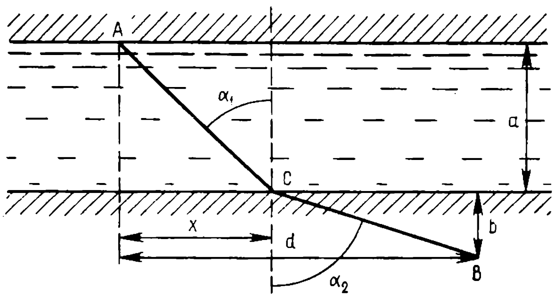


Fig. 1.3

to B is to select the angles α_1 and α_2 in such a manner that $(\sin \alpha_1/(\sin \alpha_2) = v_1/v_2$.

1.4. An object slides without friction down an inclined plane from a point B to a point C that is distant a from

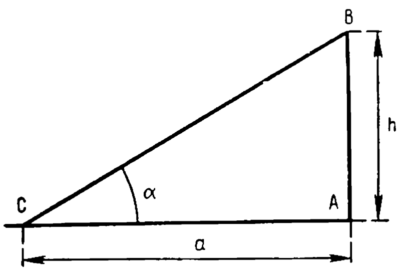


Fig. 1.4

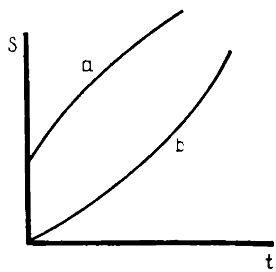


Fig. 1.5

a point A. At what height h (or at what angle α) is the sliding time minimal?

1.5. The time dependence of the lengths of the paths of two bodies moving in a straight line is given by curves a and b, respectively. What curve corresponds to accelerated motion and what curve to decelerated motion?

1.6. A material particle is moving along a straight line in such a manner that its velocity varies as shown in the figure. At which moment in time numbered successively on the time axis will the acceleration of the particle be maximal? How should one use the graph to determine the average velocity of motion over the time interval from t_1 to t_2?

1.7. The velocity of a particle moving in a straight line varies with time in such a manner that the v vs. t curve

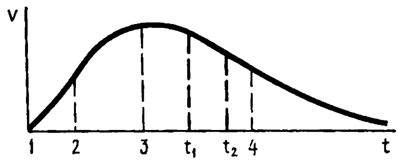


Fig. 1.6

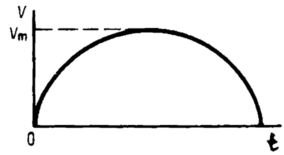


Fig. 1.7

is represented by one half of an ellipse. The maximal velocity is v_m and the total time of motion is t. What is the path traversed by the particle and the average velocity over t? Can such motion actually occur?

1.8. The velocity of a particle decreases in relation to the path traversed according to the linear law $v = v_0 - ax$. After what time will the particle get to a point B

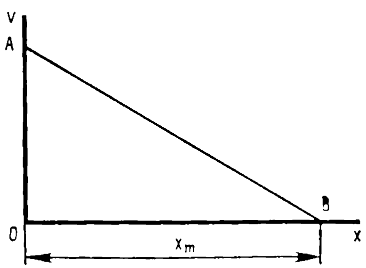


Fig. 1.8

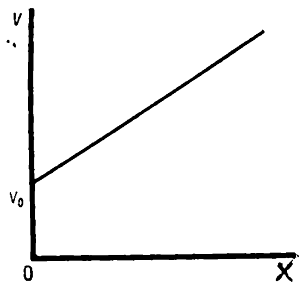


Fig. 1.9

that lies on the axis of abscissas distant x_m from the origin of coordinates?

1.9. The velocity of a particle moving in a straight line increases according to the linear law $v = v_0 + kx$. How does the acceleration change in the course of such motion? Does it increase or decrease or stay constant?

1.10. The figure shows the "timetable" of a train, the dependence of the speed of the train on the distance traveled. How can this graph be used to determine the average speed over the time interval it took the train to travel the entire distance?

1.11. A rod of length l leans by its upper end against a smooth vertical wall, while its other end leans against the floor. The end that leans against the wall moves uni-

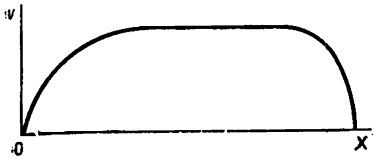


Fig. 1.10

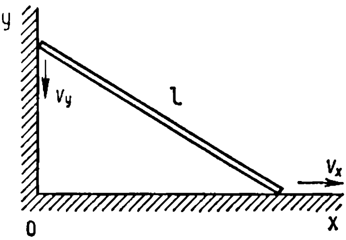
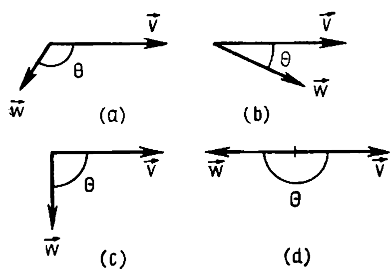


Fig. 1.11

formly downward. Will the other end move uniformly, too?

1.12. An object is thrown upward with an initial velocity v_0. The drag on the object is assumed to be proportional to the velocity. What time will it take the object to move upward and what maximal altitude will it reach?

1.13. At a certain moment in time the angle between the velocity vector **v** of a material particle and the acce-



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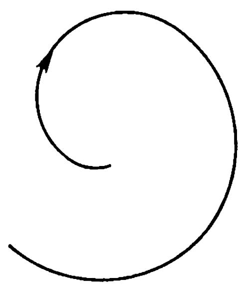


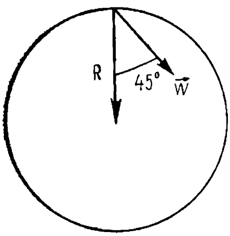
Fig. 1.14

leration vector **w** of that particle is θ. What will be the motion of the particle at this moment for different θ's: rectilinear or curvilinear, accelerated or uniform or decelerated?

1.14. A particle is moving along an expanding spiral in such a manner that the particle's normal acceleration remains constant. How will the linear and angular velocities change in the process?

1.15. A particle is moving in a circular orbit with a constant tangential acceleration. After a certain time t has elapsed after the beginning of motion, the angle between the total acceleration $\mathbf{w}$ and the direction along the radius R becomes equal to 45°. What is the angular acceleration of the particle?

1.16. An object is thrown at an angle α to the horizontal ($0° < \alpha < 90°$) with a velocity $\mathbf{v}_0$. How do the normal acceleration w_n and the tangential acceleration w_t vary in the process of ascent if the drag is ignored?



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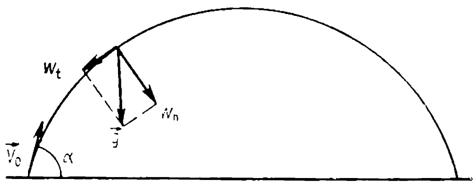


Fig. 1.16

1.17. At the foot of a hill a certain velocity is imparted to a sled, as a result of which the sled moves up the hill to a point A and then down the hill. What are the directions of the normal and tangential components of the acceleration at point A?

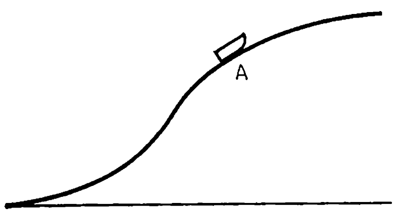


Fig. 1.17

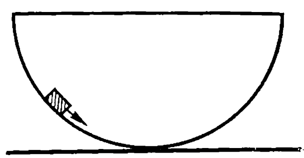


Fig. 1.18

1.18. An object moves without friction along a concave surface. What are the directions of the normal and tangential components of the acceleration at the lowest possible point?

1.19. A stunt rider on a unicycle is riding around the arena of a circus in a circle of radius R. The radius of the wheel of the unicycle is r and the angular velocity with which the wheel rotates is ω. What is the angular acceleration of the wheel? (Ignore the fact that the wheel axis is inclined.)

1.20. A liquid has been poured into a cylindrical vessel of mass M (the mass of the vessel bottom can be ighored) and height H. The linear density of the liquid, that is, the ratio of the mass of the liquid column to its height, is δ.

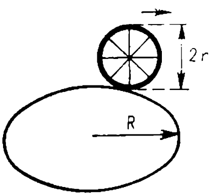


Fig. 1.19

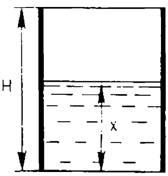


Fig. 1.20

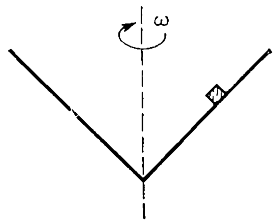
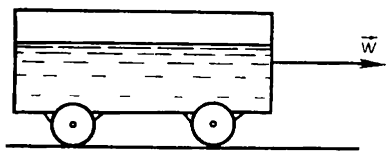


Fig. 1.21

What is the height x of the column of liquid at which the common center of gravity of the liquid plus the vessel is in the lowest position?

1.21. A cone-shaped funnel is being rotated with constant angular velocity ω. An object is placed on the inner



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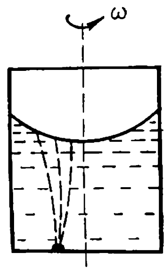


Fig. 1.24

wall of the funnel. The object can freely move along the generatrix of the cone, but during the motion of the funnel the body is in a state of equilibrium. Is this equilibrium stable or unstable?

1.22. A vessel filled with water is moving horizontally with constant acceleration w. What shape will the surface of the liquid have?

1.23. A liquid has been poured into a cylindrical vessel. What shape will the surface of the liquid have if the vessel is rotated uniformly about its axis with an angular velocity ω?

1.24. A piece of cork has been attached to the bottom of a cylindical vessel that has been filled with water and is rotating about the vertical axis with a constant angular velocity ω. At some moment the cork gets free and comes to the surface. What is the trajectory along which the cork moves to the surface: does it approach the wall or the axis or does it move vertically upward?

1.25. A force acting on a material particle of mass m first grows to a maximum value F_m and then decreases to

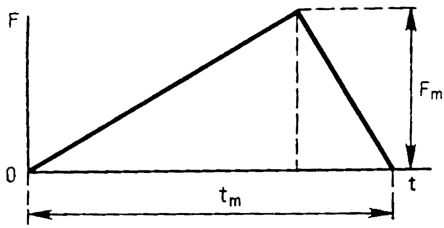
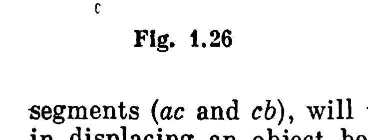


Fig. 1.25

zero. The force varies with time according to a linear law, and the total time of motion is t_m. What will be the velocity of the particle by the end of this time interval if the initial velocity is zero?

1.26. Along which of the two trajectories, the horizontal line $ac'b$ or the broken line consisting of two straight



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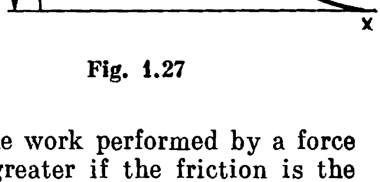


Fig. 1.27

segments (ac and cb), will the work performed by a force in displacing an object be greater if the friction is the same for all three straight segments?

1.27. An object of mass m is sliding down a hill of arbitrary shape and, after traveling a certain horizontal path, stops because of friction. The friction coefficient may be different for different segments of the entire path but it is independent of the velocity and direction of motion. Find the work that a force must perform to return the object to its initial position along the same path.

1.28. The dependence of the potential energy of an object on its position is given by the equation $W = ax^2$

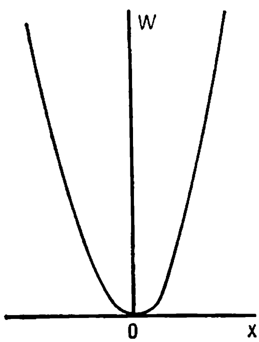


Fig. 1.28

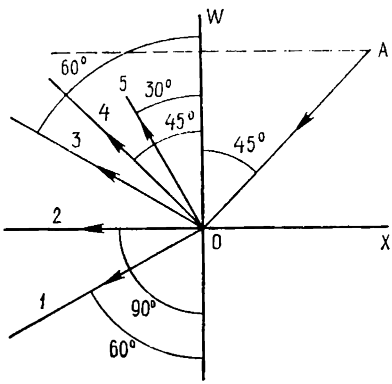


Fig. 1.29

(a parabola). What is the law by which the force acting on the object varies?

1.29. An object whose density is ρ_{ob} falls from a certain height into a liquid whose density is ρ_{liq}. In the figure the potential energy W of the object is plotted along the vertical axis and the position of the object (its altitude) is plotted along the horizontal axis. The potential energy of the object at the level of the liquid is taken zero and the positive direction of the vertical axis (the W axis) is the one pointing upward from the liquid's surface. Determine which of the five straight lines, *1-5*, corresponds to an object with the highest density and which to an object with the lowest density. Is there a straight line among these five for which $\rho_{ob} = (1/2)\,\rho_{liq}$? The arrows on the straight lines point in the direction of motion of the object.

1.30. The dependence of the potential energy W of the interaction between two objects on the distance r separating them is shown in the figure. What will be the distances between the objects that correspond to equilibrium positions? At what distance will the equilibrium be stable? (Answer the same question for unstable equilibrium.) What segments of the curve correspond to a repulsive force and what segments, to an attractive force?
1.31. A load of mass m_2 is hanging from a string. A bullet flying horizontally hits the load. Three cases are possible here, namely, (1) the bullet pierces the load and,

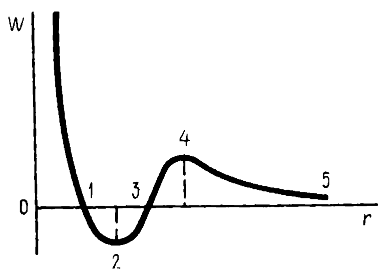


Fig. 1.30

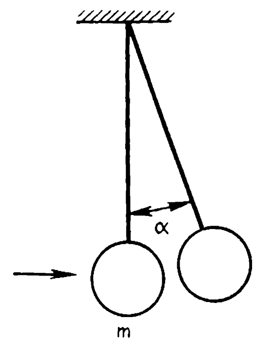


Fig. 1.31

retaining a fraction of its velocity, continues its flight, (2) the bullet gets stuck in the load, and (3) the bullet recoils from the load. In which of these three cases will the load be deflected by an angle α with the greatest magnitude and in which will it be deflected by an angle with the smallest magnitude?
1.32. Two spheres of equal mass collide, with the collision being absolutely elastic but not central. Prove that in this case the angle between the velocities after collision must be 90°.
1.33. A sphere of mass m_1 impinges with a velocity $\mathbf{v}_0$ on a sphere of mass m_2 that is at rest, with $m_1 > m_2$. The collision is absolutely elastic but not central. By what maximal angle θ will the impinging sphere be deflected?
1.34. Two spheres of equal mass are moving at right angles with velocities that are equal in magnitude. At the moment of collision the velocity vector of sphere *1* is

directed along the straight line connecting the centers of the spheres. The collision is absolutely elastic. Plot the velocity vectors before and after collision in different coordinate systems: (1) in the laboratory system (in this system the velocities of the spheres are those specified above), (2) in the coordinate system connected with the center of mass of the two spheres, and (3) and (4) in the coordinate systems linked to each of the spheres.

1.35. The centers of the spheres *1*, *2*, and *3* lie on a single straight line. Sphere *1* is moving with an (initial) velocity v_1 directed along this line and hits sphere *2*.

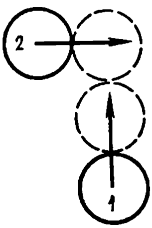
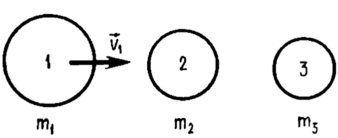


Fig. 1.34



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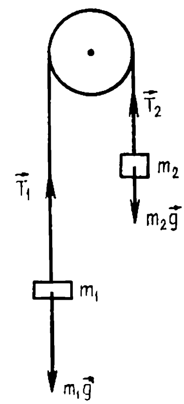


Fig. 1.37

Sphere *2*, acquiring after collision a velocity v_2, hits sphere *3*. Both collisions are absolutely elastic. What must be the mass of sphere *2* for the sphere *3* to acquire maximum velocity (the masses m_1 and m_3 of spheres *1* and *3* are known)?

1.36. A sphere of mass m_1 moving with a velocity v_0 hits a sphere of mass m_2 that is at rest. The collision is absolutely elastic and central. The velocities of the spheres after collision are u_1 and u_2, respectively. What are the mass ratios for the following values of velocities: $u_1 = 0$, $u_1 < 0$, and $u_1 > 0$?

1.37. A device often used to illustrate the laws of uniformly accelerated motion is the Atwood machine. The machine consists of two loads of mass m_1 and m_2 attached to the ends of a limp but inextensible string. The string runs over a pulley. The acceleration with which the loads move is

$$w = \frac{m_1 - m_2}{m_1 + m_2} g,$$

whereas the angular acceleration of the pulley is ignored. Is the last assumption true for exact calculations?

1.38. Strings are wound around a shaft and a sheave of equal mass, and a load is attached to the end of each string (the loads have equal mass). Which of the two loads

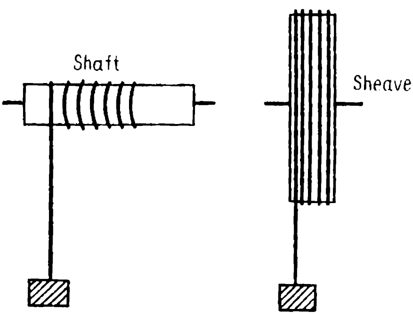


Fig. 1.38

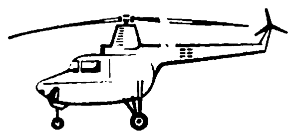


Fig. 1.40

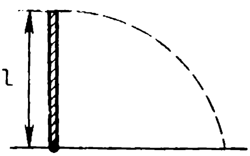


Fig. 1.41

will descend with a greater acceleration and which of the rotating objects, the shaft or the sheave, has a greater angular acceleration?

1.39. A vacuum cleaner standing on the floor turns through a small angle when switched on and then stops. Why does this happen?

1.40. A number of types of helicopters, among which are the Soviet-made "Mi" helicopters and the Westland Whirlwinds designed for use by Queen Elizabeth II, utilize one main rotor and a small vertical tail rotor. What is the function of this second rotor?

1.41. A rod whose lower end is sliding along the horizontal plane starts to topple from the vertical position. What will be the velocity of the upper end when this end hits the ground?

1.42. A thin rod of length $2R$ and mass m is standing (vertically) on a perfectly smooth floor. The state of equilibrium in which the rod is at rest is unstable, and the rod falls. Find the trajectories that the various points of the rod describe and the velocity with which the upper end of the rod hits the floor.

1.43. A homogeneous rod AB is lying on a perfectly smooth floor. A bullet hits the rod and gets stuck in it. The direction of the bullet's initial velocity $\mathbf{v}_0$ is perpendicular to the rod, and the point where the bullet hits the rod lies at a distance x from the middle of the rod. The mass of the bullet is m and the mass of the rod is M.

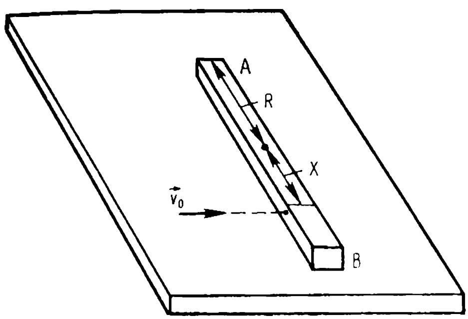


Fig. 1.43

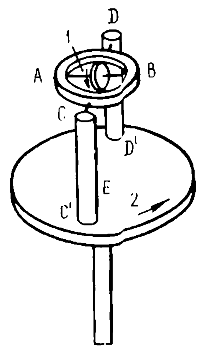


Fig. 1.44

Will a velocity directed in opposition to $\mathbf{v}_0$ be imparted to end A at the first moment after the collision?

1.44. The axis AB of a gyroscope is mounted in a frame that can rotate about the axis CD. This frame is mounted, via vertical supports CC' and DD', on a horizontal platform which, in turn, can rotate about the axis EF. At first the platform is at rest and the gyroscope is rotating in the direction designated by arrow *1*. Then the platform begins to rotate in the direction designated by arrow *2*. How will the gyroscope's axis change its position in space?

1.45. A top is spinning in the direction designated by the arrow in the figure. In what direction does the precession of the top occur?

1.46. A shaft whose diameter is d and length is l is rotating without friction in bearings with an angular velocity ω_0. A sleeve of height h and outer diameter D is fitted on the shaft (the materials of the sleeve and the shaft are the same). At first the sleeve is not connected with the shaft and is at rest. Then at some moment the sleeve is clamped to the shaft. What will be the common angular velocity of the shaft plus the sleeve?

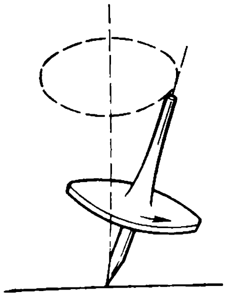


Fig. 1.45

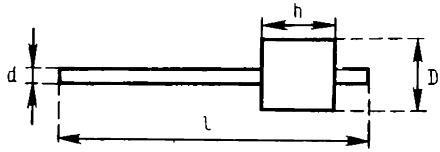


Fig. 1.46

1.47. A disk and a sphere roll off two inclined planes of the same altitude and length. Which of the two objects will get to the bottom of the respective plane first? How does the result depend on the masses and diameters of the disk and the sphere?

1.48. A spacecraft is circling the earth E along an elliptical orbit. How must the velocity of the spacecraft at

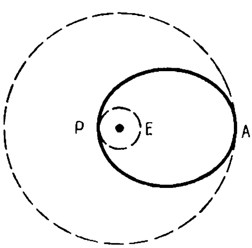


Fig. 1.48

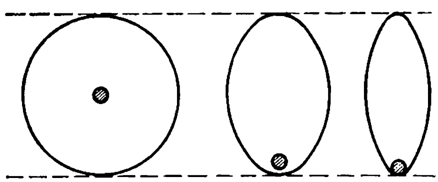


Fig. 1.50

perigee P and apogee A be changed so that the spacecraft follow a circular orbit?

1.49. Several artificial satellites of the same mass are circling the earth along circular orbits of different radii. How do the kinetic, potential, and total energies and angular momenta of the satellites depend on the radii of the orbits?

1.50. Three orbital space stations are circling the earth along different orbits: one along a circular orbit and the other two along elliptical orbits whose major axes are equal to the diameter of the circular orbit. The masses of the stations are the same. Will the energies and angular momenta of the stations coincide or will they be different?

1.51. A spacecraft is circling the earth along a circular orbit and retains its orientation with respect to the earth. Is zero gravity inside the spacecraft absolute in this case?

1.52. A comet flies into the solar system from remote outer space. The trajectory of the comet is a branch of

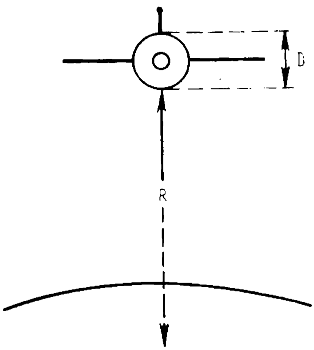


Fig. 1.51

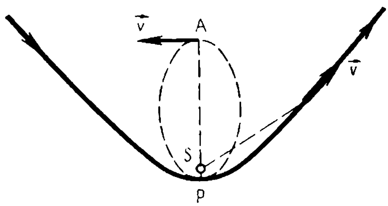


Fig. 1.52

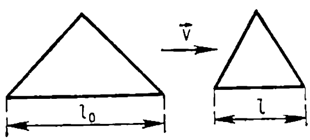


Fig. 1.54

a hyperbola. Can the comet become a satellite of the sun S if the interaction of the comet with the planets of the solar system is ignored?

1.53. What shape will a round disk have if viewed from a system of coordinates with respect to which the disk is moving with a certain velocity directed along the diameter of the disk?

1.54. An isosceles right triangle is moving with respect to a system of coordinates with a velocity **v** directed along the hypotenuse. When viewed from this system, the triangle appears to be an equilateral triangle. Find the velocity with which the triangle is moving with respect to this system.

1.55. The various relationships that exist between time intervals, coordinates, and velocities in the special theory of relativity are conveniently illustrated via a system of coordinates in which on the axes we lay off either distance and time multiplied by the speed of light or time and distance divided by the speed of light. Curves that represent motion in such systems are known as world lines. Various world lines are shown in the figure in the x/c vs. t coordinates. What does each line represent? Is there a line that contradicts the main principles of relativity theory?

1.56. A world line is directed at an angle θ to the x/c axis (see Problem 1.55). What is the ratio of the kinetic energy calculated via the formula of relativity theory to the value calculated via the formula of classical mechanics? Take the specific case of $\theta = 60°$ as an example.

1.57. Two systems are moving with respect to each other with a certain velocity. The motion of one system

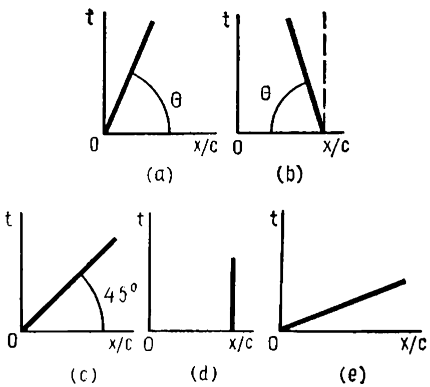


Fig. 1.55

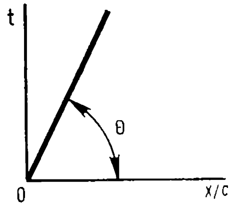


Fig. 1.57

in terms of the coordinates x/c and t of the other system is represented by a world line directed at an angle θ to the x/c axis. After a time interval T_0 reckoned from the origin of coordinates has elapsed, one system sends a signal to the other. After what time will the second system receive the signal?

1.58. Three systems, A, B and C, are moving with respect to each other in such a manner that with respect to system B the velocities of A and C coincide in magnitude and are directed toward B (Figure (a)). When system A comes alongside system B (Figure (b)), the clocks in the two systems are synchronized. At this moment system A begins emitting signals directed at B and separated by equal time intervals T_0. This continues until A comes alongside C (Figure (c)), with N signals being set over

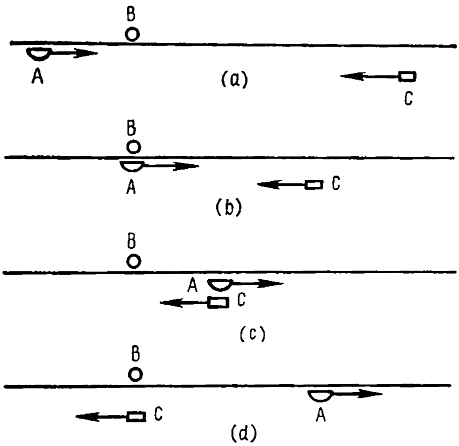


Fig. 1.58

the entire interval between the encounters. At this moment the clock in C is synchronized with the clock in A and system C starts to send signals directed at B that are separated by the same time intervals T_0. Find the difference in readings of the clock in B and C when these two systems come alongside (Figure (d)).

Answers and Solutions

1. Fundamentals of Mechanics

1.1. If $AB = AC = l$, then the times of flight from A to B and from B to A are, respectively, $l/(c - v)$ and $l/(c + v)$. The entire flight time is

$$t_1 = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2-v^2}.$$

For the second airplane to fly from A to C, its velocity must be directed at an angle to the direction of the wind

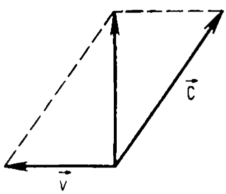


Fig. 1.1

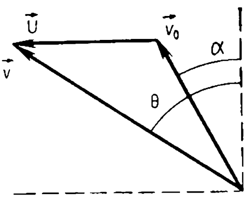


Fig. 1.2

in such a manner that the resulting velocity directed toward C is equal to $(c^2 - v^2)^{1/2}$ in magnitude. The entire flight time of this airplane will be

$$t_2 = \frac{2l}{\sqrt{c^2-v^2}}.$$

The second airplane will arrive before the first, and the flight time ratio is

$$t_2/t_1 = \sqrt{1 - v^2/c^2}.$$

1.2. The figure shows that

$$\tan\theta = \frac{v_0 \sin\alpha + u}{v_0 \cos\alpha} = \tan\alpha + \frac{u}{v_0 \cos\alpha}.$$

Velocity v can be found from the equation

$$(v_0 \sin\alpha + u)^2 + v_0^2 \cos^2\alpha = v^2,$$

which yields

$$v = v_0 \sqrt{1 + 2\frac{u}{v_0}\sin\alpha + \left(\frac{u}{v_0}\right)^2}.$$

The boat will travel directly across the river if $\theta = 0$. Under this condition, $\sin\alpha = -u/v_0$. Obviously, the boat can travel at right angles to the current only if v_0 is greater than u.

1.3. The time of travel by boat from A to C is

$$t_1 = \sqrt{x^2 + a^2}/v_1.$$

The time of travel by foot from C to B is

$$t_2 = \sqrt{(d-x)^2 + b^2}/v_2.$$

The total time of travel is

$$t = t_1 + t_2 = \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{(d-x)^2+b^2}}{v_2}.$$

The extremum condition is $dt/dx = 0$, or

$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{x^2+a^2}} - \frac{d-x}{v_2\sqrt{(d-x)^2+b^2}} = 0.$$

Since

$$\frac{x}{\sqrt{x^2+a^2}} = \sin\alpha_1 \quad \text{and} \quad \frac{d-x}{\sqrt{(d-x)^2+b^2}} = \sin\alpha_2,$$

we can write $\sin\alpha_1/v_1 = \sin\alpha_2/v_2$, whence

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2}.$$

We can easily see that the extremum corresponds to the minimum of time of travel.

1.4. The time of travel along straight line BC is determined by the length S of segment BC and the acceleration w. The figure shows that

$$S = \sqrt{a^2+h^2}, \quad w = \frac{h}{\sqrt{a^2+h^2}}g.$$

Since $S = wt^2/2$, we can write

$$\sqrt{a^2+h^2} = \frac{g}{2}\frac{h}{\sqrt{a^2+h^2}}t^2,$$

whence

$$t = \sqrt{\frac{2}{g}\frac{(a^2+h^2)}{h}}.$$

Nullifying the derivative (the extremum condition),

$$\frac{dt}{dh}=\frac{h^2-a^2}{\sqrt{2gh^3(a^2+h^2)}}=0,$$

yields $h=a$.

The same result is obtained if we express S and w in terms of α:

$$S=a/\cos\alpha,\quad w=g\sin\alpha,$$

$$t=\sqrt{\frac{2}{g}\frac{a}{\sin\alpha\cdot\cos\alpha}}.$$

Nullifying the derivative $dt/d\alpha$, we find that $\alpha=45°$.

1.5. The acceleration in rectilinear motion is the second derivative of the distance traveled with respect to time. For a concave curve the second derivative is positive, while for a convex curve the second derivative is negative, whereby curve (*a*) corresponds to decelerated motion and curve (*b*) to accelerated motion.

1.6. By definition, acceleration is the time derivative of velocity, $\mathbf{w}=d\mathbf{v}/dt$. For rectilinear motion the vector equation can be written in scalar form. The acceleration is the highest when the derivative is the greatest, that is, when the curvature of the curve is maximal. The curvature is determined by the slope of the tangent line to the particular point on the curve. This corresponds to moment *2* on the time axis. Note that for curvilinear motion the question contains an ambiguity, since to determine the acceleration we must know the radius of the trajectory at every moment in the course of the motion in addition to the magnitude of the velocity. To find the average velocity, we must know the distance traveled by the particle in the course of a definite time interval. In terms of the velocity vs. time graph, the distance traveled is the area of the figure bounded by the curve, the time axis, and the vertical straight lines passing through the initial and final moments of time on the time axis. Analytically the distance is calculated via the integral

$$S=\int_{t_1}^{t_2} v\,dt,$$

whence the average velocity is

$$v = \frac{\int_{t_1}^{t_2} v \, dt}{t_2 - t_1}.$$

1.7. In terms of the velocity vs. time graph, the distance traveled is determined by the area bounded by the curve and the time axis. This area is

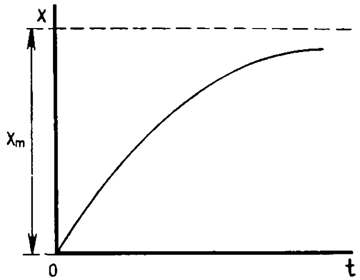


Fig. 1.8

$$S = \frac{\pi}{4} v_m t.$$

The average velocity is

$$v = \frac{S}{t} = \frac{\pi}{4} v_m.$$

Such motion cannot be realized in practical terms since at the initial and final moments of the motion the acceleration, which is dv/dt, is infinitely large in absolute value.

1.8. The particle will never get to point B but will approach it without bound. Indeed, from the equation $v = v_0 - ax$ we get

$$\frac{dx}{v_0 - ax} = dt.$$

Integration of this expression yields

$$\ln\left(\frac{x - v_0/a}{-v_0/a}\right) = -at,$$

whence

$$x = \frac{v_0}{a}(1 - e^{-at}). \qquad (1.8.1)$$

The limit value $x_m = v_0/a$ can be attained only at $t \to \infty$. The dependence of x on t defined by Eq. (1.8.1) is represented by the curve shown in the figure.

1.9. The acceleration

$$w = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = k(v_0 + kx)$$

increases with x. The same result can be obtained from the following line of reasoning: at constant acceleration the relationship between the velocity and the distance traveled is given by the formula

$$v^2 = v_0^2 + 2wx,$$

so that the velocity increases in proportion to the square root of the distance. Hence, for the velocity to increase linearly with x, the acceleration must increase.

1.10. The train covers the distance dx in the course of $dt = dx/v(x)$, where $v(x)$ is the speed with which it travels over dx. The total time of motion is

$$t = \int_0^S \frac{dx}{v(x)}.$$

The average speed is determined by dividing the distance covered by the train by the entire time of motion:

$$v_{av} = \frac{S}{\int_0^S \frac{dx}{v(x)}}.$$

If the graph cannot be represented by a formula, it can be reconstructed into the $1/v$ vs. x graph. In this case the integral in the denominator of the expression for v_{av} can be evaluated by graphical means.

1.11. The speed with which the lower end of the rod moves, $v_x = dx/dt$, can be written in the form

$$v_x = \frac{dy}{dt}\frac{dx}{dy}.$$

Since $x = \sqrt{l^2 - y^2}$, we can write

$$\frac{dx}{dy} = -\frac{y}{\sqrt{l^2-y^2}},$$

whence

$$v_x = -\frac{y}{\sqrt{l^2-y^2}}\frac{dy}{dt} = \frac{y\,|v_y|}{\sqrt{l^2-y^2}}.$$

Thus, the speed of the lower end gets smaller and smaller and vanishes at $y = 0$.

1.12. Since the drag is proportional to the velocity of the object, so is the acceleration caused by this force (with a minus sign). Hence, by Newton's second law,

$$\frac{dv}{dt} = -g - rv,$$

where r is the proportionality factor. Whence

$$\int_{v_0}^{v} \frac{dv}{v+g/r} = -r \int_0^t dt.$$

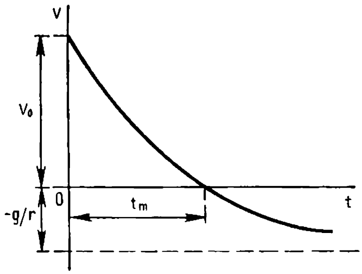


Fig. 1.12

Integration yields*

$$v = \left(v_0 + \frac{g}{r}\right) e^{-rt} - \frac{g}{r}. \qquad (1.12.1)$$

For $v = 0$ this yields

$$t_m = \frac{1}{r} \ln \left(1 + \frac{rv_0}{g}\right). \qquad (1.12.2)$$

To find the maximal altitude, we rewrite (1.12.1) in the form

$$\frac{dh}{dt} = \left(v_0 + \frac{g}{r}\right) e^{-rt} - \frac{g}{r}. \qquad (1.12.3)$$

Integrating this equation up to t, we find that

$$h = \left(v_0 + \frac{g}{r}\right) \frac{1}{r} (1 - e^{-rt}) - \frac{g}{r} t. \qquad (1.12.4)$$

Bearing in mind that at the point of greatest ascent $v = dh/dt = 0$ and combining this result with (1.12.3), we get

$$\left(v_0 + \frac{g}{r}\right) e^{-rt_m} = \frac{g}{r}. \qquad (1.12.5)$$

Combining (1.12.4) with (1.12.5) yields

$$h = \frac{v_0 - gt_m}{r}.$$

Substituting t_m from (1.12.2), we arrive at the final result

$$h = \frac{1}{r}\left[v_0 - \frac{g}{r} \ln \left(1 - \frac{rv_0}{g}\right)\right].$$

When drag is extremely low, or $rv_0/g \ll 1$, we can employ the expansion

$$\ln\left(1+\frac{rv_0}{g}\right) \approx \frac{rv_0}{g} - \frac{1}{2}\left(\frac{rv_0}{g}\right)^2.$$

This results in the well-known formula

$$h = \frac{v_0^2}{2g}.$$

* The section of the curve that lies below the t axis (see the figure) corresponds to the descent of the object after the object has reached the maximal altitude. The rate of descent asymptotically approaches the value at which the force of gravity is balanced by the drag.

1.13. The acceleration vector can be decomposed into two components, the tangential acceleration w_t, which is directed along the same straight line as the velocity of the particle, and the normal acceleration w_n, which is perpendicular to the velocity. For instance, for $\theta > 90°$ (see Figure (a) accompanying the problem) the tangential acceleration is directed opposite to the particle's velocity and the motion in this case is decelerated, $w < 0$. The presence of a nonzero normal acceleration suggests that the motion is curvilinear. The situation for the other cases is as follows: for $\theta < 90°$ (Figure (b)) the motion is curvilinear and accelerated, for $\theta = 90°$ (Figure (c)) the motion is curvilinear and uniform, and for $\theta = 180°$ (Figure (d)) the motion is rectilinear and decelerated, $w < 0$. Of course, characterizing the motion by the angle between the velocity $\mathbf{v}$ and the acceleration $\mathbf{w}$ is meaningful only for a definite moment in time. Subsequent motion may change this characteristic.

1.14. The normal acceleration is

$$w_n = v^2/R = \omega^2 R,$$

whence the linear velocity grows in proportion to the square root of the curvature radius of the spiral, while the angular velocity decreases by the same law.

1.15. When the angle between the total acceleration and the radius becomes equal to 45°, the normal acceleration becomes equal to the tangential acceleration. Since $w_n = \omega^2 R$ and $w_t = \varepsilon R$, we have $\omega^2 = \varepsilon$, and since $\omega = \varepsilon t$, we have $\varepsilon^2 t^2 = \varepsilon$, with the result that

$$\varepsilon = 1/t^2.$$

1.16. The acceleration with which the object moves is the acceleration of gravity, which at all points of the trajectory is directed vertically downward. From the figure that accompanies the problem we see that as the object ascends the tangential acceleration decreases while the normal acceleration grows. At the highest possible point the tangential acceleration is zero while the normal acceleration is equal to the acceleration of gravity.

1.17. Since at point A the sled's velocity is zero, so is the normal acceleration $w_n = v^2/R$. The tangential acceleration is directed down the hill along the tangent to the surface of the hill. The figure accompanying the answer shows the forces that act on the sled. These are the force of gravity $m\mathbf{g}$ and the reaction force $\mathbf{N}$ exerted by the surface of the hill. The resultant $\mathbf{F}$ is directed downward along the hill. According to Newton's second law, the acceleration vector points in the same direction as the resultant. If there is friction, the resultant vector does not change direction but becomes somewhat shorter, with the result that the tangential acceleration becomes smaller, too.

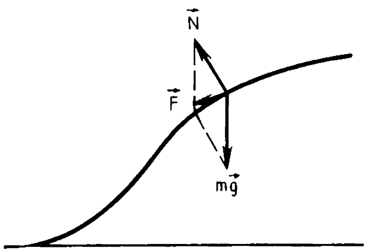


Fig. 1.17

1.18. The acceleration vector points in the direction of the resultant of the forces acting on the object. At the lowest possible point only the force of gravity and the reaction force act on the body, provided that there is no friction. This means that at this point the object experiences no tangential acceleration. Since the object is moving along a curvilinear trajectory with a certain velocity, there is a normal acceleration, which is directed toward the center of curvature of the trajectory. This acceleration is generated by the difference between the reaction force exerted by the surface and the force of gravity.

1.19. In the course of time Δt the angular velocity vector will vary from $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_2$ without changing its length. The direction of the vector will change by an angle of $\Delta\varphi$. This angle is equal, on the one hand, to $|\Delta\boldsymbol{\omega}|/\omega$ and, on the other, to $\Delta S/R$, where ΔS stands for the displacement of the center of the wheel.* This displacement is equal to $\Omega R \Delta t$, where Ω is the angular velocity of the center of the wheel. Thus,

$$|\Delta\omega|/\omega = \Omega R \Delta t/R \quad \text{and} \quad \varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \omega\Omega.$$

When the wheel is rotating, the point at which it touches the arena will shift in the course of Δt by a distance of $r\omega\Delta t$ on the wheel and by $R\Omega\Delta t$ on the arena. Hence, ω and Ω are linked by the following formula: $\omega r = \Omega R$, whence

$$\varepsilon = \omega^2 \frac{r}{R}.$$

* It is assumed that $\Delta\varphi \ll 1$ rad.

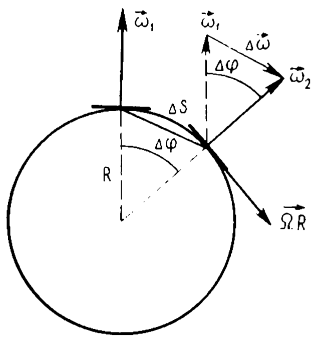


Fig. 1.19

1.20. The height of the center of mass of the vessel with the liquid is determined by the formula

$$h_c = \frac{M(H/2) + m(x/2)}{M+m}, \quad (1.20.1)$$

where m is the mass of the liquid. We rewrite (1.20.1) by replacing the mass of the liquid with δx:

$$h_c = \frac{1}{2}\,\frac{MH + \delta x^2}{M + \delta x}. \qquad (1.20.2)$$

Nullifying the derivative of h with respect to x,

$$\frac{dh_c}{dx} = \frac{1}{2}\,\frac{2\delta x(M+\delta x) - \delta(MH + \delta x^2)}{(M+\delta x)^2} = 0,$$

we get

$$x = \pm\sqrt{\frac{M^2}{\delta^2} + \frac{MH}{\delta}} - \frac{M}{\delta}. \qquad (1.20.3)$$

Of course, only the positive value of the root has physical meaning. Substituting this value into (1.20.2), we will find the position of the center of mass. After elementary transformations we get

$$h_c = \sqrt{\frac{M^2}{\delta^2} + \frac{MH}{\delta}} - \frac{M}{\delta}.$$

We have found that the position of the center of mass coincides with the level of the liquid.

Here are some particular cases:

(1) $\delta H = M$ (the liquid filling the vessel completely has a mass equal to the mass of the vessel). Then

$$h_c = x = H(\sqrt{2} - 1) \approx 0.41H.$$

(2) $\delta H \ll M$. Let us transform (1.20.3) to the form

$$x = \frac{M}{\delta}\left(\sqrt{1 + \frac{\delta H}{M}} - 1\right).$$

The fraction in the radicand is considerably less than unity. Expanding $(1 + \delta H/M)^{1/2}$ in a series and retaining only three terms, we get

$$x \approx \frac{M}{\delta}\left(1 + \frac{\delta H}{2M} - \frac{\delta^2 H^2}{8M^2} - 1\right),$$

or

$$h_c = x \approx \frac{H}{2}\left(1 - \frac{\delta H}{4M}\right).$$

The level of the liquid is below the middle of the vessel by an insignificant distance.

(3) $\delta H \gg M$. Let us transform (1.20.3) to the form

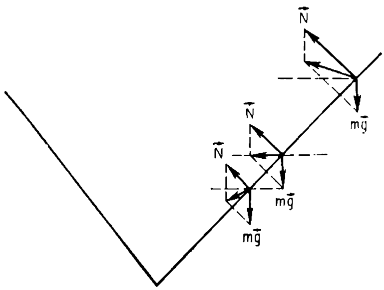
$$x = H\left(\sqrt{\frac{M^2}{\delta^2 H^2} + \frac{M}{\delta H}} - \frac{M}{\delta H}\right).$$

Bearing in mind that $(M/\delta H)^{1/2} \gg M/\delta H$, we can assume that the expression inside the parentheses in the above formula is simply $(M/\delta H)^{1/2}$, whence

$$h_c = x \approx H\,(M/\delta H)^{1/2}.$$

The level of the liquid is above the bottom of the vessel by an insignificant distance.

1.21. For the object to be in a state of equilibrium in relation to the wall of the funnel the resultant of the forces acting on the object must impart an acceleration to the object together with the funnel. These forces are the force of gravity and the reaction force exerted by the funnel. Since the force of gravity is constant in this problem and the resultant must be directed horizontally, the direction and magnitude of the reaction force are determined uniquely. But the latter has a different value at different distances from the funnel axis. At a constant angular velocity of the funnel, the greater the radius of rotation the greater the reaction force. For this reason (see the figure accompanying the answer), as the object moves farther from the funnel axis, the resultant of the force of gravity and the reaction force acquires a component directed upward, while as the object moves closer to the axis, the resultant acquires a component directed downward. In



Fg. 1.21

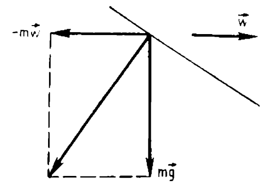


Fig. 1.22

the first case the object tends to move away from the axis still further and rises, while in the second case it tends to move toward the axis and lowers. Thus, the state of equilibrium is unstable.

1.22. It is convenient to think of the vessel with water as a noninertial system. In this case, on each particle of water there acts, in addition to the force of gravity, a force of inertia equal to the product of the particle's mass by the acceleration taken with the minus sign. The surface of water is a plane perpendicular to the vector of the resultant of these two forces. The slope of this surface in relation to the horizontal plane is

$$\tan\alpha = w/g.$$

1.23. Just like in the answer to the previous problem, we can assume the vessel with the liquid to be a noninertial system, in which a force of inertia equal to $-mw = -m\omega^2 x$ acts on every particle of mass m. The resultant of this force and the force of gravity is perpendicular to the surface of the liquid. The derivative dy/dx, equal to the slope of the line tangent to the surface at a given point, is

$$\frac{dy}{dx} = \tan\alpha = \frac{m\omega^2 x}{mg}.$$

Integrating, we find that

$$y = \frac{\omega^2}{2g} x^2.$$

The surface of the liquid is shaped in the form of a paraboloid of revolution.

1.24. Just like in the answers to Problems 1.22 and 1.23, the vessel can be assumed to be a noninertial system. In such a system, every mass element of water, say, an element whose volume is equal to the volume of the piece of cork, is in a state of equilibrium due to three forces:

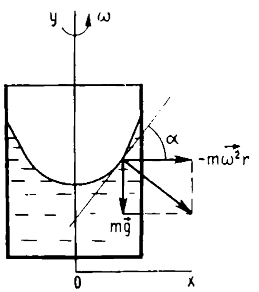


Fig. 1.23

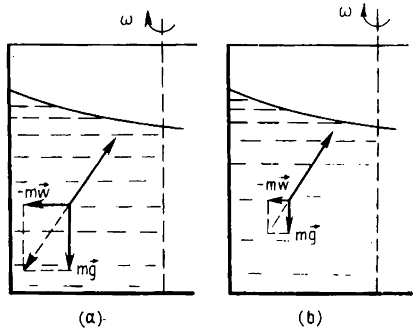


Fig. 1.24

the force of pressure of the surrounding water, the force of gravity, and the force of inertia, which is equal to the product of the element's mass by the normal acceleration of that element taken with the minus sign (Figure (a)). There are also three forces acting on the piece of cork that replaces the element of water: the force of pressure of the surrounding water is the same but the forces of gravity and inertia are lower. As Figure (b) shows, the net force (the difference between the force of pressure and the forces of gravity and inertia) make the cork rise to the surface and, at the same time, move toward the axis of the vessel.

A similar line of reasoning forces us to conclude that an object with a density greater than the density of water, when immersed into a rotating vessel with water, will sink and, in the process, move toward the wall of the vessel.

1.25. According to Newton's second law,

$$\int_0^t F\,\mathrm{d}t = \Delta mv.$$

In the case at hand,

$$\int_0^t F\,\mathrm{d}t = F_m t/2,$$

whence

$$v = F_m t/2m.$$

1.26. The work performed along ac' is

$$A_1 = ac'\,mgk.$$

The work performed against the forces of friction on the inclined segment ac is

$$A_2 = ac\,mgk\cos\alpha = \frac{ac'}{\cos\alpha}\,mgk\cos\alpha = ac'mgk.$$

We see that the two quantities coincide, and so, obviously, do the similar quantities for $c'b$ and cb. The change in the potential energy about $ac'b$ and acb is zero. Thus, the work performed against the forces of friction along $ac'b$ and that performed against the forces of friction along acb coincide.

1.27. The initial potential energy of the object with respect to the bottom of the hill, mgh, has been used up for work against the force of friction. In returning the body to its initial position, the force performs the same work and, in addition, imparts to the object the initial potential energy. As a result, the total work will be $2mgh$.

1.28. The work performed on an elementary segment of displacement is equal to the decrease in potential energy:

$$\mathrm{d}A = -\mathrm{d}W.$$

The same work can be represented as the product of force by displacement:

$$\mathrm{d}A = F_x\mathrm{d}x.$$

Hence

$$F_x = -\frac{\mathrm{d}W}{\mathrm{d}x} = -2ax.$$

Forces known as quasielastic also obey this law.

1.29. When the object is immersed in the liquid, two forces act on it: the force of gravity and Archimedes' force. If V is the volume of the object, the resultant of these two forces is

$$F = V(\rho_{ob} - \rho_{liq})g.$$

For $\rho_{ob} > \rho_{liq}$, as the object is immersed in the liquid, its potential energy continues to fall below zero, but slower than it would in air. The rate of this decrease is the higher the greater the value of ρ_{ob}. Straight line *1* in the figure accompanying the problem corresponds to an object sinking in a liquid. When $\rho_{ob} = \rho_{liq}$, the potential energy remains constant (straight line *2* coinciding with the x axis). If $\rho_{ob} < \rho_{liq}$, the potential energy of the object begins to increase when the object sinks into the liquid (straight lines *3*, *4*, *5*), and the rate of this increase is the higher the lower the value of ρ_{ob}. The potential energy, while growing, cannot exceed the initial potential energy of the object in air (the dashed horizontal line), and the object can attain this level only when the medium exerts no drag on it. If this is the case, the object will sink to a certain level in the liquid, stop, and then return to the surface with the same speed at the surface as it had when it entered the liquid. Once out of the liquid, the object will rise to the height determined by the initial potential energy. After this it drops back into the liquid, and so on. Of course, under real conditions the drag exerted by the medium will slow down the object, and the greater the viscosity of the liquid the faster this happens.

If the density of the material of the object is one-half the density of the liquid, $\rho_{ob} = (1/2)\rho_{liq}$, then

$$F = V\rho_{ob}g.$$

In this case the difference between Archimedes' force and the force of gravity is equal (in absolute value) to the latter but is directed in opposition to the force of gravity. The slope of the straight line must be the same as that of the straight line that represents the variation of the potential energy of a falling object. Straight line *4* has such a slope.

1.30. The formula that links the force acting on an object with the potential energy of the object,

$$F_r = -\frac{dW}{dr},$$

shows that equilibrium, which occurs when the force is zero, sets in when $dW/dr = 0$. There are two such points on the curve, point *2* and point *4*. Since when the object moves away from point *2* its potential energy increases while when it moves away from point *4* its potential energy decreases, at point *2* equilibrium is stable and at point *4* it is unstable. The fact that a system always tends to a state in which its potential energy is minimal implies that repulsive forces act on the *1-2* and *4-5* segments and an attractive force acts on the *4-2* segment.

1.31. Momentum conservation for the given problem can be written thus:

$$m_1 v_0 = m_1 u_1 + m_2 u_2, \tag{1.31.1}$$

where m_1 is the bullet's mass, m_2 the load's mass, v_0 the initial velocity of the bullet, u_1 the final velocity of the bullet, and u_2 the velocity acquired by the load as a result of the collision. From (1.31.1) it follows that

$$u_2 = \frac{m_1 (v_0 - u_1)}{m_2}. \tag{1.31.2}$$

If the bullet flies through the load, after it has left the load it has a velocity that is surely greater than u_2. We write $u_1 = u_2 + V$. Substituting this expression into (1.31.1), we get

$$u_2 = \frac{m_1 (v_0 - V)}{m_1 + m_2}. \tag{1.31.3}$$

If the bullet gets stuck in the load, then $u_1 = u_2$ and, hence,

$$u_2 = \frac{m_1 v_0}{m_1 + m_2}. \tag{1.31.4}$$

Finally, if the bullet recoils from the load, the velocity it acquires after collision, u_1, is negative and (1.31.2) can be written in the form

$$u_2 = \frac{m_1 (v_0 + |u_1|)}{m_2}. \tag{1.31.5}$$

A comparison of (1.31.3), (1.31.4), and (1.31.5) shows that the load acquires the highest velocity (and the greatest deflection, as a result) when the bullet recoils from it, while the lowest velocity is acquired when the bullet pierces the load.

1.32. For the sake of convenience we employ a coordinate system in which the velocity of one of the spheres prior to collision is zero. According to the energy conservation law, in the case of an absolutely elastic collision we have

$$\frac{m_1v_0^2}{2}=\frac{m_1u_1^2}{2}+\frac{m_2u_2^2}{2},$$

where m_1 and m_2 are the masses of the spheres, v_0 is the velocitiy of the first sphere prior to collision, and u_1 and u_2 are the velocities of the spheres after collision. Since the masses of the spheres are the same, we can write

$$v_0^2=u_1^2+u_2^2.$$

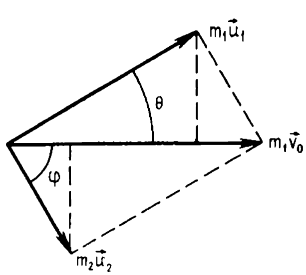


Fig. 1.33

The velocity vector $\mathbf{v}_0$ is the hypotenuse of a right triangle whose sides are the velocity vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, and hence the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$ is 90°.

1.33. Let u_1 and u_2 be the final velocities of the impinging sphere and the one that was at rest prior to collision, respectively, and θ is the angle between $\mathbf{u}_1$ and $\mathbf{v}_0$. The equations that express the laws of conservation of energy and momentum (for each projection) have the following form:

$$\frac{m_1v_0^2}{2}=\frac{m_1u_1^2}{2}+\frac{m_2u_2^2}{2}, \qquad (1.33.1)$$

$$m_1v_0=m_1u_1\cos\theta+m_2u_2\cos\varphi, \qquad (1.33.2)$$

$$m_1u_1\sin\theta+m_2u_2\sin\varphi=0. \qquad (1.33.3)$$

If m_1, m_2, and v_0 are fixed, then u_1, u_2, θ, and φ are linked through three equations. For this reason two of the four variables can be excluded and the variable θ can be expressed in terms of the third remaining variable, say, u_1. Taking $m_1u_1\cos\theta$ to the left-hand side of Eq. (1.33.2), squaring the result and Eq. (1.33.3), and adding the two squares, we get

$$m_1^2 (v_0^2 - 2u_1u_2 \cos\theta + u_1^2) = m_2^2 u_2^2.$$

Replacing u_2 with its value obtained from (1.33.1) and carrying out the necessary transformations, we arrive at a quadratic equation for u_1, namely,

$$u_1^2 - 2\frac{m_1}{m_1+m_2} v_0 \cos\theta \times u_1 + \frac{m_1-m_2}{m_1+m_2} v_0^2 = 0, \quad (1.33.4)$$

whose solution has the form

$$u_1 = \frac{m_1}{m_1+m_2}\left(\cos\theta \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\theta}\right) v_0. \quad (1.33.5)$$

This equation shows that the maximal angle θ is determined by the condition

$$\sin\theta_{\mathrm{m}} = m_2/m_1. \quad (1.33.6)$$

For values of θ smaller than θ_{m} two cases are possible, since two distinct values of u_1 correspond to one value of θ. For example, for $m_1/m_2 = 3$ and $\sin\theta = 0.2$, the velocity u_1 may have two values, $0.93v_0$ and $0.53v_0$. The first collision is commonly known as soft, while the second is commonly known as hard. The extreme case of soft collision is the grazing collision (or even the case where one sphere misses the other), while the extreme case of hard collision is the head-on collision, after which the velocity of the impinging sphere becomes

$$u_1 = \frac{m_1-m_2}{m_1+m_2} v_0.$$

Condition (1.33.6) can be obtained in another manner as well. For instance, if we express $\cos\theta$ via (1.33.4), namely,

$$\cos\theta = \frac{1}{2m_1}(m_1+m_2)\frac{u_1}{v_0} + (m_1-m_2)\frac{v_0}{u_1},$$

and nullify the derivative of $\cos\theta$ with respect to u_1, we can find the minimal value of $\cos\theta$ or the maximal value of $\sin\theta$. The motion of the impinging sphere can also be considered using the system of coordinates linked with the center of mass of the two spheres. If in the laboratory system the coordinates of the spheres are x_1 and x_2, then the coordinate of the center of mass is

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$

while the velocity of the center of mass is

$$v_c = \frac{m_1}{m_1 + m_2} v_0.$$

Correspondingly, the velocity of the impinging sphere in this system prior to collision is

$$v_0' = v_0 - v_c = v_0.$$

As a result of the collision the vector $\mathbf{v}_0'$ retains its length but turns through a certain angle depending on the distance between the center of the second sphere and

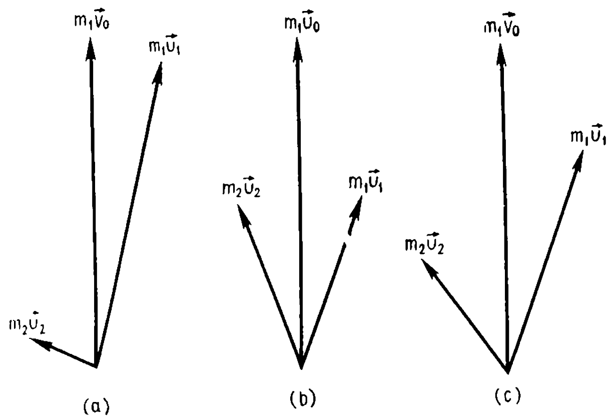


Fig. 1.33

the direction of flight of the impinging sphere prior to collision. The velocity $\mathbf{u}_1$ is equal to the sum of $\mathbf{v}_0'$ and $\mathbf{v}_c$. The momentum vectors of both spheres are shown in the figure for three cases: soft collision (Figure (a)) and hard collision (Figure (b)) for $m_1/m_2 = 3$ and $\sin\theta = 0.2$ and the case with $\sin\theta = m_2/m_1 = 1/3$ (Figure (c)). The velocity of the impinging sphere after collision is

$$u_1 = \frac{m_1 v_0}{m_1 + m_2} \cos\theta = 0.707 v_0.$$

The above-discussed problem is important for the theory of atomic collisions. For instance, if a potassium ion impinges on a helium atom ($m_1/m_2 = 10$), as a result of an elastic collision the ion may be deflected by an angle no greater than 5.7°.

1.34. We will consider each case in the order that it appears in the problem.

(1) The directions of the velocities of the spheres in the laboratory system are shown in the figure accompanying

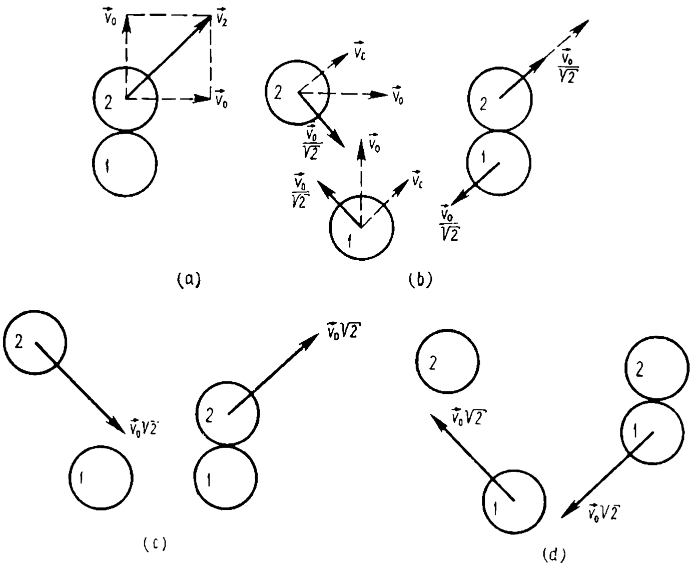


Fig. 1.34

the problem. If at the moment of collision we project the velocities of the spheres and the corresponding momenta on two axes one of which coincides with the direction of the initial velocity of sphere *1* and the other with that of the initial velocity of sphere *2*, then in the first of these two directions the spheres exchange the respective projections of the velocities, just like in a head-on elastic collision. Sphere *1* stops in the process. Since in the collision the force acts along the straight line connecting the centers of the spheres, the initial velocity of sphere *2* is conserved, with the velocity of sphere *1*, which is perpendicular to the initial velocity of sphere *2*, added to it. As a result the velocity of sphere *2* becomes equal to the geometric sum of the initial velocities of both spheres, that is, $v_0\sqrt{2}$ (Figure (a)).

(2) To determine the velocities of the spheres in the center-of-mass system, we decompose the velocity vector of each sphere into two perpendicular and equal components, v_{1a}, v_{1b} and v_{2a}, v_{2b}. The components v_{1a} and v_{2a} are equal in magnitude and point in the same direction. Obviously, the common center of mass moves in the same direction and with the same velocity, v_c, with respect to the laboratory system. Therefore, in the system linked with the center of mass there are only the velocities v_{1b} and v_{2b}. The velocities of the spheres after collision can be obtained if we subtract v_c from the velocities of the spheres in the laboratory system. The other velocities are shown in Figure (b).

(3) In the system linked with sphere *1*, the sphere, obviously, remains at rest during the entire collision process. The velocity of sphere *2* in this system can be obtained by subtracting geometrically the initial velocity of sphere *1* from the velocity of sphere *2* in the laboratory system. Since the velocity of sphere *1* after collision is equal, in the laboratory system, to zero and is also zero in the system linked with sphere *1*, the velocity of sphere *2* in this system after collision is the same as in the laboratory system (Figure (c)).

(4) In the system linked with sphere *2*, the velocity of sphere *1* is obtained by subtracting geometrically the initial velocity of sphere *2* from the velocity of sphere *1*. After collision the velocity of sphere *1* is equal, in absolute value, to the final velocity of sphere *2* in the laboratory system and points in the opposite direction (Figure (d)).

In conclusion we would like to bring the reader's attention to the fact that the angular momenta of the spheres with respect to the center of mass remain constant during the entire collision process. In collision, the center of mass is the point where the spheres touch and the angular momentum of sphere *1* is zero and remains such after collision. The angular momentum of sphere *2* is equal, prior to collision, to the product of momentum mv by the arm R. After collision the momentum of sphere *2* becomes $mv\sqrt{2}$, but the arm is now $R/\sqrt{2}$, so the product is the same and the angular momentum is conserved. Of course, since the system consisting of the spheres is isolated, the angular momentum is conserved in the entire process of motion.

1.35. After collision, sphere *2* acquires the velocity

$$u_2 = \frac{2m_1 v_1}{m_1 + m_2}. \qquad (1.35.1)$$

Sphere *3* acquires the following velocity after collision:

$$u_3 = \frac{2m_2 u_2}{m_2 + m_3}.$$

Substituting the value of u_2 from (1.35.1), we get

$$u_3 = \frac{4m_1 m_2 v_1}{(m_1 + m_2)(m_2 + m_3)}.$$

The extremal value of u_3 can be found by nullifying the derivative of u_3 with respect to m_2:

$$\frac{du_3}{dm_2} = \frac{4m_1 v_1 (m_1 m_3 - m_2^2)}{[(m_1 + m_2)(m_2 + m_3)]^2} = 0.$$

From this it follows that

$$m_2 = \sqrt{m_1 m_3}.$$

We can easily see that this value corresponds to the maximum of u_3.

Here are some particular cases.

(1) $m_1 \gg m_3$. In this case

$$u_3 \approx \frac{4m_1}{m_1 + m_2} v_1.$$

If we also assume that $m_1 \gg m_2$, then

$$u_3 \approx 4v_1.$$

If sphere *1* were to hit sphere *3* directly (without the intermediate sphere *2*), the highest velocity of sphere *3* for $m_1 \gg m_3$ would be roughly $2v_1$.

In some fantastic projects of interplanetary flight it has been suggested that the spaceship be accelerated to the necessary speed through a series of collisions with intermediate objects whose masses must be calculated in the appropriate manner.

(2) $m_1 = m_3$. In this case $m_2 = m_1 = m_3$ and $u_3 = v_1$.

(3) $m_1 \ll m_3$. Assuming that $m_2 \gg m_1$, we get

$$u_3 \approx 4v_1 m_1/m_3.$$

Here the velocity of sphere *3* is approximately double the velocity without an intermediate object, sphere *2*.

1.36. The velocities of the spheres after collision are

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2} v_0, \quad u_2 = \frac{2m_1}{m_1 + m_2} v_0.$$

Here are some particular cases.

(1) $u_1 < 0$ if $m_1 < m_2$. Since in this case $2m_1 < m_1 + m_2$, we have $0 < u_2 < v_0$.

(2) $u_1 = 0$ if $m_1 = m_2$. Then $u_2 = v_0$.

(3) $u_1 > 0$ if $m_1 > m_2$. Then $2m_1 > m_1 + m_2$ and $v_0 < u_2 < 2v_0$.

1.37. The equations of motion for the loads and the pulley can be written as follows:

$$m_1 w = m_1 g - T_1, \quad m_2 w = T_2 - m_2 g, \quad J\varepsilon = (T_1 - T_2)R, \tag{1.37.1}$$

where T_1 is the force exerted by the left end of the string on the left load, T_2 the force exerted by the right end of the string on the right load, J the moment of inertia of the pulley, w the acceleration of the loads, and ε is angular acceleration of the pulley. Dividing (1.37.1) by R, adding all the equations, and replacing ε with w/R, we arrive, after appropriate transformations, at

$$w = \frac{m_1 - m_2}{m_1 + m_2 + J/R} g. \tag{1.37.2}$$

Equation (1.37.2) shows that in exact calculations we must allow for the moment of inertia and the radius of the pulley.

If the pulley is a homogeneous disk, then instead of J we can write $m_p R^2/2$, and Eq. (1.37.2) assumes the form

$$w = \frac{m_1 - m_2}{m_1 + m_2 + m_p/2} g.$$

We see that in this case the radius of the pulley plays no role; what is important is only the mass of the pulley.

1.38. The equations of motion for the load-shaft or the load-sheave can be written as follows:

$$mg - T = mw, \quad TR = J\varepsilon,$$

where m is the mass of each load, T the force exerted by the strings attached to the loads, R the radius of the shaft or sheave, J the moment of inertia of the shaft or sheave, and ε the angular acceleration of the shaft or sheave. Eliminating T from the equations and replacing ε with w/R and the moment of inertia of the shaft or sheave with $MR^2/2$, we arrive, after simple transformations, at

$$w = \frac{m}{m + M/2} g,$$

from which it follows that the accelerations with which the two loads are lowered coincide. The angular acceleration is the greater the larger the radius, which means that the shaft has a greater angular acceleration than the sheave.

1.39. Prior to switch-on, the sum of the angular momenta of all the parts of the vacuum cleaner is zero. When the motor is switched on, a torque appears in the rotor of the motor, with the same torque (in absolute value) appearing in the stator and the casing of the vacuum cleaner fixed to the stator. Due to the latter torque, the vacuum cleaner begins to turn, but this motion dies out very soon because of friction.

1.40. When the engine of the helicopter of this type is operating, two torques appear: one is applied to the main rotor and the other (equal in magnitude to the first) is applied to the fuselage of the copter. This second torque tends to turn the fuselage in the direction opposite to that of the main rotor. The vertical tail rotor creates a torque that cancels out the torque applied to the fuselage. In toy helicopters this second rotor is fixed and the helicopter rotates in flight in a direction opposite to that of the main rotor.

1.41. The rod is in rotational motion, and so its potential energy is transformed into the kinetic energy of rotation. If the mass of the rod is m and the length is l, we have

$$\frac{mgl}{2} = \frac{J\omega^2}{2}.$$

Replacing ω with v/l and J with $ml^2/3$, we get

$$v = \sqrt{3gl}.$$

1.42. To determine the trajectories that the various points of the rod describe, we introduce a coordinate system whose origin lies at B, the lower point of the rod prior to falling, whose x axis points horizontally in the direction in which point B moves during motion, and whose y axis points upward, along the rod prior to motion. Since there are no forces that act on the rod in the horizontal direction, the rod's center of mass moves downward (from C to B). As Figure (a) shows, the coordinates of the

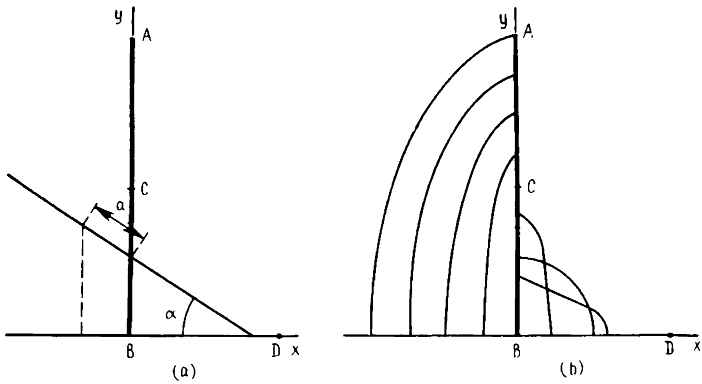


Fig. 1.42

points lying above the center of mass by a distance a are determined by the equations

$$x = -a \cos \alpha, \quad y = (R + a) \sin \alpha,$$

while the coordinates of the points lying below the center of mass by a distance a are determined by the equations

$$x = a \cos \alpha, \quad y = (R - a) \sin \alpha.$$

These equations imply that in the process of falling the rod (and that means all of its points except the center of mass) describes quarters of ellipses (Figure (b)) specified by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{(R+a)^2} = 1 \quad \text{(upper points)},$$

$$\frac{x^2}{a^2} + \frac{y^2}{(R-a)^2} = 1 \quad \text{(lower points)}.$$

When the rod is falling, its motion can be considered as rotation about an instantaneous center, D. Therefore, the velocity of the upper point (A) can be determined just like in Problem 1.41, using the law of conservation of energy. The appropriate equations yield

$$v = \sqrt{6gR}.$$

1.43. The velocity imparted to point A will be directed in opposition to $\mathbf{v}_0$ if the rod's linear velocity acquired as a result of rotation after the bullet has hit the rod is greater than the velocity of the center of mass of the rod. Moreover, for such a situation to occur, the distance x must not exceed one-half of the length of the rod. According to the law of conservation of momentum,

$$mv_0 = m(v + \omega x) + Mv. \qquad (1.43.1)$$

Here we have allowed for the fact that the velocity of the bullet after the bullet has hit the rod is the sum of the velocity of the center of mass, v, and the velocity ωx which the point that is distant x from the center of mass acquires as a result of rotational motion with angular velocity ω.

According to the law of conservation of angular momentum,

$$mv_0x = m(v + \omega x)x + J\omega, \qquad (1.43.2)$$

where J is the moment of inertia of the rod about the center of mass, $J = MR^2/3$. Multiplying (1.43.1) by x and subtracting the product from (1.43.2), we get

$$\omega = Mvx/J = 3vx/R^2.$$

The linear velocity of rotation acquired by point A (we denote this velocity by V) is

$$V = \omega R = 3vx/R.$$

The ratio V/v is greater than unity if $x > R/3$.

1.44. According to the right-hand screw rule, the vector of the angular velocity of the gyroscope is directed to the right in the figures accompanying the problem and the answer. The revolving platform applies a torque to the frame, and the vector of this torque is directed perpendicularly to the vector of the angular velocity of the gyroscope. This torque creates an angular acceleration ε, and under this acceleration the vector of angular velocity rotates in the direction shown by the arrow in the figure accompanying the answer. As a result the giroscope's axis places itself vertically and the direction of rotation of the gyroscope coincides with the direction of rotation of the platform. If the direction of rotation of the gyroscope or the direction of rotation of the platform were to change, the gyroscope's axis would point in the opposite direction.

In all cases the axis rotates in such a manner that the vector of angular velocity places itself in the direction coinciding with that of the vector of an external torque. This property of gyroscopes is used in navigation in gyrocompasses. The "platform" that applies a torque to the gyroscope is the earth in this case.

1.45. The vector of the angular velocity of the top is directed upward along the top's axis (see the figure accompanying the answer). The force of gravity applied to the top at the top's center of mass creates a torque

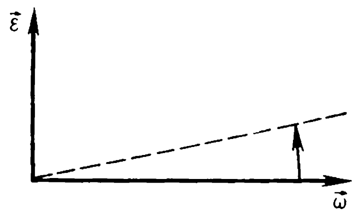


Fig. 1.44

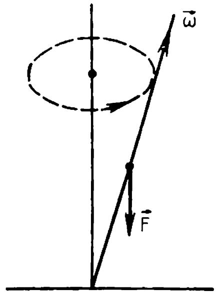


Fig. 1.45

whose vector, being perpendicular to the vector of angular velocity, is directed away from the reader. This torque does not change the magnitude of the angular velocity but creates an angular acceleration and hence changes the direction of the vector of angular velocity, just like centripetal acceleration does not change the value of the velocity but does change the direction of the velocity vector, as a result of which the body to which the centripetal acceleration is applied moves along a circle. In the case at hand the direction of the angular acceleration is such that precession occurs counterclockwise (if one views the top from above).

1.46. Since no external forces act on the shaft-sleeve system, the total angular momentum of the system remains constant:

$$J_{sh}\,\omega_0 = (J_{sh} + J_{sl})\,\omega. \tag{1.46.1}$$

The moment of inertia of the shaft is

$$J_{sh} = \rho\,\frac{\pi d^4}{32}\,l,$$

where ρ is the density of the material of the shaft and sleeve. The moment of inertia of the sleeve is

$$J_{sl} = \rho \frac{\pi (D^4 - d^4)}{32} h.$$

From (1.46.1) it follows that

$$\omega_0 d^4 l = \omega [(D^4 - d^4) h + d^4 l],$$

whence

$$\omega = \frac{d^4 l}{d^4 (l-h) + D^4 h} \omega_0 = \frac{d^4 l}{d^4 l + (D^4 - d^4) h} \omega_0$$

$$= \frac{1}{1 + \left(\frac{D^4}{d^4} - 1\right) \frac{h}{l}} \omega_0.$$

1.47. The potential energy of an object on the top of a hill, mgh, transforms into the kinetic energy of translational and rotational motion:

$$mgh = \frac{mv^2}{2} + \frac{J\omega^2}{2}.$$

Replacing ω with v/R, we get

$$mgh = \frac{mv^2}{2} + \frac{Jv^2}{2R^2}. \qquad (1.47.1)$$

The moments of inertia of the disk, J_d, and the sphere, J_{sp}, are

$$J_d = \frac{mR^2}{2} \quad \text{and} \quad J_{sp} = \frac{2}{5} mR^2,$$

respectively, with R the radius of disk or sphere. Substituting these values into (1.47.1) and dividing by m, we get

$$gh = \frac{v^2}{2} + \frac{v^2}{4} = 0.75 v^2 \qquad (1.47.2)$$

for the disk and

$$gh = \frac{v^2}{2} + \frac{v^2}{5} = 0.7 v^2 \qquad (1.47.3)$$

for the sphere. Since the left-hand sides of these equations are the same, the final velocity of the sphere is greater, and since the motion is uniformly accelerated, the sphere will get to the horizontal section earlier than the disk. Neither the masses nor the radii of the objects rolling down the inclined planes are present in (1.47.2) and

(1.47.3), with the result that the time it takes the objects to roll down is independent of these quantities.

1.48. When the spacecraft goes into a circular orbit at the perigee, it will circle the earth along a low orbit during the second half of the orbit. For this reason the spacecraft's potential energy at the new apogee will be lower than at the old one and, hence, such a maneuver requires lower kinetic energy. This means that the spacecraft must lower its velocity. Similar reasoning shows that to go into a circular orbit at the apogee, the spacecraft must increase its velocity.

1.49. The kinetic energy of a satellite is determined by the value of the orbital (or satellite) velocity. According to Newton's second law and the law of universal gravitation,

$$G\frac{Mm}{R^2}=\frac{mv^2}{R},$$

where M is the mass of the earth, m the mass of the satellite, v the velocity of the satellite, and G the gravitational constant. From this it follows that the kinetic energy

$$W_{\text{kin}}=\frac{mv^2}{2}=\frac{GMm}{2R}$$

is the smaller the higher the orbit of the satellite.

The potential energy (we take it equal to zero at infinity)

$$W_{\text{pot}}=-G\frac{Mm}{R}$$

is the greater the higher the orbit of the satellite. The same is true of the total energy:

$$W=W_{\text{kin}}+W_{\text{pot}}=-G\frac{Mm}{2R}.$$

The angular momentum also increases as we move farther away from the earth and is equal to

$$mvR=m\sqrt{GMR}.$$

1.50. Let us consider an extremely elongated orbit. In this case the distance between the foci differs little from the length of the major axis. Therefore, the force acting on a space station near the apogee can be assumed to be roughly the same for all extremely elongated orbits.

Under this force the space stations move with the same accelerations $w_c = v^2/R$, where R is the curvature radius of the trajectory, and v is the velocity at apogee. The smaller the radius of curvature, the smaller is the velocity of a space station, and the greater the elongation of the orbit, the smaller is the radius. Hence, the velocity and therefore the kinetic energy at apogee tend to zero and the space stations possess almost exclusively potential energy.

Since the total energy of a space station remains constant in flight, at all other points on the orbit it is equal to the sum of the kinetic and potential energies. The potential energy of the interaction between the earth and the station (this energy is assumed to be zero at infinity) is

$$W_{pot} = -G\frac{Mg}{a},$$

where M is the mass of the earth, m the mass of the station, G the gravitational constant, and a the distance from the center of the earth to the station (this quantity is practically equal to the length of the major axis of the orbit). When circling the earth along a circular orbit whose radius R is approximately $a/2$, the station possesses potential energy

$$W_{pot} = -2G\frac{Mm}{a}.$$

As shown in the solution to Problem 1.49, the kinetic energy of the station in this case is

$$W_{kin} = G\frac{Mm}{a},$$

while the total energy is

$$W = -G\frac{Mm}{a},$$

which means that it is the same as for an elliptical orbit.

It is convenient to determine the angular momentum of a station when the station passes through the apogee:

$$L = mva.$$

For extremely elongated orbits, a is roughly the same for all orbits, but the greater the elongation of the orbit the smaller the velocity at apogee. Hence, the angular momentum at apogee is the smaller the greater the elongation of the orbit. But since the torque of the force of attraction to the earth is zero, the angular momentum must be the same at all points of the orbit. Hence, the energy of the station in a circular orbit and that of the station in an elliptical orbit coincide, while the angular momentum is the smaller the greater the elongation of the orbit.

1.51. The fact that the spacecraft retains its orientation with respect to the earth means that all points of the spacecraft move with the same angular velocity. Suppose that the point closest to the surface of the earth moves with the orbital (satellite) velocity according to the equation

$$\omega^2 R = G\frac{M}{R^2}, \tag{1.51.1}$$

where R is the distance between this point and the center of the earth. The point of the spacecraft farthest from the earth moves with an acceleration $\omega^2(R+D)$, where D is the distance between the two points.

If we consider the spacecraft to be a noninertial system, we can assume that on an object of mass m placed at the point farthest from the earth there acts a force of inertia

$$F_1 = -m\omega^2(R+D).$$

At the same time, there is the force of gravity acting on this object:

$$F = G\frac{Mm}{(R+D)^2}.$$

The sum of these two forces plays the role of "weight" for the object, or numerically the reaction of the support exerted on the object:

$$F_{\text{w}} = m\omega^2(R+D) - G\frac{Mm}{(R+D)^2}.$$

Bearing in mind that $D \ll R$, we can replace $(R+D)^{-2}$ with $(1-2D/R)/R^2$. Thus

$$F_{\text{w}} \approx m\left[\omega^2 R\left(1+\frac{D}{R}\right) - G\frac{M}{R^2}\left(1-2\frac{D}{R}\right)\right],$$

and if we allow for (1.51.1), we get

$$F_{\text{w}} \approx G\frac{Mm}{R^2} - \frac{3D}{R}.$$

Since GMm/R^2 is equal, to a high accuracy, to the weight of the object on the surface of the earth, or mg, we get

$$F_w = \frac{3D}{R} mg.$$

This expression gives the "weight" of an object in the spacecraft at the point farthest from the earth. Assuming that D is 2.1 m and bearing in mind that $R = 6300$ km, we find that the "weight" of an astronaut whose mass is 70 kg is 6.9×10^{-4} N at the point within the spacecraft farthest from the earth.

1.52. The potential energy of the comet (equal to zero at infinity) is $-GMm/r$, where m is the comet's mass, M the mass of the sun, and r the distance between the sun and the comet. As the comet approaches the sun, this energy decreases, which means that the kinetic energy increases, with

$$\frac{mv^2}{2} - G\frac{Mm}{r}$$

remaining zero.* The angular momentum of the comet is also conserved, since the torque produced by central forces is always zero. If we take two points, one at the aphelion of the presumable closed trajectory and the other placed at the same distance from the sun on the second branch of the parabola, then the potential energies at these points must coincide (since the distances coincide), which means that the kinetic energies at these points coincide and so do the velocities. But, as follows from the figure accompanying the problem, the angular momentum at the aphelion must be higher than on the branches of the parabola, which is impossible. At the same time, at symmetrical points both the kinetic energies and the potential energies are the same, and the same is true of the angular momenta.

The above reasoning is true for both closed orbits (ellipses and circles) and open orbits (parabolas and hyperbolas) of heavenly bodies moving in the field of a single attraction center. The fact that both the energy conservation law and the angular momentum conservation law must be satisfied makes it impossible for a central force to change the nature of a trajectory.

* It is assumed that the initial kinetic energy of the comet in far-away regions of space is negligible.

1.53. If D_0 is the diameter of the disk at rest, then in the system of coordinates with respect to which the disk

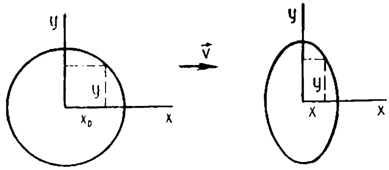


Fig. 1.53

is in motion the diameter in the direction of the velocity will be

$$D = D_0\sqrt{1 - v^2/c^2} = D_0\sqrt{1 - \beta^2}.$$

The same is true of the ratio of the halves of the chord passing at an altitude y from the center:

$$x = x_0\sqrt{1 - \beta^2}.$$

Since $x_0^2 = R^2 - y^2$, we have

$$x^2 = (R^2 - y^2)(1 - \beta)^2,$$

whence

$$\frac{x^2}{R^2(1-\beta^2)} + \frac{y^2}{R^2} = 1.$$

The moving disk appears to be an ellipse with semi-axes R and $R\sqrt{1 - \beta^2}$.

1.54. The velocity of the triangle is directed perpendicularly to the altitude, with the result that the length of the altitude is independent of the velocity. The hypotenuse is equal to twice the altitude ($l_0 = 2h$), while the length of a side of the equilateral triangle is $l = 2h \tan 30°$. Thus, for the moving triangle we have $l = l_0$ and

$$2\frac{\sqrt{3}}{3}h = 2h\sqrt{1 - \beta^2}.$$

Hence $\beta = 0.816$.

1.55. As Figure (a) accompanying this problem shows, the world line passing through the origin at an angle θ

to the x/c axis represents the motion of an object moving away from the observer (placed at the origin) with a velocity $v = c \cot \theta$. The other figures correspond to the following cases: (b) an object moving toward the observer with a velocity $v = c \cot \theta$, (c) motion with the speed of light, and (d) an object is at rest at a certain distance from the origin. Case (e) contradicts the main principles of relativity theory since it represents the motion of an object with a speed greater than that of light.

1.56. According to the theory of relativity, the kinetic energy of a moving object is given by the following formula

$$W_{\text{rel}} = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right),$$

with $\beta = v/c$. In classical mechanics,

$$W_{\text{cl}} = \frac{m_0 v^2}{2}.$$

Thus,

$$\frac{W_{\text{rel}}}{W_{\text{cl}}} = \frac{2}{\beta^2} \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right).$$

Since $\beta = \cot \theta$, we have

$$\frac{W_{\text{rel}}}{W_{\text{cl}}} = \frac{2}{\cot^2 \theta} \left(\frac{1}{\sqrt{1-\cot^3 \theta}} - 1 \right).$$

At $\theta = 60°$,

$$W_{\text{rel}}/W_{\text{cl}} = 1.37.$$

1.57. Let us assume that at $t = 0$ by the clocks in both systems, the systems were close to each other (in the figure accompanying the problem this moment corresponds to the origin). If one of the systems sends a signal after a time interval T_0 has elapsed, the second system will receive the signal after a time interval

$$T = T_0 \sqrt{\frac{1+\beta}{1-\beta}}.$$

The angle θ corresponds to a relative velocity $\beta = \cot \theta$. Thus,

$$T = T_0 \sqrt{\frac{1+\cot \theta}{1-\cot \theta}}.$$

1.58. The time interval separating the signals received by B from A is

$$T_1 = T_0 \sqrt{\frac{1+\beta}{1-\beta}}.$$

Since system C is moving toward A, its (relative) velocity is negative and, hence, the signals it sends are received by A separated by time intervals

$$T_2 = T_0 \sqrt{\frac{1-\beta}{1+\beta}}.$$

System A will register N signals from B in the course of

$$t_1 = NT_1 = NT_0 \sqrt{\frac{1+\beta}{1-\beta}},$$

while the signals from C will be registered in the course of

$$t_2 = NT_2 = NT_0 \sqrt{\frac{1-\beta}{1+\beta}}.$$

When system A meets system C, the clock in the first system will show

$$t_B = t_1 + t_2 = NT_0 \left(\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}} \right) = \frac{2NT_0}{\sqrt{1-\beta^2}}.$$

The clock in C will show the time that is the sum of the time during which system A sends N signals prior to meeting C and the time during which system C sends N signals prior to meeting system B. Thus,

$$t_C = 2NT_0.$$

The difference in the readings of the clocks will be

$$\Delta t = t_B - t_C = 2\left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) NT_0.$$

The fractional variation in the duration of the signals is

$$\frac{t_B}{t_C} = \frac{1}{\sqrt{1-\beta^2}}.$$

For example, at $\beta = 0.6$ we have

$$t_B/t_C = 1.25.$$