3

The Forced Oscillator

The Operation of i upon a Vector

We have already seen that a harmonic oscillation can be conveniently represented by the form $e^{i\omega t}$. In addition to its mathematical convenience i can also be used as a vector operator of physical significance. We say that when i precedes or operates on a vector the direction of that vector is turned through a positive angle (anticlockwise) of $\pi/2$, i.e. i acting as an operator advances the phase of a vector by 90°. The operator $-i$ rotates the vector clockwise by $\pi/2$ and retards its phase by 90°. The mathematics of i as an operator differs in no way from its use as $\sqrt{-1}$ and from now on it will play both roles.

The vector $\mathbf{r} = \mathbf{a} + i\mathbf{b}$ is shown in Figure 3.1, where the direction of $\mathbf{b}$ is perpendicular to that of $\mathbf{a}$ because it is preceded by i. The magnitude or modulus or $\mathbf{r}$ is written

$$r = |\mathbf{r}| = (a^2 + b^2)^{1/2}$$

and

$$r^2 = (a^2 + b^2) = (\mathbf{a} + i\mathbf{b})(\mathbf{a} - i\mathbf{b}) = \mathbf{r}\mathbf{r}^*,$$

where $(\mathbf{a} - i\mathbf{b}) = \mathbf{r}^*$ is defined as the complex conjugate of $(\mathbf{a} + i\mathbf{b})$; that is, the sign of i is changed.

The vector $\mathbf{r}^* = \mathbf{a} - i\mathbf{b}$ is also shown in Figure 3.1.

The vector $\mathbf{r}$ can be written as a product of its magnitude r (scalar quantity) and its phase or direction in the form (Figure 3.1)

$$\begin{aligned} \mathbf{r} &= r\,e^{i\phi} = r(\cos\phi + i\sin\phi) \\ &= \mathbf{a} + i\mathbf{b} \end{aligned}$$

showing that $a = r\cos\phi$ and $b = r\sin\phi$.

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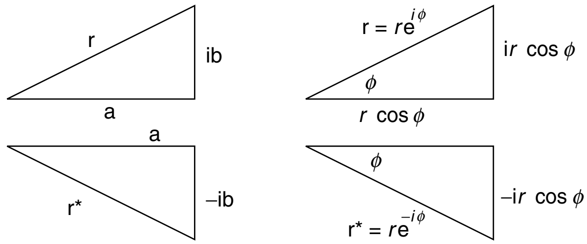

Figure 3.1 Vector representation using i operator and exponential index. Star superscript indicates complex conjugate where $-\mathrm{i}$ replaces i

It follows that

$$\cos\phi = \frac{a}{r} = \frac{a}{(a^2+b^2)^{1/2}}$$

and

$$\sin\phi = \frac{b}{r} = \frac{b}{(a^2+b^2)^{1/2}}$$

giving $\tan\phi = b/a$.

Similarly

$$\mathbf{r}^* = r\,\mathrm{e}^{-\mathrm{i}\phi} = r(\cos\phi - i\sin\phi)$$

$$\cos\phi = \frac{a}{r}, \quad \sin\phi = \frac{-b}{r} \quad \text{and} \quad \tan\phi = \frac{-b}{a} \text{ (Figure 3.1)}$$

The reader should confirm that the operator i rotates a vector by $\pi/2$ in the positive direction (as stated in the first paragraph of p. 53) by taking $\phi = \pi/2$ in the expression

$$\mathbf{r} = r\,\mathrm{e}^{\mathrm{i}\phi} = r(\cos\pi/2 + \mathrm{i}\sin\pi/2)$$

Note that $\phi = -\pi/2$ in $\mathbf{r} = r\,\mathrm{e}^{-\mathrm{i}\pi/2}$ rotates the vector in the negative direction.

Vector form of Ohm's Law

Ohm's Law is first met as the scalar relation $V = IR$, where V is the voltage across the resistance R and I is the current through it. Its scalar form states that the voltage and current are always in phase. Both will follow a $\sin(\omega t + \phi)$ or a $\cos(\omega t + \phi)$ curve, and the value of ϕ will be the same for both voltage and current.

However, the presence of either or both of the other two electrical components, inductance L and capacitance C, will introduce a phase difference between voltage and

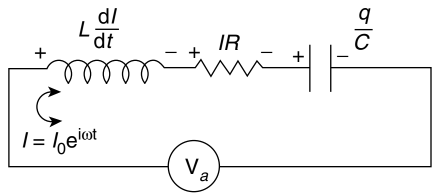

Figure 3.2a An electrical forced oscillator. The voltage V_a is applied to the series *LCR* circuit giving $V_a = L\mathrm{d}I/\mathrm{d}t + IR + q/C$

current, and Ohm's Law takes the vector form

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_e,$$

where $\mathbf{Z}_e$, called the *impedance*, replaces the resistance, and is the vector sum of the effective resistances of R, L, and C in the circuit.

When an alternating voltage V_a of frequency ω is applied across a resistance, inductance and condenser in series as in Figure 3.2a, the balance of voltages is given by

$$V_a = IR + L\frac{\mathrm{d}I}{\mathrm{d}t} + q/C$$

and the current through the circuit is given by $I = I_0\,\mathrm{e}^{\mathrm{i}\omega t}$. The voltage across the inductance

$$V_L = L\frac{\mathrm{d}I}{\mathrm{d}t} = L\frac{\mathrm{d}}{\mathrm{d}t}I_0\,\mathrm{e}^{\mathrm{i}\omega t} = \mathrm{i}\omega L I_0\,\mathrm{e}^{\mathrm{i}\omega t} = \mathrm{i}\omega L I$$

But ωL, as we saw at the end of the last chapter, has the dimensions of ohms, being the value of the effective resistance presented by an inductance L to a current of frequency ω. The product ωLI with dimensions of ohms times current, i.e. volts, is preceded by i; this tells us that the phase of the voltage across the inductance is 90° ahead of that of the current through the circuit.

Similarly, the voltage across the condenser is

$$\frac{q}{C} = \frac{1}{C}\int I\,\mathrm{d}t = \frac{1}{C}I_0\int \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{1}{\mathrm{i}\omega C}I_0\,\mathrm{e}^{\mathrm{i}\omega t} = -\frac{\mathrm{i}I}{\omega C}$$

(since $1/\mathrm{i} = -\mathrm{i}$).

Again $1/\omega C$, measured in ohms, is the value of the effective resistance presented by the condenser to the current of frequency ω. Now, however, the voltage $I/\omega C$ across the condenser is preceded by $-\mathrm{i}$ and therefore lags the current by 90°. The voltage and current across the resistance are in phase and Figure 3.2b shows that the vector form of Ohm's Law may be written $\mathbf{V} = \mathbf{I}\mathbf{Z}_e = I[R + \mathrm{i}(\omega L - 1/\omega C)]$, where the impedance $\mathbf{Z}_e = R + \mathrm{i}(\omega L - 1/\omega C)$. The quantities ωL and $1/\omega C$ are called *reactances* because they

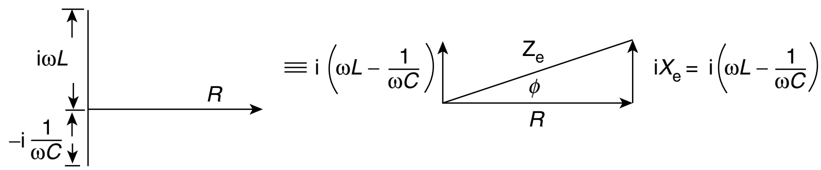

Figure 3.2b Vector addition of resistance and reactances to give the electrical impedance $\mathbf{Z}_e = R + \mathrm{i}(\omega L - 1/\omega C)$

introduce a phase relationship as well as an effective resistance, and the bracket $(\omega L - 1/\omega C)$ is often written X_e, the reactive component of $\mathbf{Z}_e$.

The magnitude, in ohms, i.e. the value of the impedance, is

$$Z_e = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}$$

and the vector $\mathbf{Z}_e$ may be represented by its magnitude and phase as

$$\mathbf{Z}_e = Z_e\,\mathrm{e}^{\mathrm{i}\phi} = Z_e(\cos\phi + \mathrm{i}\sin\phi)$$

so that

$$\cos\phi = \frac{R}{Z_e}, \quad \sin\phi = \frac{X_e}{Z_e}$$

and

$$\tan\phi = X_e/R,$$

where ϕ is the phase difference between the total voltage across the circuit and the current through it.

The value of ϕ can be positive or negative depending on the relative value of ωL and $1/\omega C$: when $\omega L > 1/\omega C, \phi$ is positive, but the frequency dependence of the components show that ϕ can change both sign and size.

The magnitude of $\mathbf{Z}_e$ is also frequency dependent and has its minimum value $Z_e = R$ when $\omega L = 1/\omega C$.

In the vector form of Ohm's Law, $\mathbf{V} = \mathbf{I}\mathbf{Z}_e$. If $\mathbf{V} = V_0\,\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathbf{Z}_e = Z_e\,\mathrm{e}^{\mathrm{i}\phi}$, then we have

$$\mathbf{I} = \frac{V_0\,\mathrm{e}^{\mathrm{i}\omega t}}{Z_e\,\mathrm{e}^{\mathrm{i}\phi}} = \frac{V_0}{Z_e}\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}$$

giving a current of amplitude V_0/Z_e which lags the voltage by a phase angle ϕ.

The Impedance of a Mechanical Circuit

Exactly similar arguments hold when we consider not an electrical oscillator but a mechanical circuit having mass, stiffness and resistance.

The mechanical impedance is defined as the force required to produce unit velocity in the oscillator, i.e. $\mathbf{Z}_m = \mathbf{F}/\mathbf{v}$ or $\mathbf{F} = \mathbf{v}\mathbf{Z}_m$.

Immediately, we can write the mechanical impedance as

$$\mathbf{Z}_m = r + \mathrm{i}\left(\omega m - \frac{s}{\omega}\right) = r + \mathrm{i}X_m$$

where

$$\mathbf{Z}_m = Z_m\,\mathrm{e}^{\mathrm{i}\phi}$$

and

$$\tan\phi = X_m/r$$

ϕ being the phase difference between the force and the velocity. The magnitude of $Z_m = [r^2 + (\omega m - s/\omega)^2]^{1/2}$.

Mass, like inductance, produces a positive reactance, and the stiffness behaves in exactly the same way as the capacitance.

Behaviour of a Forced Oscillator

We are now in a position to discuss the physical behaviour of a mechanical oscillator of mass m, stiffness s and resistance r being driven by an alternating force $F_0 \cos\omega t$, where F_0 is the amplitude of the force (Figure 3.3). The equivalent electrical oscillator would be an alternating voltage $V_0 \cos\omega t$ applied to the circuit of inductance L, capacitance C and resistance R in Figure 3.2a.

The mechanical equation of motion, i.e. the dynamic balance of forces, is given by

$$m\ddot{x} + r\dot{x} + sx = F_0 \cos\omega t$$

and the voltage equation in the electrical case is

$$L\ddot{q} + R\dot{q} + q/C = V_0 \cos\omega t$$

We shall analyse the behaviour of the mechanical system but the analysis fits the electrical oscillator equally well.

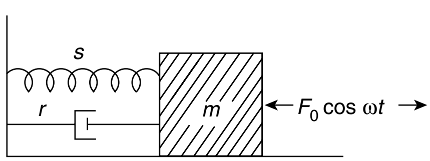

Figure 3.3 Mechanical forced oscillator with force $F_0 \cos\omega t$ applied to damped mechanical circuit of Figure 2.1

The complete solution for x in the equation of motion consists of two terms:

(1) a 'transient' term which dies away with time and is, in fact, the solution to the equation $m\ddot{x} + r\dot{x} + sx = 0$ discussed in Chapter 2. This contributes the term

$$x = C\,\mathrm{e}^{-rt/2m}\,\mathrm{e}^{\mathrm{i}(s/m - r^2/4m^2)^{1/2}t}$$

which decays with $\mathrm{e}^{-rt/2m}$. The second term
(2) is called the 'steady state' term, and describes the behaviour of the oscillator after the transient term has died away.

Both terms contribute to the solution initially, but for the moment we shall concentrate on the 'steady state' term which describes the ultimate behaviour of the oscillator.

To do this we shall rewrite the force equation in vector form and represent $\cos\omega t$ by $\mathrm{e}^{\mathrm{i}\omega t}$ as follows:

$$m\ddot{\mathbf{x}} + r\dot{\mathbf{x}} + s\mathbf{x} = F_0\,\mathrm{e}^{\mathrm{i}\omega t} \tag{3.1}$$

Solving for the vector $\mathbf{x}$ will give both its magnitude and phase with respect to the driving force $F_0\,\mathrm{e}^{\mathrm{i}\omega t}$. Initially, let us try the solution $\mathbf{x} = \mathbf{A}\,\mathrm{e}^{\mathrm{i}\omega t}$, where $\mathbf{A}$ may be complex, so that it may have components in and out of phase with the driving force.

The velocity

$$\dot{\mathbf{x}} = \mathrm{i}\omega\mathbf{A}\,\mathrm{e}^{\mathrm{i}\omega t} = \mathrm{i}\omega\mathbf{x}$$

so that

$$\ddot{\mathbf{x}} = \mathrm{i}^2\omega^2\mathbf{x} = -\omega^2\mathbf{x}$$

and equation (3.1) becomes

$$(-\mathbf{A}\omega^2 m + \mathrm{i}\omega\mathbf{A}r + \mathbf{A}s)\,\mathrm{e}^{\mathrm{i}\omega t} = F_0\,\mathrm{e}^{\mathrm{i}\omega t}$$

which is true for all t when

$$\mathbf{A} = \frac{F_0}{\mathrm{i}\omega r + (s - \omega^2 m)}$$

or, after multiplying numerator and denominator by $-\mathrm{i}$

$$\mathbf{A} = \frac{-\mathrm{i}F_0}{\omega[r + \mathrm{i}(\omega m - s/\omega)]} = \frac{-\mathrm{i}F_0}{\omega\mathbf{Z}_m}$$

Hence

$$\mathbf{x} = \mathbf{A}\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{-\mathrm{i}F_0\,\mathrm{e}^{\mathrm{i}\omega t}}{\omega\mathbf{Z}_m} = \frac{-\mathrm{i}F_0\,\mathrm{e}^{\mathrm{i}\omega t}}{\omega Z_m\,\mathrm{e}^{\mathrm{i}\phi}}$$

$$= \frac{-\mathrm{i}F_0\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}}{\omega Z_m}$$

where

$$Z_m = [r^2 + (\omega m - s/\omega)^2]^{1/2}$$

This vector form of the *steady state* behaviour of x gives three pieces of information and completely defines the magnitude of the displacement x and its phase with respect to the driving force after the transient term dies away. It tells us

1. That the phase difference ϕ exists between x and the force because of the reactive part $(\omega m - s/\omega)$ of the mechanical impedance.
2. That an extra difference is introduced by the factor $-\mathrm{i}$ and even if ϕ were zero the displacement x would lag the force $F_0 \cos \omega t$ by 90°.
3. That the maximum amplitude of the displacement x is $F_0/\omega Z_m$. We see that this is dimensionally correct because the velocity x/t has dimensions F_0/Z_m.

Having used $F_0\,\mathrm{e}^{\mathrm{i}\omega t}$ to represent its real part $F_0 \cos \omega t$, we now take the real part of the solution

$$\mathbf{x} = \frac{-\mathrm{i}F_0\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}}{\omega Z_m}$$

to obtain the actual value of **x**. (If the force had been $F_0 \sin \omega t$, we would now take that part of **x** preceded by i.)

Now

$$\begin{aligned}\mathbf{x} &= -\frac{\mathrm{i}F_0}{\omega Z_m}\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}\\ &= -\frac{\mathrm{i}F_0}{\omega Z_m}[\cos(\omega t - \phi) + \mathrm{i}\sin(\omega t - \phi)]\\ &= -\frac{\mathrm{i}F_0}{\omega Z_m}\cos(\omega t - \phi) + \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)\end{aligned}$$

The value of x resulting from $F_0 \cos \omega t$ is therefore

$$x = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)$$

[the value of x resulting from $F_0 \sin \omega t$ would be $-F_0 \cos(\omega t - \phi)/\omega Z_m$].

Note that both of these solutions satisfy the requirement that the total phase difference between displacement and force is ϕ plus the $-\pi/2$ term introduced by the $-\mathrm{i}$ factor. When $\phi = 0$ the displacement $x = F_0 \sin \omega t/\omega Z_m$ *lags* the force $F_0 \cos \omega t$ by exactly 90°.

To find the velocity of the forced oscillation in the steady state we write

$$\mathbf{v} = \dot{\mathbf{x}} = (\mathrm{i}\omega)\frac{(-\mathrm{i}F_0)}{\omega Z_m}\,\mathrm{e}^{\mathrm{i}(\omega t-\phi)}$$
$$= \frac{F_0}{Z_m}\,\mathrm{e}^{\mathrm{i}(\omega t-\phi)}$$

We see immediately that

1. There is no preceding i factor so that the velocity **v** and the force differ in phase only by ϕ, and when $\phi = 0$ the velocity and force are in phase.
2. The amplitude of the velocity is F_0/Z_m, which we expect from the definition of mechanical impedance $\mathbf{Z}_m = \mathbf{F}/\mathbf{v}$.

Again we take the real part of the vector expression for the velocity, which will correspond to the real part of the force $F_0\,\mathrm{e}^{\mathrm{i}\omega t}$. This is

$$v = \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

Thus, the *velocity is always exactly 90° ahead of the displacement in phase* and differs from the force only by a phase angle ϕ, where

$$\tan\phi = \frac{\omega m - s/\omega}{r} = \frac{X_m}{r}$$

so that a force $F_0\cos\omega t$ gives a displacement

$$x = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)$$

and a velocity

$$v = \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

(Problems 3.1, 3.2, 3.3, 3.4)

Behaviour of Velocity v in Magnitude and Phase versus Driving Force Frequency ω

The velocity amplitude is

$$\frac{F_0}{Z_m} = \frac{F_0}{[r^2 + (\omega m - s/\omega)^2]^{1/2}}$$

so that the magnitude of the velocity will vary with the frequency ω because Z_m is frequency dependent.

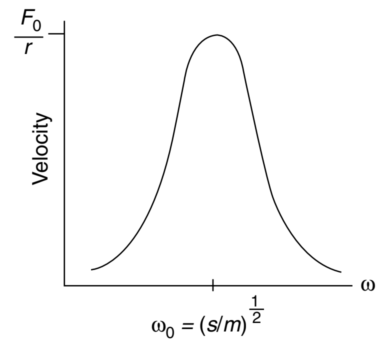

Figure 3.4 Velocity of forced oscillator versus driving frequency ω. Maximum velocity $v_{\text{max}} = F_0/r$ at $\omega_0^2 = s/m$

At low frequencies, the term $-s/\omega$ is the largest term in Z_m and the impedance is said to be *stiffness controlled*. At high frequencies ωm is the dominant term and the impedance is *mass controlled*. At a frequency ω_0 where $\omega_0 m = s/\omega_0$, the impedance has its minimum value $Z_m = r$ and is a real quantity with zero reactance.

The velocity F_0/Z_m then has its maximum value $v = F_0/r$, and ω_0 is said to be the frequency of *velocity resonance*. Note that $\tan\phi = 0$ at ω_0, the velocity and force being in phase.

The variation of the magnitude of the velocity with driving frequency, ω, is shown in Figure 3.4, the height and sharpness of the peak at resonance depending on r, which is the only effective term of Z_m at ω_0.

The expression

$$v = \frac{F_0}{Z_m} \cos(\omega t - \phi)$$

where

$$\tan\phi = \frac{\omega m - s/\omega}{r}$$

shows that for positive ϕ; that is, $\omega m > s/\omega$, the velocity v will lag the force because $-\phi$ appears in the argument of the cosine. When the driving force frequency ω is very high and $\omega \to \infty$, then $\phi \to 90°$ and the velocity lags the force by that amount.

When $\omega m < s/\omega, \phi$ is negative, the velocity is ahead of the force in phase, and at low driving frequencies as $\omega \to 0$ the term $s/\omega \to \infty$ and $\phi \to -90°$.

Thus, at low frequencies the velocity leads the force (ϕ negative) and at high frequencies the velocity lags the force (ϕ positive).

At the frequency ω_0, however, $\omega_0 m = s/\omega_0$ and $\phi = 0$, so that velocity and force are in phase. Figure 3.5 shows the variation of ϕ with ω for the velocity, the actual shape of the curves depending upon the value of r.

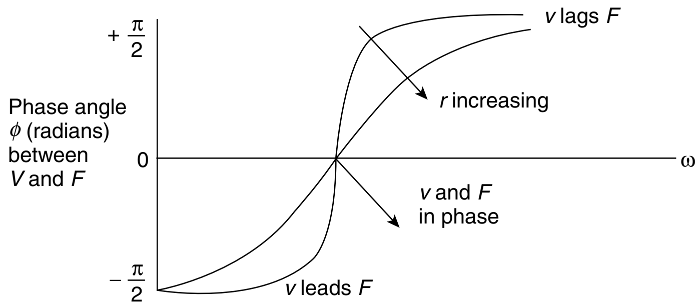

Figure 3.5 Variation of phase angle ϕ versus driving frequency, where ϕ is the phase angle between the velocity of the forced oscillator and the driving force. $\phi = 0$ at velocity resonance. Each curve represents a fixed resistance value

(Problem 3.5)

Behaviour of Displacement versus Driving Force Frequency ω

The phase of the displacement

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi)$$

is at all times exactly 90° behind that of the velocity. Whilst the graph of ϕ versus ω remains the same, the total phase difference between the displacement and the force involves the extra 90° retardation introduced by the $-i$ operator. Thus, at very low frequencies, where $\phi = -\pi/2$ rad and the velocity leads the force, the displacement and the force are in phase as we should expect. At high frequencies the displacement lags the force by π rad and is exactly out of phase, so that the curve showing the phase angle between the displacement and the force is equivalent to the ϕ versus ω curve, displaced by an amount equal to $\pi/2$ rad. This is shown in Figure 3.6.

The amplitude of the displacement $x = F_0/\omega Z_m$, and at low frequencies $Z_m = [r^2 + (\omega m - s/\omega)^2]^{1/2} \to s/\omega$, so that $x \approx F_0/(\omega s/\omega) = F_0/s$.

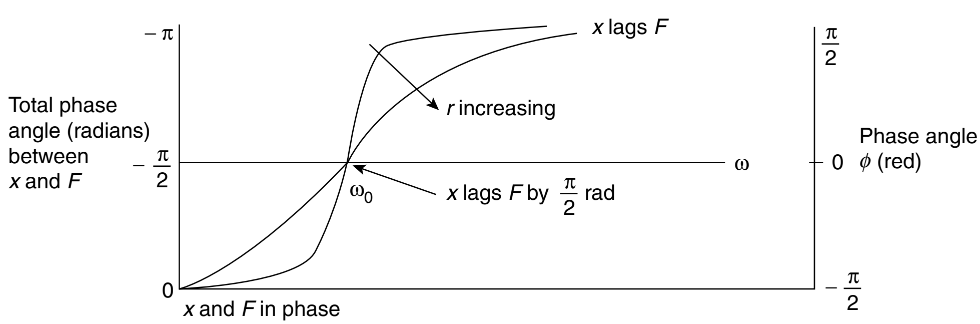

Figure 3.6 Variation of total phase angle between displacement and driving force versus driving frequency ω. The total phase angle is $-\phi - \pi/2$ rad

At high frequencies $Z_m \to \omega m$, so that $x \approx F_0/(\omega^2 m)$, which tends to zero as ω becomes very large. At very high frequencies, therefore, the displacement amplitude is almost zero because of the mass-controlled or inertial effect.

The velocity resonance occurs at $\omega_0^2 = s/m$, where the denominator Z_m of the velocity amplitude is a minimum, but the displacement resonance will occur, since $x = (F_0/\omega Z_m) \sin(\omega t - \phi)$, when the denominator ωZ_m is a minimum. This takes place when

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega Z_m) = \frac{\mathrm{d}}{\mathrm{d}\omega}\,\omega[r^2 + (\omega m - s/\omega)^2]^{1/2} = 0$$

i.e. when

$$2\omega r^2 + 4\omega m(\omega^2 m - s) = 0$$

or

$$2\omega[r^2 + 2m(\omega^2 m - s)] = 0$$

so that either

$$\omega = 0$$

or

$$\omega^2 = \frac{s}{m} - \frac{r^2}{2m^2} = \omega_0^2 - \frac{r^2}{2m^2}$$

Thus the *displacement resonance* occurs at a frequency slightly less than ω_0, the frequency of velocity resonance. For a small damping constant r or a large mass m these two resonances, for all practical purposes, occur at the frequency ω_0.

Denoting the displacement resonance frequency by

$$\omega_r = \left(\frac{s}{m} - \frac{r^2}{2m^2}\right)^{1/2}$$

we can write the maximum displacement as

$$x_{\max} = \frac{F_0}{\omega_r Z_m}$$

The value of $\omega_r Z_m$ at ω_r is easily shown to be equal to $\omega' r$ where

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2} = \omega_0^2 - \frac{r^2}{4m^2}$$

The value of x at displacement resonance is therefore given by

$$x_{\max} = \frac{F_0}{\omega' r}$$

where

$$\omega' = \left(\omega_0^2 - \frac{r^2}{4m^2}\right)^{1/2}$$

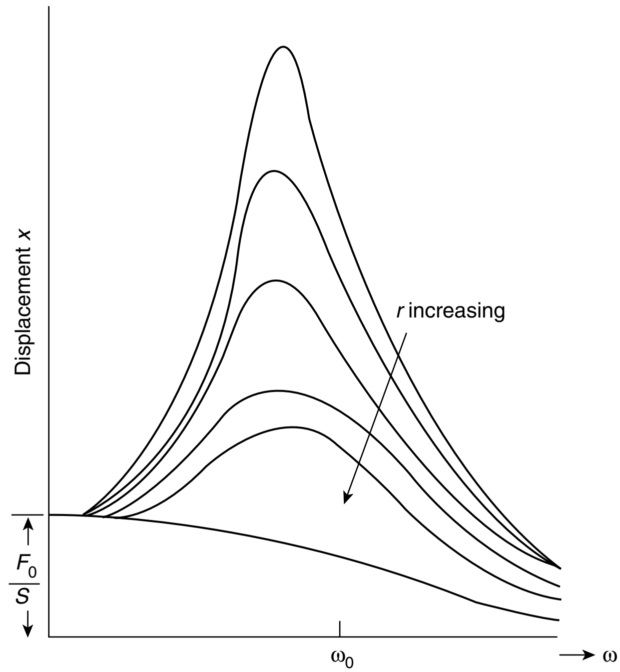

Figure 3.7 Variation of the displacement of a forced oscillator versus driving force frequency ω for various values of r

Since $x_{\max} = F_0/\omega' r$ at resonance, the amplitude at resonance is kept low by increasing r and the variation of x with ω for different values of r is shown in Figure 3.7. A negligible value of r produces a large amplification at resonance: this is the basis of high selectivity in a tuned radio circuit (see the section in this chapter on Q as an amplification factor). Keeping the resonance amplitude low is the principle of vibration insulation.

(Problems 3.6, 3.7)

Problem on Vibration Insulation

A typical vibration insulator is shown in Figure 3.8. A heavy base is supported on a vibrating floor by a spring system of stiffness s and viscous damper r. The insulator will generally operate at the mass controlled end of the frequency spectrum and the resonant frequency is designed to be lower than the range of frequencies likely to be met. Suppose the vertical vibration of the floor is given by $x = A\cos\omega t$ about its equilibrium position and y is the corresponding vertical displacement of the base about its rest position. The function of the insulator is to keep the ratio y/A to a minimum.

The equation of motion is given by

$$m\ddot{y} = -r(\dot{y} - \dot{x}) - s(y - x)$$

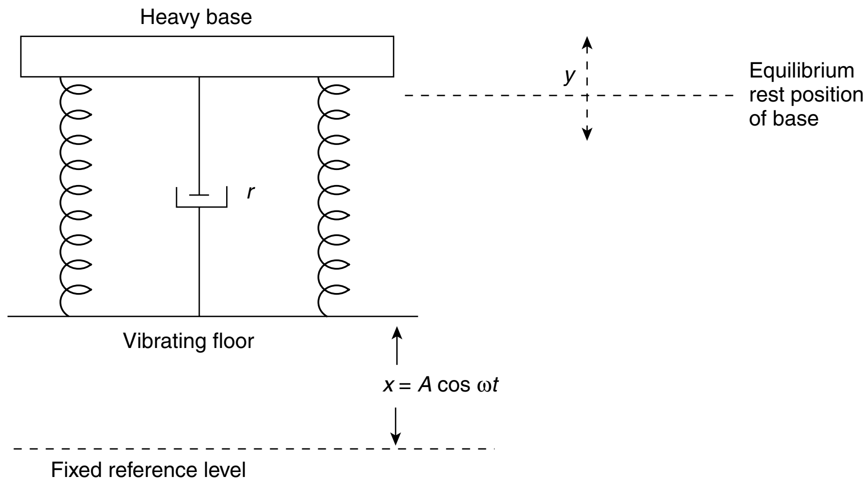

Figure 3.8 Vibration insulator. A heavy base supported by a spring and viscous damper system on a vibrating floor

which, if $y - x = X$, becomes

$$m\ddot{X} + r\dot{X} + sX = -m\ddot{x} = mA\omega^2 \cos \omega t$$
$$= F_0 \cos \omega t,$$

where

$$F_0 = mA\omega^2$$

Use the steady state solution of X to show that

$$y = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi) + A \cos \omega t$$

and (noting that y is the superposition of two harmonic components with a constant phase difference) show that

$$\frac{y_{\max}}{A} = \frac{(r^2 + s^2/\omega^2)^{1/2}}{Z_m}$$

where

$$Z_m^2 = r^2 + (\omega m - s/\omega)^2$$

Note that

$$\frac{y_{\max}}{A} > 1 \quad \text{if} \quad \omega^2 < \frac{2s}{m}$$

so that s/m should be as low as possible to give protection against a given frequency ω.

(a) Show that

$$\frac{y_{\max}}{A} = 1 \quad \text{for} \quad \omega^2 = \frac{2s}{m}$$

(b) Show that

$$\frac{y_{\max}}{A} < 1 \quad \text{for} \quad \omega^2 > \frac{2s}{m}$$

(c) Show that if $\omega^2 = s/m$, then $y_{\max}/A > 1$ but that the damping term r is helpful in keeping the motion of the base to a reasonably low level.
(d) Show that if $\omega^2 > 2s/m$, then $y_{\max}/A < 1$ but damping is detrimental.

Significance of the Two Components of the Displacement Curve

Any single curve of Figure 3.7 is the superposition of the two component curves (a) and (b) in Figure 3.9, for the displacement x may be rewritten

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi) = \frac{F_0}{\omega Z_m}(\sin \omega t \cos \phi - \cos \omega t \sin \phi)$$

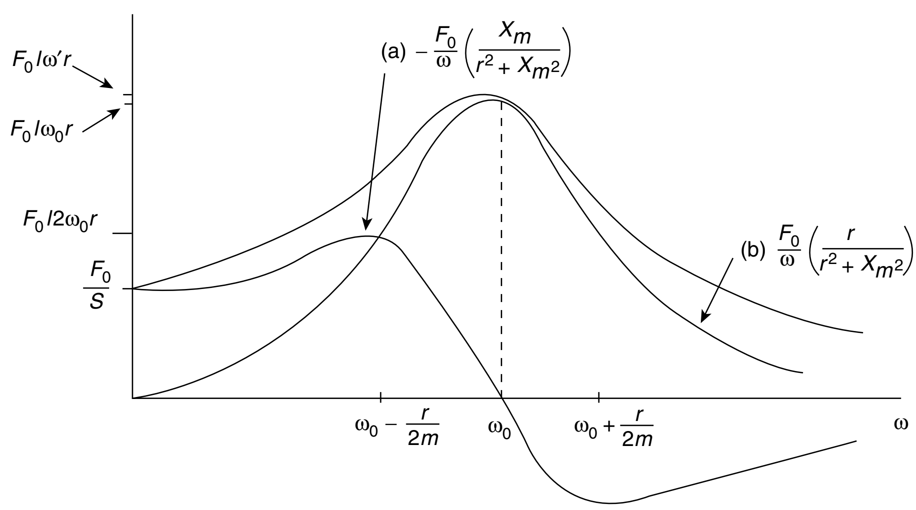

Figure 3.9 A typical curve of Figure 3.7 resolved into its 'anti-phase' component (curve (a)) and its '90° out of phase' component (curve (b)). Curve (b) represents the resistive fraction of the impedance and curve (a) the reactive fraction. Curve (b) corresponds to absorption and curve (a) to anomalous dispersion of an electromagnetic wave in a medium having an atomic or molecular resonant frequency equal to the frequency of the wave

or, since

$$\cos\phi = \frac{r}{Z_m} \quad \text{and} \quad \sin\phi = \frac{X_m}{Z_m}$$

as

$$x = \frac{F_0}{\omega Z_m}\frac{r}{Z_m}\sin\omega t - \frac{F_0}{\omega Z_m}\frac{X_m}{Z_m}\cos\omega t$$

The $\cos\omega t$ component (with a negative sign) is exactly anti-phase with respect to the driving force $F_0\cos\omega t$. Its amplitude, plotted as curve (a) may be expressed as

$$-\frac{F_0}{\omega}\frac{X_m}{Z_m^2} = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2} \tag{3.2}$$

where $\omega_0^2 = s/m$ and ω_0 is the frequency of velocity resonance.

The $\sin\omega t$ component lags the driving force $F_0\cos\omega t$ by 90°. Its amplitude plotted as curve (b) becomes

$$\frac{F_0}{\omega}\frac{r}{r^2 + X_m^2} = \frac{F_0\omega r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}$$

We see immediately that at ω_0 curve (a) is zero and curve (b) is near its maximum but they combine to give a maximum at ω where

$$\omega^2 = \omega_0^2 - \frac{r^2}{2m^2}$$

the resonant frequency for amplitude displacement.

These curves are particularly familiar in the study of optical dispersion where the forced oscillator is an electron in an atom and the driving force is the oscillating field vector of an electromagnetic wave of frequency ω. When ω is the resonant frequency of the electron in the atom, the atom absorbs a large amount of energy from the electromagnetic wave and curve (b) is the shape of the characteristic absorption curve. Note that curve (b) represents the dissipating or absorbing fraction of the impedance

$$\frac{r}{(r^2 + X_m^2)^{1/2}}$$

and that part of the displacement which lags the driving force by 90°. The velocity associated with this component will therefore be in phase with the driving force and it is this part of the velocity which appears in the energy loss term $r\dot{x}^2$ due to the resistance of the oscillator and which gives rise to absorption.

On the other hand, curve (a) represents the reactive or energy storing fraction of the impedance

$$\frac{X_m}{(r^2 + X_m^2)^{1/2}}$$

and the reactive components in a medium determine the velocity of the waves in the medium which in turn governs the refractive index n. In fact, curve (a) is a graph of the value of n^2 in a region of anomalous dispersion where the ω axis represents the value $n = 1$. These regions occur at every resonant frequency of the constituent atoms of the medium. We shall return to this topic later in the book.

(Problems 3.8, 3.9, 3.10)

Power Supplied to Oscillator by the Driving Force

In order to maintain the steady state oscillations of the system the driving force must replace the energy lost in each cycle because of the presence of the resistance. We shall now derive the most important result that:

'in the steady state the amplitude and phase of a driven oscillator adjust themselves so that the average power supplied by the driving force just equals that being dissipated by the frictional force'.

The *instantaneous power P* supplied is equal to the product of the *instantaneous driving force* and the *instantaneous velocity*; that is,

$$\begin{aligned} P &= F_0 \cos \omega t \frac{F_0}{Z_m} \cos(\omega t - \phi) \\ &= \frac{F_0^2}{Z_m} \cos \omega t \cos(\omega t - \phi) \end{aligned}$$

The *average power*

$$\begin{aligned} P_{\text{av}} &= \frac{\text{total work per oscillation}}{\text{oscillation period}} \\ \therefore P_{\text{av}} &= \int_0^T \frac{P\,\mathrm{d}t}{T} \quad \text{where } T = \text{oscillation period} \\ &= \frac{F_0^2}{Z_m T} \int_0^T \cos \omega t \cos(\omega t - \phi)\,\mathrm{d}t \\ &= \frac{F_0^2}{Z_m T} \int_0^T [\cos^2 \omega t \cos \phi + \cos \omega t \sin \omega t \sin \phi)\,\mathrm{d}t \\ &= \frac{F_0^2}{2 Z_m} \cos \phi \end{aligned}$$

because

$$\int_0^T \cos\omega t \times \sin\omega t \, dt = 0$$

and

$$\frac{1}{T}\int_0^T \cos^2\omega t \, dt = \frac{1}{2}$$

The power supplied by the driving force is not stored in the system, but dissipated as work expended in moving the system against the frictional force $r\dot{x}$.

The rate of working (instantaneous power) by the frictional force is

$$(r\dot{x})\dot{x} = r\dot{x}^2 = r\frac{F_0^2}{Z_m^2}\cos^2(\omega t - \phi)$$

and the average value of this over one period of oscillation

$$\frac{1}{2}\frac{rF_0^2}{Z_m^2} = \frac{1}{2}\frac{F_0^2}{Z_m}\cos\phi \quad \text{for} \quad \frac{r}{Z_m} = \cos\phi$$

This proves the initial statement that the power supplied equals the power dissipated.

In an electrical circuit the power is given by $VI\cos\phi$, where V and I are the instantaneous r.m.s. values of voltage and current and $\cos\phi$ is known as the *power factor*.

$$VI\cos\phi = \frac{V^2}{Z_e}\cos\phi = \frac{V_0^2}{2Z_e}\cos\phi$$

since

$$V = \frac{V_0}{\sqrt{2}}$$

(Problem 3.11)

Variation of P_{av} with ω. Absorption Resonance Curve

Returning to the mechanical case, we see that the average power supplied

$$P_{\text{av}} = (F_0^2/2Z_m)\cos\phi$$

is a maximum when $\cos\phi = 1$; that is, when $\phi = 0$ and $\omega m - s/\omega = 0$ or $\omega_0^2 = s/m$. The force and the velocity are then in phase and Z_m has its minimum value of r. Thus

$$P_{\text{av}}(\text{maximum}) = F_0^2/2r$$

A graph of P_{av} versus ω, the frequency of the driving force, is shown in Figure 3.10. Like the curve of displacement versus ω, this graph measures the response of the oscillator; the sharpness of its peak at resonance is also determined by the value of the damping constant r, which is the only term remaining in Z_m at the resonance frequency ω_0. The peak occurs at the frequency of velocity resonance when the power absorbed by the system from the driving force is a maximum; this curve is known as the absorption curve of the oscillator (it is similar to curve (b) of Figure 3.9).

The *Q*-Value in Terms of the Resonance Absorption Bandwidth

In the last chapter we discussed the quality factor of an oscillator system in terms of energy decay. We may derive the same parameter in terms of the curve of Figure 3.10, where the sharpness of the resonance is precisely defined by the ratio

$$Q = \frac{\omega_0}{\omega_2 - \omega_1},$$

where ω_2 and ω_1 are those frequencies at which the power supplied

$$P_{av} = \tfrac{1}{2} P_{av}(\text{maximum})$$

The frequency difference $\omega_2 - \omega_1$ is often called the bandwidth.

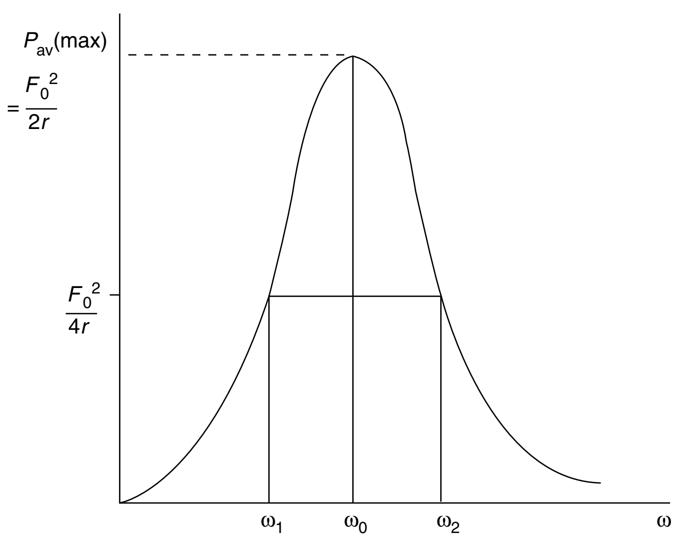

Figure 3.10 Graph of average power versus ω supplied to an oscillator by the driving force. Bandwidth $\omega_2 - \omega_1$ of resonance curve defines response in terms of the quality factor, $Q = \omega_0/(\omega_2 - \omega_1)$, where $\omega_0^2 = s/m$

Now

$$P_{\text{av}} = rF_0^2/2Z_m^2 = \tfrac{1}{2}P_{\text{av}}\ (\text{maximum}) = \tfrac{1}{2}F_0^2/2r$$

when

$$Z_m^2 = 2r^2$$

that is, when

$$r^2 + X_m^2 = 2r^2 \quad \text{or} \quad X_m = \omega m - s/\omega = \pm r.$$

If $\omega_2 > \omega_1$, then

$$\omega_2 m - s/\omega_2 = +r$$

and

$$\omega_1 m - s/\omega_1 = -r$$

Eliminating s between these equations gives

$$\omega_2 - \omega_1 = r/m$$

so that

$$Q = \omega_0 m/r$$

Note that $\omega_1 = \omega_0 - r/2m$ and $\omega_2 = \omega_0 + r/2m$ are the two significant frequencies in Figure 3.9. The quality factor of an electrical circuit is given by

$$Q = \frac{\omega_0 L}{R},$$

where

$$\omega_0^2 = (LC)^{-1}$$

Note that for high values of Q, where the damping constant r is small, the frequency ω' used in the last chapter to define $Q = \omega' m/r$ moves very close to the frequency ω_0, and the two definitions of Q become equivalent to each other and to the third definition we meet in the next section.

The *Q*-Value as an Amplification Factor

We have seen that the value of the displacement at resonance is given by

$$A_{\max} = \frac{F_0}{\omega' r} \quad \text{where} \quad \omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2}$$

At low frequencies $(\omega \rightarrow 0)$ the displacement has a value $A_0 = F_0/s$, so that

$$\left(\frac{A_{\max}}{A_0}\right)^2 = \frac{F_0^2}{\omega'^2 r^2}\frac{s^2}{F_0^2} = \frac{m^2\omega_0^4}{r^2[\omega_0^2 - r^2/4m^2]}$$
$$= \frac{\omega_0^2 m^2}{r^2[1 - 1/4Q^2]^{1/2}]} = \frac{Q^2}{[1 - 1/4Q^2]}$$

Hence:

$$\frac{A_{\max}}{A_0} = \frac{Q}{[1 - 1/4Q^2]^{1/2}} \approx Q\left[1 + \frac{1}{8Q^2}\right] \approx Q$$

for large Q.

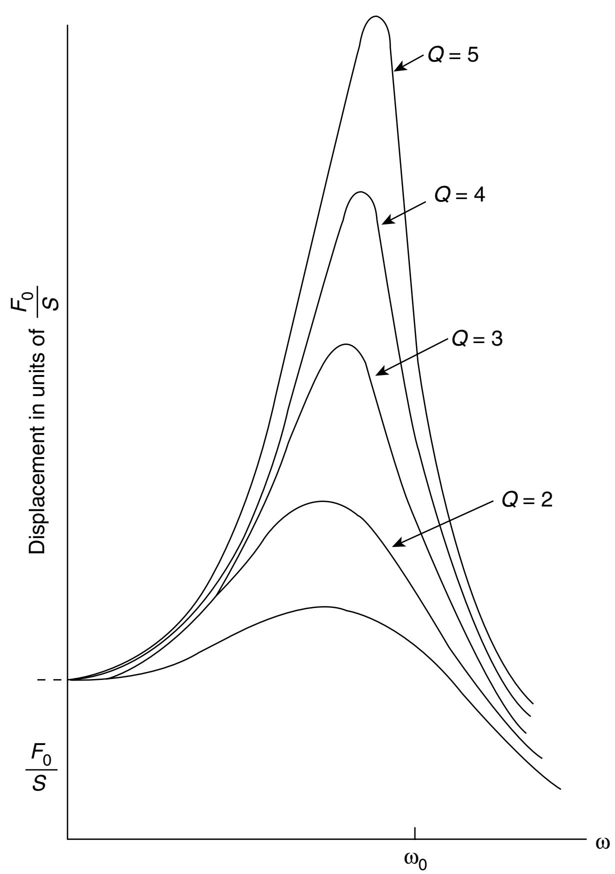

Figure 3.11 Curves of Figure 3.7 now given in terms of the quality factor Q of the system, where Q is amplification at resonance of low frequency response $x = F_0/s$

Thus, the displacement at low frequencies is amplified by a factor of Q at displacement resonance.

Figure 3.7 is now shown as Figure 3.11 where the Q-values have been attached to each curve. In tuning radio circuits, the Q-value is used as a measure of selectivity, where the sharpness of response allows a signal to be obtained free from interference from signals at nearby frequencies. In conventional radio circuits at frequencies of one megacycle,

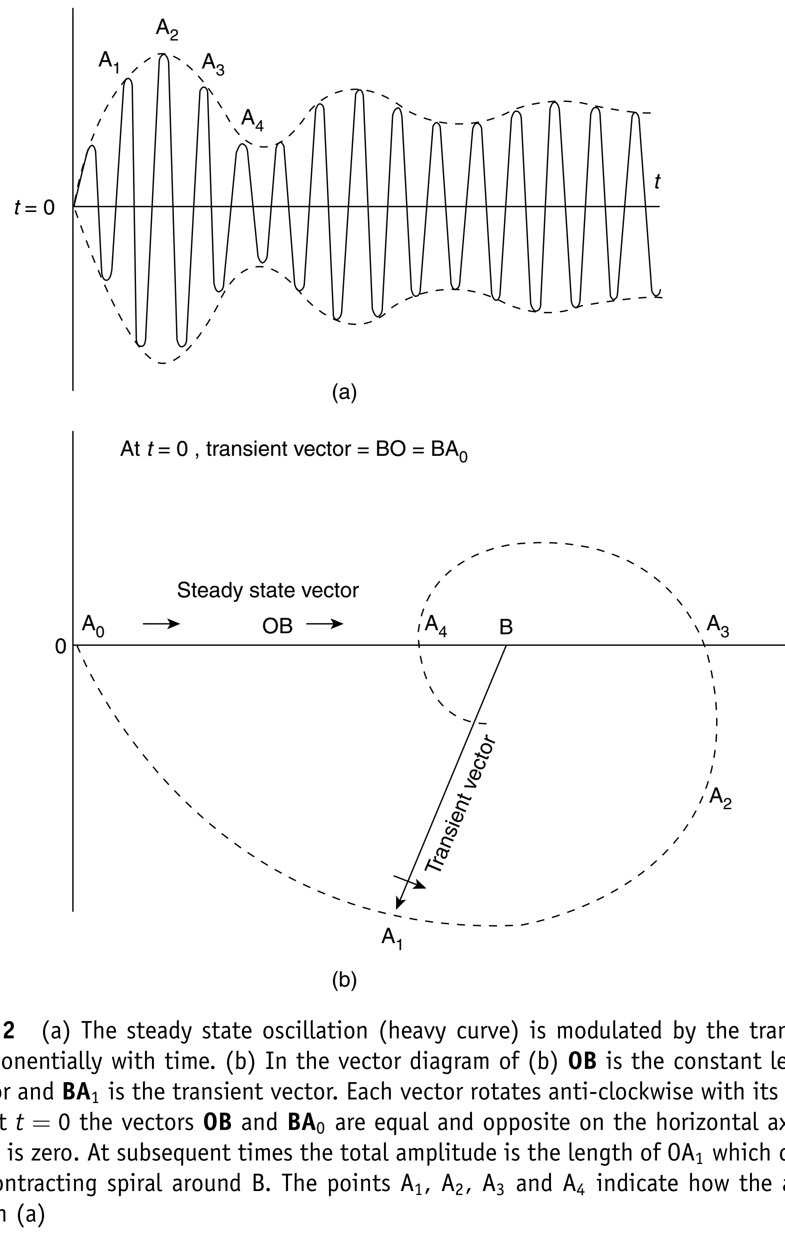

Figure 3.12 (a) The steady state oscillation (heavy curve) is modulated by the transient which decays exponentially with time. (b) In the vector diagram of (b) **OB** is the constant length steady state vector and $\mathbf{BA}_1$ is the transient vector. Each vector rotates anti-clockwise with its own angular velocity. At $t = 0$ the vectors **OB** and $\mathbf{BA}_0$ are equal and opposite on the horizontal axis and their vector sum is zero. At subsequent times the total amplitude is the length of OA_1 which changes as A traces a contracting spiral around B. The points A_1, A_2, A_3 and A_4 indicate how the amplitude is modified in (a)

Q-values are of the order of a few hundred; at higher radio frequencies resonant copper cavities have Q-values of about 30 000 and piezo-electric crystals can produce Q-values of 500 000. Optical absorption in crystals and nuclear magnetic resonances are often described in terms of Q-values. The Mössbauer effect in nuclear physics involves Q-values of 10^{10}.

The Effect of the Transient Term

Throughout this chapter we have considered only the steady state behaviour without accounting for the transient term mentioned on p. 58. This term makes an initial contribution to the total displacement but decays with time as $e^{-rt/2m}$. Its effect is best displayed by considering the vector sum of the transient and steady state components.

The steady state term may be represented by a vector of constant length rotating anticlockwise at the angular velocity ω of the driving force. The vector tip traces a circle. Upon this is superposed the transient term vector of diminishing length which rotates anti clockwise with angular velocity $\omega' = (s/m - r^2/4m^2)^{1/2}$. Its tip traces a contracting spiral.

The locus of the magnitude of the vector sum of these terms is the envelope of the varying amplitudes of the oscillator. This envelope modulates the steady state oscillations of frequency ω at a frequency which depends upon ω' and the relative phase between ωt and $\omega' t$.

Thus, in Figure 3.12(a) where the total oscillator displacement is zero at time $t = 0$ we have the steady state and transient vectors equal and opposite in Figure 3.12(b) but because $\omega \neq \omega'$ the relative phase between the vectors will change as the transient term decays. The vector tip of the transient term is shown as the dotted spiral and the total amplitude assumes the varying lengths OA_1, OA_2, OA_3, OA_4, etc.

(Problems 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18)

Problem 3.1

Show, if $F_0 e^{i\omega t}$ represents $F_0 \sin \omega t$ in the vector form of the equation of motion for the forced oscillator that

$$x = -\frac{F_0}{\omega Z_m} \cos(\omega t - \phi)$$

and the velocity

$$v = \frac{F_0}{Z_m} \sin(\omega t - \phi)$$

Problem 3.2

The displacement of a forced oscillator is zero at time $t = 0$ and its rate of growth is governed by the rate of decay of the transient term. If this term decays to e^{-k} of its original value in a time t show that, for small damping, the average rate of growth of the oscillations is given by $x_0/t = F_0/2km\omega_0$ where x_0 is the maximum steady state displacement, F_0 is the force amplitude and $\omega_0^2 = s/m$.

Problem 3.3

The equation $m\ddot{x} + sx = F_0 \sin \omega t$ describes the motion of an undamped simple harmonic oscillator driven by a force of frequency ω. Show, by solving the equation in vector form, that the steady state solution is given by

$$x = \frac{F_0 \sin \omega t}{m(\omega_0^2 - \omega^2)} \quad \text{where} \quad \omega_0^2 = \frac{s}{m}$$

Sketch the behaviour of the amplitude of x versus ω and note that the change of sign as ω passes through ω_0 defines a phase change of π rad in the displacement. Now show that the general solution for the displacement is given by

$$x = \frac{F_0 \sin \omega t}{m(\omega_0^2 - \omega^2)} + A \cos \omega_0 t + B \sin \omega_0 t$$

where A and B are constant.

Problem 3.4

In problem 3.3, if $x = \dot{x} = 0$ at $t = 0$ show that

$$x = \frac{F_0}{m} \frac{1}{(\omega_0^2 - \omega^2)} \left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t \right)$$

and, by writing $\omega = \omega_0 + \Delta\omega$ where $\Delta\omega/\omega_0 \ll 1$ and $\Delta\omega t \ll 1$, show that near resonance,

$$x = \frac{F_0}{2m\omega_0^2} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t)$$

Sketch this behaviour, noting that the second term increases with time, allowing the oscillations to grow (resonance between free and forced oscillations). Note that the condition $\Delta\omega t \ll 1$ focuses attention on the transient.

Problem 3.5

What is the general expression for the acceleration $\dot{\mathbf{v}}$ of a simple damped mechanical oscillator driven by a force $F_0 \cos \omega t$? Derive an expression to give the frequency of maximum acceleration and show that if $r = \sqrt{sm}$, then the acceleration amplitude at the frequency of velocity resonance equals the limit of the acceleration amplitude at high frequencies.

Problem 3.6

Prove that the **exact** amplitude at the displacement resonance of a driven mechanical oscillator may be written $x = F_0/\omega' r$ where F_0 is the driving force amplitude and

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2}$$

Problem 3.7

In a forced mechanical oscillator show that the following are frequency independent (a) the displacement amplitude at low frequencies (b) the velocity amplitude at velocity resonance and (c) the acceleration amplitude at high frequencies, ($\omega \to \infty$).

Problem 3.8

In Figure 3.9 show that for small r, the maximum value of curve (a) is $\approx F_0/2\omega_0 r$ at $\omega_1 = \omega_0 - r/2m$ and its minimum value is $\approx -F_0/2\omega_0 r$ at $\omega_2 = \omega_0 + r/2m$.

Problem 3.9

The equation $\ddot{x} + \omega_0^2 x = (-eE_0/m)\cos\omega t$ describes the motion of a bound undamped electric charge $-\mathrm{e}$ of mass m under the influence of an alternating electric field $E = E_0 \cos\omega t$. For an electron number density n show that the induced polarizability per unit volume (the dynamic susceptibility) of a medium

$$\chi_e = -\frac{n\,\mathrm{e}x}{\varepsilon_0 E} = \frac{n\,\mathrm{e}^2}{\varepsilon_0 m(\omega_0^2 - \omega^2)}$$

(The permittivity of a medium is defined as $\varepsilon = \varepsilon_0(1 + \chi)$ where ε_0 is the permittivity of free space. The relative permittivity $\varepsilon_r = \varepsilon/\varepsilon_0$ is called the dielectric constant and is the square of the refractive index when E is the electric field of an electromagnetic wave.)

Problem 3.10

Repeat Problem 3.9 for the case of a damped oscillatory electron, by taking the displacement x as the component represented by curve (a) in Figure 3.9 to show that

$$\varepsilon_r = 1 + \chi = 1 + \frac{n\,\mathrm{e}^2 m(\omega_0^2 - \omega^2)}{\varepsilon_0[m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2]}$$

In fact, Figure 3.9(a) plots $\varepsilon_r = \varepsilon/\varepsilon_0$. Note that for

$$\omega \ll \omega_0, \quad \varepsilon_r \approx 1 + \frac{n\,\mathrm{e}^2}{\varepsilon_0 m\omega_0^2}$$

and for

$$\omega \gg \omega_0, \quad \varepsilon_r \approx 1 - \frac{n\,\mathrm{e}^2}{\varepsilon_0 m\omega^2}$$

Problem 3.11

Show that the energy dissipated per cycle by the frictional force $r\dot{x}$ at an angular frequency ω is given by $\pi r\omega x_{\max}^2$.

Problem 3.12

Show that the bandwidth of the resonance absorption curve defines the phase angle range $\tan\phi = \pm 1$.

Problem 3.13

An alternating voltage, amplitude V_0 is applied across an LCR series circuit. Show that the voltage at current resonance across either the inductance or the condenser is QV_0.

Problem 3.14

Show that in a resonant *LCR* series circuit the maximum potential across the condenser occurs at a frequency $\omega = \omega_0(1 - 1/2Q_0^2)^{1/2}$ where $\omega_0^2 = (LC)^{-1}$ and $Q_0 = \omega_0 L/R$.

Problem 3.15

In Problem 3.14 show that the maximum potential across the inductance occurs at a frequency $\omega = \omega_0(1 - 1/2Q_0^2)^{-1/2}$.

Problem 3.16

Light of wavelength 0.6 μm (6000 Å) is emitted by an electron in an atom behaving as a lightly damped simple harmonic oscillator with a Q-value of 5×10^7. Show from the resonance bandwidth that the width of the spectral line from such an atom is 1.2×10^{-14} m.

Problem 3.17

If the Q-value of Problem 3.6 is high show that the width of the displacement resonance curve is approximately $\sqrt{3}r/m$ where the width is measured between those frequencies where $x = x_{\max}/2$.

Problem 3.18

Show that, in Problem 3.10, the mean rate of energy absorption per unit volume; that is, the power supplied is

$$P = \frac{n\mathrm{e}^2 E_0^2}{2} \frac{\omega^2 r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}$$

Summary of Important Results

Mechanical Impedance $\mathbf{Z}_m = \mathbf{F}/\mathbf{v}$ (force per unit velocity)

$$\mathbf{Z}_m = Z_m \mathrm{e}^{\mathrm{i}\phi} = r + \mathrm{i}(\omega m - s/\omega)$$

where $Z_m^2 = r^2 + (\omega m - s/\omega)^2$

$$\sin\phi = \frac{\omega m - s/\omega}{Z_m}, \quad \cos\phi = \frac{r}{Z_m}, \quad \tan\phi = \frac{\omega m - s/\omega}{r}$$

ϕ is the phase angle between the force and velocity.

Forced Oscillator

Equation of motion $m\ddot{x} + r\dot{x} + sx = F_0 \cos\omega t$
(Vector form) $m\ddot{\mathbf{x}} + r\dot{\mathbf{x}} + s\mathbf{x} = F_0 \mathrm{e}^{\mathrm{i}\omega t}$
Use $\mathbf{x} = \mathbf{A}\,\mathrm{e}^{\mathrm{i}\omega t}$ to give steady state displacement

$$\mathbf{x} = -\mathrm{i}\frac{F_0}{\omega Z_m}\mathrm{e}^{\mathrm{i}(\omega t - \phi)}$$

and velocity

$$\dot{\mathbf{x}} = \mathbf{v} = \frac{F_0}{Z_m} \mathrm{e}^{\mathrm{i}(\omega t - \phi)}$$

When $F_0 \mathrm{e}^{\mathrm{i}\omega t}$ represents $F_0 \cos \omega t$

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi)$$
$$v = \frac{F_0}{Z_m} \cos(\omega t - \phi)$$

Maximum velocity $= \dfrac{F_0}{r}$ at **velocity** resonant frequency $\omega_0 = (s/m)^{1/2}$
Maximum displacement $= \dfrac{F_0}{\omega' r}$ where $\omega' = (s/m - r^2/4m^2)^{1/2}$ *at* **displacement** resonant frequency $\omega = (s/m - r^2/2m^2)^{1/2}$

Power Absorbed by Oscillator from Driving Force

Oscillator adjusts amplitude and phase so that power supplied equals power dissipated.

Power absorbed $= \frac{1}{2}(F_0^2/Z_m)\cos\phi$ (cos ϕ is power factor)

Maximum power absorbed $= \dfrac{F_0^2}{2r}$ at ω_0

$\dfrac{\text{Maxmium power}}{2}$ absorbed $= \dfrac{F_0^2}{4r}$ at $\omega_1 = \omega_0 - \dfrac{r}{2m}$ and $\omega_2 = \omega_0 + \dfrac{r}{2m}$

Quality factor $Q = \dfrac{\omega_0 m}{r} = \dfrac{\omega_0}{\omega_2 - \omega_1}$

$$Q = \frac{\text{maximum displacement at displacement resonance}}{\text{displacement as } \omega \to 0}$$
$$= \frac{A(\text{max})}{F_0/s}$$

For equivalent expressions for electrical oscillators replace m by L, r by R, s by $1/C$ and F_0 by V_0 (voltage).