Exercise 2.8

Chapter 2 Derivatives Exercise 2.8 1E

We have been given that the volume of a cube=V
Length of the edge of the cube=x

Then $V = (\text{edge})^3$
Or $V = x^3$
Differentiating with respect to t implicitly , we get

$$\boxed{\frac{dV}{dt} = 3x^2 \frac{dx}{dt}}$$

Chapter 2 Derivatives Exercise 2.8 2E

(A)

The area of the circle is given by

$$A = \pi r^2 \quad \text{--- (1)}$$

Where r is the radius of the circle

Differentiate both sides of the equation (1) with respect to t
Where t is time in seconds

$$\boxed{\frac{dA}{dt} = 2\pi r . \frac{dr}{dt}}$$

(B)

Now rate of change in radius $\frac{dr}{dt} = 1$ m/s is given and $r = 30\text{m}$

Then rate of change in Area

$$\frac{dA}{dt} = 2\pi .30.1 = 60\pi$$

Or

$$\boxed{\frac{dA}{dt} = 60\pi = 188.496 \quad \text{m}^2/s}$$

Hence the area of the circle is increasing with the rate 188.496 m^2/s

Chapter 2 Derivatives Exercise 2.8 3E

Let the side of a square be x cm
Given that side of the square is increasing at the rate of 6 cm/s.

Let $\frac{dx}{dt}$ is the rate of change of side. So

$\frac{dx}{dt} = 6$ cm/sec

Given that area is 16 cm^2
So $A = 16$ cm
But area is x^2 cm^2
So $x^2 = 16$
$\Rightarrow x = \sqrt{16} = \pm 4$
But side is always +ve
So $x = 4$ cm

Area of the square $A = x^2$ cm^2
Differentiating both sides with respect to x

$\frac{dA}{dt} = \frac{d}{dt}(x^2)$

$\frac{dA}{dt} = 2x \cdot \frac{dx}{dt}$

Since by chain rule

Substituting $x = 4$ and $\frac{dx}{dt} = 6$ cm/sec

$\frac{dA}{dt} = 2 \times 4 \times 6$

$= 48 \text{ cm}^2/\text{sec}$

$\therefore$ Rate of change of area of square $= 48 \text{ cm}^2/\text{sec}$

Chapter 2 Derivatives Exercise 2.8 4E

Let the length and breadth of a rectangle be x cm and y cm.

We know that $\frac{dx}{dt}$ is the rate of change of its length. But it is given that the length is increasing at the rate of 8 cm/sec

So $\frac{dx}{dt} = 8$ cm/sec

Also $\frac{dy}{dt}$ is the rate of change of breadth. But it is given that the breadth is increasing at the rate of 3 cm/sec

So $\frac{dy}{dt} = 3$ cm/sec

It is given that $x = 20$ cm
$y = 10$ cm

Area of the rectangle $A = xy$
(Since area of rectangle is $length \times breadth$)
Differentiation both sides with respect to t

$\frac{dA}{dt} = \frac{d}{dt}(xy)$

Using product rule gives $= x\frac{dy}{dt} + y \cdot \frac{dx}{dt}$

By substituting given values $\frac{dA}{dt} = 20 \times 3 + 10 \times 8$

$= 60 + 80$

$= 140 \text{ cm}^2/\text{sec}$

$\therefore$ Rate of change of area of rectangle $= 140 \text{ cm}^2/\text{sec}$

Chapter 2 Derivatives Exercise 2.8 5E

Consider the rate of water filled to the cylindrical tank with radius 5 m is $3\ \text{m}^3/\text{min}.$

Find the rate of height of the water.

Let h be the height of cylinder and r be the radius of the cylinder, and V be the volume of the cylinder.

Then, volume of the cylinder is,

$$V = \pi r^2 h$$

The rates of change are derivatives.

Here, the water is filled to the cylindrical tank at rate 3 $\text{m}^3/\text{min}.$

So, $\frac{dV}{dt} = 3$

And $r = 5.$

Differentiate the equation $V = \pi r^2 h$ with respect to the time *t* to obtain that,

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \qquad \text{Since the radius } r \text{ is the constant.}$$

$$\frac{dh}{dt} = \frac{1}{\pi r^2} \cdot \frac{dV}{dt}$$

Plug the values $\frac{dV}{dt} = 3$, and $r = 5$ in the above equation.

$$\frac{dh}{dt} = \frac{1}{\pi (5)^2} \cdot 3$$

$$= \frac{3}{25\pi}$$

Therefore, the rate of height of the water is $\frac{dh}{dt} = \boxed{\frac{3}{25\pi}\ \text{m}/\text{min}}.$

Chapter 2 Derivatives Exercise 2.8 6E

The radius of a sphere is increasing at a rate of 4 mm/s.

Let *r* and *V* be the radius and volume of the sphere.

So, $\frac{dr}{dt} = 4\ \text{mm/s}.$

The objective is to find the rate of change of the volume of the sphere when the diameter is 80 mm.

When the diameter is 80 mm, then radius is 40 mm.

So, need to find $\frac{dV}{dt}$ at $r = 40\ \text{mm/s}.$

The volume formula for a sphere of radius r is given by,

$$V=\frac{4}{3}\pi r^3.$$

Differentiate both sides with respect to time t to obtain that,

$$\frac{d}{dt}[V]=\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

Using the Chain Rule,

$$\frac{dV}{dt}=\frac{4}{3}\pi\left(3r^2\right)\cdot\frac{dr}{dt}$$
$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

Substitute the values $r=40\text{ mm},\frac{dr}{dt}=4\text{ mm/s}$ in the above equation to obtain that,

$$\begin{aligned}\frac{dV}{dt}&=4\pi(40)^2(4)\\&=4\cdot\pi\cdot1600\cdot4\\&=25600\pi\end{aligned}$$

Thus, the volume of the sphere increasing in the rate of $\boxed{25{,}600\pi\text{ mm}^3/\text{s}}$.

Chapter 2 Derivatives Exercise 2.8 7E

Consider the function:

$$y=\sqrt{2x+1}$$

(a).

The objective is to find $\frac{dy}{dt}$ when $x=4$, and $\frac{dx}{dt}=3$.

First, differentiate both sides of the equation with respect to t, by using the chain rule.

$$\begin{aligned}\frac{dy}{dt}&=\frac{d}{dt}\left[(2x+1)^{\frac{1}{2}}\right]\\&=\frac{d}{dx}\left[(2x+1)^{\frac{1}{2}}\right]\frac{dx}{dt}\\&=\frac{1}{2}(2x+1)^{\frac{1}{2}-1}\frac{d}{dx}[2x+1]\frac{dx}{dt}\\&=\frac{1}{2}(2x+1)^{\frac{1}{2}-1}\left(2\frac{d}{dx}[x]+\frac{d}{dx}[1]\right)\frac{dx}{dt}\end{aligned}$$

$$=\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2\cdot1+0)\frac{dx}{dt}$$
$$=\frac{2}{2\sqrt{2x+1}}\frac{dx}{dt}$$

Therefore,

$$\frac{dy}{dt}=\frac{1}{\sqrt{2x+1}}\frac{dx}{dt}\quad\ldots\ldots(1)$$

Substitute the values $x=4,\frac{dx}{dt}=3$ in $\frac{dy}{dt}=\frac{1}{\sqrt{2x+1}}\frac{dx}{dt}$

$$\begin{aligned}\frac{dy}{dt}&=\frac{1}{\sqrt{2(4)+1}}(3)\\&=\frac{3}{\sqrt{9}}\\&=\frac{3}{3}\\&=1\end{aligned}$$

Therefore, $\frac{dy}{dt}=\boxed{1}$.

(b).

Here the objective is to find $\frac{dx}{dt}$ when $x=12$, and $\frac{dy}{dt}=5$.

From (1), we can write

$$\frac{dy}{dt}=\frac{1}{\sqrt{2x+1}}\frac{dx}{dt}$$
$$\Rightarrow\frac{dx}{dt}=\sqrt{2x+1}\frac{dy}{dt}$$

Now substitute the values $x=12$ and $\frac{dy}{dt}=5$ in $\frac{dx}{dt}=\sqrt{2x+1}\ \frac{dy}{dt}$

$$\begin{aligned}\frac{dx}{dt}&=\sqrt{2(12)+1}\cdot(5)\\&=\sqrt{25}\cdot(5)\\&=5\cdot5\\&=25\end{aligned}$$

Therefore, $\frac{dx}{dt}=\boxed{25}$.

Chapter 2 Derivatives Exercise 2.8 8E

Suppose $4x^2+9y^2=36$, where x and y are functions of t.

Then, differentiate both sides with respect to t and get;

$$8x\frac{dx}{dt}+18y\frac{dy}{dt}=0 \quad ...... (1)$$

(a)

Here

$$\frac{dy}{dt}=\frac{1}{3}$$
$$x=2$$
$$y=\frac{2}{3}\sqrt{5}$$

Then putting values in equation (1), we get

$$8(2)\frac{dx}{dt}+(18)\frac{2}{3}\sqrt{5}\frac{1}{3}=0$$
$$16\frac{dx}{dt}+4\sqrt{5}=0$$
$$\frac{dx}{dt}=\frac{-4\sqrt{5}}{16}$$
$$=-\frac{\sqrt{5}}{4}$$

Therefore, $\boxed{\frac{dx}{dt}=\frac{-\sqrt{5}}{4}}$

(b)

Here

$$\frac{dx}{dt} = 3$$
$$x = -2$$
$$y = \frac{2}{3}\sqrt{5}$$

Then putting values in equation (1)

$$8(-2)(3) + 18\left(\frac{2}{3}\sqrt{5}\right)\frac{dy}{dt} = 0$$
$$-48 + 12\sqrt{5}\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = \frac{48}{12\sqrt{5}}$$
$$= \frac{4}{\sqrt{5}}$$

Therefore, $\boxed{\frac{dy}{dt} = \frac{4}{\sqrt{5}}}$

Chapter 2 Derivatives Exercise 2.8 9E

Suppose $x^2 + y^2 + z^2 = 9$, where x , y and z are functions of t.

Then $2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0$

Here $x = 2$, $y = 2$, $z = 1$, $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 4$

Then $4(5) + 4(4) + 2\frac{dz}{dt} = 0$

$\Rightarrow \quad 20 + 16 + 2\frac{dz}{dt} = 0$

$\Rightarrow \quad 36 + 2\frac{dz}{dt} = 0$

$\Rightarrow \quad \frac{dz}{dt} = -18$

$\boxed{\therefore \frac{dz}{dt} = -18}$

Chapter 2 Derivatives Exercise 2.8 10E

A particle is moving along a hyperbola $xy = 8$ as it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s.

Then $\frac{dx}{dt}y + x\frac{dy}{dt} = 0$

$\Rightarrow \quad \frac{dx}{dt}(2) + 4(-3) = 0$

$\Rightarrow \quad \frac{dx}{dt} = +\frac{12}{2} = 6 \text{ cm/s}$

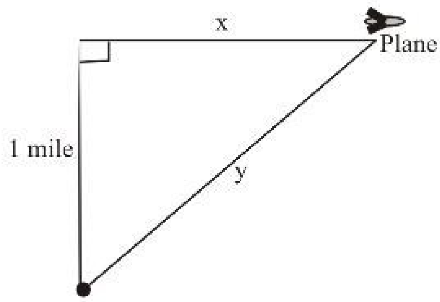
$\boxed{\therefore x- \text{ coordinate of the point changing at that instant is } 6cm/s}$

Chapter 2 Derivatives Exercise 2.8 11E

(A) Altitude and speed of plane are given

(B) The rate at which the distance from the plane to the station is increasing.

(C)



(D) Let the Horizontal distance from the plane from the station be x, and the distance from the plane to the station be y

From the figure, by the Pythagorean Theorem, we have

$$\boxed{y^2 = 1 + x^2}$$

(E) We have $y^2 = 1 + x^2$ .... (1)

Differentiating with respect to t implicitly

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

Or $y\frac{dy}{dt} = x\frac{dx}{dt}$

Or $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$ ..... (2)

Now, given that y =2 miles, and speed of the plane = dx/dt = 500 mi/h

Then from (1), we have $x^2 = y^2 - 1$

Or $x^2 = 2^2 - 1 = 3$

Or, $x = \sqrt{3}$ miles

Therefore from (2), we have

$$\frac{dy}{dt} = \frac{\sqrt{3}}{2}.(500) \text{ mi/h}$$

Or, $\frac{dy}{dt} = 250\sqrt{3}$ mi/h

Thus rate at which distance from the plane to the station is increasing is $\boxed{250\sqrt{3} \text{ mi/hr}}$

Chapter 2 Derivatives Exercise 2.8 12E

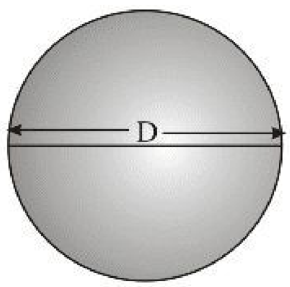
(A) Given that the rate of decrease of surface area is $1 \text{ cm}^2/\text{min}$

Let the surface area be S, then $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$

(B) We have to find the rate at which diameter decreases.

Let the diameter be D then we have to find dD/dt.

(C)



(D) If the radius of the ball is R then surface area of the ball is

$S = 4\pi R^2$

$\Rightarrow \quad S = 4\pi\left(\frac{D}{2}\right)^2$

$\Rightarrow \quad S = \pi D^2$

(E) Now we have $S = \pi D^2$

Differentiating with respect to t implicitly, we get

$$\frac{dS}{dt} = 2\pi D \frac{dD}{dt}$$

Or, $$\frac{dD}{dt} = \frac{1}{2\pi D} \cdot \frac{dS}{dt}$$

Now, $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$ and $D = 10$ cm.

Then $$\frac{dD}{dt} = \frac{1}{2\pi \times 10 \text{ cm}}\left(-1 \text{ cm}^2/\text{min}\right)$$

Or $$\boxed{\frac{dD}{dt} = \frac{-1}{20\pi} \text{ cm/min}} \approx -0.0159 \text{ cm/min}$$

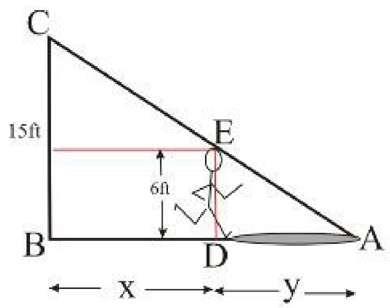
Thus diameter decreases at the rate of $\frac{1}{20\pi}$ cm/min

Chapter 2 Derivatives Exercise 2.8 13E

(A) Height of the pole = 15 ft
Height of the man = 6 ft
Speed of the man = 5 ft/s

(B) We have to find the rate at which the tip of the man's shadow is moving when he is 40 ft from the pole.

(C)



(D) From the figure,
Height of the pole BC = 15 ft
Height of the man DE = 6 ft
Length of the shadow DA = y ft
At time t, distance from the pole to the man BD = x ft
Then dx/dt = 5 ft/s
Since the triangles ΔADE and ΔABC are similar, then

$\Rightarrow \quad \frac{AB}{BC} = \frac{DA}{DE}$

$\Rightarrow \quad \frac{x+y}{15} = \frac{y}{6}$

$\Rightarrow \quad 15y = 6x + 6y$

$\Rightarrow \quad 9y = 6x$

$\Rightarrow \quad 3y = 2x$

$\Rightarrow \quad y = 2x/3$

Then distance from the tip of the shadow to the pole L = x+y
Or L= x+2x/3
Or L=5x/3 .....(1)

(E) Differentiating (1) with respect to t implicitly

Speed of the tip of the shadow is $\frac{dL}{dt} = \frac{5}{3}\frac{dx}{dt}$

Or $\frac{dL}{dt} = \frac{5}{3}(5)$

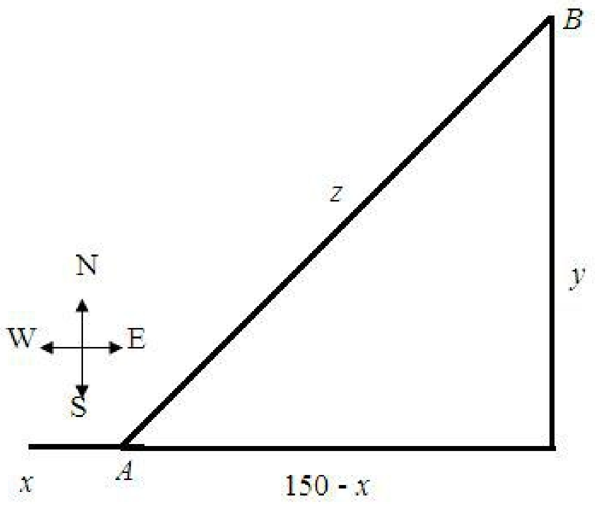
Or $\frac{dL}{dt} = 25/3$ ft/s

Then the tip of the shadow is moving with the speed of $\boxed{25/3 \text{ ft/s}}$

It is given that Ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h.

The objective is to find the rate at which the distance between the ships changing at 4:00 PM.

Sketch a diagram with the given information.



The distance between *A* and *B* is initially 150 km.

Ship *A* is sailing east at 35 km/h.

Ship *B* is sailing north at 25 km/h.

Let *z* be the distance between the ships at 4 P.M.

Then find $\frac{dz}{dt}$ at $t = 4$.

Here *t* is the time in hours.

Initially the distance between the ships is 150.

Let ship *A* be traveled *x* km for 4 hours.

Then the distance between *A* and initial point of *B* is $150 - x$.

Let ship *B* is traveled *y* km for 4 hours.

From the diagram, observe that the triangle is a right triangle.

From the Pythagoras theorem,

$$\text{hyp}^2 = \text{opp}^2 + \text{adj}^2$$

Apply the Pythagoras theorem for the data.

$$z^2 = (150 - x)^2 + y^2$$

Differentiate both sides with respect to *t*.

$$2z\frac{dz}{dt} = -2(150 - x)\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$z\frac{dz}{dt} = -(150 - x)\frac{dx}{dt} + y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left((x - 150)\frac{dx}{dt} + y\frac{dy}{dt}\right) \quad \text{...... (1)}$$

Since ship *A* is sailing east at 35 km/h,

$$\frac{dx}{dt}=35$$

And *A* is traveled 35 km per hour.

Since ship *A* is traveled *x* km for 4 hours,

$$\begin{aligned} x &= 4(35) \\ &= 140\text{km} \end{aligned}$$

And

Since ship *B* is sailing north at 25 km/h,

$$\frac{dy}{dt}=25$$

And *B* is traveled 25 km per hour.

Since ship *B* is traveled *y* km for 4 hours,

$$\begin{aligned} y &= 4(25) \\ &= 100\text{km} \end{aligned}$$

Substitute $x=140, y=100$ in

$$z^2=(150-x)^2+y^2$$

$$\begin{aligned} z^2 &= (150-140)^2+(100)^2 \\ &= 100+10000 \\ &= 10100 \\ z &= \sqrt{10100} \end{aligned}$$

Substitute, $x=140, y=100, z=\sqrt{10100}$, $\frac{dx}{dt}=35$ and $\frac{dy}{dt}=25$ in (1) to find the required rate $\frac{dz}{dt}$.

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{z}\left((x-150)\frac{dx}{dt}+y\frac{dy}{dt}\right) \\ &= \frac{1}{\sqrt{10100}}\left((140-150)(35)+(100)(25)\right) \\ &= \frac{1}{\sqrt{10100}}\left(-10(35)+2500\right) \\ &= \frac{1}{\sqrt{10100}}\left(-350+2500\right) \\ &= \frac{2150}{\sqrt{10100}} \\ &\approx 21.4 \end{aligned}$$

Hence, the rate at which the distance between the ships is changed at 4 P.M is $\boxed{21.4\,\text{km/h}}$.

Chapter 2 Derivatives Exercise 2.8 15E

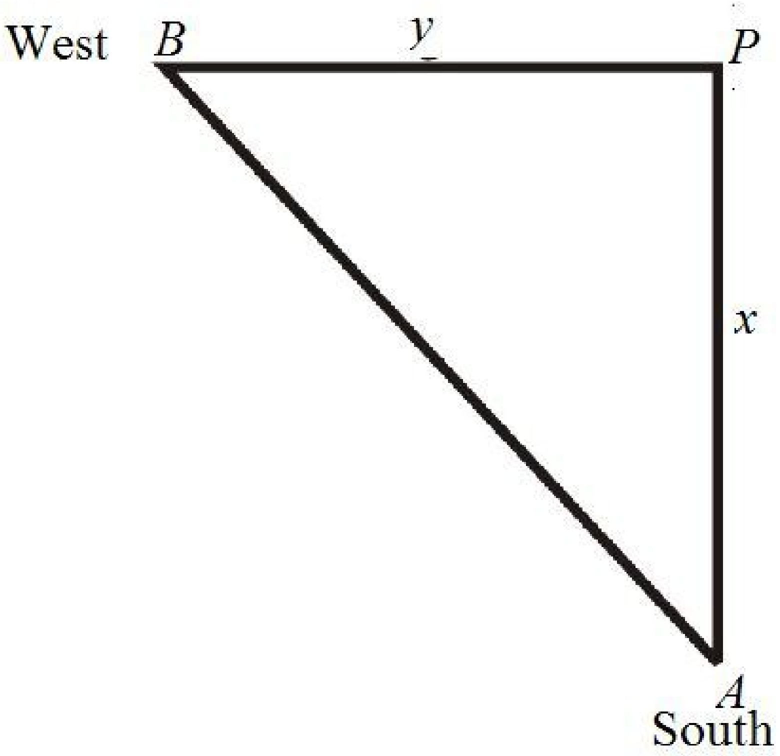
Consider the data,

Initially the two cars are at the same point.

Speed of first car moving towards south is 60 mi/hr

Speed of second car moving towards west is 25 mi/hr

Sketch the figure showing the figure direction of the cars.



From the figure, initially the cars at *P*. later the first car moved towards *A*. and the second car moved towards *B*.

After *t* hours the first car is at the point A and second car is at B

Speed of the first car is 60 mi/h, then $\frac{dx}{dt} = 60 \text{ mi/h}$

Speed of the second car is 25 mi/h, then $\frac{dy}{dt} = 25 \text{ mi/h}$

For 1 hour the distance covered by first car is=60 mi

After 2 hours the distance covered by the first car is

$$\begin{aligned} x &= 2 \cdot 60 \\ &= 120 \text{ mi} \end{aligned}$$

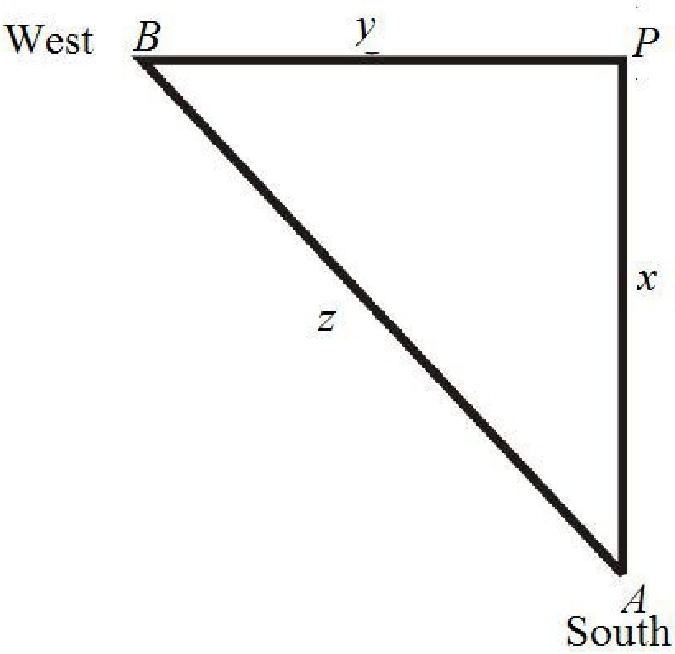
For 1 hour the distance covered by second car is=25 mi

After 2 hours the distance covered by the second car is

$$\begin{aligned} y &= 2 \cdot 25 \\ &= 50 \text{ mi} \end{aligned}$$

Let the distance between the cars be z at time t.

Represent this z in the right triangle formed above.



Because the figure represents a right triangle use Pythagorean Theorem

$z^2 = x^2 + y^2$ .... (1)

When $x = 120$ miles and $y = 50$ miles

Then

$z^2 = (120)^2 + (50)^2$

$z^2 = 16900$

$z = 130$ miles

Differentiate (1) with respect to t implicitly

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

Plug the values of x, y, z, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

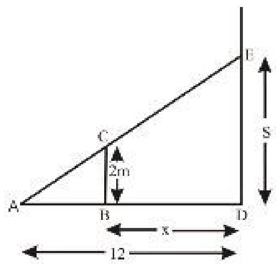
Hence, the rate of change of the distance between the cars is

$$\frac{dz}{dt} = \frac{1}{130}\left[120(60) + 50(25)\right]$$

$$\frac{dz}{dt} = 65 \text{ mi/h}$$

Thus, the distance between the cars is increasing with the rate of 65 miles /hour

Chapter 2 Derivatives Exercise 2.8 16E



By the property of similar triangle, we have

$$\frac{AB}{BC}=\frac{AD}{DE}$$

Or $$\frac{12-x}{2}=\frac{12}{S}$$

$\Rightarrow$ $$S=\frac{24}{(12-x)} \qquad \ldots\ldots(1)$$

Let the spotlight be at the point A, which is 12 m away from the wall DE.
Length of the man be BC = 2 m
Let the man be at x meters away from the wall at time t and at this time the length of the wall be DE= S meters
We have been given that the speed of the man is 1.6 m/s toward the wall.

Then $$\frac{dx}{dt}=-1.6 \text{ m/s}$$

By the property of similar triangle, we have

$$\frac{AB}{BC}=\frac{AD}{DE}$$

Or $$\frac{12-x}{2}=\frac{12}{S}$$

$\Rightarrow$ $$S=\frac{24}{(12-x)} \qquad \ldots\ldots(1)$$

Differentiating equation (1) with respect to t implicitly, we have

$$\frac{dS}{dt}=\frac{24}{(12-x)^2}\cdot\frac{dx}{dt}$$

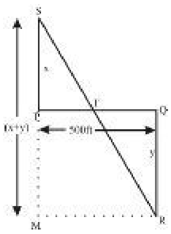
When $\frac{dx}{dt}=-1.6$ m/s and $x=4$ m, then

$$\frac{dS}{dt}=\frac{24}{(12-4)^2}\cdot(-1.6)$$

$$=-\frac{24}{64}(1.6)=\boxed{-0.6 \text{ m/s}}$$

Length of the shadow decreases at the rate of 0.6 m/s.

Chapter 2 Derivatives Exercise 2.8 17E



Let after time t seconds, the man reaches at the point S and the distance covered by him is x.
Let after time t seconds, the woman reaches at the point R and distance covered by her is y.
The point Q is 500ft away from the point P, and the woman starts from Q

Speed of man is 4 ft/s then $\frac{dx}{dt}=4$ ft/s

Speed of woman is 5 ft/s then $\frac{dy}{dt}=5$ ft/s

From the figure

$$MS=x+y$$

Let $$RS=D \text{ feet}$$

Then by the Pythagorean Theorem

$$D^2=(x+y)^2+(500)^2 \qquad \ldots..(1)$$

Differentiating with respect to t, implicitly

$$2D\frac{dD}{dt}=2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

Or $$\frac{dD}{dt}=\frac{(x+y)}{D}\left(\frac{dx}{dt}+\frac{dy}{dt}\right) \qquad \ldots(2)$$

Distance covered by woman after 15 minutes is

$$y = \text{speed} \times \text{time}$$

$\Rightarrow$ $$y = (5 \text{ ft/s}) \times (15 \text{min} \times 60 \text{ s/min}) \qquad [1 \text{ minute} = 60 \text{ s}]$$

$\Rightarrow$ $$y = 4500 \text{ ft}$$

When the woman starts, the man already walked for five minutes, thus the man walks for 20 minutes.

Then the distance covered by the man, 15 minutes after the woman starts walking, is

$$x = \text{speed} \times \text{time}$$

$\Rightarrow$ $$x = (4 \text{ ft/s}) \times (20 \text{min} \times 60 \text{ s/min}) \qquad [1 \text{ minute} = 60 \text{ s}]$$

$\Rightarrow$ $$x = 4800 \text{ ft}$$

From (1), the distance between man and woman is

$$D^2 = (4800 + 4500)^2 + (500)^2$$

Or $$D^2 = 86740000$$

Or $$D \approx 9313.43 \text{ ft}$$

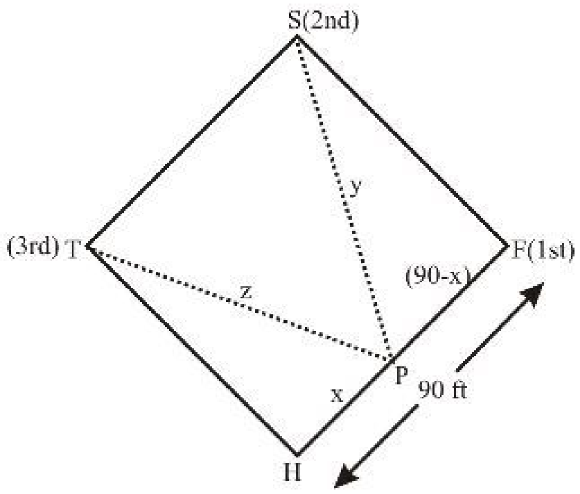
Then from (1), the rate of increase of the distance between the man and the woman is

$$\frac{dD}{dt} \approx \frac{(4800+4500)}{9313.43}(4+5)$$

Or $$\frac{dD}{dt} \approx \frac{83700}{9313.43}$$

Or $$\boxed{\frac{dD}{dt} \approx 8.99 \text{ ft/s}}$$

Chapter 2 Derivatives Exercise 2.8 18E



(A) Form the figure, the man hits the ball at point H and runs towards the point F (first base)

Let at time t the man is at the point P, which is x feet away from the point H.

Since the speed of the man is 24 ft /s, then $\frac{dx}{dt} = 24$ ft/s

We have been given that HF=FS=ST=TH=90 ft

Then the distance between the man and the second base at time t is

$$SP^2 = FS^2 + PF^2$$

$\Rightarrow$ $$y^2 = (90)^2 + (90 - x)^2 \qquad \text{.....(1)}$$

Differentiating with respect to time t implicitly, we get

$$2y\frac{dy}{dt} = -2(90 - x)\frac{dx}{dt}$$

Or $$\frac{dy}{dt} = -\frac{(90-x)}{y}\left(\frac{dx}{dt}\right) \qquad \text{... (2)}$$

When the man is halfway to the first base, $x = 45$ ft

From (1), we have $$y^2 = (90)^2 + (90-45)^2$$

Or $$y^2 = 10125$$

Or $$y = \sqrt{10125} = 45\sqrt{5} \text{ ft}$$

Therefore, from (2), we have

$$\frac{dy}{dt} = -\frac{(90-45)}{45\sqrt{5}}(24)$$

Or $$\boxed{\frac{dy}{dt} = -\frac{24}{\sqrt{5}} \text{ ft/s}}$$

Thus the distance from the second base is decreasing with the rate of $\frac{24}{\sqrt{5}}$ ft/s

(B) Again from the figure, we have

$$TP^2 = TH^2 + HP^2$$

Or $$z^2 = (90)^2 + x^2 \quad \text{.....(3)}$$

Differentiating with respect to time t implicitly

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$$

Or $$\frac{dz}{dt} = \left(\frac{x}{z}\right)\frac{dx}{dt} \quad \text{...(4)}$$

When $x = 45$ ft , then from (3)

$$z^2 = (90)^2 + 45^2$$

Or $$z^2 = 10125$$

Or $$z = \sqrt{10125} = 45\sqrt{5} \text{ ft}$$

Therefore from (4), the rate of change of the distance between the third base and the man is

$$\frac{dz}{dt} = \left(\frac{45}{45\sqrt{5}}\right)(24)$$

Or $$\boxed{\frac{dz}{dt} = \frac{24}{\sqrt{5}} \text{ ft/s}}$$

Thus the distance from the third base is increasing with the rate of $\frac{24}{\sqrt{5}}$ ft/s.

Chapter 2 Derivatives Exercise 2.8 19E

Let the altitude be h and the base be b.
Then the area of the triangle is

$$A = \frac{1}{2}bh \quad \text{.....(1)}$$

Differentiating with respect to the time t implicitly, we get

$$\frac{dA}{dt} = \frac{1}{2}\left\{b\frac{dh}{dt} + h\frac{db}{dt}\right\} \quad \text{[Product rule]}$$

......(2)

We have $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm^2/min

When $h = 10$ cm and $A = 100$ cm^2

Then from (1)

$$100 = \frac{10}{2}b \Rightarrow b = 20 \text{ ft}$$

Substituting the values in the equation (2)

$$2 = \frac{1}{2}\left\{20(1) + (10)\frac{db}{dt}\right\}$$

Or $$20 + 10\frac{db}{dt} = 4$$

Or $$10\frac{db}{dt} = -16$$

Or $$\boxed{\frac{db}{dt} = -1.6 \text{ cm/min}}$$

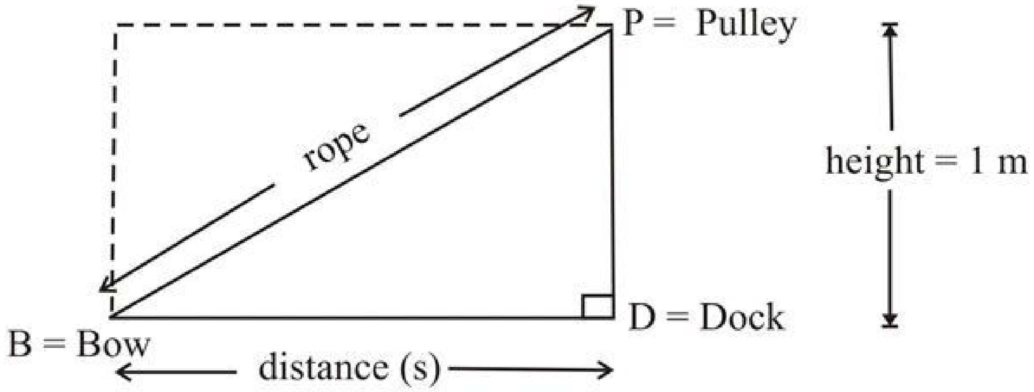
Thus the base is decreasing with the rate of 1.6 cm/min.

Chapter 2 Derivatives Exercise 2.8 20E

Suppose that the bow of the boat be at point *B* and dock is at point *D* and the bow is at a distance *s* from the point *B*.

Pulley is fixed at *P* which is 1 m higher than *B*.

This information is sown below:



Points *B* and *D* are on the same straight line, so from *D*, the height of *P* is 1m.

Then the points *B*, *D*, and *P* form a right angle triangle, where length of the rope is acting as the hypotenuse of the right triangle.

So by the property of right triangle, we have

$$(BP)^2 = PD^2 + BD^2$$

Here height *PD* is constant = 1m

So, we have

$$(BP)^2 = 1 + BD^2 \quad \ldots\ldots (1)$$

Differentiate both sides of the equation (1) with respect to time *t* (seconds).

$$2(BP)\frac{d(BP)}{dt} = 2(BD)\times\frac{d(BD)}{dt}$$

$$(BP)\cdot\frac{d(BP)}{dt} = (BD)\cdot\frac{d(BD)}{dt} \quad \ldots\ldots (2)$$

Now the rope is pulled in at a rate of 1 m/s.

So the length of rope is decreasing at a rate of 1 m/s.

Therefore,

$$\frac{d(\mathrm{BP})}{dt} = -1 \text{ m/s}$$

Here negative sign shows that length of *BP* is decreasing.

Given that, $s = 8$ m

$$BD = 8 \text{ m}$$

So the length of the rope *BP* is given by using Pythagorean Theorem,

$$(BP)^2 = 1 + BD^2$$

$$BP = \sqrt{1 + BD^2}$$

$$BP = \sqrt{1+8^2}$$

$$BP = \sqrt{65}$$

$$\approx 8.062 \text{ m}$$

Then rate of change in the distance of dock from the bow can be calculated by the equation (2).

$$(BP)\cdot\frac{d(BP)}{dt}=(BD)\cdot\frac{d(BD)}{dt}$$

$$8.062\cdot(-1)=8\cdot\frac{d(BD)}{dt}$$

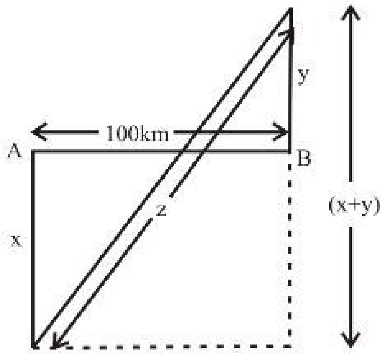
$$\frac{d(BD)}{dt}=\frac{-8.062}{8}$$

$$\approx -1.0078 \text{ m/s}$$

This means that the distance $BD = s$ of dock from the bow is decreasing at a rate $1.0078\,\text{m/s}$.

In other words, the boat is approaching the dock with rate of $\boxed{1.0078 \text{ m/s}}$

Chapter 2 Derivatives Exercise 2.8 21E



Let at time t the first ship A is at the distance x from its starting point and the ship B is at the distance y from its starting point.

At time t, the distance between the ships is

$$z^2 = 100^2 + (x+y)^2 \qquad \ldots(1) \qquad \text{[By the Pythagorean Theorem]}$$

Differentiating with respect to time t implicitly, we get

$$2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

Or $$\frac{dz}{dt} = \frac{(x+y)}{z}\left(\frac{dx}{dt}+\frac{dy}{dt}\right) \qquad \ldots(2)$$

The speed of the ship A is 35 km/h then $\frac{dx}{dt} = 35$ km/h

The speed of the ship B is 25 km/h then $\frac{dy}{dt} = 25$ km/h

At 4.00 PM, the distance covered by the ship A is = $x = 35 \text{ km/h} \times 4 \text{ h} = 140 \text{ km}$

And, the distance covered by the ship B is = $y = 25 \text{ km/h} \times 4 \text{ h} = 100 \text{ km}$

Then from (1)

$$z^2 = 100^2 + (140+100)^2$$

Or $$z^2 = 67600$$

Or $$z = 260 \text{ km}$$

Substituting the known values in equation (2), we have

$$\frac{dz}{dt} = \frac{(140+100)}{260}(35+25)$$

Or $$\frac{dz}{dt} = \frac{14400}{260}$$

Or $$\boxed{\frac{dz}{dt} \approx 55.38 \text{ km/h}}$$

Thus the distance between the ships is increasing wit the rate of 55.38 km/h

Chapter 2 Derivatives Exercise 2.8 22E

Consider a particle moving along the path $y = 2\sin\left(\frac{\pi x}{2}\right)$.

Let (x, y) be a point on the curve, and L be the distance from the point to origin.

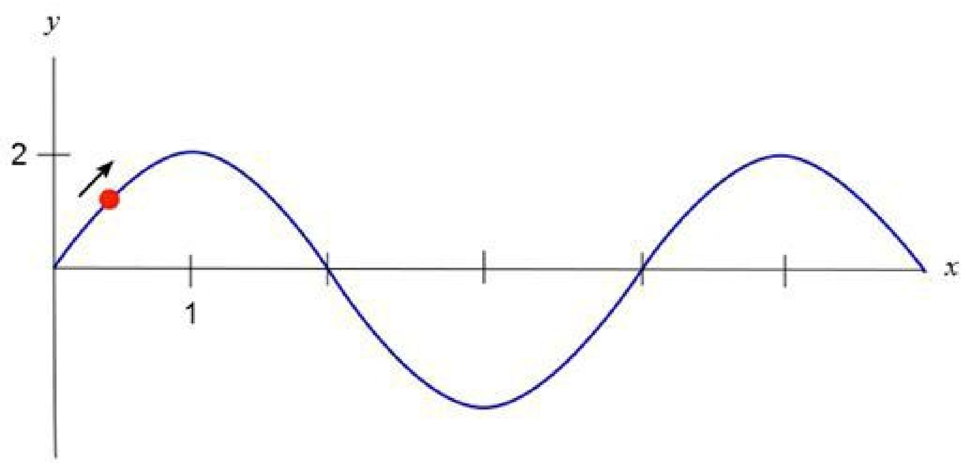
Then, by using the formula for distance between two points is found as follows:

$$L = \sqrt{(x-0)^2 + (y-0)^2}.$$

The change in $x-$coordinate is $\sqrt{10}$ cm/s, when the particle passes through the point $\left(\frac{1}{3}, 1\right)$.

That is, $\frac{dx}{dt} = \sqrt{10}$ cm/s.

Sketch the visual representation of the problem as follows:



If the particle is moving along the curve, then both x and y are changing while the distance is changing.

To find the rate of change of the distance r, differentiate each side of $L^2 = x^2 + y^2$ with respect to t.

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$L\frac{dL}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

As $y = 2\sin\left(\frac{\pi x}{2}\right)$, differentiate y with respect to t.

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= 2\cos\left(\frac{\pi x}{2}\right) \times \frac{\pi}{2} \times \frac{dx}{dt} \\ &= \pi\cos\left(\frac{\pi x}{2}\right)\frac{dx}{dt}\end{aligned}$$

Compute the distance from point $\left(\frac{1}{3},1\right)$ to origin as follows:

$$L=\sqrt{\left(\frac{1}{3}-0\right)^2+(1-0)^2}$$
$$=\sqrt{\frac{1}{9}+1}$$
$$=\frac{\sqrt{10}}{3}\text{ cm/s}$$

Substitute $L=\frac{\sqrt{10}}{3},\frac{dx}{dt}=\sqrt{10}$, and $\frac{dy}{dt}=\pi\cos\left(\frac{\pi x}{2}\right)\frac{dx}{dt}$ in $L\frac{dL}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$.

$$\frac{\sqrt{10}}{3}\frac{dL}{dt}=x\sqrt{10}+y\pi\cos\left(\frac{\pi x}{2}\right)\sqrt{10}$$
$$\frac{dL}{dt}=3x+3y\pi\cos\left(\frac{\pi x}{2}\right) \quad \text{Multiply both sides by } \frac{3}{\sqrt{10}}$$

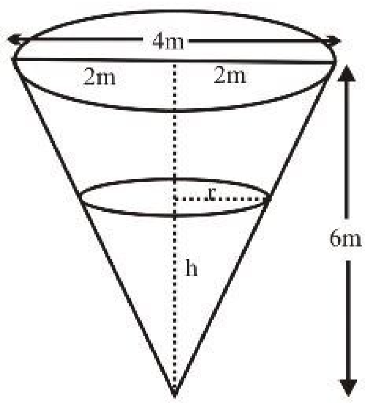
At the moment if the particle is at $\left(\frac{1}{3},1\right)$, then the rate of increase of distance is computed as follows:

$$\frac{dL}{dt}=3\left(\frac{1}{3}\right)+3(1)\pi\cos\left(\frac{\pi\left(\frac{1}{3}\right)}{2}\right) \quad \text{Replace } x=\frac{1}{3} \text{ and } y=1$$
$$=1+3\pi\cos\left(\frac{\pi}{6}\right) \quad \text{Since } \cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$
$$=1+3\pi\left(\frac{\sqrt{3}}{2}\right)$$
$$=1+\frac{3\sqrt{3}}{2}\pi$$

Therefore, the distance of the particle from the origin is increasing at a rate of

$\boxed{1+\frac{3\sqrt{3}\pi}{2}\text{ cm/s}}$.

Chapter 2 Derivatives Exercise 2.8 23E



Given that

The diameter of the tank $=4\text{ m}=400\text{ cm}$

Then the radius of the tank is $=2\text{ m}=200\text{ cm}$

Height of the tank is $=6\text{ m}=600\text{ cm}$

The rate of increase of the height of the water level $=20$ cm/ min

Then $\frac{dh}{dt}=20$ cm/min

Water is leaking out at a rate of $10000\text{ cm}^3/\text{min}$

Then Rate out $=10000\text{ cm}^3/\text{min}$

Let at time t the height of the water level be h and the radius be r.

And the volume of the water in the tank be V.

By the property of similar triangle, we have

$$\frac{h}{r}=\frac{6}{2}\Rightarrow r=h/3$$

Volume of the water is

$$V=\frac{1}{3}\pi r^2 h$$

Or $$V=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2(h)=\frac{1}{27}\pi h^3 \quad \ldots.(1)$$

Differentiating (1) with respect to time t implicitly, we get

$$\frac{dV}{dt}=\frac{1}{9}\pi h^2\frac{dh}{dt}$$

Since $\frac{dV}{dt}=$ rate in - rate out $=$ rate in -10000

Then $$\text{rate in }-10000=\frac{1}{9}\pi h^2\frac{dh}{dt}$$

Or $$\text{rate in }=10000+\frac{1}{9}\pi h^2\frac{dh}{dt} \quad \ldots..(2)$$

Substituting the known values in equation (2), we have

We have $$\text{rate in }=10000+\frac{1}{9}\pi(200)^2(20) \quad \text{cm}^3/\text{min}$$

Or $$\text{rate in }=10000+\frac{800000\pi}{9}$$

Or $$\boxed{\text{rate in }\approx 289253\ \text{cm}^3/\text{min}}$$

Chapter 2 Derivatives Exercise 2.8 24E

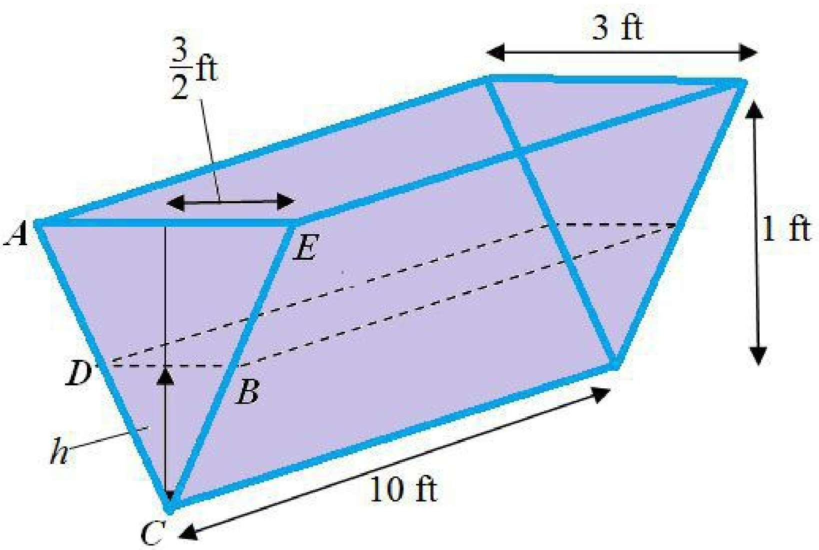
Consider the following data:

A trough is of length 10 ft. and the top base of it 3 ft. and ends of this trough are in the form of isosceles triangles.

Rate of change of water to be filled in trough is $12\ \text{ft}/\text{min}$

Water level is rising by 6 inches=0.6 ft. deep.

The trough with the given data is as follows:



Suppose the water level is at height h seen by the dotted lines.

Since the trough is being filled at a rate of $12\ \text{ft}^3/\text{min}$.

That is to say, if V is the volume of the water in the trough then $\frac{dV}{dt}=12\ \text{ft}^3/\text{min}$

Determine change in water level that is the rate of change in height $\frac{dh}{dt}$ when

$h=6\text{ inches}=\frac{1}{2}\text{ ft}$.

To find the volume of the water at height h, use the geometry formula for any Right Cylinder. Then the volume is:

$$V = (\text{area of face of cylinder}) \cdot (\text{length of cylinder}).$$

In this case, a Right Isosceles Triangular Cylinder is of length 10 ft. and face is right isosceles triangle.

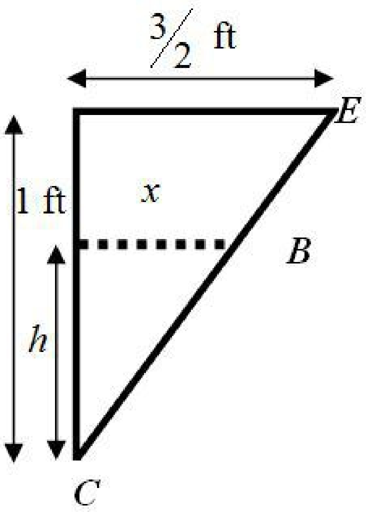
So,

$$V = (\text{ Area of triangle } DBC\ ) \cdot (10 \text{ ft })$$

But, a triangle has Area formula $\frac{1}{2}$ base time's height. So, the triangle has height h, then the volume is:

$$V = \left(\frac{1}{2}\overline{DB} \cdot h\right)(10) \text{ ...... (1)}$$

The figure the shows the triangle is as follows:



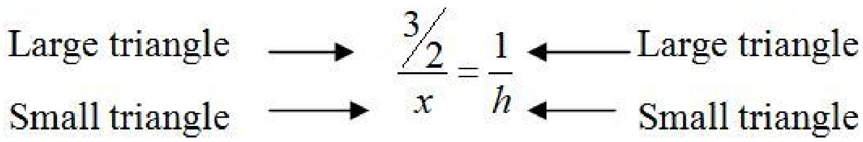
Use half the triangle AEC which is a right triangle to find the area of triangle *DBC*. See that x is equivalent to the quantity $\frac{1}{2}\overline{DB}$. This is clear from the above picture.

Hence equation (1) becomes:

$$V = (x \cdot h)(10) \text{ ...... (2)}$$

But, it is needed that V in terms of only the variable h so let's find x in terms of h.

Using a proportion of large right triangle to small right triangle:



So, the value of *x* is:

$$x = \frac{3}{2}h.$$

Substitute the value $x = \frac{3}{2}h$ into equation (2). Then,

$$\begin{aligned} V &= \left(\frac{3}{2}h \cdot h\right)(10) \\ &= \frac{30}{2}h^2 \\ &= 15h^2 \end{aligned}$$

So, the volume is given by equation:

$$V = 15h^2 \text{ ...... (3)}$$

Take $\frac{d}{dt}$ of both sides of equation (3) and use the Chain Rule. Compute the rate of change of volume as follows:

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left[15h^2\right] \\ &= 15\frac{d}{dh}\left[h^2\right]\frac{dh}{dt} \\ &= 15(2h)\frac{dh}{dt} \\ &= 30h\frac{dh}{dt}\end{aligned}$$

Solve the equation for $\frac{dh}{dt}$.

$$\frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}$$

Lastly, plug in all the known values $h = \frac{1}{2}$ ft, $\frac{dV}{dt} = 12$ ft^3/min to get:

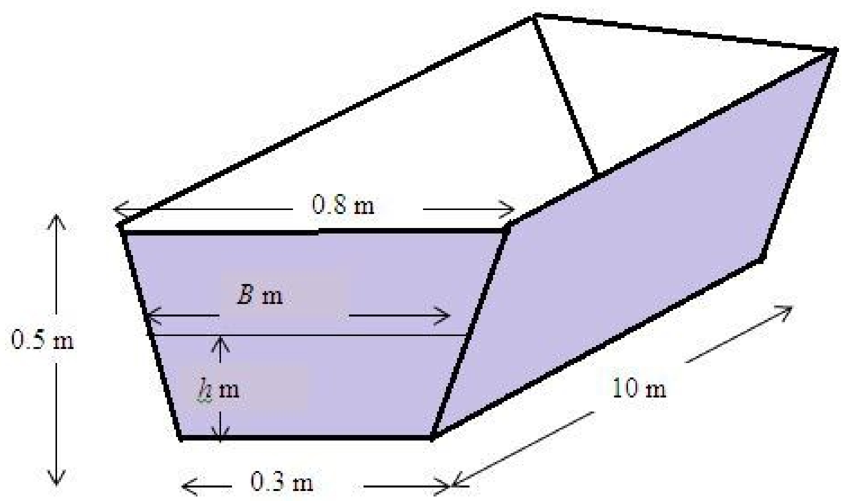
$$\begin{aligned}\frac{dh}{dt} &= \frac{1}{30\left(\frac{1}{2}\right)}(12) \\ &= \frac{12}{15} \\ &= \frac{4}{5}\end{aligned}$$

Thus, the water level is rising at a speed of $\boxed{\frac{4}{5}\text{ ft/min}}$ at the instant the water level is at height 6 inches.

Chapter 2 Derivatives Exercise 2.8 25E

Consider a trough is 10 m long and a cross-section has the shape of an isosceles trapezoid with 30 cm wide at bottom, 80 cm wide at top and has height 50 cm.

Draw the diagram and label all measurements in same units.



Suppose the trough is being filled with water at a rate of $0.2\text{ m}^3/\text{min}$.

So, $\frac{dV}{dt}=0.2$

Here, V is the volume of the water in trough.

Find the change of the height of water in trough $\frac{dh}{dt}$.

The volume of trough is,

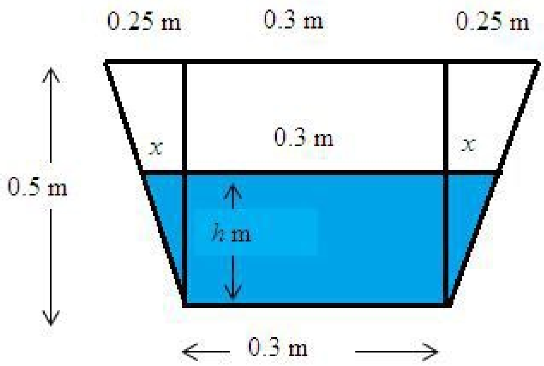
$$V=\frac{1}{2}l\left(b_1+b_2\right)h$$

where l is the length of the trough, h is the height of the trough, and b_1, b_2 are the wide of trough at bottom and top respectively.

As the length of trough is $l=10$, and $b_1=0.3, b_2=b$.

So, $V=\frac{1}{2}10\left(0.3+b\right)h$

Estimate the value of b using similar triangles.



Here, $b=0.3+2x$

By the similar triangles,

$$\frac{h}{x}=\frac{0.5}{0.25}$$
$$\frac{h}{x}=2$$
$$h=2x$$

So, $b=0.3+h$

Plug in $b=0.3+h$ in the equation $V=\frac{1}{2}10\left(0.3+b\right)h$ to obtain that,

$$\begin{aligned}V&=\frac{1}{2}\cdot10\left(0.3+0.3+h\right)h\\&=5\left(0.6+h\right)h\\&=3h+5h^2\end{aligned}$$

Thus, $V=3h+5h^2$

As the volume of the water and height of the water are changes with the time t.

So, they are the functions of time t.

Differentiate the equation $V=3h+5h^2$ with respect to t.

$$\begin{aligned}\frac{dV}{dt}&=3\frac{dh}{dt}+5\left(2h\frac{dh}{dt}\right)\\&=\left(3+10h\right)\frac{dh}{dt}\end{aligned}$$

Plugin $h = 0.3\text{ m}$ (water level) and $\frac{dV}{dt} = 0.2$ in the above equation $\frac{dV}{dt} = (3+10h)\frac{dh}{dt}$ to obtain that,

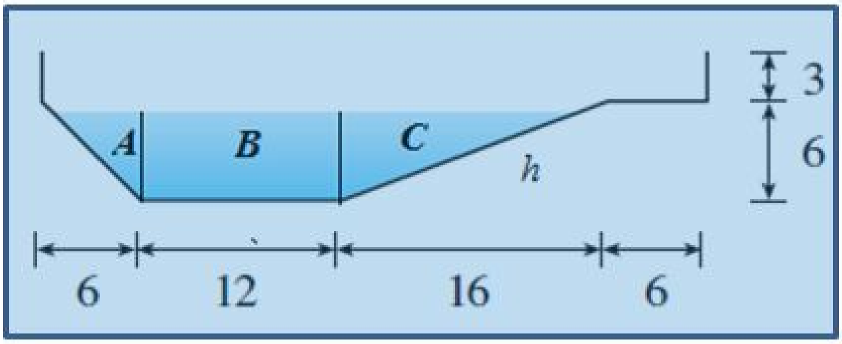
$$0.2 = (3+10(0.3))\frac{dh}{dt}$$
$$0.2 = 6\frac{dh}{dt}$$
$$\frac{0.2}{6} = \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{2}{60}$$
$$= \frac{1}{30}$$

Thus, the water raising is,

$$\frac{dh}{dt} = \frac{1}{30}\text{meter/min}$$
$$= \frac{1}{30}\times 100\,\text{cm/min}$$
$$= \boxed{\frac{10}{3}\text{cm/min}}.$$

Chapter 2 Derivatives Exercise 2.8 26E

Begin by naming variables. Let *h* be the water level in the pool, and let *V* be the volume of water in the pool. The pool is 20 ft. wide, and a cross-section is shown below.



The given quantity is the rate, at which the pool is being filled,

$$\frac{dV}{dt} = 0.8\text{ ft}^3/\text{min}$$

The unknown is the rate at which the water level is rising $\frac{dh}{dt}$, when $h = 5\text{ ft}$.

Since $\frac{dh}{dt}$ is unknown and $\frac{dV}{dt}$ is known, the next step is to find an equation relating the variables *h* and *V*. Since the pool is 20 ft. wide, the volume of water is 20 times the cross-sectional area of water. The cross-section can be divided into regions *A*, *B* and *C*, as in the picture. Region *A* is a right triangle. Since the slanted line on its boundary has a total height of 6 and a total width of 6, the base and height of triangle *A* are equal, so

$$\begin{aligned}\text{Area of } A &= \frac{1}{2}bh \\ &= \frac{1}{2}(h)h \\ &= \frac{1}{2}h^2\end{aligned}$$

Region *B* is a rectangle of width 12 and length *h*, so

$$\begin{aligned}\text{Area of } B &= lb \\ &= (h)(12) \\ &= 12h\end{aligned}$$

Finally, region *C* is a right triangle. Since the slanted line on its boundary has a total height of 6 and a total width of 16, the ratio of its height to width is 6:16, so

$$\begin{aligned}\text{Area of } C &= \frac{1}{2}bh \\ &= \frac{1}{2}\left(\frac{16}{6}h\right)h \\ &= \frac{4}{3}h^2\end{aligned}$$

The total cross-sectional area of the water is the sum of these areas,

$$\begin{aligned}A^* &= \text{Area of } A + \text{Area of } B + \text{Area of } C \\ &= \frac{1}{2}h^2 + 12h + \frac{4}{3}h^2 \\ &= \frac{11}{6}h^2 + 12h\end{aligned}$$

The total volume of water is

$$\begin{aligned}V &= \text{wide} \times A^* \\ &= 20A^* \\ &= 20\left(\frac{11}{6}h^2 + 12h\right) \\ &= \frac{110}{3}h^2 + 240h\end{aligned}$$

To get an equation which involves the rates of change, differentiate both sides with respect to *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{110}{3}h^2 + 240h\right)$$

$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{110}{3}h^2\right) + \frac{d}{dt}(240h)$ By using the sum rule

$= \frac{110}{3}\frac{d}{dt}(h^2) + 240\frac{d}{dt}(h)$ By using the constant multiple rule

$= \frac{110}{3}\left(2h\frac{dh}{dt}\right) + 240\frac{dh}{dt}$ By using the power rule: $\frac{d}{dx}(x^n) = nx^{n-1}, n \in \mathbb{R}$

$= \left(\frac{220}{3}h + 240\right)\frac{dh}{dt}$

Now plug in the known values to solve for the unknown.

$$0.8=\left(\frac{220}{3}(5)+240\right)\frac{dh}{dt}$$ Substitute $\frac{dV}{dt}=0.8, h=5$

$$\frac{8}{10}=\left(\frac{1100}{3}+240\right)\frac{dh}{dt}$$

$$\frac{4}{5}=\left(\frac{1820}{3}\right)\frac{dh}{dt}$$

$$\frac{1}{5}=\left(\frac{455}{3}\right)\frac{dh}{dt}$$

So, $\frac{dh}{dt}=\boxed{\frac{3}{2275}\text{ ft/min}}$

Chapter 2 Derivatives Exercise 2.8 27E

The volume of the pile is $v=\frac{1}{3}\pi r^2 h$, where r is the radius and h is the height of the pile (conical shape).

Also, given that, $r=\frac{h}{2}$ (Diameter = height)

Then $$v=\frac{1}{3}\pi\frac{h^2}{4}h=\frac{\pi}{12}h^3 \quad ..(1)$$

Differentiating (1) with respect to t implicitly , we get

$$\frac{dv}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}$$

But the rate of increase of the volume is $\frac{dv}{dt}=30\ \text{ft}^3/\text{min}$

Then $$30=\frac{\pi}{4}h^2\frac{dh}{dt}$$

Or $$\frac{dh}{dt}=\frac{120}{\pi h^2}$$

At $h=10\,\text{ft}$,

$$\frac{dh}{dt}=\frac{120}{\pi(10^2)}$$

$$\approx \boxed{0.382\,\text{ft/min}}.$$

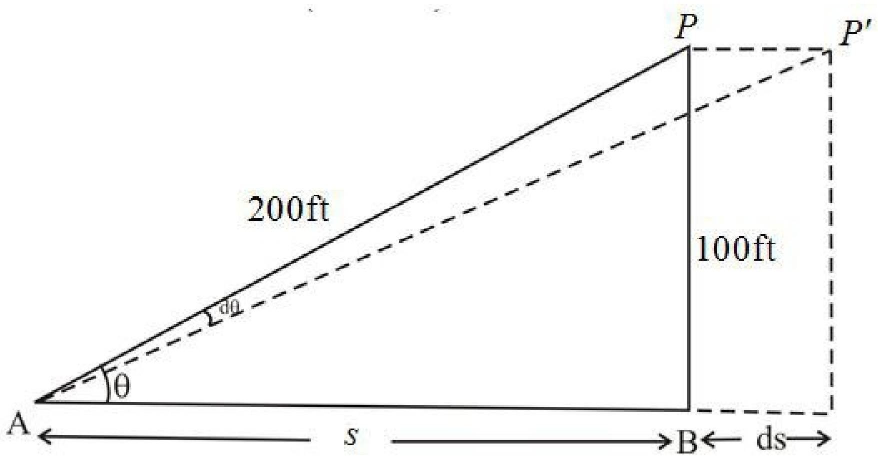
Thus the height of the pile increases with the rate of $\boxed{0.382\,\text{ft/min}}$.

Chapter 2 Derivatives Exercise 2.8 28E

Suppose that the kite is at P when the length of the string is $AP=200\,\text{ft}$.

The height of the kite is $BP=100\ \text{ft}(\text{constant})$.

This information is shown as below:



The angle between string and horizontal is,

$$\sin\theta = \frac{BP}{AP}$$
$$= \frac{100}{200}$$
$$= \frac{1}{2}$$

Then, $\theta = \frac{\pi}{6}$ radian

Then horizontal distance from A to B (opposite to the position of kite) is,

$$s = AB$$
$$= AP \cdot \cos\theta \quad \text{Since } \cos\theta = \frac{AB}{AP}$$
$$= 200\left(\cos\frac{\pi}{6}\right)$$
$$= 200\left(\frac{\sqrt{3}}{2}\right)$$
$$s = 100\sqrt{3} \text{ ft}$$

Now the relation between horizontal and height is,

$$\tan\theta = \frac{BP}{AB}$$

Or $\tan\theta = \frac{100}{s}$

Differentiate both sides of the equation of this equation with respect to time t (in seconds).

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-100}{s^2} \cdot \frac{ds}{dt}$$

Substitute $\theta = \frac{\pi}{6}$, $s = 100\sqrt{3}$ ft and $\frac{ds}{dt} = 8$ ft/s in this equation, to get

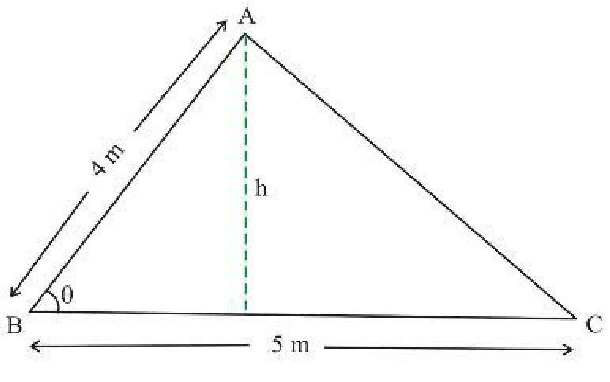
$$\sec^2\frac{\pi}{6} \cdot \frac{d\theta}{dt} = \frac{-100}{\left(100\sqrt{3}\right)^2} \cdot 8$$

$$\left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = \frac{-8}{300} \quad \text{Since } \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

$$\frac{4}{3} \cdot \frac{d\theta}{dt} = \frac{-8}{300}$$
$$\frac{d\theta}{dt} = \frac{-8}{300} \cdot \frac{3}{4}$$
$$\frac{d\theta}{dt} = \frac{-2}{100}$$
$$\frac{d\theta}{dt} = -0.02 \text{ rad/s}$$

Therefore the angle between string and horizontal is decreasing at a rate of $\boxed{0.02 \text{ rad/s}}$

Chapter 2 Derivatives Exercise 2.8 29E



In triangle ABC, AB = 4 m and BC = 5 m are given. If the angle between AB and BC is θ then

Altitude of the triangle $h = AB.\sin\theta$ or $\boxed{h = 4\sin\theta}$ --- (1)

The area of the triangle $= \frac{1}{2}$ base $\times$ altitude

$$A = \frac{1}{2} \times BC \times h$$

$$= \frac{1}{2} 5.(4\sin\theta)$$

$$\boxed{A = 10\sin\theta} \qquad \text{--- (2)}$$

Differentiate both sides of the equation (2) with respect to t, we have
The rate of change in area, i.e.,

$$\frac{dA}{dt} = 10\cos\theta.\frac{d\theta}{dt}$$

We have

$$\theta = \frac{\pi}{3}$$

and $\frac{d\theta}{dt} = 0.06$ rad/s

Then

$$\frac{dA}{dt} = 10.\cos\frac{\pi}{3}(0.06)$$

$$= 0.6.\frac{1}{2}$$

$$= 0.3\ m^2/s$$

Then the area of the triangle is increasing with the rate $\boxed{0.3\ m^2/s}$

Chapter 2 Derivatives Exercise 2.8 30E

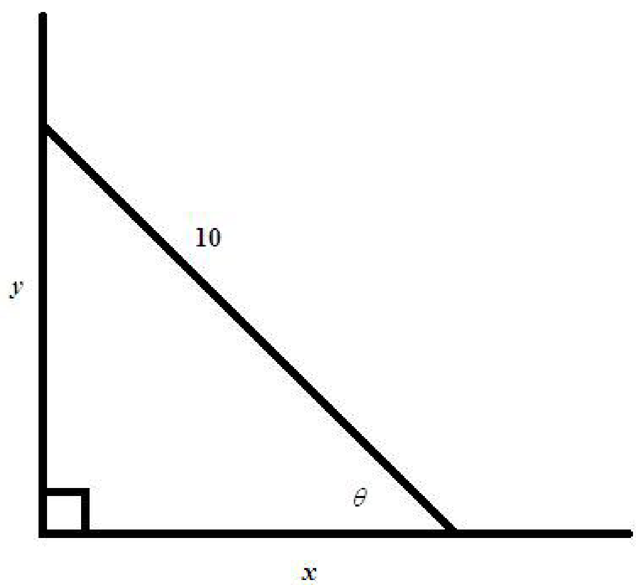
Consider a ladder 10 ft long rests against a vertical wall.

Let x be the horizontal distance from the wall to the ladder, and let y be the vertical distance from the ladder to the ground.

Let θ be the angle of ladder to the ground at any time.

The objective is to find $\frac{d\theta}{dt}$.

Sketch the diagram as shown below:



The rate of travel of bottom of the ladder is 1 ft/s.

So, $\frac{dx}{dt} = 1 \text{ ft/s}$

The rate of change of angle in radian per second is $\frac{d\theta}{dt}$.

From, the above triangle,

$$\cos\theta = \frac{x}{10}$$
$$x = 10\cos\theta$$

Differentiate the above equation $x = 10\cos\theta$ to obtain that,

$$\frac{dx}{dt} = 10\left(-\sin\theta \cdot \frac{d\theta}{dt}\right)$$
$$\frac{d\theta}{dt} = \frac{1}{-10\sin\theta} \cdot \frac{dx}{dt}$$

When the bottom of the ladder $x = 6$:

By the Pythagorean Theorem:

$$x^2 + y^2 = 10^2$$
$$6^2 + y^2 = 10^2$$
$$y^2 = 10^2 - 6^2$$
$$y^2 = 100 - 36$$
$$y^2 = 64$$
$$y = \pm 8$$
$$y = 8 \qquad \text{Take positive only.}$$

Thus,

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse.}}$$
$$= \frac{8}{10}$$

Plugin $\frac{dx}{dt} = 1 \text{ ft/s}$ and $\sin\theta = \frac{8}{10}$ in the equation $\frac{d\theta}{dt} = \frac{1}{-10\sin\theta} \cdot \frac{dx}{dt}$

$$\frac{d\theta}{dt} = \frac{1}{-10\left(\frac{8}{10}\right)} \cdot 1$$
$$= -\frac{1}{8}$$

Therefore, the rate of angle between bottom the ladder to the ground is $\frac{d\theta}{dt} = \boxed{-\frac{1}{8} \text{ ft/s}}$.

Here, the negative sign represents the angle between the bottoms of the ladder to the ground is decreasing.

Chapter 2 Derivatives Exercise 2.8 31E

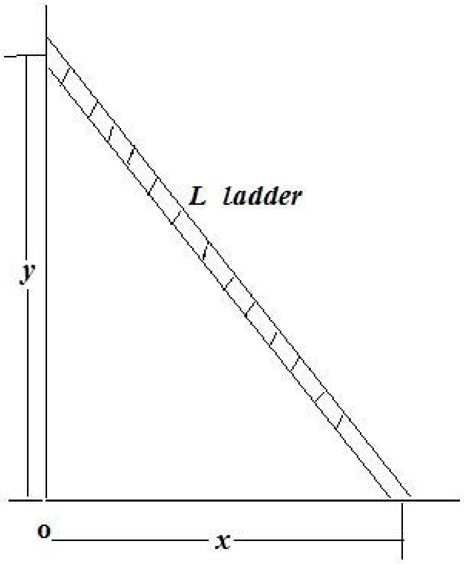
Rate of change of sliding of ladder (decreasing height) is 0.15 m/s and at this rate ladder is 3 m from the wall.

Rate of change of ladder to move away from wall (increasing distance) is 0.2 m/s.

The objective is to find the length of the ladder.

Let *x* be the horizontal distance from the wall to the ladder, let *y* be the vertical distance from the ladder to the ground, and let *L* be the unknown length of the ladder.

Sketch the figure as follows.



Rate of change the height is 0.15. So,

$$\frac{dy}{dt} = -0.15 \text{ m/s}$$

With the quantity being negative since the position *y* is decreasing in time.

Also, given that when $x = 3 \text{ m}$, then $\frac{dx}{dt} = 0.2 \text{ m/s}$.

Positive since *x* is increasing in time.

Determine the length of the ladder *L*.

Use the Pythagorean Theorem on the right triangle seen in the picture above.

$$x^2 + y^2 = L^2$$
$$L = \sqrt{x^2 + y^2}$$

Squaring on both sides to obtain that,

$$L^2 = x^2 + y^2$$

Differentiate the above equation with respect to *t* and use the Chain Rule.

$$\frac{d}{dt}\left(L^2\right) = \frac{d}{dt}\left(x^2 + y^2\right)$$
$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Note that, the length *L* is constant in time its time derivative is 0.

Plug in the known values at the moment in question gives,

$$0 = 2(3)(0.2) + 2y(-0.15)$$
$$0.3y = 1.2$$
$$y = 4 \text{ m}$$

Plug in the known values of *x* and *y* at the given time to equation (1), to obtain that,

$$\begin{aligned} L^2 &= x^2 + y^2 \\ &= 3^2 + 4^2 \\ &= 25 \end{aligned}$$

Since the length must be a positive number, the length of the ladder is,

$$\begin{aligned} L &= \sqrt{25} \\ &= 5 \end{aligned}$$

Therefore, length of the ladder is $\boxed{5 \text{ m}}$.

Chapter 2 Derivatives Exercise 2.8 32E

The bowel is being filled at a rate of $2\,\text{L/min} = 2000\ \text{cm}^3/\text{min}$.

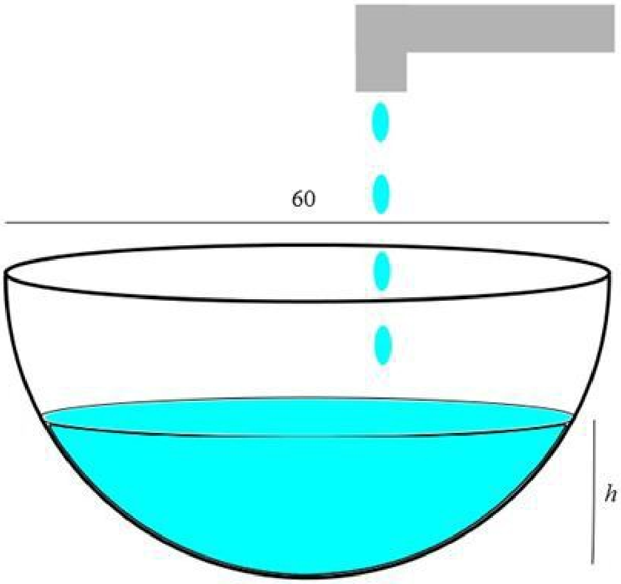
Let h be the water level in the basin, and let V be the volume of water in the basin.

So,

$$\frac{dV}{dt} = 2000\ \text{cm}^3/\text{min}.$$

The objective is to find $\frac{dh}{dt}$ when the height of the water reaches 15 cm.

Sketch the figure as shown below:



The volume of the water at height h is given by the formula,

$$V = \pi\left(rh^2 - \frac{1}{3}h^3\right)$$

where r is the radius. In this case, the diameter is 60 cm, so the radius is 30 cm.

Therefore,

$$V = \pi\left(30h^2 - \frac{1}{3}h^3\right)$$

Differentiate with respect to t to obtain that,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\pi\left(30h^2 - \frac{1}{3}h^3\right)\right)$$

$$\frac{dV}{dt} = \pi\left(60h - h^2\right)\cdot\frac{dh}{dt}$$

Solve this for $\frac{dh}{dt}$ to obtain that,

$$\pi\left(60h - h^2\right)\cdot\frac{dh}{dt} = \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi\left(60h - h^2\right)}\frac{dV}{dt}$$

Plug in the given values $h = 15\text{ cm}, \frac{dV}{dt} = 2000\text{ cm}^3/\text{min}$ to obtain that,

$$\frac{dh}{dt} = \frac{1}{\pi\left[60\cdot(15)-(15)^2\right]}\cdot 2000$$
$$= \frac{2000}{\pi(675)}$$
$$= \frac{80\cdot 25}{\pi(27\cdot 25)}$$
$$= \frac{80}{27\pi}$$

Thus, the rate at which the water is the water is rising in the basin when it is half full is

$$\frac{dh}{dt} = \boxed{\frac{80}{27\pi}\text{ cm/min}}.$$

Chapter 2 Derivatives Exercise 2.8 33E

Boyle's law is

$PV = C$ --- (1)

Where P is pressure and V is the volume of gas and C is a constant.

Now differentiate both sides of the equation (1) with respect to t, we have

$$\frac{d}{dt}(PV) = \frac{d}{dt}(C)$$

By product law

$$\boxed{P.\frac{dV}{dt} + V.\frac{dP}{dt} = 0} \quad \text{--- (1) [where C is a constant]}$$

When $V = 600 Cm^3$, $P = 150\ kPa$ and $\frac{dP}{dt} = 20\ kPa/\min$ then from the equation (1), we have

$$150.\frac{dV}{dt} + 600\times 20 = 0$$
$$\Rightarrow 150.\frac{dV}{dt} = -12000$$
$$\Rightarrow \frac{dV}{dt} = \frac{-12000}{150}\ \text{Cm}^3/\min$$
$$\Rightarrow \frac{dV}{dt} = -80\ \text{Cm}^3/\min$$

Thus the volume is decreasing with the rate $\boxed{80\,\text{Cm}^3/\min}$

Chapter 2 Derivatives Exercise 2.8 34E

Consider the following equation:

$PV = C$ ......(1)

Here, C is a constant and P is pressure V is the volume of the gas.

The volume is $V = 600\text{ cm}^3$.

The pressure is $P = 150\text{ kPa}$.

The pressure is increasing at the rate of 20 kPa/min.

That is $\frac{dP}{dt} = 20\text{ kPa/min}$

The objective is to find rate of decrease of volume.

Differentiate both sides of the equation (1) with respect to t.

$$V\frac{dP}{dt}+P\frac{dV}{dt}=0$$

Now, substitute the values $V=600\text{ Cm}^3$, $P=150\ kPa$ and $\frac{dP}{dt}=20\ kPa/\min$ in the above equation.

$$600(20)+(150)\frac{dV}{dt}=0$$
$$12000+150\frac{dV}{dt}=0$$
$$150\frac{dV}{dt}=-12000$$
$$\frac{dV}{dt}=-\frac{12000}{150}$$
$$=-80$$

Therefore, the volume is decreasing at the rate about $\boxed{80\ \text{Cm}^3/\min}$.

Chapter 2 Derivatives Exercise 2.8 35E

We have $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$

Then $R=\frac{R_1R_2}{R_1+R_2}$

Then $$\frac{dR}{dt}=\frac{(R_1+R_2)\frac{d}{dt}(R_1R_2)-(R_1R_2).\frac{d}{dt}(R_1+R_2)}{(R_1+R_2)^2}$$

Or $$\frac{dR}{dt}=\frac{(R_1+R_2)\left[R_1\frac{dR_2}{dt}+R_2\frac{dR_1}{dt}\right]-(R_1R_2)\left[\frac{dR_1}{dt}+\frac{dR_2}{dt}\right]}{(R_1+R_2)^2}$$

We have $R_1=80\,\Omega$, $R_2=100\,\Omega$, $\frac{dR_1}{dt}=0.3\,\Omega/s$ and $\frac{dR_2}{dt}=0.2\,\Omega/s$

Then $$\frac{dR}{dt}=\frac{(80+100)[80.(0.2)+(0.3)100]-[8000][0.3+0.2]}{(180)^2}$$
$$=\frac{(180)(16+30)-4000}{32400}$$
$$=\frac{4280}{32400}$$
$$\boxed{\frac{dR}{dt}=\frac{107}{810}\Omega/s}$$

Or $\boxed{\frac{dR}{dt}\approx 0.132\,\Omega/s}$, $\frac{dR}{dt}$ is the rate of change in R

Chapter 2 Derivatives Exercise 2.8 36E

We have $W=0.12L^{2.53}$ --- (1)

Over 10 million years the average length of a certain species of fish evolved from 15 Cm to 20 Cm at a constant rate

Then

$$\frac{dL}{dt}=\frac{20-15}{10\ \text{million}}=\frac{5}{10^7}=5\times10^{-7}\text{cm/year} \qquad [1\text{ million}=10^{+6}]$$

From equation (1)

$$W=0.12L^{2.53}$$

Then $$\frac{dW}{dt}=0.12(2.53).L^{1.53}\frac{dL}{dt}$$
$$=0.3036L^{1.53}\frac{dL}{dt}$$
$$=0.3036L^{1.53}(5\times10^{-7})$$
$$=1.518.L^{1.53}\times10^{-7}\text{ grams/year}$$

Now we have $B = 0.007 W^{2/3}$

Then $\dfrac{dB}{dt} = (0.007)\dfrac{2}{3} W^{-1/3} \cdot \dfrac{dW}{dt}$

$$= (0.007)\frac{2}{3}\frac{1}{\sqrt[3]{0.12L^{2.53}}}.(1.518)L^{1.53} \times 10^{-7}$$

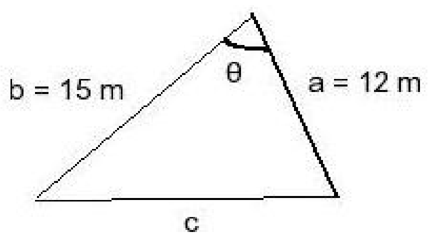
When L = 18cm then

$$\frac{dB}{dt} = \frac{(0.007)2\times(1.518)(18)^{1.53}}{3\sqrt[3]{0.12(18)^{2.53}}}.10^{-7}$$

$$\boxed{\frac{dB}{dt} \approx 1.045\times10^{-8} \text{grams/year}}$$

Chapter 2 Derivatives Exercise 2.8 37E

Step 1:



Given:

$$\frac{d\theta}{dt} = 2\,\frac{\text{degrees}}{\text{minute}}$$

Want to know:

$\dfrac{dc}{dt}$ when θ = 60 degrees

Step 2:

By the Law of Cosines:

$$c^2 = b^2 + c^2 - 2\text{abcos}(\theta)$$

$$c^2 = 15^2 + 12^2 - 2 \cdot 12 \cdot 15 \cdot \cos(60)$$

$$c = 13.74$$

Step 3:

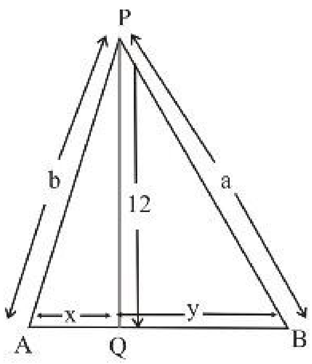
$$2c \cdot \frac{dc}{dt} = 360 \cdot \sin(\theta) \cdot \frac{d\theta}{dt}$$

$$\frac{dc}{dt} = \frac{2\pi \sin(60)}{13.74}$$

$$\frac{dc}{dt} = .396 \frac{\text{meter}}{\text{minute}}$$

Note: Make sure your calculator is in the correct mode (degree), or your calculator will give you the incorrect answer for c.

Chapter 2 Derivatives Exercise 2.8 38E



Let AP = b and PB = a then a + b = 39 because total length of the rope is 39 ft.
Then b = 39 − a --- (1)
Now let AQ = x and BQ = y and PQ = 12 ft the length of altitude of ABP
From (1) we have

$$B = 39 - a$$

$$\Rightarrow \sqrt{12^2 + x^2} = 39 - \sqrt{12^2 + y^2} \quad \text{--- (2)}$$

(Using Pythagoras theorem in triangles PQA and PQB)

Differentiating both sides of the equation (2) with respect to t, we get

$$\frac{1}{2} \frac{2x}{\sqrt{12^2 + x^2}} \cdot \frac{dx}{dt} = -\frac{1}{2} \frac{2y}{\sqrt{12^2 + y^2}} \frac{dy}{dt}$$

$$\Rightarrow \frac{x}{\sqrt{12^2 + x^2}} \frac{dx}{dt} = \frac{-y}{\sqrt{12^2 + y^2}} \frac{dy}{dt}$$

So we have

$$\boxed{\frac{dy}{dt} = \frac{-\sqrt{12^2 + y^2}}{y} \cdot \frac{x}{\sqrt{12^2 + x^2}} \cdot \frac{dx}{dt}} \quad \text{--- (3)}$$

When AQ = x = 5 ft then from equation (2)

$$\sqrt{12^2+5^2} = 39 - \sqrt{12^2+y^2}$$
$$\Rightarrow 13 = 39 - \sqrt{12^2+y^2}$$
$$\Rightarrow \sqrt{12^2+y^2} = 39-13$$
$$\Rightarrow \sqrt{12^2+y^2} = 26$$

Squaring both sides

$$12^2+y^2 = 26^2$$
$$\Rightarrow y^2 = 26^2 - 12^2 = 532 \Rightarrow y = 23.06$$
$$\Rightarrow \boxed{y \approx 23 ft}$$

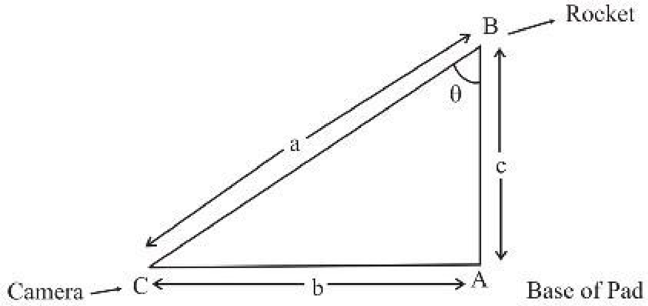
Now from the equation (3), when $y \approx 23$ and $\frac{dx}{dt} = 2$ ft/s,

$$\frac{dy}{dt} \approx \frac{-26}{23} \cdot \frac{5}{\sqrt{169}} \cdot 2$$
$$\approx \frac{-26}{23} \cdot \frac{10}{13}$$
$$\approx -\frac{20}{23} ft/second \approx -0.87 \text{ ft/s}$$

Then cart B moving with speed about $\boxed{\frac{20}{23} \text{ ft/s} \approx 0.87 \text{ ft/s}}$ towards Q

(The negative sign indicates the decreases in distance y as cart B move toward Q

Chapter 2 Derivatives Exercise 2.8 39E



(A) Let the camera is located at C and base of the rocket launching pad is at A, B is the position of rocket. Then we have a triangle ABC

Where AC = b = 4000 ft is given

Now by Pythagoras theorem,

$$BC^2 = AC^2 + AB^2$$
$$\Rightarrow a^2 = b^2 + c^2 \qquad \text{--- (1)}$$

When we have b = 4000 ft and c = 3000 ft

Then

$$a^2 = (4000)^2 + (3000)^2$$
$$= 16000000 + 9000000$$
$$a^2 = 25000000$$
$$\Rightarrow \boxed{a = 5000}$$

Now differentiate equation (1) with respect to t

$$2a\frac{da}{dt} = 0 + 2c\frac{dc}{dt} \qquad \text{[Because b = 4000 is a constant]}$$

Or $\boxed{\frac{da}{dt} = \frac{c}{a} \cdot \frac{dc}{dt}}$

We have c = 3000 ft, a = 5000 ft $\frac{dc}{dt} = 600 ft/s$

Then $\frac{da}{dt} = \frac{3000}{5000} \cdot 600 ft/s$

$$= \frac{3}{5} \cdot 600$$

Or $\boxed{\frac{da}{dt} = 360 ft/s}$

Hence the distance between camera and rocket is increasing by the rate $\boxed{360 ft/s}$

(B) Let angle of elevation of the camera at the point B is θ .
when c = 3000 ft , a = 5000 ft

$$\sin\theta = \frac{AC}{BC} = \frac{b}{a} = \frac{4000}{5000} = \frac{4}{5}$$

And $$\cos\theta = \frac{AB}{BC} = \frac{c}{a} = \frac{3000}{5000} = \frac{3}{5}$$

Now we have $\cos\theta = \frac{c}{a}$

Then $c = a\cos\theta$

Differentiate both sides with respect to t

$$\frac{dc}{dt} = -a\sin\theta.\frac{d\theta}{dt} + \cos\theta\frac{da}{dt}$$

Then $$a\sin\theta\frac{d\theta}{dt} = \cos\theta.\frac{da}{dt} - \frac{dc}{dt}$$

$$\Rightarrow 5000.\frac{4}{5}.\frac{d\theta}{dt} = \frac{3}{5}.360 - 600$$

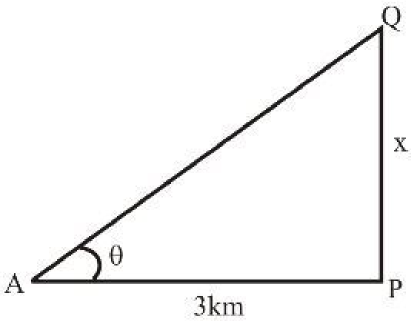
$$\Rightarrow 4000.\frac{d\theta}{dt} = 216 - 600 = -384$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-384}{4000} = -0.096\,rad/s$$

Then the angle is changing with the rate $\boxed{0.096\,rad/s}$

[Negative sign indicates that the angle decreasing as rocket moves upwards]

Chapter 2 Derivatives Exercise 2.8 40E



Q is a point on shoreline, x distance from the point P.
A is the lighthouse.
Now light Beam makes 4 revolution/min = $4 \times 2\pi$ radians / min .

Then $$\frac{d\theta}{dt} = 8\pi \text{ rad/ min} \quad .....(1)$$

From the triangle APQ, we have

$$x = 3\tan\theta$$

$$\Rightarrow \frac{dx}{dt} = 3\sec^2\theta\frac{d\theta}{dt} \quad ....(2)$$

When x = 1 km then $3\tan\theta = 1 \Rightarrow \tan\theta = 1/3$

$$\Rightarrow \sec^2\theta = 1 + \tan^2\theta = 1 + 1/9 = 10/9$$

Therefore from (2)

$$\Rightarrow \frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi)$$

$$= \frac{80}{3}\pi \text{ km/min}$$

$$\approx \boxed{83.78 \text{ km/min.}}$$

Chapter 2 Derivatives Exercise 2.8 41E

Let: x km. be the horizontal distance of the plane from the tracking station,

θ be the angle of elevation

Then

$$x = 5\cot(\theta)$$

Differentiating each side with respect to t, we have

$$\frac{dx}{dt} = -5\left[\csc^2\theta\right]\frac{d\theta}{dt}$$

The given

$\theta = \frac{\pi}{3}$ and $\frac{d\theta}{dt} = -\frac{\pi}{6}$ rad/min (because the angle is decreasing)

==>

$$\frac{dx}{dt} = -5\left[\csc^2\left(\frac{\pi}{3}\right)\right]\left(-\frac{\pi}{6}\right)$$

$$= \left(\frac{5\pi}{6}\right)\left(\frac{4}{3}\right) = \frac{10\pi}{9}\left(\text{km}/\text{min}\right)$$

Chapter 2 Derivatives Exercise 2.8 42E

We have period of circular motion t=2 minutes so the frequency is

$$N = \frac{1}{t} = \frac{1}{2}$$

The angular speed $\omega = 2\pi N = \pi$, so angle of rotation is

$$\theta = \omega t = \pi t$$

Let h be the height, h is given from equality

$$R - h = R\cos\theta \Rightarrow h = R - R\cos\theta = R(1 - \cos\theta)$$

==>

$$h(t) = 10(1 - \cos(\pi t)) \quad (1)$$

Differentiating with respect to t

$$\frac{dh}{dt} = 10\pi\sin\left(\pi t\right) \quad (2)$$

At h=16 m, we have

$$16 = 10(1 - \cos(\pi t)) \Rightarrow 16 = 10 - 10\cos(\pi t) \Rightarrow 10\cos(\pi t) = -6$$

so

$$\cos\left(\pi t\right) = -\frac{6}{10} = -0.6$$

therefore

$$\sin\left(\pi t\right) = \sqrt{1 - \cos^2\left(\pi t\right)} = \sqrt{1 - (-0.6)^2} = \sqrt{0.64} = 0.8$$

so

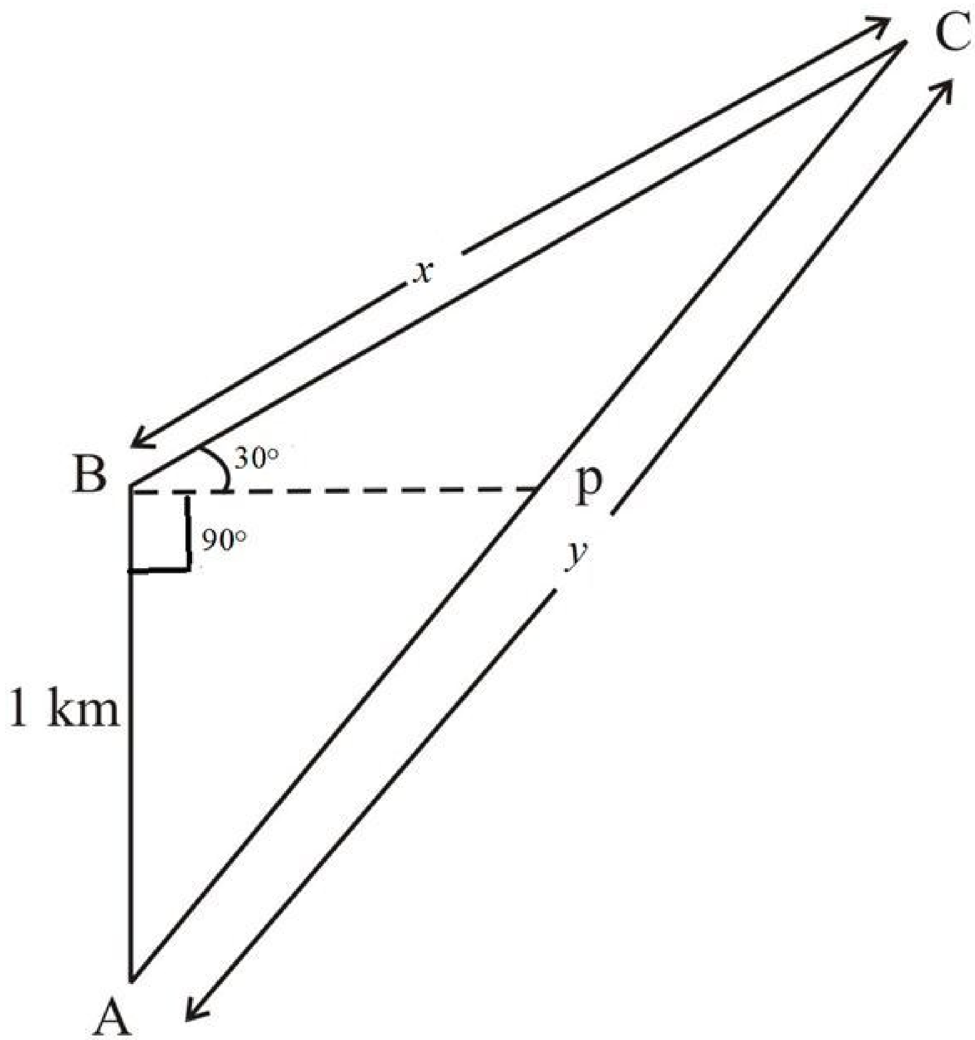
$$\frac{dh}{dt} = 10\pi\sin\left(\pi t\right) = 10\pi\left(0.8\right) = 8\pi \text{ (m/min)}$$

The rider rising at 8π m/min when his seat is 16 m above ground level

Chapter 2 Derivatives Exercise 2.8 43E

Consider a plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°.

To find rate, the distance from the plane to the radar station increasing a minute later:



Let the radar be at the point A and plane be at B. Then AB = 1 km.

A plane flying with a constant speed of 300 km/h passes over a ground radar station means

$$\frac{dx}{dt} = 300$$

The plane is climbing at an angle of $30°$

Then angle $\text{ABC} = 90^\circ + 30^\circ$

$$= 120^\circ$$

By the Law of Cosines,

$$y^2 = x^2 + 1^2 - 2(x)(1)\cos 120^\circ$$

$$= x^2 + 1^2 - 2 \cdot x \cdot \left(-\frac{1}{2}\right) \qquad \text{Since } \cos 120^\circ = -\frac{1}{2}$$

$$= x^2 + 1^2 + x$$

$y^2 = x^2 + x + 1$ ...... (1)

Now differentiating the equation (1) with respect to *t*,

$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$ Power Rule

$$= (2x+1)\frac{dx}{dt}$$

$\frac{dy}{dx} = \frac{(2x+1)}{2y}\frac{dx}{dt}$ ...... (2)

Distance climbed after 1 minute or $\frac{1}{60}\text{ h}$

$$x = 300 \times \frac{1}{60} \text{ km}$$

$$= 5 \text{ km}$$

Put $x = 5$ in equation (1),

$$y^2 = x^2 + x + 1$$

$$y^2 = 5^2 + 5 + 1$$

$$= 31$$

Thus, $y = \sqrt{31} \text{ km}$

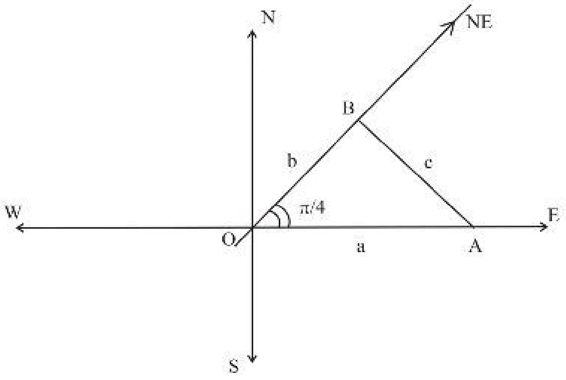
Put $x = 5, \frac{dx}{dt} = 300 \text{ km/h}, \text{and } y = \sqrt{31}$ in (2),

$$\frac{dy}{dt} = \frac{(2 \times 5 + 1)}{2 \cdot \sqrt{31}} \times 300$$

$$= \frac{11 \times 300}{2\sqrt{31}}$$

Then the rate of change of the distance of the plane from radar is

$$\boxed{\frac{dy}{dt} = \frac{1650}{\sqrt{31}} \text{ km/h or } 296 \text{ km/h}}$$



Let the first person starting from the point O and walking in the east, reaches at the point A after time "t." And let the second person starting from the point O reaches point B after time "t." Let OA=a miles and let OB=b miles. Let the distance between AB be c miles.

Then from the properties of triangles, we have

$$\cos\frac{\pi}{4} = \frac{a^2+b^2-c^2}{2ab}$$
$$\frac{1}{\sqrt{2}} = \frac{a^2+b^2-c^2}{2ab}$$
$$\sqrt{2}ab = a^2+b^2-c^2$$
$$\boxed{c^2 = a^2+b^2-\sqrt{2}ab} \quad \text{--- (1)}$$

Distance covered by first person in 15 minutes

$$3\times\frac{1}{4} = \frac{3}{4}\text{ miles}$$

And distance covered by second person in 15 minutes, i.e., in $\frac{1}{4}$ hr is

$$= 2\times\frac{1}{4} = \frac{2}{4}\text{ miles}$$

Then from the equation (1), we have

$$c^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}.\frac{3}{4}.\frac{2}{4}$$
$$= \frac{9}{16}+\frac{4}{16}-\frac{6\sqrt{2}}{16}$$
$$= \frac{13}{16}-\frac{6\sqrt{2}}{16}$$

Or $$c = \frac{\sqrt{13-6\sqrt{2}}}{4}\text{ miles}$$

Now differentiate the equation (1) with respect to t, we get

$$2c\frac{dc}{dt} = 2a.\frac{da}{dt}+2b\frac{db}{dt}-\sqrt{2}a.\frac{db}{dt}-\sqrt{2}b.\frac{da}{dt}$$

We have after 15 minutes

$$a=\frac{3}{4}, b=\frac{2}{4}, \frac{da}{dt}=3, \frac{db}{dt}=2, \text{ and } c=\frac{\sqrt{13-6\sqrt{2}}}{4}$$

Then

$$\frac{\sqrt{13-6\sqrt{2}}}{2}\frac{dc}{dt} = 2.\left(\frac{3}{4}\right).3+2.\left(\frac{2}{4}\right).2-\sqrt{2}.\left(\frac{3}{4}\right).2-\sqrt{2}.\left(\frac{2}{4}\right).3$$
$$= \frac{9}{2}+2-\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}$$
$$= \frac{13-6\sqrt{2}}{2}$$
$$\frac{dc}{dt} = \frac{2}{\sqrt{13-6\sqrt{2}}}\left[\frac{13-6\sqrt{2}}{2}\right]$$
$$= \sqrt{13-6\sqrt{2}}$$
$$\approx 2.125$$

Then

$$\frac{dc}{dt} \approx 2.125 \text{ miles/h}$$

$$\Rightarrow \boxed{\text{rate} \approx 2.125 \text{ miles/h}}$$

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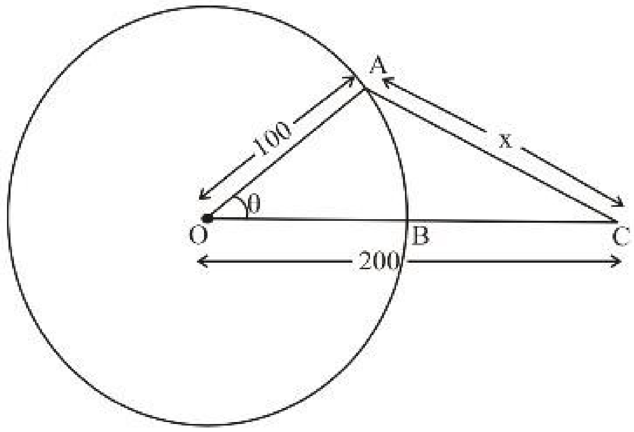

Let the runner be at the point A, OA is the radius of the circular track that is r = 100 m (given) and let the friend be standing at the point C then OC = 200m (given)
Suppose the distance of the runner from his friend is AC = x meter and angle $ABO = \theta$
Then the length of arc AB is

$$L = r\theta$$

Then $\frac{dL}{dt} = r\frac{d\theta}{dt}$

We have $\frac{dL}{dt} = 7 \text{ m/s}$ and r = 100m

Then

$$\boxed{\frac{d\theta}{dt} = \frac{7}{100}\text{rad/s}} \qquad \text{--- (1)}$$

If two sides and angle between these sides of triangle are given then the third side can be calculated by the properties of triangles

In triangle OAC

$$AC^2 = OA^2 + OC^2 - 2OA.OC\cos\theta$$

We have

$$x^2 = (100)^2 + (200)^2 - 2(100).(200)\cos\theta$$

$$x^2 = 50000 - 40000.\cos\theta \qquad \text{--- (2)}$$

When x =200 m

$$200^2 = 50000 - 40000.\cos\theta$$
$$40000 = 50000 - 40000.\cos\theta$$

Then $\boxed{\cos\theta = \frac{1}{4}}$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

Then $= \sqrt{1-\frac{1}{16}}$

$$= \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

Or $\boxed{\sin\theta = \frac{\sqrt{15}}{4}}$

Now differentiate both sides of the equation (2) with respect to t

$$2x\frac{dx}{dt} = 40000.\sin\theta.\frac{d\theta}{dt}$$

Or $\frac{dx}{dt} = \frac{20000}{x}.\sin\theta.\frac{d\theta}{dt}$

We have $x = 200$ m, $\sin\theta = \frac{\sqrt{15}}{4}$ and $\frac{d\theta}{dt} = \frac{7}{100}$ rad/s

Then

$$\frac{dx}{dt} = \frac{20000}{200}.\frac{\sqrt{15}}{4}.\frac{7}{100}$$

Or $\boxed{\frac{dx}{dt} = 7.\frac{\sqrt{15}}{4}} \approx 6.78$ m/s

Thus the distance between runner and his friend is changing with the rate $\boxed{6.78 \text{ m/s}}$

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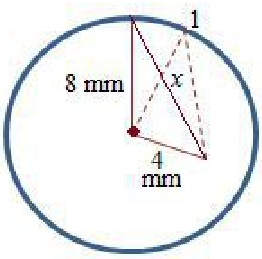
Consider a watch with 8mm hour hand and 4mm minute hand.

Use law of cosines formula, to find the rate of change of distance between the tips of the hands at one o'clock.

Law of Cosines Formula:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

The visual representation of the problem is as follows:



Let x be the distance between the tips of the hands, and θ be the angle from the minute hand to the hour hand.

Here $a = 8, b = 4, c = x$ and $C = \theta$.

Plug in the values to the law of cosines formula.

$$\begin{aligned} x^2 &= 8^2 + 4^2 - 2(8)(4)\cos\theta \\ &= 64 + 16 - 64\cos\theta \\ &= 80 - 64\cos\theta. \end{aligned}$$

Differentiate the equation $x^2 = 80 - 64\cos\theta$ with respect to t, in order to get the rate of change of the distance between the tips of the hands at one o'clock.

$$\begin{aligned} 2x\frac{dx}{dt} &= -64(-\sin\theta)\frac{d\theta}{dt} \\ x\frac{dx}{dt} &= 32\sin\theta\frac{d\theta}{dt} \end{aligned}$$

At one o'clock the angle θ is $\frac{2\pi}{12} = \frac{\pi}{6}.$

Plug in $\theta = \frac{\pi}{6}$ to the equation $x^2 = 80 - 64\cos\theta.$

$$x^2 = 80 - 64\cos\left(\frac{\pi}{6}\right)$$

$$x^2 = 80 - 64\left(\frac{\sqrt{3}}{2}\right) \qquad \text{since } \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$x^2 = 80 - 32\sqrt{3}$$

$$x = \sqrt{80 - 32\sqrt{3}}$$

Thus, the distance between the hands is $x = \sqrt{80 - 32\sqrt{3}}.$

The minute hand goes an angle of 2π radians per hour, and the hour hand goes an angle of $\frac{\pi}{6}$ radians per hour in the same direction.

So, the angle between them changes at a rate of $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ radians per hour.

At one o'clock, the angle between the hands is decreasing so the rate of change of the angle is $\frac{d\theta}{dt} = -\frac{11\pi}{6}$ radians per hour.

Plug in $x = \sqrt{80 - 32\sqrt{3}}, \frac{d\theta}{dt} = -\frac{11\pi}{6},$ and $\theta = \frac{\pi}{6}$ to the equation $x\frac{dx}{dt} = 32\sin\theta\frac{d\theta}{dt}.$

$$\sqrt{80 - 32\sqrt{3}}\,\frac{dx}{dt} = 32\sin\left(\frac{\pi}{6}\right)\times\left(-\frac{11\pi}{6}\right)$$

$$\sqrt{80 - 32\sqrt{3}}\,\frac{dx}{dt} = -32\left(\frac{1}{2}\right)\left(\frac{11\pi}{6}\right) \qquad \text{Since } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\sqrt{80 - 32\sqrt{3}}\,\frac{dx}{dt} = -\frac{88}{3}\pi \qquad \text{Simplify}$$

$$\frac{dx}{dt} = -\frac{88}{3\sqrt{80 - 32\sqrt{3}}}\pi$$

$$\approx -18.590$$

Therefore, the distance between the tips of the hands at one o'clock is decreasing approximately $\boxed{18.6 \text{ mm per hour}}$.