ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર-ક્રમાંક
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MATHEMATICS

Standard 9

(Semester I)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

Gujarat State Board of School Textbooks
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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is pleasure for the Gujarat State Board of School Textbooks, to place before the students this textbook of **Mathematics for Standard 9 (Semester I)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions, from people interested in education, to improve the quality of the textbook.

Dr. Bharat Pandit
Director
Date : 03-03-2015

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

CONTENTS

About This Textbook...

The Gujarat Secondary and Higher Secondary Education Board has prepared a new syllabus for school curriculum with the help of Gujarat State Textbook Board, learned school teachers, college teachers and university teachers in order to make a student of Gujarat State secure his leading position at national level in present scenario. This syllabus is equivalent to NCERT syllabus. NCF 2005 was kept in mind while preparing this syllabus. Then a panel of subject experts was formed to prepare textbook based on this syllabus.

In a very short period the first draft of the textbook was prepared in English and then translated into Gujarati. Considering global thinking, it was decided to prepare the textbook first in English.

A panel of experts from school, colleges and universities held a workshop of three days and thoroughly discussed the content and made some suggestions. Amendments were carried out in the English as well as Gujarati draft accordingly. A professor of English also was helpful in suggesting some changes in English language. After that there was again a workshop of four days for the Gujarati draft of the textbook. Again suggestions were obtained from experts and amendments were carried out.

The final draft was thus prepared. The experts and members of syllabus committee along with authors reviewed the textbook in the office of Gujarat State Secondary and Higher Secondary Board.

This textbook is prepared according to new syllabus. The NCERT textbook based on NCERT syllabus is having classical approach to geometry. But Gujarat State is using modern approach using set theory for last four decades. Thus the first chapter on set theory is additional compared to the NCERT textbook. The whole curriculum is divided into two semesters. Therefore this book is also divided into two parts. Since standard 8 is included in primary education, content of geometry as specified in the NCERT textbook for standard 8 at present has to be covered in this book.

In the first semester set theory, number system, polynomials, coordinate geometry, graphs of linear equations along with four chapters of geometry are included. All chapters are explained in a lucid language and yet logical consistency is preserved, plenty of illustrations are used to explain the concepts. We have kept in mind that student at the remote end of the state also can study with his own skill. Concept, illustrations and exercises are formed accordingly. Printing in two colours and attractive title in four colours are an added assets to the textbook.

Four chapters of geometry, calculations of area and volume and statistics are included in the textbook of second semester. Also keeping in view the need of a science student, chapter on logarithm is given at the end.

Some teachers of Central Board schools also reviewed the draft of the textbook. They also found the draft of the textbook very useful and they praised the explanation and the illustrations.

In the golden jubilee year of the birth of Gujarat State, we have tried to see that the students of the state get a textbook equivalent to curriculum specified by NCERT. So that students may maintain leading position at national level using the textbook.

In both the semester books detailed explanation is given using figures, diagrams and graphs.

Complete syllabus of NCERT is covered by the textbook. Moreover examples and exercises are introduced to be equally useful to the teachers and students. The main aim of this textbook is to make student of Gujarat face challenges at national level and still study with interest and without burden.

- Author

CHAPTER 1

SET OPERATIONS

- *The essence of mathematics lies in its freedom.*
- *I see it but I don't believe it !*

– Georg Cantor

1.1 Introduction

The concept of a set is a base for all branches of mathematics. The theory of sets was developed by mathematician **Georg Cantor** (1845-1918 A.D). We have learnt some important and primary facts about sets in std. VIII.

In day-to-day life, we often talk about a group of same kind of objects; e.g., a herd of cows, a pack of cards, a team of players etc. This type of a **well-defined collection of objects is considered as a set.**

The father of set theory

The main inventor of set theory was the mathematician Georg Cantor. He was born on 3rd March, 1845 in St. Petesburg, Russia. He took his school education in St. Petesburg. In 1856, he moved to Germany. He was president of **Berlin Mathematical Society** (1864-1865). He achieved doctorate degree in 1867. He taught at a girls school in Berlin. In 1872, he was promoted as an extraordinary professor in Halle. He was a friend of Dedekind. He got some very surprising results in Mathematics. In 1873, he proved that rational numbers are countable. The birth of set theory dates to 1873, when Georg Cantor proved the uncountability of real line, actually December 7, 1873. Hilbert described Cantor's work as the finest product of mathematical genius and one of the supreme achievement of purely intellectual human activity. Some powerful people who disagreed with him severely criticized him for this. But today while those who troubled him are forgotten, Georg Cantor is remembered and widely respected. He died on 6th January, 1918 in Halle, Germany.

1.2 Important Points for Revision

- **A set is a well-defined collection of objects.**
- **A set without any member (element) is called a null set or an empty set.**
- **A set having only one member is called a singleton.**
- **$\in$ (belongs to) is an undefined symbol.**
- **If x is a member of the set A, we write $x \in A$**
- **If x is not a member of the set A, we write $x \notin A$.**
- **A set total number of members of which is a positive integer is called a finite set and a set which is not finite is called an infinite set. Null set is considered to be a finite set.**
- **If all the elements of a set A are present in the set B, then the set A is called a subset of the set B. This fact is denoted by $A \subset B$.**

Important points about subsets :

(1) **Empty set is a subset of every set. Thus, for any set A, $\emptyset \subset A$.**

(2) **Every set is a subset of itself. Thus, for any set A, $A \subset A$.**

(3) **If a set A has n elements, then number of its subsets is 2^n.**

(4) **$N \subset Z \subset Q \subset R$.**

- Generally, while dealing with a problem, we consider some definite set and its subsets. Such a definite set is called the **universal set** with reference to that problem. The **universal set** is denoted as U.

  A set which is a universal set for one problem need not be a universal set for another problem. For example, In Geometry, space or plane is a universal set. For interrelations of integers, set of integers Z is a universal set. For the solution of linear equations, the set of real numbers is a universal set.

- The set of all the elements which are in U but not in the given set A is called the **Complement of the set A**. It is denoted by A'.

  Thus, $A' = \{x \mid x \in U, x \notin A\}$

  so from the above definition, we get the following results.

  (1) $A \cup A' = U$ and (2) $A \cap A' = \emptyset$

- If two sets have same elements, they are said to be **equal sets. If every member of set A is a member of set B and every member of set B is a member of set A, then set A and set B are called equal sets. If A and B are equal sets we write A = B.** For equal sets A and B, $A \subset B$ and $B \subset A$.

i.e. if A $\subset$ B and B $\subset$ A, then A = B.

For example let A = $\{x \mid x \in N, x < 5\}$ and B = {1, 2, 3, 4} be two sets.
Then both the sets A and B have the same members {1, 2, 3, 4}.
So, we say that A = B

- If every member of set A corresponds to one and only one member of set B and every member of set B corresponds to one and only one member of set A then the sets A and B are said to be in one-one correspondence with each other and the sets A and B are called **equivalent sets**. If set A is equivalent to set B, we write A ~ B.
- Thus, if two finite sets are in one-one correspondence with each other, then they should have the same number of elements.
- **Equal sets are always equivalent sets but equivalent sets need not be equal sets.**

For example, if A = {1, 2, 3}, B = $\{a, b, c\}$ then A ~ B but A $\neq$ B.
But for infinite sets, situation is different.
If, E = {2, 4, 6, 8,...}, then N ~ E. Because for every element of N, A unique number n is related to the number $2n$ belonging E and for every element of E, a unique number m is related to the number $\frac{m}{2} \in$ N. But E $\subset$ N.

EXERCISE 1.1

1. Classify the following sets in (a) as empty set or singleton set and in (b) as equal sets or equivalent sets :

(a) (1) A = $\{x \mid x \in Z, x + 1 = 0\}$

(2) B = $\{x \mid x \in N, x^2 - 1 = 0\}$

(3) C = $\{x \mid x \in$ N, x is a prime number between 13 and 17$\}$

(b) (1) A = $\{x \mid x \in N, x \leq 7\}$,

B = $\{x \mid x \in Z, -3 \leq x \leq 3\}$

(2) A = $\{x \mid x \in$ N, x is a multiple of 2, $x < 10\}$,

B = $\{x \mid x \in$ N, x is an even natural number with a single digit$\}$

2. Find the number of subsets of the set A = {1, 2, 3}. Also write all the subsets of the set A.

3. If A = $\{x \mid x \in Z, x^2 - x = 0\}$, B = $\{x \mid x \in N, 1 \leq x \leq 4\}$, then can we say that A $\subset$ B ? Why ?

4. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 2, 4, 6, 8}, then find A' and also verify that A $\cup$ A' = U.

5. If A = {1, 2, 3}, B = {3, 4, 6}, then find all possible non-empty sets X which satisfy the following conditions :

(1) $X \subset A, X \not\subset B$ (2) $X \subset B, X \not\subset A$ (3) $X \subset A, X \subset B$

6. Examine whether the following statements are true or false :

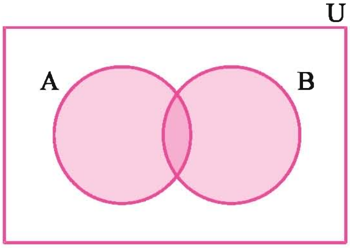
(1) $\{1, 2, 3\} \subset \{1, 2, 3\}$ (2) $\{a, b\} \not\subset \{b, c, a\}$

(3) $\emptyset \notin \{\emptyset\}$ (4) $\{3\} \subset \{1, 2, \{3\}, 4\}$

*

1.3 Properties of the Union Operation

Union set : For any two sets A and B, the set consisting of all the elements which are either in A or in B (or in both) is called the union set of the sets A and B and it is denoted by $A \cup B$. The process of finding the union of two set is called the union operation.



Venn diagram of $A \cup B$

Figure 1.1

Thus, in symbol, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. Venn-diagram is useful in understanding various relations between sets. In Venn-diagram 1.1 the coloured region describes $A \cup B$.

Example 1 : Let A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8} be two sets. Find $A \cup B$.

Solution : $A \cup B = \{1, 3, 5, 7, 9\} \cup \{2, 4, 6, 8\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Example 2 : If α = The letters of the word AHMEDABAD
and β = The letters of the word BARODA are two sets, then find $\alpha \cup \beta$.

Solution : Here α = {A, B, D, E, H, M} and
β = {A, B, D, O, R}
$\therefore \alpha \cup \beta$ = {A, B, D, E, H, M} $\cup$ {A, B, D, O, R}
= {A, B, D, E, H, M, O, R}

Properties : Following are some rules followed by union operation. We will verify them with the help of illustrations.

(1) Union is a Binary Operation : For any two sets A and B, if $A \subset U$ and $B \subset U$, then $(A \cup B) \subset U$.

Suppose $U = \{x \mid x \in N, 1 \le x \le 6\}$, A = {1, 2, 3}, B = {2, 3, 4, 5}

Here, U = {1, 2, 3, 4, 5, 6}

So, $A \subset U$ and $B \subset U$

Now, $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

Clearly, all members of $A \cup B$ are in U.

So, $(A \cup B) \subset U$. This result says union is a binary operation.

(2) Commutative Law : For any two sets A and B, $A \cup B = B \cup A$.

Let $A = \{c, d, e, f\}$, $B = \{p, q, r, s, t\}$ be any two sets.

Then,

$A \cup B = \{c, d, e, f\} \cup \{p, q, r, s, t\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(i)**

Now, $B \cup A = \{p, q, r, s, t\} \cup \{c, d, e, f\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(ii)**

Thus from (i) and (ii), $A \cup B$ and $B \cup A$ have the same elements.

Therefore, $\mathbf{A \cup B = B \cup A}$

This law is known as commutative law for union, i.e. union is a commutative operation.

(3) Associative Law :

For any three sets A, B and C, $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $A = \{x \mid x \in N, 1 \leq x \leq 5\}$, $B = \{x \mid x \in N, x \text{ is an even number}, x < 10\}$,

$C = \{x \mid x \in N, x \text{ is a multiple of } 3, x < 10\}$

Now, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 6, 9\}$ are given sets.

So, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{2, 4, 6, 8\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

$\therefore$ $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\} \cup \{3, 6, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(i)**

Now, $B \cup C = \{2, 4, 6, 8\} \cup \{3, 6, 9\}$

$= \{2, 3, 4, 6, 8, 9\}$

$\therefore$ $A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{2, 3, 4, 6, 8, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(ii)**

$\therefore$ By results (i) and (ii) we verify that union is associative.

From Venn-diagram 1.2 it can be verified that,

$\mathbf{(A \cup B) \cup C = A \cup (B \cup C)}$

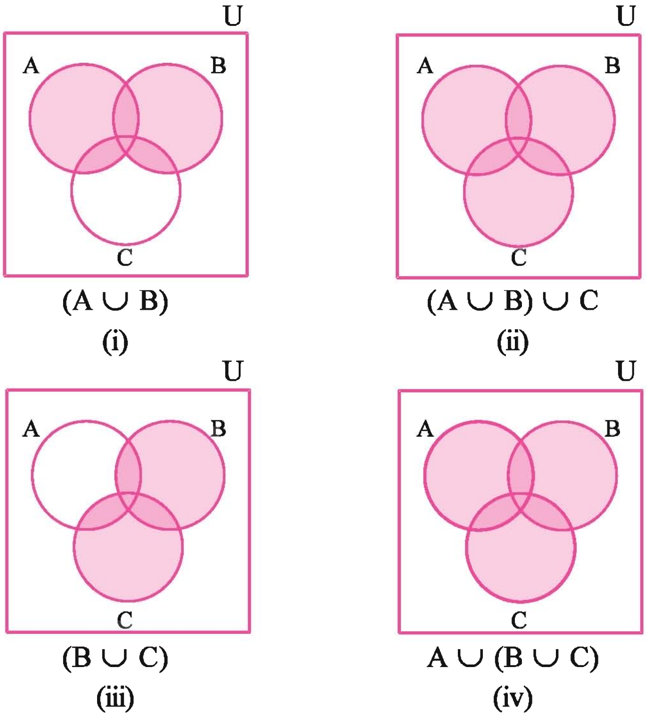


Figure 1.2

In Venn diagram 1.2 coloured region describes the set mentioned below the Venn-diagram.

This law is known as the associative law for union.

i.e. **union is an associative operation.**

(4) For any two sets A and B, $A \subset (A \cup B)$ and $B \subset (A \cup B)$

Let $A = \{x \mid x \in Z, x^2 - 4 = 0\}$, $B = \{x \mid x \in N, x \leq 5\}$ be two sets.

$\therefore$ Here $A = \{-2, 2\}$, $B = \{1, 2, 3, 4, 5\}$

Now $A \cup B = \{-2, 2\} \cup \{1, 2, 3, 4, 5\}$

$= \{-2, 1, 2, 3, 4, 5\}$

Clearly $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

Look at the Venn-diagram 1.3. Here A and B are two sets. The set A consists of the regions R_1 and R_2; the set B consists of the regions R_2 and R_3. So, $A \cup B$ consists of the regions R_1, R_2 and R_3. Thus, the regions R_1 and R_2 are included in the regions R_1, R_2 and R_3 together. i.e. the set A is contained in the set $A \cup B$, so $A \subset (A \cup B)$.

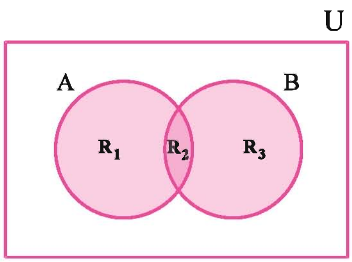


Figure 1.3

Similarly the regions R_2 and R_3 are included in the regions R_1, R_2 and R_3 together. That means the set B is contained in the set A ∪ B. i.e. B ⊂ (A ∪ B)

In general, **for any two sets A and B, A ⊂ (A ∪ B) and B ⊂ (A ∪ B).**

(5) If A ⊂ B, then A ∪ B = B

Let us try to understand the above property by following example.

Example 3 : If α = The set of the letters of the word GATE and β = The set of the letters of the word LOCGATE are two sets, then verify that α ⊂ β and α ∪ β = β

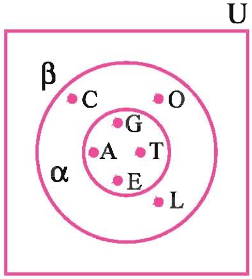


Figure 1.4

Solution : Here α = {G, A, T, E},

β = {L, O, C, G, A, T, E}

∴ Here all the elements of the set α are present in β.

∴ α ⊂ β

Now, α ∪ β = {G, A, T, E} ∪ {L, O, C, G, A, T, E}

= {L, O, C, G, A, T, E}

∴ α ∪ β = β

(6) A ∪ U = U and A ∪ ∅ = A

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} be the universal set and

A = {x | x ∈ N, x is a prime number less than 10}

be a given set

∴ A = {2, 3, 5, 7}

Thus, A ∪ U = {2, 3, 5, 7} ∪ {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} = U

A ∪ ∅ = {2, 3, 5, 7} ∪ ∅

= {2, 3, 5, 7}

= A

Thus, we say that A ∪ U = U and A ∪ ∅ = A.

1.4 Properties of the Intersection Operation

Now we will study some rules about operation of intersection and verify them with the help of illustrations.

Intersection set : For any two sets A and B, the set consisting of all the elements which belong to both the sets A and B is called the intersection set of two sets A and B and it is denoted by A ∩ B.

Thus, in symbols $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

In Venn-diagram 1.5 the coloured region describes $A \cap B$.

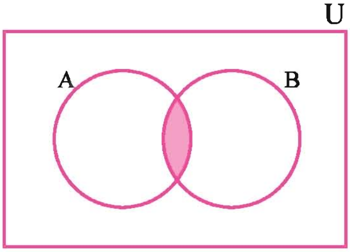


Figure 1.5

Example 4 : Let $A = \{x \mid x \in N, x \text{ is a multiple of } 3, x \leq 15\}$

$B = \{x \mid x \in Z, 0 < x < 10\}$ be two sets.

Find $A \cap B$.

Solution :

Here $A = \{x \mid x \in N, x \text{ is a multiple of } 3, x \leq 15\}$

$= \{3, 6, 9, 12, 15\}$

$B = \{x \in Z, 0 < x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\therefore \quad A \cap B = \{3, 6, 9, 12, 15\} \cap \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{3, 6, 9\}$

Let us verify some properties of intersection by examples.

Properties :

(1) Intersection is a Binary Operation : For two sets A and B, if $A \subset U$, $B \subset U$, then $(A \cap B) \subset U$.

Suppose $U = \{x \mid x \in N, 1 \leq x \leq 25\}$, $A = \{1, 4, 9, 16, 25\}$

$B = \{4, 8, 12, 16, 20\}$

Since $U = \{1, 2, 3, \ldots, 25\}$, $A \subset U$, $B \subset U$

Now $A \cap B = \{1, 4, 9, 16, 25\} \cap \{4, 8, 12, 16, 20\} = \{4, 16\}$

Clearly, each member of $A \cap B$ is in U.

$\therefore \quad (A \cap B) \subset U$

So, intersection is a binary operation.

(2) Commutative Law : For any two sets A and B, $A \cap B = B \cap A$

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 6, 8\}$ be any two sets.

Then $A \cap B = \{1, 2, 3, 4, 5\} \cap \{3, 4, 6, 8\} = \{3, 4\}$ **(i)**

and $B \cap A = \{3, 4, 6, 8\} \cap \{1, 2, 3, 4, 5\}$

$= \{3, 4\}$ **(ii)**

Thus, $A \cap B = B \cap A$ **(from (i) and (ii))**

This law is known as commutative law for intersection i.e. intersection is a commutative operation.

(3) Associative Law : For any three sets A, B and C,

$(A \cap B) \cap C = A \cap (B \cap C)$

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$, $C = \{1, 4, 7\}$

$\therefore \quad A \cap B = \{3, 4, 5\}$

$\therefore \quad (A \cap B) \cap C = \{4\}$ (i)

$B \cap C = \{4\}$

$\therefore \quad A \cap (B \cap C) = \{4\}$ (ii)

Thus, (A ∩ B) ∩ C = A ∩ (B ∩ C) **((i) and (ii))**

Now, Let us verify the law with the help of Venn-diagram.

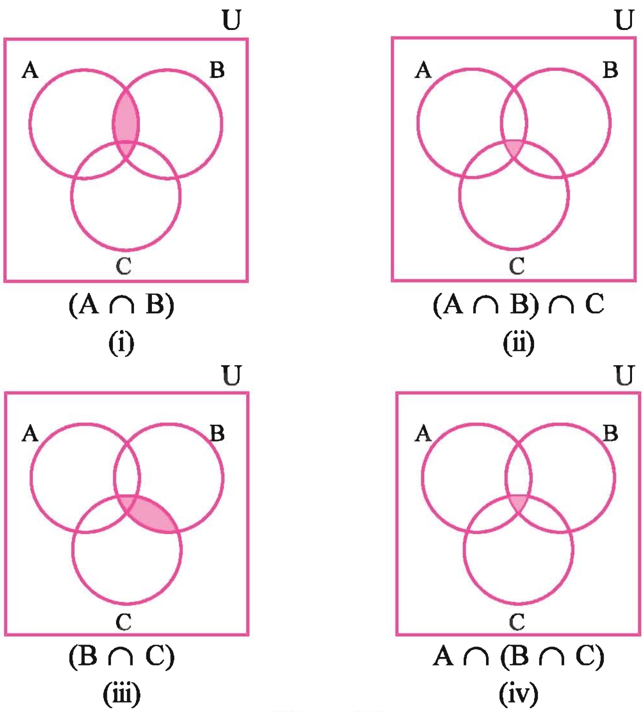


Figure 1.6

In Venn-diagram 1.6, coloured region describes the set mentioned below the Venn-diagram.

It can be seen from the Venn-diagram 1.6 that, (A ∩ B) ∩ C = A ∩ (B ∩ C)

In general, **for any three sets A, B, and C, (A ∩ B) ∩ C = A ∩ (B ∩ C)**

This rule is known as the associative law for the operation of intersection.

(4) (A ∩ B) ⊂ A and (A ∩ B) ⊂ B.

All the elements of A ∩ B belong to the sets A and B.

Hence (A ∩ B) ⊂ A and (A ∩ B) ⊂ B

Look at the Venn-diagram 1.7. The set A consists of the regions R_1 and R_2. The set B consists of the regions R_2 and R_3. The region R_2 is common to both A and B. Thus, A ∩ B consists of the region R_2. The region R_2 is contained in A as well as in B. Thus, (A ∩ B) ⊂ A and (A ∩ B) ⊂ B.

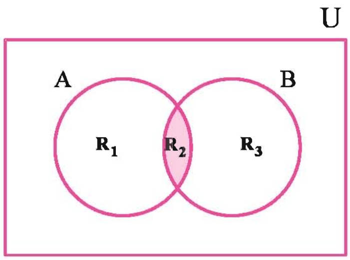


Figure 1.7

(5) If A ⊂ B, then A ∩ B = A

Let us verify this property by an example.

Example 5 : Let A = $\{x \mid x \in N, x^2 - 9 = 0\}$ and B = $\{x \mid x \in N, x < 5\}$ be two given sets. Verify that $A \subset B$ and $A \cap B = A$

Solution : Here $x^2 - 9 = 0$

$\therefore \quad x^2 - 3^2 = 0$

$\therefore \quad (x + 3)(x - 3) = 0$

$\therefore \quad x = -3 \quad$ or $\quad x = 3$

as $x \in N$, $x = -3$ is not possible.

$\therefore \quad x = 3$

Hence A = {3} **(i)**

Now, B = $\{x \mid x \in N, x < 5\}$

= {1, 2, 3, 4} **(ii)**

Hence by the results (i) and (ii), we can say that $A \subset B$.

Now, $A \cap B = \{3\} \cap \{1, 2, 3, 4\} = \{3\} = A$

$\therefore$ If $A \subset B$, then $A \cap B = A$

Similarly If $B \subset A$, then $A \cap B = B$. (Verify it by yourself)

(6) $A \cap \emptyset = \emptyset$ and $A \cap U = A$

Let U = {1, 2, 3, 4, 5}, A = {2, 3}

Then obviously $A \cap U = \{2, 3\} = A$ and $A \cap \emptyset = \emptyset$

Disjoint sets : For any two non-empty sets A and B, if $A \cap B = \emptyset$ then the sets A and B are said to be disjoint

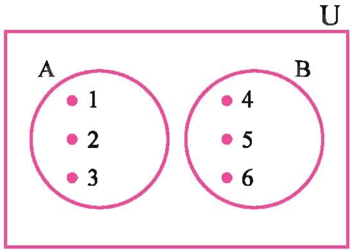
Example 6 : If A = {1, 2, 3}, B = $\{x \mid x \in N, 3 < x < 7\}$ are two sets, are they disjoint ?

Solution : Here B = {4, 5, 6}

Hence $A \cap B = \{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset$

So there is no element common to both A and B. Hence we say that A and B are disjoint sets.

By Venn-diagram 1.8 we can also understand the above definition very easily.



$A \cap B = \emptyset$

Figure 1.8

1.5 Distributive Laws

We are familiar with the distributive law of multiplication over addition for real numbers.

For all $a, b, c \in R$, $\quad a \times (b + c) = (a \times b) + (a \times c)$

For example if $a = 3$, $b = 4$, $c = 5$, then

L.H.S. = $a \times (b + c) = 3 \times (4 + 5) = 3 \times 9 = 27$

R.H.S. = $(a \times b) + (a \times c) = (3 \times 4) + (3 \times 5) = 12 + 15 = 27$

$\therefore$ L.H.S. = R.H.S.

But the converse does not hold. Thus, the distributive law of addition over multiplication is not satisfied.

i.e. $a + (b \times c) \neq (a + b) \times (a + c)$. For set operations, the situation is different. Union is distributive over intersection and intersection is distributive over union.

(1) Distributivity of Union over Intersection : For any three sets A, B and C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $A = \{p, q, r, s\}$, $B = \{q, r\}$, $C = \{r, s, t\}$ are three sets.

Taking $B \cap C = \{q, r\} \cap \{r, s, t\} = \{r\}$

$\therefore$ L.H.S. $= A \cup (B \cap C) = \{p, q, r, s\} \cup \{r\} = \{p, q, r, s\}$ **(i)**

Now $A \cup B = \{p, q, r, s\} \cup \{q, r\} = \{p, q, r, s\}$

$A \cup C = \{p, q, r, s\} \cup \{r, s, t\} = \{p, q, r, s, t\}$

$\therefore$ R.H.S. $= (A \cup B) \cap (A \cup C) = \{p, q, r, s\} \cap \{p, q, r, s, t\}$

$= \{p, q, r, s\}$ **(ii)**

Thus, from (i) and (ii), it is clear that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

This law can be verified using Venn-diagram as follows :

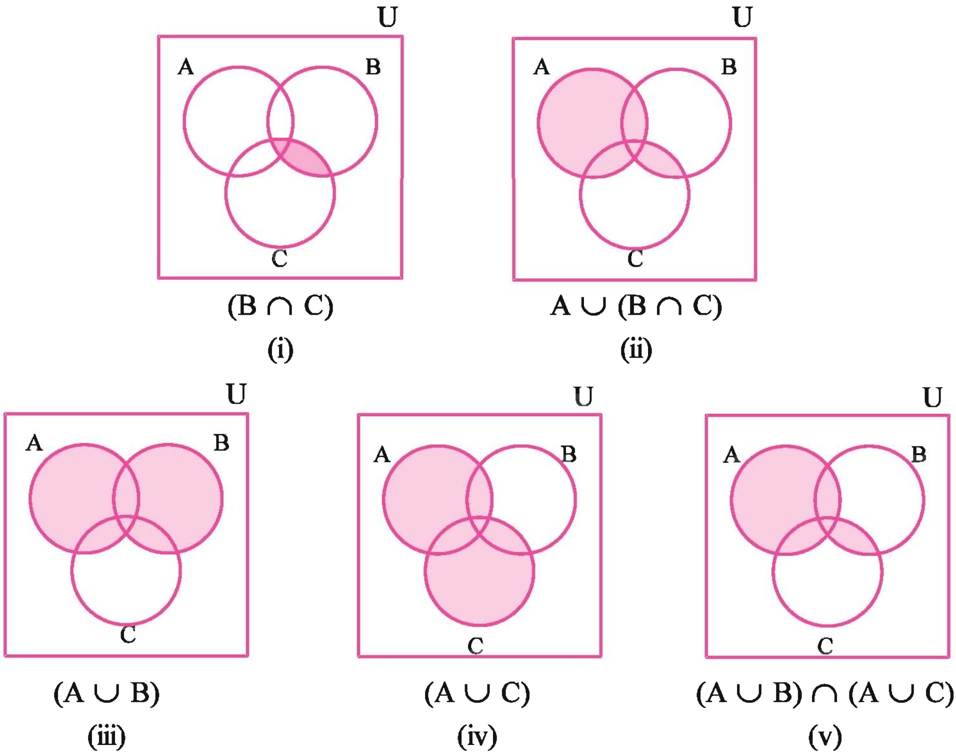


Figure 1.9

In Venn-diagram 1.9, coloured region describes the set mentioned below the Venn-diagram.

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

In words; **Union is distributive over intersection**

(2) Distributivity of Intersection over Union :

For any three sets A, B and C, we have

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let A = {1, 2, 3, 4}, B = {2, 3, 4}, C = {3, 4, 5, 6} be three sets. Then $B \cup C = \{2, 3, 4\} \cup \{3, 4, 5, 6\} = \{2, 3, 4, 5, 6\}$

Now $A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5, 6\} = \{2, 3, 4\}$ **(i)**

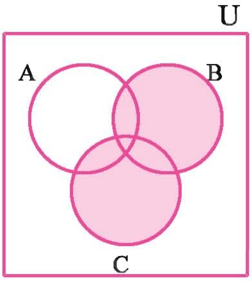
$A \cap B = \{1, 2, 3, 4\} \cap \{2, 3, 4\} = \{2, 3, 4\}$ and

$A \cap C = \{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}$, we get

$(A \cap B) \cup (A \cap C) = \{2, 3, 4\} \cup \{3, 4\} = \{2, 3, 4\}$ **(ii)**

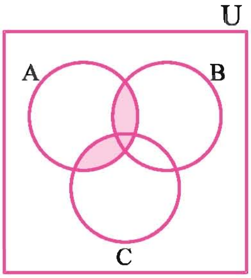
Thus, by (i) and (ii), **the distributivity of intersection over union is verified.**

We can also verify this by following Venn-diagrams :



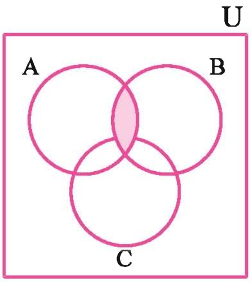
$(B \cup C)$

(i)



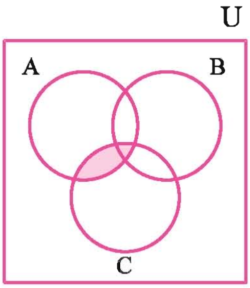
$A \cap (B \cup C)$

(ii)



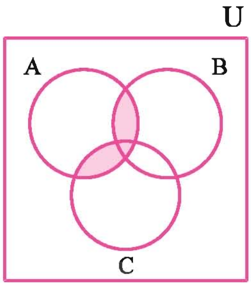
$(A \cap B)$

(iii)



$(A \cap C)$

(iv)



$(A \cap B) \cup (A \cap C)$

(v)

Figure 1.10

In Venn-diagram 1.10, coloured region describes the set mentioned below the Venn-diagram.

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

In general, for any three sets A, B and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

In words, **intersection is distributive over union.**

EXERCISE 1.2

1. Verify that $A \cup B = B \cup A$ for the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.
2. If $A = \{x \mid x \in N, x$ is a factor of $12\}$ and $B = \{x \mid x \in N, 2 < x < 7\}$, then find $A \cap B$.
3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 3, 4, 5\}$, $B = \{4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then verify the distributivity of union over intersection.
4. If $A = \{x \mid x \in N, x$ is a prime factor of $12\}$ and $B = \{x \mid x \in N, x$ is a prime factor of $20\}$ are given sets, then find $A \cap B$.
5. Let $A = \{x \mid x \in N, x < 10\}$, $B = \{x \mid x \in N, x$ is a multiple of 3; x less then $15\}$, $C = \{x \mid x \in Z, -4 < x < 4\}$ be three sets, then verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
6. If $A = \{1, 2, 3, 4\}$, $B = \{x \mid x \in N, 4 \le x \le 6\}$ are given sets, then find $A \cap B$. Are they disjoint sets ?

*

1.6 Properties of Complement of a Set

Complement of a set : The set consisting of all the elements of U which are not in the given set A is called the complement of a set A. It is denoted by A'. The process of finding the complement of a set is called complementation.

Thus, in symbols, $A' = \{x \mid x \in U, x \notin A\}$

From the definition, it is clear that,

(1) A member of U which is not in the set A is in set A', and a member of U which is not in A' is in the set A. Thus each member of U is either in A or in A'. So, $A \cup A' = U$.

(2) A member of U which is in A, is not in A', and a member of U which is in A', is not in A. Thus A and A' have no common members. This means that $A \cap A' = \emptyset$.

If an element of U is in A, it cannot be in A' and hence it must be in (A')' and vice-versa. Thus, $(A')' = A$.

Let us understand the above results by the following example.

Example 7 : Let $U = \{-1, 0, 1, 2, 3, 4, 5\}$ be the universal set and $A = \{-1, 0, 1\}$ is a given set. Then verify

(1) $A \cup A' = U$ (2) $A \cap A' = \emptyset$ (3) $(A')' = A$

Solution : Here A = {−1, 0, 1}. U = {−1, 0, 1, 2, 3, 4, 5}

∴ A' = {2, 3, 4, 5}

(1) Now, $A \cup A'$ = {−1, 0, 1} ∪ {2, 3, 4, 5}

= {−1, 0, 1, 2, 3, 4, 5} = U

(2) $A \cap A'$ = {−1, 0, 1} ∩ {2, 3, 4, 5} = ∅

because there is no common element. So they are disjoint.

(3) A' = {2, 3, 4, 5}, (A')' = {−1, 0, 1} = A

According to the definition, the members of U which are not in A are in A'. But there is no member of U which is not in U. So there is no member in U'. So U' is the null set. i.e. U' = ∅.

Similarly, **∅' consists of all members of U, which are not in ∅. But there is no member in ∅. Therefore ∅' = U.**

1.7 De Morgan's Laws

For $A \subset U$ and $B \subset U$ we have the following results which are known as De Morgan's laws

(i) $(A \cap B)' = A' \cup B'$ (ii) $(A \cup B)' = A' \cap B'$.

By Venn-diagram, we can verify the above laws.

Verification of $(A \cup B)' = A' \cap B'$

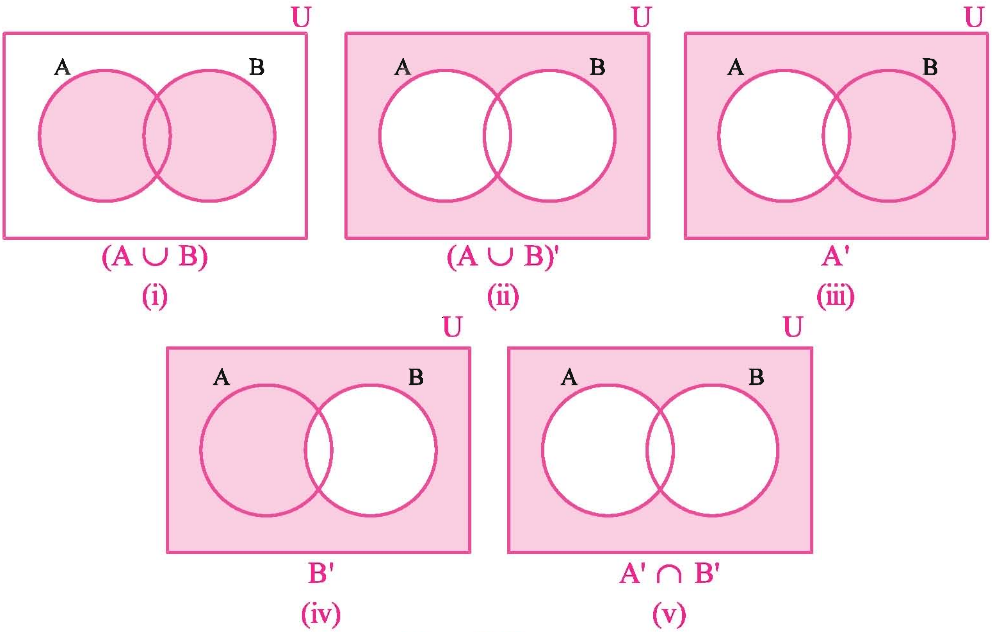


Figure 1.11

In Venn-diagram 1.11, coloured region describes the set mentioned below the Venn-diagram.

From the Venn-diagram 1.11, it is clear that $(A \cup B)' = A' \cap B'$

Similarly, we can verify the other law $(A \cap B)' = A' \cup B'$ by Venn-diagram (do it by yourself).

EXERCISE 1.3

1. If $U = \{x \mid x \in N, x < 10\}$, $A = \{2x \mid x \in N, x < 5\}$ and $B = \{1, 3, 4, 5\}$, then find $(A \cup B)'$ and $(A \cap B)'$.
2. If $U = \{a, b, c, d, e, f, g, h\}$, $P = \{b, c, d, e, f\}$, $Q = \{a, c, d, e, g\}$, then verify De Morgan's laws.
3. Let $U = N$. If $B \subset A$, then find $A' \cap B'$ and A'. What do you conclude ?
4. If $A = \{x \mid x \in Z, x^3 = x\}$, $B = \{x \mid x \in Z, x^2 = x\}$, $C = \{x \mid x \in N, x^2 = x\}$, then considering $U = \{-1, 0, 1, 2\}$, verify the following results

   (1) $(B \cup C)' = B' \cap C'$ (2) $(C')' = C$ (3) $(B \cap C)' = B' \cup C'$
5. If $U = \{x \mid x \in N, (x + 1)^2 < 40\}$, $A = \{x \mid x \in N, x < 4\}$ and $B = \{2x \mid x \in N, x < 3\}$, then find A', B' and verify De Morgan's laws.

EXERCISE 1

1. Examine from the following which are the subsets of the other set ?

   $A = \{-1, 5\}$, $B = \{-2, -1, 0, 1, 2\}$, $C = \{2, 4, 6, 8, 10, 12\}$

   $D = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$
2. If $A = \{x \mid x \in Z, 4 \leq (x + 1)^2 < 25\}$, $B = \{-2, -1, 0, 1, 2\}$, then find $A \cup B$ and $A \cap B$.
3. If $A \subset B$, then can A and B be disjoint, when $A \neq \emptyset$, $B \neq \emptyset$? Why ?
4. If $A = \{x \mid x \in N, x \text{ is a factor } 18\}$, $B = \{x \mid x \in N, x \text{ is a multiple of } 3, x < 20\}$ and $U = \{x \mid x \in N, x \leq 20\}$, then verify De Morgan's Laws.
5. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{x \mid x \in N, x < 10\}$, $C = \{3x \mid x \in N, x < 20\}$ and $U = \{1, 2, 3, ... 20\}$, then verify the distributive laws.
6. If $U = \{1, 3, 4\}$, $A = \{1\}$, then find the complement of the set A.
7. If $U = Z$, then find the complements of

   (1) $A = \{2x \mid x \in Z\}$ and (2) $B = \{2x - 1 \mid x \in Z\}$
8. Write all the subsets of the set $A = \{-1, 0, 1, 2\}$
9. If $A = \{6, 8, 10, 12, 14\}$, $B = \{8, 9, 10, 11, 12, 13\}$, $C = \{7, 8, 9, 10, 12, 14\}$, then prove that $(A \cap C) \cup B = (A \cup B) \cap (B \cup C)$.

10. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If U = $\{x \mid x \in N, x < 5\}$, A = $\{x \mid x \in N, x \leq 2\}$ then A' = ... ☐

(a) $\{1, 2\}$ (b) $\{1, 2, 3, 4, 5\}$ (c) $\{3, 4\}$ (d) $\{3, 4, 5\}$

(2) $\emptyset$ ...... $\{\emptyset\}$ ☐

(a) $\subset$ (b) $\notin$ (c) = (d) $\not\subset$

(3) If A = $\{1, 2, 3\}$, B = $\{3, 4, 5\}$, then A $\cup$ B = ... ☐

(a) $\{1, 2, 3, 4, 5\}$ (b) $\{3\}$ (c) $\{1, 2\}$ (d) $\emptyset$

(4) If A = $\{x \mid x \in N, x \leq 7\}$ and B = $\{2, 4, 6\}$, then B ...... A. ☐

(a) = (b) $\subset$ (c) $\not\subset$ (d) $\sim$

(5) If A = $\{1, 2, 3\}$, B = $\{2, 3, 4\}$, C = $\{3, 4, 5\}$, then (A $\cap$ B) $\cap$ C' = ... where U = $\{1, 2, 3, 4, 5\}$ ☐

(a) $\{1\}$ (b) $\{2\}$ (c) $\{1,2\}$ (d) $\{2,3\}$

(6) If A = $\{x \mid x \in N, x \leq 3\}$, B = $\{1, 2, 3\}$, U = N, then A and B are ...... sets. ☐

(a) equal (b) singleton (c) null (d) complements of each other

(7) If A = $\{1, 2, 3, 4\}$ .... is a correct statement. ☐

(a) $3 \notin A$ (b) $\{1\} \in A$ (c) $\{2\} \in A$ (d) $\{3, 4\} \subset A$

(8) If A = $\{1, 2, 3, 4\}$, then number of subsets of A are = ... ☐

(a) 2 (b) 4 (c) 8 (d) 16

(9) ...... is a singleton. ☐

(a) A = $\{x \in R : x^2 - x = 0\}$

(b) B = $\{x \mid x \in N, 2x = 3\}$

(c) C = $\{x \mid x \in R : x^2 = -4\}$

(d) D = $\{x \mid x \in Z, x$ is neither positive nor negative$\}$

(10) If A = $\{0, 1, 2, 4\}$, B = $\{1, 3, 5, 7, 9\}$, C = $\{0, 1, 4, 3, 9\}$, then (A $\cap$ B) $\cup$ C = ... ☐

(a) A (b) B (c) C (d) A $\cup$ B

(11) If A $\cup$ B = $\emptyset$, then ...... ☐

(a) A $\neq \emptyset$ and B $\neq \emptyset$ (b) A = $\emptyset$ and B $\neq \emptyset$

(c) A $\neq \emptyset$ and B = $\emptyset$ (d) A = $\emptyset$ and B = $\emptyset$

(12) If A = $\{x \mid x \in N, x \leq 4\}$, B = $\{-1, 0, 1, 2, 3\}$, C = $\{0, 1, 2\}$, then (A $\cup$ B) $\cap$ (A $\cup$ C) = ... ☐

(a) $\{1, 2, 3, 4\}$ (b) $\{0, 1, 2\}$

(c) $\{0, 1, 2, 3, 4\}$ (d) $\{-1, 0, 1, 2, 3, 4\}$

(13) If A = {1, 2, 3, 4}, B = {2, 4, 5, 6}, U = {1, 2, 3, 4, 5, 6, 7}, then $A' \cap B'$ = ... ☐

(a) $\emptyset$ (b) {1, 2, 3, 4, 5, 6} (c) {7} (d) {3, 4, 5, 6}

(14) $\emptyset \cap U'$ = ... ☐

(a) $\emptyset$ (b) U (c) {U} (d) { $\emptyset$ }

(15) $(A \cap B')'$ = ... ☐

(a) $A \cup B'$ (b) $A' \cup B$ (c) $A \cup B$ (d) $A \cap B$

*

Summary

In this chapter,

1. Revision of set theory learnt in standard 8 is given.
2. Union operation, Intersection operation, Complement operation are defined and their properties are studied.
3. Distributive laws and De-Morgan's laws are explained.

●

Set theory is the branch of mathematics that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics.

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

The language of set theory could be used in the definitions of nearly all mathematical objects, such as functions, and concepts of set theory are integrated throughout the mathematics curriculum. Elementary facts about sets and set membership can be introduced in primary school, along with Venn and Euler diagrams, to study collections of commonplace physical objects. Elementary operations such as set union and intersection can be studied in this context.

CHAPTER 2

NUMBER SYSTEM

Give me a place to stand and I will move the earth. - **Archimedes**

Numbers are the free creation of the human mind. - **R. Dedekind**

2.1 Introduction

In our earlier classes we have learnt about the number line. Let us review it.

We take a line and select one point on it and call it O and associate 0 with it. Now if we move towards the right side of O we get numbers like 1, 2, 3,... . Picking all such numbers and collecting in one bag, can you think about the total number of numbers we collected ? Of course, infinitely many. This collection is denoted by N. Thus, N = {1, 2, 3,...}.

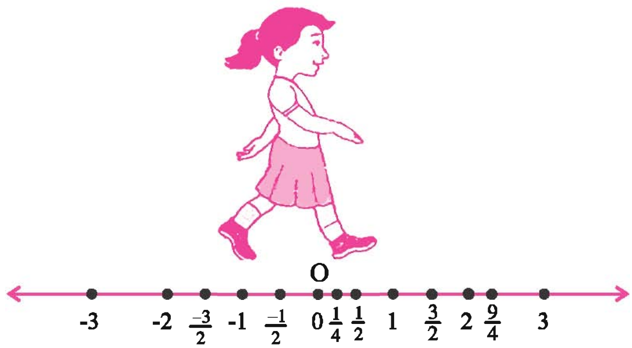


Figure 2.1

Now again, we return to O and if we pick up 0 and put it also in the same bag, this new collection is denoted by W. Thus W is the set of whole numbers.

i.e., W = {0, 1, 2, 3,...}.

We observed that, $N \subset W$.

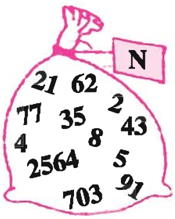


Figure 2.2

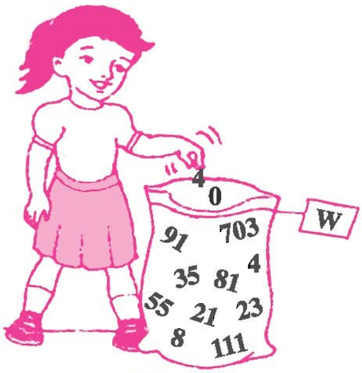


Figure 2.3

Now let us enrich our collection by moving towards the left side of O on the line. What kind of numbers we get ? of course, negative integers −1, −2, −3,.... We add them to the same bag. This collection is denoted by Z. Thus Z is the set of integers.

$Z = \{..., -3, -2, -1, 0, 1, 2, 3,...\}$. $W \subset Z$

Figure 2.4

$\therefore \quad N \subset W \subset Z$

Still there are numbers left on the number line like $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, -\frac{1}{2}$, etc. If we put all such numbers in the same bag, then we have a collection of numbers called **rational numbers** and it is denoted by Q. Rational numbers can be expressed in the form of $\frac{p}{q}$, where $p \in Z$, $q \in N$ and p and q have no common factors other than 1. $Q = \left\{\frac{p}{q} \mid p \in Z, q \in N\right\}$. For any $a \in Z$ we can write 'a' as $a = \frac{a}{1}$ so $a \in Q$. For example, −31 can be written as $\frac{-31}{1}$. Thus all integers are also rational, so $Z \subset Q$. Thus, $N \subset W \subset Z \subset Q$.

Here, 'Rational' comes from the word 'ratio' and Q comes from the word 'quotient'.

We also know that rational numbers do not have a unique representation in the form $\frac{p}{q}$. For example, $\frac{1}{3} = \frac{2}{6} = \frac{5}{15} = \frac{10}{30} = \frac{35}{105}$, and so on. These are equivalent

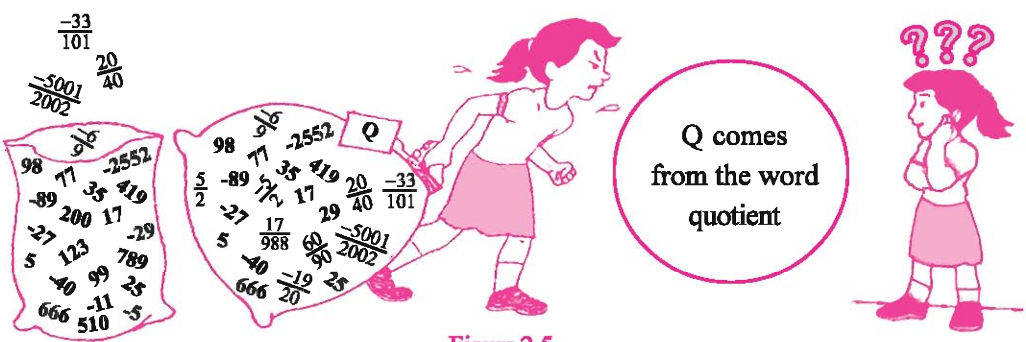


Figure 2.5

rational numbers. If we want to represent a rational number $\frac{p}{q}$ on the number line, then we assume that p and q have no common factors other than 1, i.e. p and q are co-prime. In short $\frac{p}{q}$ should be in its simplest form.

Example 1 : Are the following statements true or false ? Give the reason.

(1) Every whole number is an integer.
(2) Every rational number is a whole number.
(3) Every integer is a rational number.

Solution :

(1) True, because $W \subset Z$
(2) False, because $\frac{3}{4} \in Q$ but $\frac{3}{4} \notin W$.
(3) True, because $Z \subset Q$

Now let us see how to find a rational number or rational numbers between two given rational numbers.

Example 2 : Find four rational numbers between 2 and 3.

Solution : To find a rational number between given numbers 2 and 3, we find their average $\frac{2+3}{2} = \frac{5}{2}$ which is a rational number between 2 and 3. Similarly other rational numbers can be obtained by successive averaging as follows.

$$\frac{\frac{5}{2}+2}{2} = \frac{9}{4}, \quad \frac{\frac{9}{4}+2}{2} = \frac{17}{8}, \quad \frac{\frac{5}{2}+3}{2} = \frac{11}{4}$$

$\frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \frac{11}{4}$ are four rational numbers between 2 and 3. A general method is described below.

Method 1 : Let $a, b \in Q$. We want to find a rational number between a and b. As such there are infinitely many numbers between a and b, but we think of getting one number between a and b in a convenient way. As $a < \frac{a+b}{2} < b$, $\frac{a+b}{2}$ is a rational number between a and b. The same method can be applied for getting more numbers between a and b.

Method 2 : To get n rational numbers between a and b $(a < b)$, let $d = \frac{b-a}{n+1}$, then $a + d, a + 2d, \ldots, a + (n-1)d, a + nd$ are n rational numbers between a and b.

Between two rational numbers there are infinitely many rational numbers. This is an important property of Q. This property is called "The rational numbers exhibit gap".

EXERCISE 2.1

1. Are 3, −5 and −3.5 rational numbers ? If yes, then express them in the form of $\frac{p}{q}$, where $p \in$ Z, $q \in$ N.
2. Find six rational numbers between −2 and 5.
3. Find three rational numbers between 4 and 6.
4. Find two rational numbers between $\frac{2}{7}$ and $\frac{4}{9}$.
5. Find four rational numbers between $\frac{2}{5}$ and $\frac{3}{7}$.

*

2.2 Irrational Numbers

Now, our bag is very heavy, so it seems that no numbers are left out. But no ! there are still many numbers left on the number line. Which are they ? They are called **Irrational Numbers**. So let us pick them up and put in our bag. This collection is denoted by R, the set of real numbers. R includes rational and irrational numbers. Thus $Q \subset R$, i.e. $N \subset W \subset Z \subset Q \subset R$.

Thus **real numbers which are not rational are called irrational numbers.** The set of irrational numbers is denoted by I. $I \cap Q = \emptyset$ and $I \cup Q = R$.

Let us study something more about irrational numbers.

Now we will see how to represent some irrational numbers on number line.

Example 3 : Represent $\sqrt{2}$ on the number line.

Solution : Consider a square OABC (Figure 2.6), with each side having length 1 unit. Here by the Pythagoras theorem OB = $\sqrt{1^2+1^2} = \sqrt{2}$; Place O on the number line in such a way that vertex O coincides with zero and A with 1. (figure 2.7)

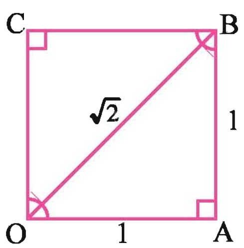


Figure 2.6

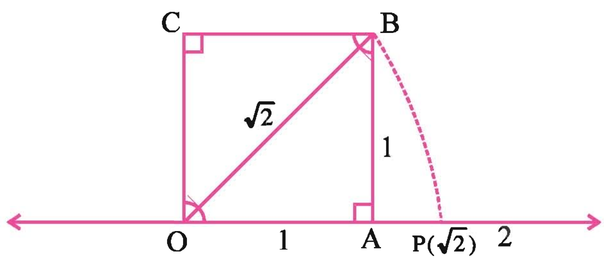


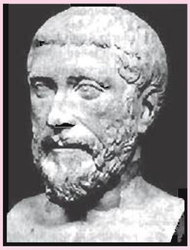
Figure 2.7

We know that OB = $\sqrt{2}$. Draw an arc with center O and radius OB, intersecting the number line at the point P. Then point P corresponds to $\sqrt{2}$ on the number line.

Pythagoras studied properties of numbers which would be familiar to mathematician today. Such as even and odd numbers, triangular numbers and perfect numbers. The theorem known as Pythagoras's theorem was known to the Babylonians one thousand years earlier. He may have been the first to prove it.

Some theorems attributed to Pythagorians are

(1) The sum of the angles of a triangle is equal to two right angles. (2) Constructing the figures of a given area and geometrical algebra e.g. they solve the equation such as $a \cdot (a - x) = x^2$ by geometrical means. (3) 5-regular solids. Pythagoras himself knew how to construct first three. (4) In Astronomy Pythagoras taught that the earth was a sphere at the centre of the universe. (5) He also recognised that the orbit of the moon was inclined to the equator of the earth.



Pythagoras
569 BC - 475 BC

Example 4 : Represent $\sqrt{3}$ on the number line.

There are two methods to locate $\sqrt{3}$ on the number line.

Solution 1 : Let us return to figure 2.7

Construct $\overline{PD}$ perpendicular to $\overline{OP}$ having unit length (figure. 2.8(i))

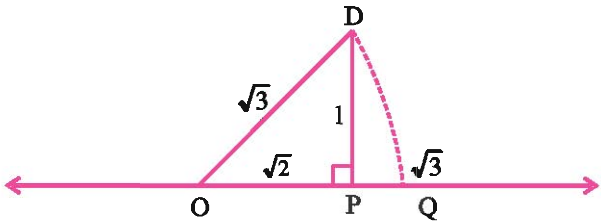


Figure 2.8 (i)

By Pythagoras theorem, OD = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$. Draw an arc with centre O and radius OD, intersecting the number line at the point Q. Then point Q corresponds to $\sqrt{3}$ on the number line.

Solution 2 : Let us return to figure 2.7

Construct $\overline{BD}$ having unit length perpendicular to $\overline{OD}$. (figure 2.8(ii))

By Pythagoras theorem, OB = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$, Draw an arc with centre O and radius OB, intersecting the number line at the point Q. Then point Q corresponds to $\sqrt{3}$ on the number line.

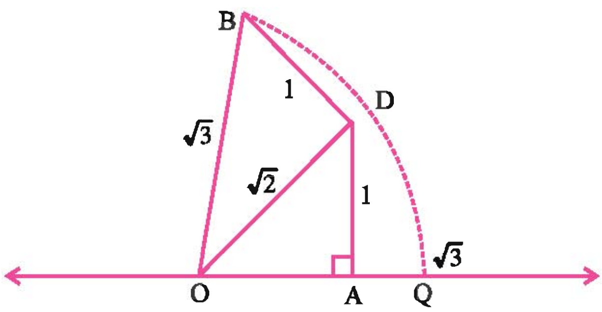
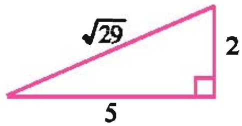


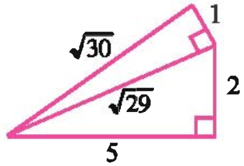
Figure 2.8 (ii)

In the same manner, we can represent $\sqrt{n+1}$ for any positive integer n after $\sqrt{n}$ has been represented or put $n + 1$ as sum of the squares of two or three or four numbers and using Pythagoras theorem we can get $\sqrt{n+1}$.

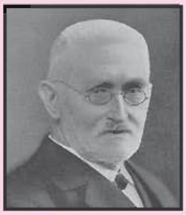
For example : $\sqrt{29} = \sqrt{5^2 + 2^2}$

$\sqrt{30} = \sqrt{5^2 + 2^2 + 1^2}$





Now we can say that, every point on the number line represents a unique real number and every real number represents a unique point on the number line. i.e., there is one-one correspondence between set of real numbers and the set of points on the number line.



R. Dedekind
(1831-1916)

Julius Wilhelm Richard Dedekind

Born : 6th October, 1831 in Braunschweig, duchy of Braunschweig (now Germany).

Died : 12th February, 1916 in Braunschweig, duchy of Braunschweig (now Germany).

- Dedekind did his doctorat work under Gauss' supervision.
- He attended the courses by Dirichlet on the theory of numbers.
- His remarkable piece of work was redefinition of irrational numbers in terms of Dedekind cuts.

EXERCISE 2.2

1. State whether the following statements are true or false. Give reasons for your answer :

(1) Every rational number is a real number.

(2) Every integer is an irrational number.

(3) $\sqrt{4}$ is an irrational number.

(4) There is a real number whose square is -3.

2. Represent $\sqrt{5}$ on the number line.

3. Represent $\sqrt{17}$ on the number line.

*

Classroom Activity : (Construction of the "Square root Spiral") :

We construct square root spiral in the following way.

Take point O and draw $\overline{OA}$ of unit length. Draw $\overline{AB}$ of unit length perpendicular to $\overline{OA}$ (figure 2.9). Now draw $\overline{BC}$ of unit length perpendicular to $\overline{OB}$. Then draw $\overline{CD}$ of unit length perpendicular to $\overline{OC}$. Continuing in this manner, we can get a spiral known as square root spiral. Here OB = $\sqrt{2}$, OC = $\sqrt{3}$, OD = $\sqrt{4}$,...

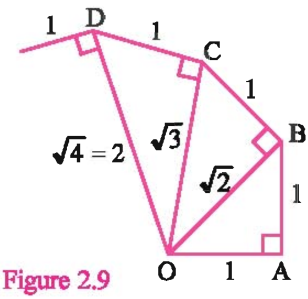


Figure 2.9

2.3 Real Number and Their Decimal Expression

In this section we will obtain the decimal expression of real numbers and using them we will distinguish between rational and irrational numbers.

Let us take some examples of rational numbers; $\frac{1}{3}, \frac{3}{8}, \frac{8}{7}$.

$$\begin{array}{r|l} & 0.333 \\ \hline 3 & 1.0 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 1 \end{array}$$

Remainders : 1, 1, 1

Divisor : 3

$$\begin{array}{r|l} & 0.375 \\ \hline 8 & 3.0 \\ & \underline{24} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{40} \\ & 00 \end{array}$$

Remainders : 6, 4, 0

Divisor : 8

$$\begin{array}{r|l} & 1.142857 \\ \hline 7 & 8 \\ & \underline{7} \\ & 10 \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 1 \end{array}$$

Remainders : 1, 3, 2, 6, 4, 5, 1

Divisor : 7

From the above examples, we observe the following :

(1) The remainder becomes 0 or the remainders start recurring themselves.

(2) Remainders are less than the divisor and they form a recurring string.

(In case of $\frac{1}{3}$, remainder is recurring and it is less than the divisor 3. In case of $\frac{8}{7}$ there are six digits 1, 3, 2, 6, 4, 5 recurring in order and they are less than the divisor 7.)

(3) If the remainder is recurring, then some digit or a group of digits of quotient are also recurring (In case of $\frac{1}{3}$, 3 is recurring and in case of $\frac{8}{7}$ group 142857 is recurring in the quotient).

These observations are true for all the rational number in the form $\frac{p}{q}$. If we divide p by q, then remainder becomes zero or never becomes zero and the digits of remainder are recurring after some stage.

Now we will observe them individually.

(1) The remainder becomes zero.

In the example $\frac{3}{8}$, we have observed that remainder becomes zero after some stage, its decimal expression is $\frac{3}{8} = 0.375$. Few other examples are $\frac{3}{4} = 0.75$, $\frac{73}{25} = 2.92$.

Here the decimal expression terminates or ends after a certain number of steps. These types of decimal expressions are called terminating.

(2) The remainder never becomes zero.

In the example $\frac{1}{3}$ and $\frac{8}{7}$, remainder is recurring after a certain stage and the decimal expression will go on and on. The digits of quotient of such expression are recurring after certain stage. For example in case of $\frac{1}{3}$ = 0.333..., digit 3 repeats and in case of $\frac{8}{7}$ = 1.142857142857142857..., digits 1,4,2,8,5,7 repeat in the same order. These types of expressions are non terminating recurring in the same order. In the decimal expression of $\frac{1}{3}$, digit 3 is repeated in its quotient, So we write $\frac{1}{3}$ as $0.\overline{3}$ or $0.\dot{3}$. Similarly for $\frac{8}{7}$, the digits 1,4,2,8,5,7 are repeating in the same order. So we write it as $\frac{8}{7} = 1.\overline{142857}$ or $1.\dot{1}4285\dot{7}$. Similarly if we have an expression 2.7323232..., then we write it as $2.7\overline{32}$. All these expressions are non-terminating and recurring.

Thus **the decimal expression of rational numbers are either terminating or non-terminating recurring. Conversely, if the decimal expression of a number is either terminating or non-terminating recurring, then the number is a rational number.**

In fact if in $\frac{p}{q}$, $q = 2^m \cdot 5^n$, $m, n \in N \cup \{0\}$, then $\frac{p}{q}$ has terminating expression and not otherwise. (Why ? Can you explain ?)

Let us consider a terminating decimal number.

Example 5 : Show that 2.1321 is a rational number.

Solution 1 : We can write $2.1321 = \frac{21321}{10000}$ and it is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$ and hence 2.1321 is a rational number.

Now let us consider a non terminating recurring decimal expression.

Example 6 : Show that $0.666... = 0.\overline{6}$ can be expressed in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

Solution 1 : Let $x = 0.\overline{6}$

$\therefore \quad x = 0.666...$

$\therefore \quad 10x = 6.666...$

$\therefore \quad 10x = 6 + 0.666...$

$\therefore \quad 10x = 6 + x$

$\therefore \quad 9x = 6$

$\therefore \quad x = \frac{6}{9}$

$\therefore \quad x = \frac{2}{3}$ is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

$\therefore \quad 0.\overline{6}$ represents a rational number.

Note : In fact here 0.666....= 0.6 + 0.06 + 0.006... and this is called an 'infinite geometric series'. If a number is $0.ppp...$ then it is equal to $\frac{p}{9}$.

Similarly, $0.pqpq... = \frac{pq}{99}$; $0.pqrpqr... = \frac{pqr}{999}$ etc... Can you explain $0.\overline{9} = 1$?

Example 7 : Find the $\frac{p}{q}$ form of $2.\overline{237}$.

Solution : Let $x = 2.\overline{237}$

$\therefore \quad x = 2.237237237...$

$\therefore \quad 1000x = 2237.237237...$

$\therefore \quad 1000x = 2235 + 2.237237...$

$\therefore \quad 1000x = 2235 + x$

$\therefore \quad 999x = 2235$

$\therefore \quad x = \frac{2235}{999}$ is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

[**Note :** Straight away $2.\overline{237} = 2 + 0.\overline{237} = 2 + \frac{237}{999} = \frac{2235}{999}$]

Example 8 : Prove that $3.1\overline{23}$ is a rational number and obtain its $\frac{p}{q}$ form.

Solution : Given number $3.1\overline{23}$ is non-terminating recurring, so it is a rational number.

Let $x = 3.1\overline{23}$

$\therefore \quad x = 3.1232323...$

$\therefore \quad 100x = 312.32323...$

$\therefore \quad 100x = 309.2 + 3.12323...$

$\therefore \quad 100x = 309.2 + x$

$\therefore \quad 99x = \frac{3092}{10}$

$\therefore \quad x = \frac{3092}{990} = \frac{1546}{495}$, is in the form $\frac{p}{q}$ where $p \in Z$, $q \in N$.

$\therefore \quad 3.1\overline{23}$ is a rational number.

[**Note :** $x = 3.1 + 0.0232323... = 3.1 + (0.1)(0.232323...)$

$= 3.1 + \left(\frac{1}{10}\right)\left(\frac{23}{99}\right) = \frac{31}{10} + \frac{23}{990}$ etc.]

From above examples we can observe that in example 6, the number is $0.\overline{6}$, here one digit is repeated, so we have multiplied the number by 10. Similarly in example 7, the number is $2.\overline{237}$. Here three digits are repeated. So we have multiplied it by 10^3. In general, if number of digits repeated in the given number is n, then we multiply the number by 10^n, $n \in N$.

Express $0.\overline{9}$ in the form $\frac{p}{q}$. What is your answer ? Why is it so ?

Thus, a number is rational if and only if its decimal expression is terminating or non-terminating recurring. So obviously, a number is an irrational number if and only if its decimal expression is non-terminating and non-recurring.

A decimal expression like 0.303303330... is non-terminating and non-recurring and so it is an irrational number. We can write infinitely many such irrational numbers.

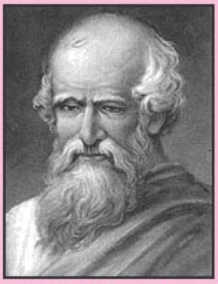
The decimal expression of $\sqrt{2}$, $\sqrt{3}$ can be obtained by the division method. Such square roots are non-terminating and non-recurring.

Let us look at the decimal expression of irrational numbers $\sqrt{2}$ and π.

$\sqrt{2} = 1.414213562373...$

$\pi = 3.14159265358979323846264338327950...$

- His famous work is measurement of the circle.
- He got value of *pi* between two fractions $3\frac{10}{71}$ and $3\frac{1}{7} = \frac{22}{7}$.
- He got his information by inscribing and circumscribing a circle with a 96-sided regular polygon.
- He proved that volume of an inscribed sphere is $\frac{2}{3}$rd the volume of a circumscribed cylinder.
- He requested that this formula be inscribed on his tomb.
- He discovered density and specific gravity.
- He invented the machine called Archimedes Screw, which is a mechanical water pump.



Archimedes
(287 BC - 212 BC)

Note : In calculation of area, volume, etc, we take π as $\frac{22}{7}$ but π is irrational while $\frac{22}{7}$ is rational, so $\pi \neq \frac{22}{7}$. But $\frac{22}{7}$ or 3.14 are approximate values of π. (Which of 3.14 or $\frac{22}{7}$ is nearer to π ?)

Now we will learn how to obtain an irrational number between two rational numbers.

Example 9 : Find an irrational number between $\frac{2}{7}$ and $\frac{3}{7}$.

Solution : First of all we will find the decimal expression of $\frac{1}{7}$.

$$
\begin{array}{r|l}
 & 0.142857 \\
\hline
7 & 1.0 \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{35} \\
 & 50 \\
 & \underline{49} \\
 & 1
\end{array}
$$

$\therefore \quad \frac{1}{7} = 0.\overline{142857}$. To find $\frac{2}{7}$ we multiply $0.\overline{142857}$ by 2.

So we will have $\frac{2}{7} = 0.\overline{285714}$. Similarly $\frac{3}{7} = 0.\overline{428571}$

Now to find irrational number between $\frac{2}{7}$ and $\frac{3}{7}$, we will write one number which is non - terminating and non - recurring between their expressions. 0.350350035000... is one such required number.

Example 10 : Find three different irrational numbers between $\frac{4}{7}$ and $\frac{8}{11}$.

Solution : First of all we will find the decimal expression of $\frac{1}{7}$ and $\frac{8}{11}$.

$\frac{1}{7} = 0.\overline{142857}$ (from example 9) and

$\therefore \quad \frac{1}{11} = 0.\overline{09}$

$$\begin{array}{r|l} & 0.09 \\ \hline 11 & 1.00 \\ & \underline{\ \ 99} \\ & \ \ \ 1 \end{array}$$

$\therefore \quad \frac{4}{7} = 0.\overline{571428}$ and $\frac{8}{11} = 0.\overline{72}$

$\therefore$ The different irrational numbers between $\frac{4}{7}$ and $\frac{8}{11}$ are,

0.590590059000..., 0.606606660...

(Write one more irrational number by yourself.)

Note : we can obtain infinitely many such irrational numbers between any two given rational numbers.

EXERCISE 2.3

1. Classify the following numbers as rational or irrational.

(1) $\sqrt{25}$ (2) $\sqrt{331}$ (3) 0.41757575... (4) 7.808808880... (5) $\frac{\pi}{7}$ (6) $0.\overline{98}$

2. Convert following rational numbers in decimal form and state the kind of its decimal expression.

(1) $\frac{43}{1000}$ (2) $\frac{33}{5}$ (3) $\frac{5}{6}$ (4) $1\frac{2}{7}$ (5) $\frac{157}{300}$ (6) $\frac{14}{11}$

3. Using $\frac{16}{99} = 0.\overline{16}$, obtain decimal form of $\frac{32}{99}$ and $\frac{80}{99}$.

4. Using $\frac{1}{7} = 0.\overline{142857}$, obtain decimal form of $\frac{3}{7}$ and $\frac{5}{7}$.

5. Express the following in the form of $\frac{p}{q}$, where $p \in Z$, $q \in N$.

(1) $0.\overline{23}$ (2) $0.\overline{1437}$ (3) $3.\overline{47}$

6. Find three different irrational numbers between the rational numbers $\frac{3}{5}$ and $\frac{5}{8}$.

*

2.4 Representing Real Numbers on the Number Line

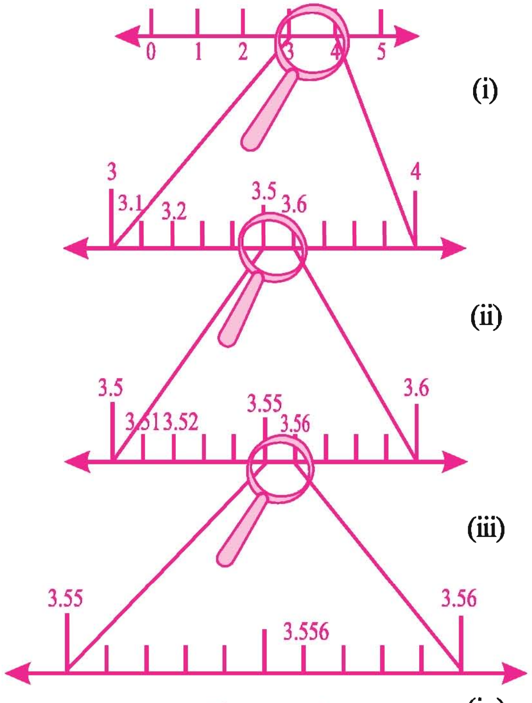


Figure 2.10

In section 2.3 we have seen that we can express any real number in a decimal expression form. After writing the real number into decimal expression form, it is easy to represent it on the number line.

Suppose we want to represent 3.556 on the number line.

We know that 3.556 lies between 3 and 4, so we look for portion between 3 and 4 (Fig. 2.10(i)) on the number line. We divide this portion in 10 equal parts and we mark them. The first mark on the right of 3 represents 3.1, the second 3.2,... as shown in figure 2.10(ii).

Now the number 3.556 lies between 3.5 and 3.6 (figure 2.10(iii)) we magnify this portion by magnifying glass and again we divide the portion between 3.5 and 3.6 in 10 equal parts and mark them. Then the first mark on the right of 3.5 represents 3.51, the second 3.52,... as shown in figure 2.10(iii).

Now we know that number 3.556 lies between 3.55 and 3.56, we visualize this portion on the number line, we magnify it and again we divide it in 10 equal parts. The first mark nearer to 3.55 is 3.551, the second 3.552,... the sixth mark is 3.556 (figure 2.10(iv)). In this way we can locate 3.556 on the number line. This method of representing a real number on the number line by magnifying the portion with magnifying glass is known as the **process of successive magnification.**

Thus the real number with a terminating decimal expression can be represented on the number line by successive magnification.

Now we will take one example to visualize the portion of a real number with a non-terminating recurring decimal and will represent it on the number line by the process of successive magnification method.

Example 11 : Represent $4.2\overline{3}$ on the number line up to 4 decimal places, i.e. up to 4.2333.

Solution : We proceed by successive magnification. We know that $4.2\overline{3}$ lies between 4 and 5. Then we locate $4.2\overline{3}$ between 4.2 and 4.3 (figure 2.11(i)) on the number line. We divide this portion in 10 equal parts and we mark them. The first mark on the right of 4.2 represents 4.21, the second 4.22,... as shown in as shown in figure 2.11(ii).

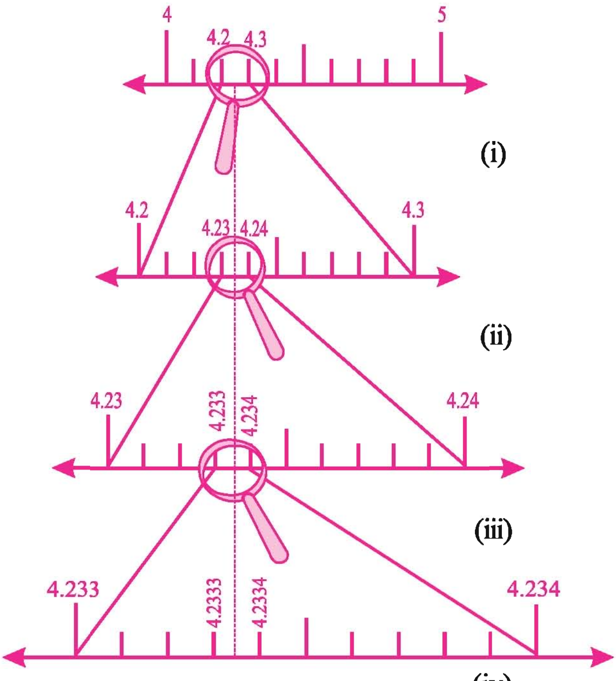


Figure 2.11

Now the number $4.2\overline{3}$ lies between 4.23 and 4.24 (figure 2.11(ii)). We magnify this portion by magnifying glass and again we divide the portion between 4.23 and 4.24 in 10 equal parts and will mark them. Then the first mark on the right of 4.23 represents 4.231, the second 4.232,... as shown in figure 2.11 (iii).

Now we know that number $4.2\overline{3}$ lies between 4.233 and 4.234, we visualize this portion on the number line, we magnify it and again we divide it in 10 equal parts. The first mark nearer to 4.233 is 4.2331, the second 4.2332, the third mark is 4.2333 (figure 2.11(iv)). In this way we can locate 4.2333 on the number line. We notice that $4.2\overline{3}$ is closer to 4.2333 than 4.2334.

We can proceed in this manner endlessly. As we proceed step by step, the length between two consecutive marks decreases and we can be closer and closer to the given number. We visualize it by the magnifying glass. Thus we can locate the number more accurately whose decimal expression is non-terminating recurring.

To visualize a real number on number line whose decimal expression is non-terminating non-recurring, we adopt the same procedure of successive magnification.

Thus, **corresponding to every real number we get a unique point on the number line and conversely corresponding to every point of the number line we get a unique real number. This line is called the real number line.**

Activity

1. Represent 4.572 on the number line, using successive magnification.
2. Represent $1.\overline{3}$ on the number line, using successive magnification, up to 3 decimal places.

*

2.5 Operation on Real Number

We have learnt in our earlier classes that for the set of rational numbers Q, the commutative laws, associative laws for addition and multiplication, distributive law of multiplication over addition are valid.

Now we note one important property of Q. The sum of rational numbers is a rational number. This property is called the closure property for addition for Q. It can also be said that Q is closed under addition. Similarly Q is also closed under subtraction, multiplication and division (when divisor is non-zero.)

Irrational numbers also satisfy commutative and associative laws for addition and multiplication and distributive law for multiplication over addition are valid. But the sum, difference, product and quotient of irrational numbers may not always be irrational.

For example : Results of $\sqrt{5} + (-\sqrt{5})$, $\sqrt{3} - \sqrt{3}$, $(2\sqrt{2}) \times (3\sqrt{2})$, $\frac{5\sqrt{7}}{2\sqrt{7}}$ are rationals.

Let us think about addition of an irrational number and a rational number, e.g. $\sqrt{2} + 2$ is irrational, and $\sqrt{2} \times 3 = 3\sqrt{2}$ is also an irrational. Because, decimal expression of $\sqrt{2}$ is non-terminating non-recurring. So $2 + \sqrt{2}$ and $3\sqrt{2}$ have also non-terminating non-recurring decimal expression.

*n*th Root of Positive Real Numbers

We know that $2^3 = 8$ and the cube-root of 8 is 2. We also write $\sqrt[3]{8} = 2$. Similarly, $3^4 = 81$ and 3 is the fourth root of 81. This is written as $\sqrt[4]{81} = 3$.

Now, $81 = (-3)^4$ also. So while considering the fourth root of 81 the question arises whether to take 3 or -3 ? We also know that the square of any real number is non-negative. Thus -3 cannot be square of any real number. Thus there is no square-root of (-3). Thus the dilemma arises while defining $\sqrt[n]{a}$ where n is even. For positive a, there are two real nth roots of a. For negative a, there is no real nth root. In order to resolve this dilemma, we define the positive nth root of a positive real number a.

If a is a positive real number and $n \in$ N, then there is one and only one positive real number x, so that $x^n = a$ (we accept this). **This number x is called the positive nth root of a and it is expressed as $x = \sqrt[n]{a}$. Also $0^n = 0$ and we take $\sqrt[n]{0} = 0$.**

If $a > 0$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$. We note that $\sqrt[n]{a}$ denotes positive nth root of positive number a. Thus $\sqrt{36} = 6$ and not -6. Although $6^2 = 36$ and $(-6)^2 = 36$ and we say 36 has two square roots 6 and -6, we write $\sqrt{36} = 6$ when using symbols.

$(-4)^3 = -64$ and hence cube root of -64 is -4, but we can not write $\sqrt[3]{-64} = -4$ as the symbol $\sqrt[3]{a}$ is defined for $a > 0$ only.

Thus, $\sqrt{81} = 9$, $\sqrt[3]{27} = 3$, $\sqrt[4]{16} = 2$, $\sqrt[3]{-27}$ is not defined. But cube root of -27 is -3. Square roots of 81 are 9 and -9.

Explanation for $\sqrt{x^2}$: It is wrong to write $\sqrt{x^2} = \pm x$. According to definition, $\sqrt{x^2}$ is that unique non-negative number whose square is x^2.

If $x \geq 0$, x is the non-negative number whose square is x^2.

$\therefore$ If $x \geq 0$, then $\sqrt{x^2} = x = |x|$

If $x < 0$ then $-x > 0$. Also $(-x)^2 = [(-1) \cdot x]^2 = (-1)^2 \cdot x^2 = x^2$.

Thus, $-x$ is the non-negative number whose square is x^2.

$\therefore$ $\sqrt{x^2} = -x = |x|$ as $x < 0$.

$\therefore$ In any case for every $x \in R$, $\sqrt{x^2} = |x|$.

Thus $\sqrt{7^2} = |7| = 7$ and $\sqrt{(-7)^2} = |-7| = 7$. We take $0^n = 0$. ($\sqrt[n]{0} = 0$)

Let $x, y \in R^+$, then (R^+ is the set of all positive real numbers.)

(1) $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ (2) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

(3) $(\sqrt{a} + \sqrt{b})(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay} + \sqrt{bx} + \sqrt{by}$

(4) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$

(5) $(\sqrt{x} \pm \sqrt{y})^2 = x \pm 2\sqrt{xy} + y$

Let us solve examples using above identities for square roots.

Example 12 : Simplify.

(1) $(\sqrt{7} + \sqrt{3})(\sqrt{5} - \sqrt{3})$ (2) $(\sqrt{13} + \sqrt{5})(\sqrt{13} - \sqrt{5})$

(3) $(2 + \sqrt{3})(2 - \sqrt{3})$ (4) $(\sqrt{6} + \sqrt{3})^2$ (5) $(6 - \sqrt{7})^2$

Solution : (1) $(\sqrt{7} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = \sqrt{7 \times 5} - \sqrt{7 \times 3} + \sqrt{3 \times 5} - \sqrt{3 \times 3}$

$= \sqrt{35} - \sqrt{21} + \sqrt{15} - 3$

(2) $(\sqrt{13} + \sqrt{5})(\sqrt{13} - \sqrt{5}) = (\sqrt{13})^2 - (\sqrt{5})^2$

$= 13 - 5$

$= 8$

(3) $(2 + \sqrt{3})(2 - \sqrt{3}) = (2)^2 - (\sqrt{3})^2$

$= 4 - 3$

$= 1$

(4) $(\sqrt{6} + \sqrt{3})^2 = (\sqrt{6})^2 + 2\sqrt{6 \times 3} + (\sqrt{3})^2$

$= 6 + 2\sqrt{18} + 3$

$= 6 + 2\sqrt{9 \times 2} + 3$

$= 9 + 2(3)\sqrt{2} = 9 + 6\sqrt{2}$

(5) $(6 - \sqrt{7})^2 = (6)^2 - 2(6)(\sqrt{7}) + (\sqrt{7})^2$

$= 36 - 12\sqrt{7} + 7$

$= 43 - 12\sqrt{7}$

Example 13 : Add.

(1) $3\sqrt{5} + 2\sqrt{2}$ to $\sqrt{5} - \sqrt{2}$ (2) $3 - 2\sqrt{7}$ to $2 + 2\sqrt{7}$

Solution : (1) $(3\sqrt{5} + 2\sqrt{2}) + (\sqrt{5} - \sqrt{2})$

$= (3\sqrt{5} + \sqrt{5}) + (2\sqrt{2} - \sqrt{2})$

$= (3 + 1)\sqrt{5} + (2 - 1)\sqrt{2}$

$= 4\sqrt{5} + \sqrt{2}$

(2) $(2 + 2\sqrt{7}) + (3 - 2\sqrt{7}) = (2 + 3) + (2\sqrt{7} - 2\sqrt{7})$

$= 5 + 0$

$= 5$

Example 14 : Substract.

(1) $3\sqrt{5} + \sqrt{3}$ from $5\sqrt{5} - 2\sqrt{3}$ (2) $4 - 3\sqrt{3}$ from $2 - 3\sqrt{3}$

Solution : (1) $(5\sqrt{5} - 2\sqrt{3}) - (3\sqrt{5} + \sqrt{3})$

$= (5\sqrt{5} - 3\sqrt{5}) + (-2\sqrt{3} - \sqrt{3})$

$= 2\sqrt{5} - 3\sqrt{3}$

(2) $(2 - 3\sqrt{3}) - (4 - 3\sqrt{3}) = (2 - 4) + (-3\sqrt{3} + 3\sqrt{3})$

$= -2 + 0$

$= -2$

Example 15 : Multiply : (1) $(3 + \sqrt{7}) \times (4 - 2\sqrt{7})$ (2) $2\sqrt{7} \times 5\sqrt{7}$

(3) $(\sqrt{3} + \sqrt{2}) \times (\sqrt{3} - \sqrt{2})$

Solution : (1) $(3 + \sqrt{7}) \times (4 - 2\sqrt{7}) = 3(4 - 2\sqrt{7}) + \sqrt{7}(4 - 2\sqrt{7})$

$= (12 - 6\sqrt{7}) + (4\sqrt{7} - 14)$

$= (12 - 14) + (-6\sqrt{7} + 4\sqrt{7})$

$= -2 - 2\sqrt{7}$

$= -(2 + 2\sqrt{7})$

(2) $(2\sqrt{7}) \times (5\sqrt{7}) = 2 \times 5 \times \sqrt{7} \times \sqrt{7}$

$= 10 \times 7$

$= 70$

(3) $(\sqrt{3} + \sqrt{2}) \times (\sqrt{3} - \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2$

$= 3 - 2$

$= 1$

Example 16 : Divide : (1) $4\sqrt{21}$ by $2\sqrt{7}$ (2) $3\sqrt{11}$ by $6\sqrt{11}$

Solution : (1) $4\sqrt{21} \div 2\sqrt{7} = \frac{4\sqrt{3} \times \sqrt{7}}{2\sqrt{7}}$

$= 2\sqrt{3}$

(2) $3\sqrt{11} \div 6\sqrt{11} = \frac{3\sqrt{11}}{6\sqrt{11}}$

$= \frac{1}{2}$

Above examples lead to the following facts :

(1) The sum or difference or product of a rational number with an irrational number is irrational. (for product rational number must be non-zero.)

(2) Quotient of a non zero rational number by an irrational number is irrational.

(3) Sum, difference, product or division of two irrational numbers may be a rational number or an irrational number.

Now we will see how to represent square roots of real numbers on number line.

We have studied in section 2.2, how to represent $\sqrt{n}$ on the number line for $n \in$ R.

Now we show that, how to represent $\sqrt{a}$ geometrically, when a is a positive real number.

We do one geometric construction to represent $\sqrt{a}$, where $a \in R^+$. ($a > 1$)

Steps of Construction :

(1) Draw $\overrightarrow{AX}$.

(2) Mark B on $\overrightarrow{AX}$ in such a way that AB = a units.

(3) Mark C on $\overrightarrow{AX}$ in such a way that BC = 1 unit.

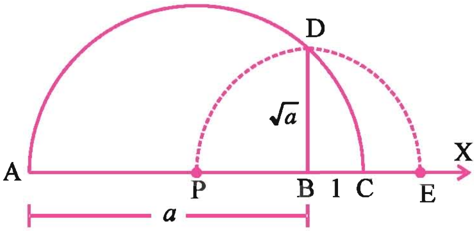


Figure 2.12

(4) Let P be the mid-point of $\overline{AC}$.

(5) Draw a semicircle with centre P and radius AP.

(6) Draw perpendicular at B to $\overline{AC}$ intersecting the semicircle in D.

(7) BD = $\sqrt{a}$

Justification : Radius of semicircle is $\frac{a+1}{2}$ units.

$\therefore$ PC = PD = PA = $\frac{a+1}{2}$ units **(Radii)**

$\therefore$ PB = PC − BC = $\frac{a+1}{2} - 1 = \frac{a-1}{2}$ units

In the right ΔPBD, by Pythagoras' theorem,

$$PD^2 = PB^2 + BD^2$$

$$\therefore \quad \left(\frac{a+1}{2}\right)^2 = \left(\frac{a-1}{2}\right)^2 + BD^2$$

$$\therefore \quad BD^2 = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 = a$$

$$\therefore \quad BD = \sqrt{a}$$

This construction shows us that for all $a \in R^+$, $\sqrt{a}$ exists.

Now to represent $\sqrt{a}$, $a \in R^+$, on the number line, take $\overleftrightarrow{AX}$ as number line and point B corresponding to zero. If we draw an arc with centre B and having radius BD, the arc will intersect $\overrightarrow{BX}$ at some point say E. Then E represents the number $\sqrt{a}$.

Example 17 : Find $\sqrt{2.5}$ geometrically and represent it on the number line.

Draw $\overrightarrow{AX}$. Mark B on $\overrightarrow{AX}$ such that $AB = 2.5$ units. Mark C on $\overrightarrow{BX}$ such that $BC = 1$ unit. Draw perpendicular bisector of $\overline{AC}$ which intersects $\overline{AC}$ at P. Draw a semicircle with centre P and radius AP. Draw perpendicular at B to $\overline{AC}$ intersecting the semicircle at D.

Thus $BD = \sqrt{2.5}$

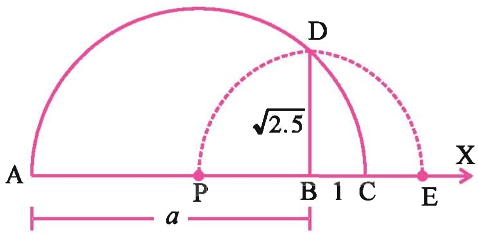


Figure 2.13

Now with B as centre and radius BD, draw an arc which intersects $\overrightarrow{BX}$ at E. Then E represents $\sqrt{2.5}$ on number line. (here B corresponds to zero.)

Now we will extend our idea of square roots to cube roots, fourth roots and so on. We extend it up to *n*th root of a positive real number, where $n \in N$.

In the end let us think of plotting $\frac{1}{\sqrt{3}}$ on the number line. Here the denominator is $\sqrt{3}$, an irrational number. So we can not divide unit length of number line into $\sqrt{3}$ number of equal parts. If the denominator is a rational number 3, then division process is possible. There is one process known as **rationalization** which can make this possible.

Rationalization : If an irrational number is multiplied by some suitable irrational number which can make the product a rational number, then such a process is known as rationalization.

The suitable multiplier irrational number is called a **rationalizing factor** and the given number is said to be **rationalized**. **Rationalization factor is not unique.**

$\sqrt{3} \cdot \sqrt{3} = 3$, $\sqrt{3}(2\sqrt{3}) = 6$...

So, a rationalizing factor of $\sqrt{3}$ could be $\sqrt{3}$, $2\sqrt{3}$, $3\sqrt{3}$, etc.

For example : $\frac{1}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}$

Here denominator $\sqrt{3}$ is rationalized and $\sqrt{3}$ (need not be the same every time) is a rationalizing factor.

Now to represent $\frac{1}{\sqrt{3}}$ on the number line, we represent $\frac{\sqrt{3}}{3}$ on the number line. This number is between 0 and $\sqrt{3}$.

We divide the portion between 0 and $\sqrt{3}$ in three equal parts, the first part near to 0 is the point on the number line which corresponds to number $\frac{1}{\sqrt{3}}$.

Example 18 : Rationalize the denominator of $\frac{4}{\sqrt{5}+\sqrt{2}}$.

Solution : Here we multiply and divide $\frac{4}{\sqrt{5}+\sqrt{2}}$ by $\sqrt{5} - \sqrt{2}$.

$$\frac{4}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = 4\frac{\sqrt{5}-\sqrt{2}}{5-2} = \frac{4}{3}(\sqrt{5} - \sqrt{2}).$$

Example 19 : Rationalize the denominator of $\frac{1}{2-\sqrt{7}}$.

Solution : We get $\frac{1}{2-\sqrt{7}} = \left(\frac{1}{2-\sqrt{7}}\right)\left(\frac{2+\sqrt{7}}{2+\sqrt{7}}\right)$

$$= \frac{2+\sqrt{7}}{4-7} = \frac{2+\sqrt{7}}{-3} = -\frac{1}{3}(2+\sqrt{7})$$

Example 20 : Rationalize the denominator of $\frac{2}{3\sqrt{2}-2\sqrt{3}}$.

Solution : $\frac{2}{3\sqrt{2}-2\sqrt{3}} = \left(\frac{2}{3\sqrt{2}-2\sqrt{3}}\right)\left(\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}\right) = \frac{2(3\sqrt{2}+2\sqrt{3})}{18-12} = \frac{2(3\sqrt{2}+2\sqrt{3})}{6}$

$$= \frac{1}{3}(3\sqrt{2} + 2\sqrt{3})$$

EXERCISE 2.4

1. Classify the following numbers as rational or irrational :

(1) $3 + \sqrt{5}$ (2) $(5 - \sqrt{21}) + (3 + \sqrt{21})$ (3) $(4 - \sqrt{7}) + (4 + \sqrt{7})$

(4) $\frac{1}{\sqrt{2}-1}$ (5) $-\frac{\sqrt{48}}{\sqrt{27}}$ (6) $\frac{\pi+3}{2\pi}$

2. Simplify each of the following expressions :

(1) $(3 - \sqrt{7})(5 + \sqrt{3})$ (2) $(\sqrt{6} + \sqrt{3})^2$

(3) $(\sqrt{18} - \sqrt{5})(\sqrt{2} - \sqrt{15})$ (4) $(1 + \sqrt{8})(1 - 2\sqrt{2})$

(5) $(3 - \sqrt{5})^2$ (6) $(\sqrt{5})^3 - (\sqrt{2})^3$

3. Represent $\sqrt{4.2}$ on the number line.

4. Give a rationalizing factor of the following :

(1) $\frac{1}{\sqrt{8}}$ (2) $\frac{4}{\sqrt{5}}$ (3) $\frac{8+\sqrt{7}}{5}$ (4) $\frac{23}{-\sqrt{3}-\sqrt{2}}$ (5) $4-\sqrt{11}$

5. Rationalize the denominators of the following :

(1) $\frac{3}{\sqrt{15}}$ (2) $\frac{1}{4-\sqrt{7}}$ (3) $\frac{1}{-2-\sqrt{3}}$ (4) $\frac{1}{\sqrt{11}-1}$ (5) $\frac{1}{\sqrt{14}-\sqrt{7}}$

*

2.6 Laws of Exponents for Real Numbers

In earlier classes, we have learnt the following laws of exponents, when the base and exponents are natural numbers.

(1) $a^m \cdot a^n = a^{m+n}$

(2) (i) $\frac{a^m}{a^n} = a^{m-n}$; $m > n$ (ii) $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$; $m < n$ (iii) $\frac{a^m}{a^n} = 1$, $m = n$

(3) $(a^m)^n = a^{mn}$ (4) $(ab)^m = a^m \cdot b^m$ (5) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

We define $a^0 = 1$, $a^{-n} = \frac{1}{a^n}$, $a \in R - \{0\}$, $n \in Z$

Then we extend above laws for negative exponents also.

For example :

(1) $7^8 \times 7^{-11} = 7^{8-11} = 7^{-3}$

(2) (a) $\frac{5^4}{5^7} = 5^{4-7} = 5^{-3}$ (b) $\frac{5^{-4}}{5^7} = 5^{-4-7} = 5^{-11}$

(3) (a) $(3^{-4})^2 = 3^{-4(2)} = 3^{-8}$ (b) $(3^4)^{-2} = 3^{4(-2)} = 3^{-8}$

(4) $3^{-2} \times 7^{-2} = (3 \times 7)^{-2} = 21^{-2}$

(5) $\frac{9^{-4}}{11^{-4}} = \left(\frac{9}{11}\right)^{-4}$

Again we extend the laws of exponents when base is a positive real number and the exponents are rational numbers.

We defined $\sqrt[n]{a}$, where $a \in R^+$ and $n \in N$ as follows.

If $b^n = a$ then $\sqrt[n]{a} = b$, where $b \in R^+$. We will write $\sqrt[n]{a}$ as $a^{\frac{1}{n}}$.

We can perform calculations of $4^{\frac{3}{2}}$ type of examples in two ways.

$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$ or $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8$

We can define $a^{\frac{m}{n}}$ where $a \in R^+$, $m \in Z$, $n \in N$; m, n are co-prime.

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ (Both results seem to be different, but they are same)

If base is a positive real number and exponents are rational numbers, we can write the laws of exponents as follows :

For $a, b \in R^+$ and $p, q \in Q$, we have

(1) $a^p \cdot a^q = a^{p+q}$ (2) $\frac{a^p}{a^q} = a^{p-q}$ (3) $(a^p)^q = a^{pq}$

(4) $(ab)^p = a^p \cdot b^p$ (5) $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

Example 21 : Simplify.

(1) (a) $5^{\frac{3}{4}} \cdot 5^{\frac{1}{4}}$ (b) $7^{\frac{3}{5}} \cdot 7^{\frac{2}{3}}$ (2) (a) $\frac{11^{\frac{1}{2}}}{11^{\frac{5}{2}}}$ (b) $\frac{3^{\frac{4}{5}}}{3^{\frac{1}{3}}}$

(3) (a) $(2^3)^{\frac{1}{5}}$ (b) $(2^{\frac{3}{4}})^{\frac{2}{3}}$ (4) $17^{\frac{1}{7}} \cdot 5^{\frac{1}{7}}$

(5) (a) $\frac{14^{\frac{5}{6}}}{7^{\frac{5}{6}}}$ (b) $\frac{15^{\frac{3}{4}}}{20^{\frac{3}{4}}}$

Solution :

(1) (a) $5^{\frac{3}{4}} \cdot 5^{\frac{1}{4}} = 5^{\left(\frac{3}{4}+\frac{1}{4}\right)} = 5^{\frac{4}{4}} = 5^1 = 5$ (b) $7^{\frac{3}{5}} \cdot 7^{\frac{2}{3}} = 7^{\left(\frac{3}{5}+\frac{2}{3}\right)} = 7^{\frac{19}{15}}$

(2) (a) $\frac{11^{\frac{1}{2}}}{11^{\frac{5}{2}}} = 11^{\left(\frac{1}{2}-\frac{5}{2}\right)} = 11^{-2} = \frac{1}{11^2} = \frac{1}{121}$ (b) $\frac{3^{\frac{4}{5}}}{3^{\frac{1}{3}}} = 3^{\left(\frac{4}{5}-\frac{1}{3}\right)} = 3^{\frac{7}{15}}$

(3) (a) $(2^3)^{\frac{1}{5}} = 2^{3 \times \frac{1}{5}} = 2^{\frac{3}{5}}$ (b) $(2^{\frac{3}{4}})^{\frac{2}{3}} = 2^{\frac{3}{4} \times \frac{2}{3}} = 2^{\frac{1}{2}} = \sqrt{2}$

(4) $17^{\frac{1}{7}} \cdot 5^{\frac{1}{7}} = (17 \times 5)^{\frac{1}{7}} = 85^{\frac{1}{7}}$

(5) (a) $\frac{14^{\frac{5}{6}}}{7^{\frac{5}{6}}} = \left(\frac{14}{7}\right)^{\frac{5}{6}} = 2^{\frac{5}{6}}$ (b) $\frac{15^{\frac{3}{4}}}{20^{\frac{3}{4}}} = \left(\frac{15}{20}\right)^{\frac{3}{4}} = \left(\frac{3}{4}\right)^{\frac{3}{4}}$

EXERCISE 2.5

1. Find : (1) $(225)^{\frac{1}{2}}$ (2) $(81)^{\frac{1}{4}}$ (3) $(625)^{\frac{1}{2}}$ (4) $(64)^{\frac{1}{6}}$

2. Find : (1) $(9)^{\frac{5}{2}}$ (2) $(125)^{\frac{5}{3}}$ (3) $(16)^{-\frac{3}{4}}$ (4) $(243)^{-\frac{3}{5}}$

3. Simplify : (1) $3^{\frac{3}{2}} \cdot 3^{\frac{4}{5}}$ (2) $16^{\frac{4}{3}} \times 4^{\frac{2}{3}}$ (3) $3^{\frac{1}{2}} \cdot 12^{\frac{1}{2}}$ (4) $\frac{5^{\frac{4}{5}}}{(25)^{\frac{3}{2}}}$

EXERCISE 2

1. Find five rational numbers between $\frac{1}{5}$ and $\frac{3}{4}$.
2. Find four rational numbers between $-\frac{1}{7}$ and $-\frac{2}{5}$.
3. Represent $\sqrt{8}$ on the number line.
4. Represent $\sqrt{6}$ on the number line.
5. Convert $\frac{-3}{13}$ and $\frac{15}{4}$ in decimal form and state the kind of its decimal expression.
6. Express in the form of $\frac{p}{q}$; $p \in Z$, $q \in N$: (1) $0.3\overline{2}$ (2) $1.4\overline{73}$ (3) $0.\overline{271}$ (4) $0.\overline{35}$
7. Visualize 5.341 on the number line, using successive magnification. **(Activity)**
8. Visualize $2.\overline{7}$ on the number line using successive magnification up to 3 decimal places. **(Activity)**
9. Simplify :
   (1) $(\sqrt{3} - \sqrt{7})(3 + \sqrt{5})$ (2) $(\sqrt{15} - \sqrt{5})^2$ (3) $(\sqrt{7} + \sqrt{2})(\sqrt{14} - \sqrt{8})$
10. Rationalize the denominator of the following :
    (1) $\frac{1}{\sqrt{3} - \sqrt{15}}$ (2) $\frac{5}{3 + \sqrt{2}}$ (3) $\frac{3}{\sqrt{5} - 2}$ (4) $\frac{1}{-8 - \sqrt{6}}$
11. If $\sqrt[3]{a} \cdot \sqrt{b} = (x)^{\frac{1}{6}}$, then find x. $(a > 0, b > 0)$
12. Prove that $(\sqrt{x} + 1) \cdot (\sqrt[4]{x} + 1) \cdot (\sqrt[8]{x} + 1) \cdot (\sqrt[8]{x} - 1) = x - 1$, $(x \in R^+)$
13. Simplify : $(a^{\frac{1}{2}} \cdot b^{\frac{1}{3}})^{\frac{1}{4}} \cdot (a^{\frac{1}{3}} \cdot b^{\frac{1}{4}})^{\frac{1}{2}} \cdot (a^{\frac{1}{4}} \cdot b^{\frac{1}{2}})^{\frac{1}{3}}$
14. Find the value of : $\frac{(81)^{\frac{1}{4}}}{(625)^{\frac{1}{4}}} + \frac{(216)^{\frac{1}{3}}}{(8)^{\frac{1}{3}}} - (729)^{\frac{1}{6}}$
15. Simplify : (1) $\sqrt{\sqrt[4]{256}}$ (2) $\frac{1}{4}\sqrt[3]{128}$
16. Rationalize the denominator and simplify the following :
    (1) $\frac{7 + 3\sqrt{5}}{7 - 3\sqrt{5}}$ (2) $\frac{3\sqrt{2} - \sqrt{5}}{3\sqrt{3} + 2\sqrt{2}}$
17. Prove that $\frac{1}{\sqrt{9} - \sqrt{8}} - \frac{1}{\sqrt{8} - \sqrt{7}} + \frac{1}{\sqrt{7} - \sqrt{6}} - \frac{1}{\sqrt{6} - \sqrt{5}} + \frac{1}{\sqrt{5} - \sqrt{4}} - \frac{1}{\sqrt{4} - \sqrt{3}} + \frac{1}{\sqrt{3} - \sqrt{2}} - \frac{1}{\sqrt{2} - 1} = 2$
18. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :
    (1) Set of all natural number is denoted by ...... ☐
    (a) N (b) W (c) Z (d) R

(2) Set of all whole number is denoted by ......

(a) N (b) W (c) Z (d) Q

(3) Set of all integers is denoted by ......

(a) N (b) W (c) Z (d) Q

(4) Set of all rational number is denoted by ......

(a) N (b) W (c) Z (d) Q

(5) ...... is a true statement.

(a) Every whole number is a natural number

(b) Every integer is a rational number

(c) Every rational number is an integer

(d) Every real number is an irrational number.

(6) The number $\frac{3}{4}$ is ......

(a) a natural number (b) an integer

(c) a whole number (d) a rational number

(7) A pair of equivalent rational numbers is ...

(a) $\frac{4}{7}$ and $\frac{104}{182}$ (b) $\frac{5}{2}$ and $\frac{155}{64}$ (c) $\frac{144}{169}$ and $\frac{169}{225}$ (d) $\frac{8}{27}$ and $\frac{125}{216}$

(8) .... is rational number between 10 and 11.

(a) $\frac{21}{4}$ (b) $\frac{87}{8}$ (c) $\frac{97}{8}$ (d) $\frac{47}{4}$

(9) $\sqrt{9}$ = ...

(a) 3 (b) −3

(c) 3 and −3 (d) all a, b, c are true

(10) There are ...... rational numbers between two given numbers.

(a) two (b) can't say (c) finitely many (d) infinitely many

(11) $\sqrt{2}$ belongs to ....

(a) the set of whole numbers (b) the set of rational numbers

(c) the set of irrational numbers (d) the set of natural numbers

(12) The collection of rational numbers and irrational numbers together is called .....

(a) the set of whole numbers (b) the set of real numbers

(c) the set of natural numbers (d) the set of integers

(13) $\sqrt{16}$ is not ......

(a) a natural number (b) a real number

(c) an irrational number (d) a whole number

(14) The decimal expansion of $\frac{7}{4}$ is ......

(a) terminating (b) non terminating recurring

(c) non-terminating non-recurring (d) infinite

(15) 44.7232323... can be written as ......

(a) $44.\overline{723}$ (b) $44.7\overline{23}$ (c) $4\overline{4.723}$ (d) $4\overline{4}.7\overline{23}$

(16) The number 0.235 is ......

(a) a natural number (b) an integer

(c) an irrational number (d) a rational number

(17) The $\frac{p}{q}$ form of $0.\overline{35}...$ is ......

(a) $\frac{16}{45}$ (b) $\frac{176}{495}$ (c) $\frac{35}{99}$ (d) $\frac{16}{495}$

(18) The $\frac{p}{q}$ form of $0.\overline{01}$ is ......

(a) $\frac{1}{99}$ (b) $\frac{10}{99}$ (c) $\frac{100}{99}$ (d) $\frac{101}{99}$

(19) ...... is an irrational number.

(a) 0.3786 (b) $\sqrt{225}$ (c) 1.010010001... (d) 0.2353535...

(20) If $\frac{2}{7} = 0.\overline{285714}$, then $\frac{6}{7}$ =......

(a) $0.\overline{571428}$ (b) $0.\overline{142857}$ (c) $0.\overline{857142}$ (d) $0.\overline{095235}$

(21) $(\sqrt{6}) + (-\sqrt{6})$ is ......

(a) a natural number (b) an irrational number

(c) a whole number (d) an infinite number

(22) $\left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)$ is ......

(a) an irrational number (b) a rational number

(c) a whole number (d) a natural number

(23) $\sqrt{3} \cdot \sqrt{6}$ is ......

(a) a whole number (b) a natural number

(c) an irrational number (d) a rational number

(24) $\sqrt{5} + 29$ is ......

(a) an integer (b) an irrational number

(c) a whole number (d) a rational number

(25) $\sqrt{3} + \sqrt{3}$ is ......

(a) an integer (b) an irrational number

(c) a rational number (d) a whole number

(26) $6\sqrt{5} \cdot 3\sqrt{5}$ is not ......

(a) a natural number (b) an irrational number

(c) a whole number (d) a rational number

(27) $8\sqrt{8} \div 3\sqrt{2}$ is ......

(a) an integer (b) a rational number

(c) a whole number (d) an irrational number

(28) $8\sqrt{15} \div 2\sqrt{5}$ is ......

(a) an irrational number (b) an integer

(c) a whole number (d) a rational number

(29) $(\sqrt{10} - \sqrt{3})(\sqrt{10} - \sqrt{3}) =$ ......

(a) 0 (b) $13 - 2\sqrt{30}$ (c) $7 - 2\sqrt{30}$ (d) $7 + 2\sqrt{30}$

(30) $(7 + \sqrt{7})(7 - \sqrt{7}) =$ ......

(a) 0 (b) $2\sqrt{7}$ (c) $7\sqrt{7}$ (d) 42

(31) $(\sqrt{5} - \sqrt{2})^2$ is ......

(a) a natural number (b) an irrational number

(c) a whole number (d) a rational number

(32) $\frac{3}{2 - \sqrt{5}}$ is rationalized by ......

(a) -3 (b) $2 - \sqrt{5}$ (c) $2 + \sqrt{5}$ (d) $-2 + \sqrt{5}$

(33) An equivalent expression of $\frac{5}{7 + 4\sqrt{5}}$ after rationalizing the denominator is .....

(a) $\frac{20\sqrt{5} - 35}{31}$ (b) $\frac{20\sqrt{5} - 35}{129}$ (c) $\frac{35 - 20\sqrt{5}}{31}$ (d) $\frac{35 - 20\sqrt{5}}{129}$

(34) If $\sqrt[n]{a^2} = b$, then $b^{2n} =$ ...... $(a,\ b > 0,\ n \in \mathrm{N})$

(a) a (b) $(a)^{\frac{n}{2}}$ (c) a^{2n} (d) a^4

(35) $\sqrt[3]{\sqrt{64}} =$ ......

(a) 8 (b) 4 (c) 2 (d) not possible

(36) $\frac{\pi}{4}$ is ......

(a) a natural number (b) an irrational number

(c) a rational number (d) a whole number

*

Summary

In this chapter we have studied the following points :

1. A number which can be written in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$ is called a rational number. The set of rational numbers is denoted by Q.
2. A number which is not a rational is called an irrational number.
3. The collection of rational numbers and irrational numbers is the set of real numbers, denoted by R.
4. The decimal expression of rational number is either terminating or non-terminating recurring. Conversely, if the decimal expression is either terminating or non-terminating recurring, then the number is a rational number.
5. The decimal expression of an irrational number is non-terminating non-recurring. Conversely, if the decimal expression is non-terminating non-recurring, then the number is an irrational number.
6. A real number (however small) can be visualized on the number line by the process of successive magnification.
7. Each real number corresponds to a point on the number line and each point on the number line corresponds to a real number.
8. For $x, y \in R^+$, the following properties are true :

   (1) $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ (2) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

   (3) $(\sqrt{a} + \sqrt{b})(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay} + \sqrt{bx} + \sqrt{by}$

   (4) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$ (5) $(\sqrt{x} \pm \sqrt{y})^2 = x \pm 2\sqrt{xy} + y$
9. If $p \in Q$, $p \neq 0$ and q is an irrational, then $p + q$, $p - q$, $p \times q$ and $\frac{p}{q}$ are irrational numbers.
10. To rationalize denominator of a number like $\frac{1}{\sqrt{a} + \sqrt{b}}$ or $\frac{1}{\sqrt{a} + b}$ we have to multiply or divide by $\sqrt{a} - \sqrt{b}$ or $\sqrt{a} - b$ respectively.
11. Laws of exponents for real numbers :

    For $a, b \in R^+$ and $p, q \in Q$, we have

    (1) $a^p \cdot a^q = a^{p+q}$ (2) $\frac{a^p}{a^q} = a^{p-q}$ (3) $(a^p)^q = a^{pq}$

    (4) $(ab)^p = a^p \cdot b^p$ (5) $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

•

CHAPTER 3

POLYNOMIALS

3.1 Introduction

In earlier classes we have studied about algebraic operations on polynomials like addition, subtraction and multiplication of polynomials. They are like the operations on the integers. In this chapter, we shall learn about the zeroes and factorization of polynomials. We know about unknown numbers. On the basis of that we also have an idea of a term and polynomials.

3.2 Polynomials in One Variable

We begin with some terminology.

Variable : A symbol which takes different numerical values is called a variable and it is denoted by *x,y,z* etc.

Algebraic expressions : An algebraic expression is a combination of a variable and constants connected by the operations like addition, subtraction, multiplication and division. e.g. $2x + 3$, $5 - 7x$, $\frac{x}{2}$, etc. If we use a, b, c for constants, then algebraic expressions can be written as ax, bx,... etc.

Let us try to understand this by an example.

If we consider an equilateral triangle with sides of a unit length, then its perimeter is $1 + 1 + 1 = 3$ units. Look at the figure 3.1 where AB = BC = AC = 1 unit.

Thus the perimeter of ΔABC = AB + BC + AC.

Similarly if we consider the length of the sides of ΔABC as AB = c, BC = a and AC = b, then perimeter of ΔABC = BC + AC + AB = $a + b + c$.

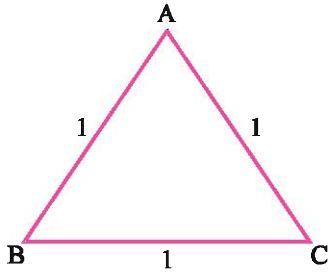


Figure 3.1

Let us think about the perimeter of a square. We know that for a square, length of each side is same. So if we take the length of a side of a square as x, then the perimeter of the square will be $x + x + x + x = 4x$. Here we say that $4x$ is an algebraic expression. If we consider the length of a rectangle as x and breadth as y, then the area of the rectangle is xy. It is also an algebraic expression. We are also familiar with other algebraic expressions like $x^3 - x^2 + x + 5$, $x^2 - 2x + 7$, etc. which are known as polynomials in one variable. Thus, we notice that the algebraic expressions which have only whole numbers as the exponents of the variable is a polynomial in one variable. So we can define a polynomial formally as follows.

Polynomial : An expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + .. + a_0$, $a_n \neq 0$, $a_0, a_1, a_2, ... a_n \in$ R is called a polynomial in variable x, where $n \in$ N $\cup$ {0}. Polynomials are denoted by P(x), Q(x), $p(x)$, $q(x)$,...etc.

a_nx^n, $a_{n-1}x^{n-1}$,..., a_0 are called **terms** of the polynomial.

Here, a_i (i = 0, 1, 2,..., n) is the coefficient of x^i. Further x^n is the highest power of x in this polynomial. Thus **a_nx^n is called the leading term of the polynomial and a_n is called the leading coefficient**. As such we can write the terms of a polynomial in any order but when the terms in a polynomial are written in order of decreasing power of x, then we say that it is written in the standard form. It can be seen that when a polynomial is written in the standard form, **the first term is the leading term and exponent of the variable in the leading term is called the degree of the polynomial.**

To determine degree of a polynomial, leading term and leading coefficient of a given polynomial, the terms of the polynomial have to be rearranged so as to write it in the standard form. Here **the coefficient a_0 of x^0 is called the constant term in the polynomial. The polynomial P(x) = a_0 is called a constant polynomial. The constant polynomial 0 is called the zero polynomial. If $a_0 \neq 0$, the constant polynomial a_0 has degree 0. The zero polynomial has no degree.**

The polynomial having only one term is called a monomial. If a polynomial has two terms, it is called a binomial and a polynomial having three terms is called a trinomial.

A polynomial can have any finite number of terms. In general, a polynomial of degree n has at most $(n + 1)$ terms with non-zero coefficients. For example $x^{2010} + x^{2009} + x^{2008} + ... + x^2 + x + 1$ is a polynomial of degree 2010 and it has 2011 terms.

The variable x can take any real value. Taking $x = \alpha$ in a polynomial $p(x)$, we get $p(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2} + ... + a_0$, called the value of the polynomial $p(x)$ at $x = \alpha$.

Example 1 : Are $\frac{x^2}{3} - 2x + 5$ and $x^{\frac{3}{2}} + 5x^2 - 7x + \frac{3}{4}$ polynomials in variable x ?

Solution : Here $\frac{x^2}{3} - 2x + 5$ is a polynomial in variable x, but $x^{\frac{3}{2}} + 5x^2 - 7x + \frac{3}{4}$ is not a polynomial in variable x because the exponent of x in the term $x^{\frac{3}{2}}$ is $\frac{3}{2}$ which is not a non-negative integer. In case of $\frac{x^2}{3} - 2x + 5$ variable x occurs to exponent 2, 1, 0 in different terms which are non-negative integers. ($5 = 5x^0$)

Example 2 : Find the degree of the polynomial.

(a) $2x + 7$ (b) $-3x^2 + 7x + 6$ (c) $3x^4 + x^5 - 7x^3 + x - 1$

Solution : Polynomial in (a) has degree 1, because index of x in the term of the highest power of x is 1.

Polynomial in (b) has degree 2, because the exponent of x in the term of the highest power of x is 2.

In case of (c) rewriting the polynomial in standard form $x^5 + 3x^4 - 7x^3 + x - 1$, the term of the highest power in it is x^5. Therefore the degree of polynomial is 5.

Linear polynomial : A polynomial of degree 1 is called a linear polynomial. So we can write the general form of a linear polynomial in one variable as $p(x) = ax + b$ where $a \neq 0$ and $a, b \in$ R.

$3x - 7$, $x + \sqrt[3]{7}$, $7x$ have degree 1. Hence they are linear polynomials.

Quadratic polynomial : A polynomial having degree 2 is known as a quadratic polynomial So, the general form of a quadratic polynomial in one variable is $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in$ R.

Now, $3x^2 - 11$, $\frac{3}{7}x - x^2$, $\sqrt{3}x^2 + 9$ have degree two and they are quadratic polynomials.

Cubic polynomial : A polynomial having degree 3 is known as a cubic polynomial. So, the general form of a cubic polynomial in one variable is $p(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, $a, b, c, d \in$ R.

The polynomials like $x^3 - 4x^2 + 7$, $\sqrt{8}x^2 - 3x^3 + 15x + 17$, $x^3 + x^2 + x + 1$, have degree 3 and hence they are cubic polynomials.

Thus, if we continue this process upto degree n of variable we get,

$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$

where, $a_n \neq 0$. $a_0, a_1, a_2 \ldots a_n$ are constants i.e. $a_i \in$ R ($i = 0, 1, 2, 3, \ldots, n$)

This is a polynomial of degree n in variable x.

EXERCISE 3.1

1. Write the degree of the following polynomials.

(1) $p(x) = 3x^7 - 6x^5 + 4x^3 - x^2 - 5$

(2) $p(x) = x^{100} - (x^{10})^{20} + 3x^{50} + x^{25} + x^5 - 7$

(3) $p(x) = 7x - 3x^2 + 4x^3 + x^4$

(4) $p(x) = 3.14x^2 + 1.57x + 1$

2. Write the coefficient of x^3 in the following polynomials.

(1) $p(x) = 4x^3 + 3x^2 + 2x + 1$ (2) $p(x) = x^2 + 2x + 1$

(3) $p(x) = x^2 - \sqrt{3}x^3 + 4x^7 + 6$

3. Classify the following polynomials as linear, quadratic or cubic.

(1) $p(x) = x^2 + 27$ (2) $p(x) = 2010x + 2009$

(3) $p(x) = 4x^2 + 7x^3 + 3$ (4) $p(x) = (x - 1)(x + 1)$

4. Verify whether following algebraic expressions are polynomial or not.

(1) $p(x) = x^7 + 10x^5 + 4x^3 + 3x + 1$ (2) $p(x) = x^{\frac{-5}{2}} + 10x + 4$

(3) $p(x) = x + \frac{1}{x}$

5. Give an example of each of a monomial of degree 10, a binomial of degree 20 and a trinomial of degree 25.

*

3.3 Zeros of a Polynomial

Let us consider the polynomial $p(x) = x^3 - 5x^2 + 6x$. If we take $x = 0$, we get $p(0) = (0)^3 - 5(0)^2 + 6(0) = 0$. If we take $x = 1$, we get $p(1) = (1)^3 - 5(1)^2 + 6(1) = 2$. Similarly for $x = 2$, we get $p(2) = 0$. For $x = 3$, $p(3) = 0$. For $x = 4$, $p(4) = 8$. Here we observe that the value of given polynomial is 0 at $x = 0$, 2 and 3. These values $x = 0, 2, 3$ where $p(x) = 0$ are called zeros of the polynomial $p(x)$. **Thus if for some $x \in R$, $p(x) = 0$, then x is called a zero of the polynomial $p(x)$. The zeros of a polynomial $p(x)$, if they exist, are called roots (solution) of the polynomial equation $p(x) = 0$.** For example, for no real number x, $x^2 + 1 = 0$. Hence the polynomial $p(x) = x^2 + 1$ has no zero, i.e. polynomial equation $x^2 + 1 = 0$ has no real roots.

Now, if we consider a constant polynomial $c(c \neq 0)$, then what can we say about its zeroes ? It is clear that it has no zeros because replacing x with any number in cx^0 we get the result c, where c is a non-zero constant.

Example 3 : Find the value of each of the following polynomials at the values of variable mentioned.

(1) $p(x) = 3x^2 - 7x + 5$ at $x = 2$

(2) $p(x) = (x^2 - 9)(x^3 + 7)$ at $x = 3, -1$

(3) $p(x) = 3x^4 - 2x^3 + 7x^2 - 5x + 9$ at $x = 3$

Solution : (1) $p(x) = 3x^2 - 7x + 5$

Let $x = 2$.

Hence we get $p(2) = 3(2)^2 - 7(2) + 5$

$= 12 - 14 + 5 = 3$

$\therefore$ $p(2) = 3$

(2) $p(x) = (x^2 - 9)(x^3 + 7)$

$= x^2(x^3 + 7) - 9(x^3 + 7)$

$= x^5 + 7x^2 - 9x^3 - 63$

$= x^5 - 9x^3 + 7x^2 - 63$

$\therefore$ $p(x)$ is a polynomial.

Let $x = -1$.

$\therefore$ $p(-1) = -1 - 9(-1) + 7 - 63$

$= -1 + 9 + 7 - 63$

$= -48$

$\therefore$ $p(-1) = -48$

Now if we consider $x = 3$, in $x^5 - 9x^3 + 7x^2 - 63$

$\therefore$ $p(3) = 3^5 - 9 \cdot 3^3 + 7 \cdot 3^2 - 63$

$= 3^5 - 3^5 + 63 - 63$

$= 0$

$\therefore$ $p(3) = 0$

So we can say that 3 is a zero of $p(x)$.

(3) $p(x) = 3x^4 - 2x^3 + 7x^2 - 5x + 9$

$p(3) = 3(3)^4 - 2(3)^3 + 7(3)^2 - 5(3) + 9$

$= 3(81) - 2(27) + 7(9) - 15 + 9$

$= 243 - 54 + 63 - 15 + 9 = 246$

Example 4 : For $p(x) = x^2 - 4$, find the value of $p(x)$ for $x = 1, 2, 3, 4$. From these what information do we get about the zeros of $p(x)$?

Solution : $p(1) = 1^2 - 4 = -3$ $\quad$ $p(2) = 4 - 4 = 0$

$p(3) = 9 - 4 = 5$ $\quad$ $p(4) = 16 - 4 = 12$

As $p(2) = 0$, $x = 2$ is a zero of $p(x)$.

Example 5 : Find the zeros of linear polynomial $p(x) = ax + b$, $a \neq 0$, $a, b \in$ R.

Solution : Let $p(x) = ax + b = 0$

$\therefore \quad ax + b = 0$

$\therefore \quad ax = -b$

$\therefore \quad x = \frac{-b}{a}$ $\quad$ **($a \neq 0$)**

$\frac{-b}{a}$ is the zero of the linear polynomial $p(x) = ax + b$.

$\therefore$ We can say that a linear polynomial in one variable has one and only one zero. e.g. $x + 3$ has zero -3 and $2x + 7$ has zero $\frac{-7}{2}$.

Now some important results for the zeros of polynomial are as follows :

(1) 0 may be a zero of a polynomial but a zero of polynomial need not be 0

(2) A linear polynomial has one and only one zero.

(3) A polynomial can have more than one zero.

EXERCISE 3.2

1. Verify whether 3 and 0 are the zeros of $p(x) = x^3 - x$.

2. Find the value of the following polynomials at values of x specified.

(1) $p(x) = x^4 + 2x^3 - x + 5$, at $x = 2$

(2) $p(x) = 3x^3 - 5x^2 + 6x - 9$, at $x = 0, -1$

(3) $p(x) = 5x^3 + 11x^2 + 10$, at $x = -2$

3. Find $p(0)$, $p(1)$, $p(2)$ for each of the following polynomials.

(1) $p(x) = x^7$ $\quad$ (2) $p(x) = (x - 1)(x + 3)$

(3) $p(x) = x^2 - 2x$

4. Find the zeros of the following polynomials.

(1) $p(x) = 3x + 2$ $\quad$ (2) $p(x) = 5x - 3$

(3) $p(x) = 3$

*

3.4 Remainder Theorem

Let us consider two numbers 21 and 8. If we divide 21 by 8, we get the quotient 2 and remainder 5. This fact can be expressed as.

$21 = (8 \times 2) + 5$

Here, 21 is called the dividend, 8 is called the divisor, 2 is the quotient and 5 is the remainder.

We observe that the remainder 5 is less than the divisor 8.

If we divide 16 by 8, then we get $16 = (8 \times 2) + 0$.

Here, the remainder is 0 and we say that 8 is a factor of 16 or 16 is a multiple of 8.

As seen above, we write dividend = (divisor)(quotient) + remainder for integers

Thus, when an integer a is divided by a non-zero integer b, we get an expression like $a = bq + r$ (where $|r| < |b|$), q is called the quotient and r is called the remainder, a is called the dividend, b is called the divisor and if $r = 0$, then b is called a factor of a or a divisor of a.

In analogy with this, if a polynomial $p(x)$ is divided by a non-zero polynomial $d(x)$, we get $p(x) = d(x) \cdot q(x) + r(x)$.

Here, $p(x)$ = dividend polynomial, $d(x)$ = divisor polynomial, $q(x)$ = quotient polynomial and $r(x)$ = remainder polynomial. Thus,

Dividend polynomial =

Divisor polynomial × quotient polynomial + remainder polynomial

Here degree of the remainder polynomial is less than the degree of the divisor polynomial or the remainder is zero.

Now consider the polynomial $p(x) = x^2 + x + 1$

$= x(x + 1) + 1$

If we regard x as a divisor then $(x + 1)$ is the quotient and 1 is the remainder. Degree of constant polynomial 1 is zero and degree of divisor polynomial x is 1.

To understand the division of a polynomial $p(x)$ by a polynomial $d(x)$, we take the following example.

Example 6 : Divide $p(x) = x^3 + 13x^2 + 32x + 20$ by $d(x) = x + 2$.

Solution : First we understand, how to solve this example using the following steps.

Step 1 : Rewrite the polynomial $p(x)$, $d(x)$ in standard form.

Step 2 : Divide the leading term of $p(x)$ by the leading term of $d(x)$. In this case we divide x^3 by x, which gives us the first term of the quotient $q(x)$.

i.e. $\frac{x^3}{x} = x^2$ = first term of quotient.

Step 3 : Multiply this first term of the quotient with the divisor and subtract this result from the dividend. i.e. multiply x^2 with $(x + 2)$. Hence we get $x^3 + 2x^2$. We subtract this result from $x^3 + 13x^2 + 32x + 20$. So we get $11x^2 + 32x + 20$.

$$
\begin{array}{r|l}
 & x^2 \\ \hline
x + 2 & x^3 + 13x^2 + 32x + 20 \\
 & x^3 + \;\; 2x^2 \\
 & - \quad - \\ \hline
 & \qquad 11x^2 + 32x + 20
\end{array}
$$

Thus, we get $x^3 + 13x^2 + 32x + 20 = x^2 (x + 2) + 11x^2 + 32x + 20$.

Since the degree of $11x^2 + 32x + 20$ is more than the degree of $x + 2$, we proceed in a similar manner as in step 2, taking $11x^2 + 32x + 20$ as 'new dividend'.

Step 4 : To find the next term of quotient, we divide the first term of new dividend by the first term of the same divisor i.e. $\frac{11x^2}{x} = 11x$ = second term of quotient.

Step 5 : Now we multiply second term $11x$ with the divisor and subtract the result from new dividend. Here we multiply $11x$ with $(x + 2)$ we get $11x^2 + 22x$. Subtract it from $11x^2 + 32x + 20$ and we get new dividend i.e. $10x + 20$.

$$
\begin{array}{r|l}
 & 11x \\ \hline
x + 2 & 11x^2 + 32x + 20 \\
 & 11x^2 + 22x \\
 & - \quad\;\; - \\ \hline
 & \qquad\quad 10x + 20
\end{array}
$$

This process is continued till the remainder is 0 or degree of 'new dividend' is less than the degree of the divisor.

Step 6 : Now here, $10x + 20$ is the new dividend, first term of which $10x$ is to be divided by the first term of divisor i.e. x. Hence we get 10. Now we multiply this divisor by 10 and subtract the result from $10x + 20$ and we get the remainder 0.

$$
\begin{array}{r|l}
 & 10 \\ \hline
x + 2 & 10x + 20 \\
 & 10x + 20 \\
 & - \quad\; - \\ \hline
 & \qquad\quad 0
\end{array}
$$

Finally we get $x^3 + 13x^2 + 32x + 20 = (x^2 + 11x + 10)(x + 2) + 0$

Hence, in this $q(x) = x^2 + 11x + 10$ and $r(x) = 0$.

This result can be written as...

Dividend = (Divisor) (quotient) + remainder i.e. $p(x) = d(x) \cdot q(x) + r(x)$

Here remainder is $r(x)$ and $r(x) = 0$ or degree of $r(x)$ is less than degree of $d(x)$ and $d(x)$ is the divisor, $q(x)$ is the quotient, $p(x)$ is the dividend.

This process can be written as long division also as illustrated below.

Thus, the above example, can be solved as follows : (by long division)

$$
\begin{array}{r|l}
 & +x^2 + 11x + 10 \\ \hline
x + 2 & +x^3 + 13x^2 + 32x + 20 \\
 & +x^3 + 12x^2 \\
 & - \quad - \\ \hline
 & \quad 11x^2 + 32x + 20 \\
 & \quad 11x^2 + 22x \\
 & \quad - \quad\; - \\ \hline
 & \qquad 10x + 20 \\
 & \qquad 10x + 20 \\
 & \qquad - \quad - \\ \hline
 & \qquad\qquad 0
\end{array}
$$

Here $x^3 + 13x^2 + 32x + 20 = (x + 2)\,(x^2 + 11x + 10)$ and the remainder is zero. In this case, we say $x + 2$ is a factor of $x^3 + 13x^2 + 32x + 20$. In general if $r(x) = 0$. $d(x)$ is a factor of $p(x)$

Let us take some more examples :

Example 7 : Divide $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $(x - 3)$ using long division.

Solution :

$$
\begin{array}{r|l}
 & +x^3 + x^2 - 4x - 4 \\ \hline
x - 3 & +x^4 - 2x^3 - 7x^2 + 8x + 12 \\
 & +x^4 - 3x^3 \\
 & - \quad + \\ \hline
 & \quad x^3 - 7x^2 + 8x + 12 \\
 & \quad 2x^3 - 3x^2 \\
 & \quad - \quad + \\ \hline
 & \qquad - 4x^2 + 8x + 12 \\
 & \qquad - 4x^2 + 12x \\
 & \qquad + \quad - \\ \hline
 & \qquad\quad - 4x + 12 \\
 & \qquad\quad - 4x + 12 \\
 & \qquad\quad + \quad - \\ \hline
 & \qquad\qquad\quad 0
\end{array}
$$

$\therefore \quad (x^4 - 2x^3 - 7x^2 + 8x + 12) = (x - 3)\,(x^3 + x^2 - 4x - 4)$

Here also $(x - 3)$ is a factor of $x^4 - 2x^3 - 7x^2 + 8x + 12$.

Example 8 : Divide $x^3 + 3x^2 + 4x + 7$ by $x^2 + x$

Solution :

$$\begin{array}{r|l} & x + 2 \\ \hline x^2 + x & x^3 + 3x^2 + 4x + 7 \\ & x^3 + x^2 \\ & - \quad - \\ \hline & \quad 2x^2 + 4x \\ & \quad 2x^2 + 2x \\ & \quad - \quad - \\ \hline & \qquad 2x + 7 \end{array}$$

Here degree of $2x + 7$ is less than the degree of $x^2 + x$. Hence $2x + 7$ is the remainder and $x + 2$ is the quotient. ($2x + 7$ has degree 1 and $x^2 + x$ has degree 2.)

Example 9 : Divide $4x^3 - 7x + 10$ by $x - 2$ and find the remainder.

Solution :

$$\begin{array}{r|l} & 4x^2 + 8x + 9 \\ \hline x - 2 & 4x^3 - 7x + 10 \\ & 4x^3 - 8x^2 \\ & - \quad + \\ \hline & \quad 8x^2 - 7x + 10 \\ & \quad 8x^2 - 16x \\ & \quad - \quad + \\ \hline & \qquad 9x + 10 \\ & \qquad 9x - 18 \\ & \qquad - \quad + \\ \hline & \qquad\quad 28 \end{array}$$

If we divide $p(x) = 4x^3 - 7x + 10$ by the linear polynomial $x - 2$, we get quotient polynomial $4x^2 + 8x + 9$ and remainder is 28.

$\therefore \quad p(x) = (x - 2)(4x^2 + 8x + 9) + 28$

Here also degree of constant polynomial 28 is zero and it is less than degree of divisor $x - 2$.

Now, if we find $p(2)$, then $p(2) = 4(2)^3 - 7(2) + 10 = 32 - 14 + 10 = 28$

$\therefore \quad p(2) = 28$

Here $r(x) = p(2) = 28$

If we divide the polynomial $p(x)$ by linear polynomial $x - a$ then we have the remainder $p(a)$. The proof of this statement will now be given. This theorem is known as **remainder theorem**.

Remainder Theorem : If a polynomial $p(x)$ of degree greater than or equal to 1 is divided by linear polynomial $x - a$, the remainder is $p(a)$. ($a \in$ R)

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1 Suppose the dividend $p(x)$ is divided by the divisor $(x - a)$. Let the quotient be $q(x)$ and remainder be $r(x)$. So we get,

$$p(x) = d(x)\, q(x) + r(x)$$
$$= (x - a)\, q(x) + r(x)$$

Degree of divisor $x - a$ is 1.

Degree of $r(x) <$ degree of divisor or $r(x) = 0$.

$\therefore$ degree of $r(x) < 1$ or $r(x) = 0$

$\therefore$ degree of $r(x) = 0$ or $r(x) = 0$

$\therefore$ $r(x) = r$, a non-zero constant or $r(x) = 0$.

$\therefore$ $r(x) = r$, a real number not depending upon value of x.

$\therefore$ $p(x) = (x - a)\, q(x) + r$.

In particular if $x = a$, this identity becomes,

$$p(a) = (a - a)\, q(a) + r$$
$$= r$$

$\therefore$ The remainder r is $p(a)$.

This proves the theorem. Let us use this theorem in the following example.

Example 10 : Find the remainder when $y^3 - 2y^2 - 29y - 42$ is divided by $(y - 3)$.

Solution :

Here $p(y) = y^3 - 2y^2 - 29y - 42$ and the divisor is $y - 3$. So $a = 3$.

$$p(3) = (3)^3 - 2(3)^2 - 29(3) - 42$$
$$= 27 - 18 - 87 - 42$$
$$= -120$$

So, by the remainder theorem, -120 is the remainder when $p(y)$ divided by $y - 3$.

It means that if any polynomial is to be divided by a linear polynomial $(x - a)$, then find the zero of the linear polynomial a and substitute its value in $p(y)$. Then we get the remainder $p(a)$ directly.

Example 11 : Find the value of c, if $y - 1$ is a factor of $p(y) = 4y^3 + 3y^2 - 4y + c$.

Solution : As $y - 1$ is a factor of $p(y)$, the remainder $p(1) = 0$

$\therefore$ $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + c$

$\therefore$ $0 = 4 + 3 - 4 + c$

$\therefore$ $c = -3$

Example 12 : Verify whether the polynomial $2x^4 + x^3 - 14x^2 - 19x - 6$ is divisible by $x + 1$ or not.

Solution : Now, here we consider the zero of linear polynomial (i.e. divisor) $x + 1$. So we get $x = -1$. Substitute this value in the given polynomial.

$\therefore \quad p(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$

$= 2 - 1 - 14 + 19 - 6$

$= 21 - 21$

$= 0$

$\therefore$ The remainder when $2x^4 + x^3 - 14x^2 - 19x - 6$ is divided by $x + 1$ is zero.

$\therefore$ $p(x)$ is divisible by $(x + 1)$

EXERCISE 3.3

1. Divide the following polynomials by $(x - 1)$ and find the quotient and remainder.
   (1) $p(x) = x^5 - 1$
   (2) $p(x) = x^4 + 4x^3 - 3x^2 - x + 1$
2. Find the remainder when the polynomial $p(t) = 2t^4 - 7t^3 - 13t^2 + 63t - 45$ is divided by the following polynomials.
   (1) $(t - 1)$ (2) $t - 3$ (3) $2t - 5$ (4) $t + 3$ (5) $2t + 3$
3. What is the remainder when $p(x) = x^4 - 4x^3 + 3x - 1$ is divided by $d(x) = x + 2$?
4. What should be added to $p(y) = 12y^3 - 39y^2 + 50y + 97$ so that the resulting polynomial is divisible by $y + 1$?
5. What should be subtracted from $p(x) = x^4 + 85$ so that the resulting polynomial is divisible by $x + 3$?
6. The product of two polynomial is $x^3 - 8x - 12 + x^2$. If one of the polynomial is $x + 2$, then find the other $\left[\textbf{Hint : Other polynomial} = \dfrac{\textbf{Product of two polynomials}}{\textbf{One polynomial}}\right]$
7. Carry out the following division and find the remainder :
   (1) $(x^4 + x^3 + 3x^2 + 2x + 2) \div (x^2 + 2)$
   (2) $(x^3 - 15x^2 - 54x + 23) \div (x^2 + 3x)$
   (3) $(x^4 + 4x^3 + 10x^2 + 12x + 15) \div (x^2 + 2x + 3)$
8. If the polynomial $ax^5 - 23x^3 + 47x + 1$ is divided by $x - 2$, then remainder is 7. Find the value of a.

*

3.5 Factorization of Polynomials

When discussing zeros of a polynomial, we have already seen that if $p(a) = 0$, then the polynomial $p(x)$ is exactly divisible by divisor $(x - a)$ and remainder is zero. In such a case the divisor is considered as its factor. The same procedure is shown in the example 12, where the given polynomial is divisible by $(x + 1)$. Here we say that $(x + 1)$ is a factor of the polynomial. We have the following theorem, which is known as **Factor Theorem**.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and $a \in R$ then

(1) If $p(a)$ is zero, then $x - a$ is a factor of $p(x)$ and

(2) If $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Proof : By the remainder theorem, we have $p(x) = (x - a)\ q(x) + p(a)$

So, (1) If $p(a) = 0$, then $p(x) = (x - a)\ q(x)$ which shows that $(x - a)$ is a factor of $p(x)$

(2) Since $(x - a)$ is a factor of $p(x)$, we have $p(x) = (x - a)\ g(x)$ for some polynomial $g(x)$.

Hence $p(a) = (a - a)\ g(a) = 0.\ g(a) = 0$.

Let us understand the above theorem by following example.

Example 13 : Examine whether $x^2 - 3x + 2$ has the factor $x - 2$.

Solution : For divisor polynomial $x - 2$, $a = 2$.

Subsititute $x = 2$ in the given polynomial $p(x)$

$$p(2) = (2)^2 - 3(2) + 2$$
$$= 4 - 6 + 2 = 0.$$

$\therefore$ By factor theorem, $p(a) = p(2) = 0$. Hence we can say that $x - 2$ is a factor of given polynomial $p(x)$.

In the above example, given polynomial has degree 2. So this polynomial is in the form of a quadratic polynomial like $ax^2 + bx + c$. We can factorize this polynomial as a product of linear polynomials. So, this polynomial can be factorized as $(px + l)\ (qx + m)$. Thus,

$$ax^2 + bx + c = (px + l)\ (qx + m) = pqx^2 + pmx + qlx + ml$$
$$= pqx^2 + (pm + lq)x + ml$$

Hence, $a = pq,\ b = pm + lq,\ c = ml.$

This shows that b is the sum of two numbers pm and lq and product of them $= (pm)\ (lq) = (pq)\ (lm) = ac$

$\therefore$ To factorize $ax^2 + bx + c$, we have to write b as the sum of two numbers product of which is ac. We can see this procedure in the following example.

Note : For the factorization, we assume that a, b, c are integers.
i.e. we factorize the polynomials on the set of integers.

$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. We do not take this type of factors.

We also do not think about the factorization of the polynomials like

$x^2 - (\sqrt{2} + \sqrt{3})x + \sqrt{6} = (x - \sqrt{2})(x - \sqrt{3})$.

Example 14 : Factorize $5x^2 + 9x + 4$ by splitting the middle term and by using factor theorem.

Solution : Here, given quadratic polynomial is $5x^2 + 9x + 4$. By splitting middle term, we have to find two numbers whose sum is 9 and product is 20. Such numbers are 5 and 4.

Here 5 and 4 are two numbers such that $5 + 4 = 9$ and $5 \times 4 = 20$

$$5x^2 + 9x + 4 = 5x^2 + 5x + 4x + 4$$
$$= 5x(x + 1) + 4(x + 1)$$

$\therefore$ $5x^2 + 9x + 4 = (x + 1)(5x + 4)$

Example 15 : If $3x^3 - x^2 - 27x + k$ has a factor $3x - 1$, then find the constant k.

Solution : $3x - 1$ is a factor of given polynomial $p(x) = 3x^3 - x^2 - 27x + k$. Hence, by considering $3x - 1 = 0$, we get $x = \frac{1}{3}$. Thus $p\left(\frac{1}{3}\right) = 0$ **(by the factor theorem).** substitute $x = \frac{1}{3}$ in the given polynomial and we get,

$$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 27\left(\frac{1}{3}\right) + k = 0$$

$\therefore$ $3\left(\frac{1}{27}\right) - \left(\frac{1}{9}\right) - 9 + k = 0$

$\therefore$ $\frac{1}{9} - \frac{1}{9} - 9 + k = 0$

$\therefore$ $k = 9$

Example 16 : Verify that $(x - 1)$ is a factor of $15x^3 - 20x^2 + 13x - 8$ and hence factorize $15x^3 - 20x^2 + 13x - 8$.

Solution : Let $p(x) = 15x^3 - 20x^2 + 13x - 8$ be given. Since $(x - 1)$ is a factor, $p(1)$ should be zero.

$p(1) = 15 - 20 + 13 - 8 = 0$

$\therefore$ $(x - 1)$ is a factor of $p(x)$

Let, $15x^3 - 20x^2 + 13x - 8$ be divided by $(x - 1)$

$$\begin{array}{r|l} & 15x^2 - 5x + 8 \\ \hline x - 1 & 15x^3 - 20x^2 + 13x - 8 \\ & 15x^3 - 15x^2 \\ & - \quad + \\ \hline & \quad - 5x^2 + 13x - 8 \\ & \quad - 5x^2 + 5x \\ & \quad + \quad - \\ \hline & \qquad 8x - 8 \\ & \qquad 8x - 8 \\ & \qquad - \quad + \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore \quad 15x^3 - 20x^2 + 13x - 8 = (x - 1)(15x^2 - 5x + 8)$

So, $p(x) = (x - 1)$ g(x), where g$(x) = 15x^2 - 5x + 8$. Here $a = 15$, $b = -5$, $c = 8$

Now we have to find two numbers sum of which is -5 and product is 120 which is not possible. Hence the factors of $p(x)$ are $(x - 1)$ and $(15x^2 - 5x + 8)$.

Infact $120 = 2 \times 60 = 3 \times 40 = 4 \times 30 = 5 \times 24 = 6 \times 20 = 8 \times 15 = 10 \times 12$

Hence in any pair of factors, the sum is at least 22 in absolute value.

According to factor theorem how can we know whether $(x - 1)$ or $(x + 1)$ are factors of the given polynomial $p(x)$. Thus to understand this we take the example of a cubic polynomial. These results are true for any polynomials to have $(x - 1)$ as a factor.

(1) Criterion for $x - 1$ to be a factor of $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in$ R, $a \neq 0$

By the remainder theorem, $(x - 1)$ is a factor of $p(x)$ if and only if $p(1) = 0$.

$p(x) = ax^3 + bx^2 + cx + d$

Now, $p(1) = a(1)^3 + b(1)^2 + c(1) + d$

$= a + b + c + d$

Thus, $p(1) = a + b + c + d = 0$ if and only if $(x - 1)$ is a factor.

Now, $a + b + c + d$ is the sum of the coefficients of $p(x)$.

Thus, **$(x - 1)$ is a factor of $p(x)$ if and only if the sum of all the coefficients of $p(x)$ is zero.**

[**Note :** In general, let $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$

Then $p(a) = a_n + a_{n-1} + ... + a_0$

= sum of the coefficients of $p(x)$

$\therefore \quad (x - 1)$ is a factor of $p(x)$ if and only if $p(1) = 0$.

i.e. sum of coefficients is zero.]

Let us understand this by the following example :

Example 17 : Factorize $p(x) = x^3 + 5x^2 + 2x - 8$.

Solution : The sum of the coefficients of $p(x) = 1 + 5 + 2 - 8 = 0$

$\therefore$ $(x - 1)$ is a factor of $p(x)$.

Now we shall factorise $p(x)$ by considering $(x - 1)$ as a factor.

$p(x) = x^3 + 5x^2 + 2x - 8$

$= x^3 - x^2 + 6x^2 - 6x + 8x - 8$

$= x^2(x - 1) + 6x(x - 1) + 8(x - 1)$

$= (x - 1)(x^2 + 6x + 8)$

$= (x - 1)(x^2 + 4x + 2x + 8)$ **(factors of 8 whose sum is 6)**

$= (x - 1)[x(x + 4) + 2(x + 4)]$

$= (x - 1)(x + 2)(x + 4)$

(2) Criterion for $x + 1$ to be a factor of $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$, $a \neq 0$

$(x + 1)$ is a factor of $p(x)$ if and only if $p(-1) = 0$.

Substitute $x = -1$ in $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = a(-1)^3 + b(-1)^2 + c(-1) + d$

$= -a + b - c + d$

$p(-1) = 0$ if and only if $-a + b - c + d = 0$.

$\therefore$ $a + c = b + d$

Here $a + c$ is the sum of the coefficients of odd powers of x in $p(x)$ and $b + d$ is the sum of the coefficients of even powers of x in $p(x)$.

Thus, $(x + 1)$ is a factor of $p(x)$ if and only if the sum of the coefficients of odd powers of x in $p(x)$ is equal to the sum of the coefficients of even powers of x in $p(x)$. This is true in general for a polynomial of any degree greater than or equal to 1 also.

Let us understand this by the following example :

Example 18 : Factorize $x^3 + 4x^2 + 4x + 1$.

Solution : Let $p(x) = x^3 + 4x^2 + 4x + 1$

The sum of the coefficients of $p(x) = 1 + 4 + 4 + 1 = 10 \neq 0$

$\therefore$ $(x - 1)$ is not a factor of $p(x)$.

The sum of the coefficients of odd powers of x in $p(x) = 1 + 4 = 5$

and the sum of the coefficients of even powers of x in $p(x) = 4 + 1 = 5$

Since both are equal, $(x + 1)$ is a factor of $p(x)$.

$p(x) = x^3 + 4x^2 + 4x + 1$

$= x^3 + x^2 + 3x^2 + 3x + x + 1$

$= x^2(x + 1) + 3x(x + 1) + 1(x + 1)$

$= (x + 1)(x^2 + 3x + 1)$

(Here $x^2 + 3x + 1$ can not be factored further as there is no pair of factors of 1 whose sum is 3).

EXERCISE 3.4

1. From the following polynomials, find out which of them has $(x - 1)$ as a factor :

(1) $2x^3 - 3x^2 + 3x - 2$ (2) $4x^3 + x^4 - x + 1$

(3) $5x^4 - 4x^3 - 2x + 1$ (4) $3x^3 + x^2 + x + 11$

2. By using factor theorem, find the other factor of the given polynomial $p(x)$, where $d(x)$ is a given factor.

(1) $p(x) = 21x^2 + 26x + 8$, $d(x) = 3x + 2$

(2) $p(x) = x^3 + 10x^2 + 23x + 14$, $d(x) = x + 1$

(3) $p(x) = x^3 - 9x^2 + 20x - 12$, $d(x) = x - 6$

3. If $p(x) = ax^3 + 3x^2 + 7x + 13$ is divided by $(x + 3)$, then the remainder is -8. Find the value of a.

4. Factorise the following polynomials :

(1) $3x^2 + 7x + 4$ (2) $15x^2 + 16x + 4$ (3) $-21x^2 + 16x + 5$

5. From the following polynomials, decide which has $(x + 1)$ or $(x - 1)$ as a factor.

(1) $p(x) = 3x^3 - 7x^2 + 5x - 1$ (2) $p(x) = 21x^3 + 16x^2 + 4x + 9$

(3) $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 2$ (4) $p(x) = x^3 + 13x^2 + 32x + 20$

6. If $x - 4$ is a factor of $p(x) = ax^4 - 7x^3 - 3x^2 - 2x - 8$, then find the value of a.

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3.6 Algebraic Identities

In previous classes, we have already seen the following algebraic identities. An algebraic identity is true for all values of the variables occuring in it. These identities are as follows. R.H.S. is the expansion of the expression on L.H.S.

(1) $(a + b)^2 = a^2 + 2ab + b^2$

(2) $(a - b)^2 = a^2 - 2ab + b^2$

(3) $(a - b)(a + b) = a^2 - b^2$

(4) $(x + a)(x + b) = x^2 + (a + b)x + ab$

(5) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

More identities can be obtained from the above identities as follows :

$$(a+b)^3 = (a+b)\cdot(a+b)^2$$
$$= (a+b)\cdot(a^2+2ab+b^2)$$
$$= a(a^2+2ab+b^2) + b(a^2+2ab+b^2)$$
$$= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$$
$$= a^3+3a^2b+3ab^2+b^3$$

$\therefore$ $\mathbf{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$ **(6)**

Similarly, $\mathbf{(a-b)^3 = a^3-3a^2b+3ab^2-b^3}$ **(7)**

The above two identities can also be writen as follows :

$\mathbf{(a+b)^3 = a^3+b^3+3ab(a+b)}$ **(8)**

$\mathbf{(a-b)^3 = a^3-b^3-3ab(a-b)}$ **(9)**

$\mathbf{(a+b+c)(a^2+b^2+c^2-ab-bc-ac) = a^3+b^3+c^3-3abc}$ **(10)**

Identity (10) can be proved by multiplication.

R.H.S. in the above 10 identities are called the expansions of the algebraic expressions on L.H.S. Let us understand above identities by following examples.

Example 19 : Find : $(3x-4y)^2$

Solution : Here $a = 3x$, $b = 4y$. By using expansion of $(a-b)^2$,

we get $(3x-4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$
$$= 9x^2 - 2(12xy) + 16y^2$$
$$= 9x^2 - 24xy + 16y^2$$

Example 20 : Find : 105×95 by using an appropriate identity.

Solution : Here appropriate identity is $(a+b)(a-b) = a^2 - b^2$

Taking $a = 100$, $b = 5$

$\therefore$ $105 \times 95 = (100+5)(100-5)$
$$= (100)^2 - (5)^2$$
$$= 10000 - 25$$
$$= 9975$$

Example 21 : Find : 107×102 by using an appropriate identity.

Solution : Here appropriate identity is $(x+a)(x+b) = x^2 + (a+b)\,x + ab$

Here $x = 100$, $a = 7$, $b = 2$

$\therefore$ $107 \times 102 = (100+7)(100+2)$
$$= (100)^2 + (7+2)(100) + (7)(2)$$
$$= 10000 + 9(100) + 14$$
$$= 10000 + 900 + 14$$
$$= 10914$$

Example 22 : Find : $(2x - 3y + 4z)^2$

Solution : By using the expansion of $(a + b + c)^2$, where $a = 2x$, $b = -3y$, $c = 4z$, we get,

$$(2x - 3y + 4z)^2 = (2x)^2 + (-3y)^2 + (4z)^2 + 2(2x)(-3y) + 2(-3y)(4z) + 2(2x)(4z)$$

$$= 4x^2 + 9y^2 + 16z^2 + 2(-6xy) + 2(-12yz) + 2(8zx)$$

$$\therefore \quad (2x - 3y + 4z)^2 = 4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$$

Example 23 : Find : $\left(\frac{2x}{3}+\frac{y}{5}\right)^2$

Solution : Let $a = \frac{2x}{3}$, $b = \frac{y}{5}$ in the expansion of $(a + b)^2$.

$$\left(\frac{2x}{3}+\frac{y}{5}\right)^2 = \left(\frac{2x}{3}\right)^2 + 2\left(\frac{2x}{3}\right)\left(\frac{y}{5}\right) + \left(\frac{y}{5}\right)^2$$

$$= \frac{4x^2}{9} + 2\left(\frac{2xy}{15}\right) + \frac{y^2}{25}$$

$$= \frac{4x^2}{9} + \frac{4xy}{15} + \frac{y^2}{25}$$

Example 24 : Find : $\left(\frac{x}{3}-\frac{2y}{5}\right)^3$

Solution : Let $a = \frac{x}{3}$, $b = \frac{2y}{5}$ in the expansion of $(a - b)^3$

We get, $$\left(\frac{x}{3}-\frac{2y}{5}\right)^3 = \left(\frac{x}{3}\right)^3 - 3\left(\frac{x}{3}\right)\left(\frac{2y}{5}\right)\left(\frac{x}{3}-\frac{2y}{5}\right) - \left(\frac{2y}{5}\right)^3$$

$$= \frac{x^3}{27} - \frac{3(2xy)}{15}\left(\frac{x}{3}-\frac{2y}{5}\right) - \frac{8y^3}{125}$$

$$= \frac{x^3}{27} - \frac{2xy}{5}\left(\frac{x}{3}-\frac{2y}{5}\right) - \frac{8y^3}{125}$$

$$= \frac{x^3}{27} - \frac{2x^2y}{15} + \frac{4xy^2}{25} - \frac{8y^3}{125}$$

Example 25 : Find the value of $(110)^3$

Solution : $(110)^3 = (100 + 10)^3$

Let $a = 100$, $b = 10$, in the expansion of $(a + b)^3$

$$(110)^3 = (100)^3 + 3(100)(10)(100 + 10) + (10)^3$$

$$= 1000000 + 3000(110) + 1000$$

$$= 1000000 + 330000 + 1000$$

$$= 1331000$$

Example 26 : Find the value of $(997)^3$

Solution : $997 = 1000 - 3$

$\therefore \quad (997)^3 = (1000 - 3)^3$

Let, $a = 1000$, $b = 3$ in the expansion of $(a - b)^3$

$(997)^3 = (1000)^3 - 3(1000)(3)(1000 - 3) - (3)^3$

$= 1000000000 - 9000000 + 27000 - 27$

$= 991026973$

The above identities are in expansion form. If we interchange their L.H.S. and R.H.S. then they are considered to be in factor form as follows :

Expression on L.H.S. represents expansion of expression on R.H.S. and expression on R.H.S. represents factors of expression on L.H.S.

(1) $a^2 + 2ab + b^2 = (a + b)^2$

(2) $a^2 - 2ab + b^2 = (a - b)^2$

(3) $a^2 - b^2 = (a - b)(a + b)$

(4) $x^2 + (a + b)x + ab = (x + a)(x + b)$

(5) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2$

(6) $a^3 + b^3 + 3a^2b + 3ab^2 = (a + b)^3$

(7) $a^3 - b^3 - 3a^2b + 3ab^2 = (a - b)^3$

(8) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Hence if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

Example 27 : Factorize : $27x^3 + 189x^2y + 441xy^2 + 343y^3$

Solution : This expression can be writen as

$(3x)^3 + (7y)^3 + 3(3x)^2 (7y) + 3(3x)(7y)^2$

$= (3x)^3 + (7y)^3 + 3(3x)(7y)(3x + 7y)$

$= (3x + 7y)^3$

Example 28 : Factorize : $x^3 + 8y^3 - 27z^3 + 18xyz$

Solution : $x^3 + 8y^3 - 27z^3 + 18xyz = (x)^3 + (2y)^3 + (-3z)^3 - 3(x)(2y)(-3z)$

$= (x + 2y - 3z)[(x)^2 + (2y)^2 + (-3z)^2 - (x)(2y) - (2y)(-3z) - (x)(-3z)]$

$= (x + 2y - 3z)(x^2 + 4y^2 + 9z^2 - 2xy + 6yz + 3zx)$

Example 29 : Factorize : $9x^2 - 30xy + 25y^2$

Solution : $9x^2 - 30xy + 25y^2$

$= (3x)^2 - 2(3x)(5y) + (5y)^2$

$= (3x - 5y)^2$

Example 30 : Factorize : $a^4 - 81b^4$

Solution : $a^4 - 81b^4 = (a^2)^2 - (9b^2)^2$

$= (a^2 - 9b^2)(a^2 + 9b^2)$

$= [(a)^2 - (3b)^2](a^2 + 9b^2)$

$= (a - 3b)(a + 3b)(a^2 + 9b^2)$

Example 31 : Factorize : $x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6zx$

Solution : $x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6zx$

$= (x)^2 + (2y)^2 + (3z)^2 + 2(x)(2y) + 2(2y)(3z) + 2(x)(3z)$

$= (x + 2y + 3z)^2$

EXERCISE 3.5

1. Use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$ to find the value of the following product :

(1) $(x - 7)(x - 12)$ (2) $(5 - 4x)(7 - 4x)$

(3) $\left(x+\frac{3}{2}\right)\left(2x+\frac{5}{3}\right)$ (4) $\left(3x+\frac{3}{2}\right)\left(3x+\frac{5}{2}\right)$

2. Evaluate by using $(a^2 - b^2) = (a - b)(a + b)$

(1) 97×103 (2) 57×63 (3) 34×26

3. Factorize the following by using appropriate identities.

(1) $16x^2 - 40xy + 25y^2$ (2) $\frac{x^2}{9} + \frac{4xy}{15} + \frac{4y^2}{25}$

(3) $9a^2 + 25b^2 + 49c^2 - 30ab + 70bc - 42ac$

(4) $16a^4 - 625b^4$ (5) $\frac{8x^3}{27} + \frac{27y^3}{64} + \frac{64z^3}{125} - \frac{6}{5}xyz$

(6) $125a^3 + 600a^2b + 960ab^2 + 512b^3$ (7) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

4. Evaluate by using the identities :

(1) 105×102 (2) $(92)^2$ (3) $(8)^3 - (4)^3$

5. If $a + b + c = 0$, by using the identity $a^3 + b^3 + c^3 = 3abc$, find the value of $(-28)^3 + (15)^3 + (13)^3$.

EXERCISE 3

1. If for $p(x) = x^3 + kx^2 - 4x + 5$, $p(3) = 0$, then find the value of k.

2. Divide the following polynomials by $(x + 2)$ and find the quotient and the remainder :

(1) $p(x) = x^4 + 2x^3 + 7x^2 + x - 5$ (2) $p(x) = 2x^3 - 5x^2 + 11x + 19$

(3) $p(x) = 5x^3 + 9x^2 + 8x + 20$

3. If a student of std. IX - A, distributes equal number of chocolates from $x^4 - 3x^3 + 5x^2 + 8x + 5$ chocolates to all of his friends, then each friend gets $(x^2 - 1)$ chocolates and remain 26 chocolates there with him for his teachers. Find how many chocolates did the boy have ? How many chocolates does each friend get ? How many friends did the boy have ?

4. In a class ₹ $(2x + 3)$ were collected from each student for relief fund. If the total sum collected was ₹ $(2x^3 + x^2 - 5x - 3)$, find the number of students in the class.

5. The product of two polynomials is $x^4 - 3x^3 + 8x^2 - 9x + 15$. If one of the polynomials is $(x^2 - 3x + 5)$, then find the other.

6. If $x - 4$ is a factor of $x^3 - 6x^2 + 4x + 16$, then find the other factor.

7. Evaluate $(107)^2$ by using appropriate identity.

8. Find the value of $(-7)^3 + (12)^3 + (-5)^3$ by using appropriate identity.

9. Factorize : $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20zx$

10. Find the quotient and the remainder when following divisions are carried out

(1) $(2x^3 + x^2 + 9x + 17) \div (x - 2)$

(2) $(x^5 + 1) \div (x + 1)$

(3) $(3x^4 + 7x^3 - 6x^2 + 5x - 9) \div (x - 1)$

(4) $(7x^3 - 11x^2 + 3x - 49) \div (x^2 + x + 3)$

11. If $a + b + c = 6$ and $a^2 + b^2 + c^2 = 60$, then find $ab + bc + ca$ and $a^3 + b^3 + c^3 - 3abc$.

12. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If $p(3) = 0$, then a factor of $p(x)$ is ...... ☐

(a) $(x - 3)$ (b) $(x - 2)$ (c) $(x + 3)$ (d) $(x + 2)$

(2) If $x^3 + 2x^2 - 6x + 9$ is divided by $x - 2$, then ...... is the remainder. ☐

(a) -13 (b) 13 (c) 9 (d) -16

(3) The degree of the polynomial $x^5 + 3x^3 - 7x^2 + 9x + 11$ is .... ☐

(a) 1 (b) 2 (c) 3 (d) 5

(4) If $x - 2$ is a factor of $3x^4 - 2x^3 + 7x^2 - 21x + k$, then the value of k is ...... ☐

(a) 2 (b) 9 (c) 18 (d) -18

(5) The zero of $7x - 3$ is ...... ☐

(a) $\frac{-3}{7}$ (b) $\frac{3}{7}$ (c) $\frac{7}{3}$ (d) $\frac{-7}{3}$

(6) If $x^2 + 6x + 7$ is divided by $x + 1$, then the remainder is ......

(a) 1 (b) 2 (c) 5 (d) 7

(7) Factors of $y^2 + 10y + 21$ are ......

(a) $(y + 3)$ and $(y - 7)$ (b) $(y - 3)$ and $(y + 7)$

(c) $(y - 3)$ and $(y - 7)$ (d) $(y + 3)$ and $(y + 7)$

(8) If $a - b = 2$ and $ab = 3$, then $a^3 - b^3 =$ ......

(a) 8 (b) 27 (c) 26 (d) 6

(9) If $a = b = c$ then $a^3 + b^3 + c^3 - 3abc =$ ......

(a) a^3 (b) $2a^3$ (c) $3a^3$ (d) 0

(10) If one factor of the polynomial $x^3 + 4x^2 - 3x - 18$ is $x + 3$, then the other factor is ......

(a) $x^2 + x$ (b) $x^2 + x + 6$ (c) $x^2 + x - 6$ (d) $x^2 - x + 6$

(11) If $(x^3 + 28)$ is divided by $(x + 3)$, then the remainder is ......

(a) 0 (b) 1 (c) -1 (d) 2

(12) ...... should be added to $x^3 - 76$ so that the resulting polynomial is divisible by $x - 4$.

(a) 5 (b) -5 (c) 12 (d) -12

(13) If $25x^2 - 49y^2$ has one factor $(5x - 7y)$, then the other factor is ......

(a) $7x + 5y$ (b) $-7x - 5y$ (c) $5x + 7y$ (d) $-5x + 7y$

(14) If $p(x) = x^3 - 2x^2 + 7x - 6$, then a zero of $p(x)$ is ......

(a) 0 (b) 1 (c) 2 (d) 3

(15) If the cost of one mathematics textbook is ₹ $(x + 4)$, then ...... textbook can be purchased by ₹ $(x^3 + 64)$.

(a) $x^2 + 8x + 16$ (b) $x^2 - 8x - 16$ (c) $x^2 - 4x + 16$ (d) $x^2 - 4x - 16$

(16) $(4x - 7y)^3 =$ ......

(a) $4x^3 - 7y^3 + 84xy$ (b) $16x^3 + 49y^3 + 84xy$

(c) $64x^3 - 343y^3 - 336x^2y + 588xy^2$ (d) $64x^3 + 343y^3 + 336x^2y - 588xy^2$

*

Summary

1. An expression of form $a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where $a_0, a_1, a_2, ... a_n \in$ R; $a_n \neq 0$ is called a polynomial in variable x. ($n \in$ N $\cup$ {0}). n is called the degree of the polynomial.
2. If a polynomial has degree zero, then it is a constant polynomial.
3. Constant polynomial 0 is known as zero polynomial.
4. According as a polynomial has one, two or three terms, then it is known as a monomial, a binomial or a trinomial respectively.
5. If the degree of a polynomial is 1, it is linear polynomial.
6. According as a polynomial has the degree 2 or 3, it is known as a quadratic or a cubic polynomial respectively.
7. If $a \in$ R and $p(a) = 0$ then a is a zero of polynomial $p(x)$. a is also called the root of polynomial equation $p(x) = 0$.
8. **Remainder Theorem :** If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.
9. **Factor Theorem :** If $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$ and if $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$.
10. $(x - 1)$ is a factor of a polynomial, if the sum of its coefficients is zero.
11. $(x + 1)$ is a factor of a polynomial, if the sum of coefficients of odd power of x equals, sum of coefficients of even power of x.
12. $(a + b)^2 = a^2 + 2ab + b^2$
13. $(a - b)^2 = a^2 - 2ab + b^2$
14. $(a + b)(a - b) = a^2 - b^2$
15. $(x + a)(x + b) = x^2 + (a + b)x + ab$.
16. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
17. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)$
18. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)$
19. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
20. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

•

CHAPTER 4

COORDINATE GEOMETRY

'I think, therefore I am.' - **René Déscartes**

4.1 Introduction

A very beautiful and important branch of mathematics known as **Coordinate Geometry**, was initially developed by the French philosopher and mathematician **René Déscartes** (1596-1650).

We know how to describe the position of a point on the real line. Every real number is represented by a unique point on the number line and also, every point on the number line represents a unique real number. In other words there is a one-to-one correspondence between the points on the line and the set of all real numbers.

The real (number) line is given here :

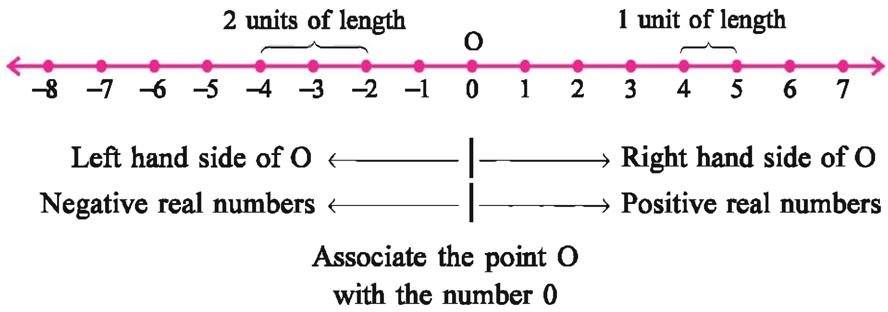

Figure 4.1

In a plane to describe the exact position of a point we need the reference of more than one line. For example, consider the following situation.

In figure 4.2, there is a main road running in the East-West direction and streets with numbering from West to East and house numbers from 1 to 5 are marked on each street.

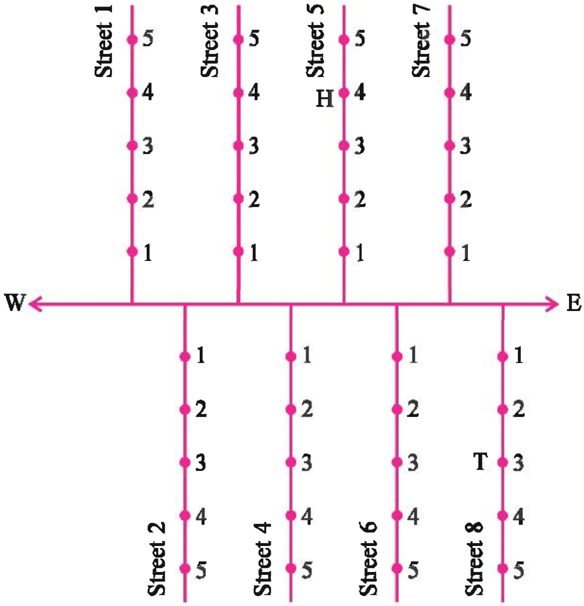

Figure 4.2

To look for a friend's house here, we need to know two points of information about it, namely the number of the street on which it is situated and the house number. If we want to reach the house, which is situated in the 5th street and has the number 4, first of all we would identify the 5th street and then the house numbered 4 on it. In figure. 4.2., H shows the location of the house. Similarly, T shows the location of the house corresponding to street number 8 and house number 3.

Two points are to be noted from this illustration :

To reach a definite place (1) one should start from a fixed point, and (2) the house number must be given on a specific perpendicular street.

Thus, we observe that to describe the position of any object lying in a plane, we need two perpendicular lines. This simple idea has given rise to a very important branch of mathematics known as Coordinate Geometry.


René Dēscartes
(1596-1650)

René Déscartes was born on 31st March, 1596 in LaHaye in South of France, the great French Mathematician of the seventeenth century, liked to lie in bed and think ! One day, when resting in bed, he solved the problem of describing the position of a point in a plane. His method was development of the older idea of latitude and longitude. He is credited as the father of analytic geometry. In honour of René Déscartes, the system used for describing the position of a point in a plane is also known as the *Cartesian system*. He died on 11th February, 1650, Stockholm, Sweden.

In honour of René Dēscartes, the system used for describing the position of a point in a plane is known as the cartesian coordinate system. In this chapter, we shall learn about it.

4.2 Cartesian Coordinate System

If A and B are non-empty subsets of the set R of real numbers, then the Cartesian product of non-empty sets A and B, symbolically represented by A × B (to be read : A cross B) is the set of all ordered pairs (a,b), where $a \in$ A, $b \in$ B. We write, $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Here, A × B, B × A, A × A and B × B all are subsets of R × R. Also, they can be represented by a graph. This type of representation depends on a graph drawn in a plane. So, first of all we shall study the method of sketching a graph in a plane.

Draw two perpendicular lines in the plane; one horizontal and another vertical. The point of their intersection O is called the **origin**.

The horizontal line is called the X-axis and the vertical line is called the Y-axis. Both the axes are called the coordinate axes. The plane is called the **co-ordinate plane** or the **cartesian plane**, or the **XY-plane**.

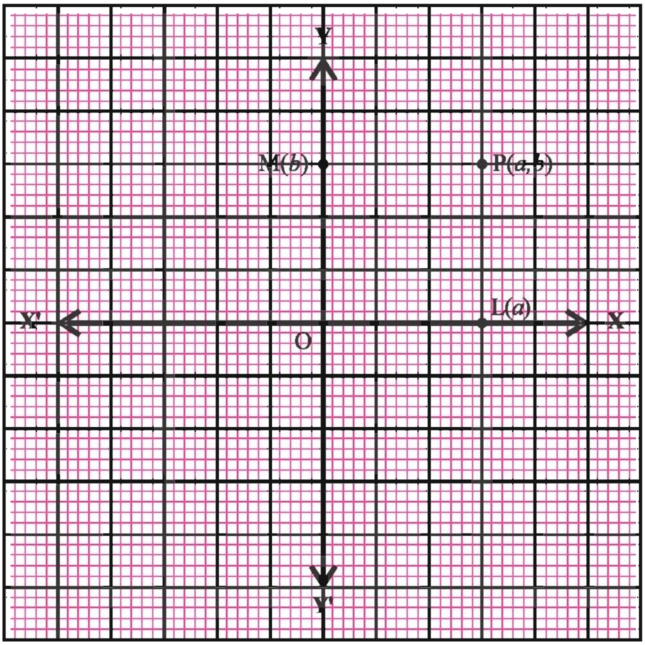

Figure 4.3

We know that there is a one-to-one correspondence between a line and the set of all real numbers.

Associate the point O on the X-axis with the number 0 (zero). Associate the points on the right hand side of O on the X-axis with positive real numbers. Associate the points on the left hand side of O on the X-axis with negative real numbers. Thus, corresponding to each point on the X-axis, there is a unique real number and conversely, corresponding to each real number, there is a unique point on the X-axis.

Similarly O on Y-axis corresponds to zero and points on Y-axis above semi plane of X-axis, correspond to positive real numbers and points on Y-axis below semi plane of X-axis correspond to negative real numbers. Thus, to every real number corresponds a point on Y-axis and conversely.

Now corresponding to each ordered pair (a, b) of R × R, we get points L(a) and M(b) on X-axis and Y-axis respectively. If the lines perpendicular to X-axis from L and perpendicular to Y-axis from M intersect in P, then P is called the point corresponding to the ordered pair (a,b). It is denoted as P(a,b). Conversely, for each point P of the plane, we get L(a) and M(b) on the coordinate axes (by drawing perpendiculars). (See figure 4.3)

Thus, corresponding to each ordered pair (a, b) there is a unique point P in the plane and corresponding to each point P in the plane, there is a unique ordered pair (a, b) in R × R. a is called the ***x*-coordinate or abscissa** of P and b is called the ***y*-coordinate or ordinate** of P.

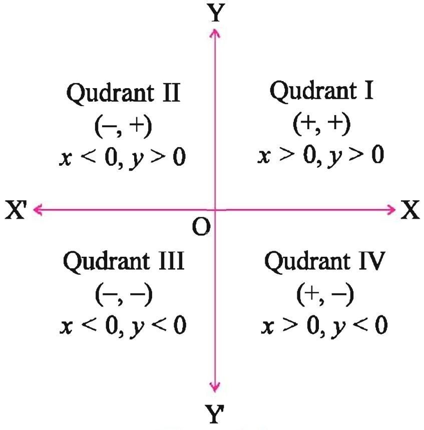

Figure 4.4

Quadrant : In the cartesian coordinate system, the perpendicular axes (i.e. coordinate axes) partition the plane into four parts. The plane is the union of points on axes and points in each subset (part). Each part is known as a quadrant. Each quadrant is named by numbers I, II, III and IV in anti-clockwise direction starting from $\overrightarrow{OX}$ (see Fig. 4.4)

The plane is the union set of the axes and four quadrants.

Considering the association of real numbers with coordinate axes, we get the following property of each quadrant :

Quadrant	part	x-co-ordinate	y-co-ordinate
First (I)	Interior of $\angle XOY$	+	+
Second (II)	Interior of $\angle YOX'$	–	+
Third (III)	Interior of $\angle X'OY'$	–	–
Fourth (IV)	Interior of $\angle Y'OX$	+	–

An ordered pair corresponding to a point in the plane : Suppose P is a point in the plane. The feet of perpendiculars from P to X-axis and Y-axis are M and N respectively. The unique real numbers associated with M and N are 3 and 4 respectively. So, the x-coordinate of P is 3 and the y-coordinate is 4. Thus, corresponding to P, there is a unique ordered pair (3, 4) of real numbers. (see figure 4.5)

Similarly by considering another point Q in the plane, the unique ordered pair corresponding to Q is (–3, –4). (See figure 4.5)

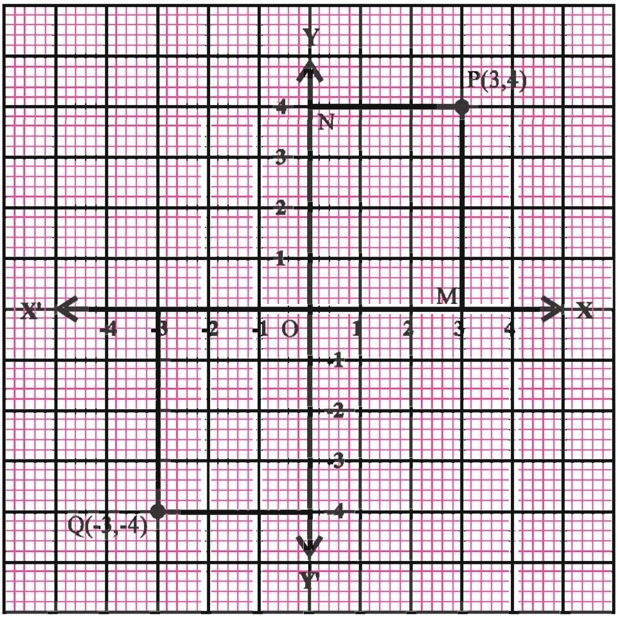

Figure 4.5

From this illustration, we can say that corresponding to each point in the plane there is a unique ordered pair of real numbers.

Coordinates of the origin are (0, 0). For a point on the X-axis with corresponding number a (called x coordinate or abscissa) y-coordinate is always 0. Thus, on the X-axis the points are of the form $(a, 0)$; a is a real number. Similarly, the points on the Y-axis are of the form $(0, b)$. Here, b is called y-coordinate or ordinate. b is a real number.

Example 1 : (a) Write the x-coordinate (abscissa) and the y-coordinate (ordinate) of the points A, B, C, D, P, Q, R, U, V and W in the figure 4.6. (b) Write the coordinates of all the points T, M, N and S from the figure 4.6

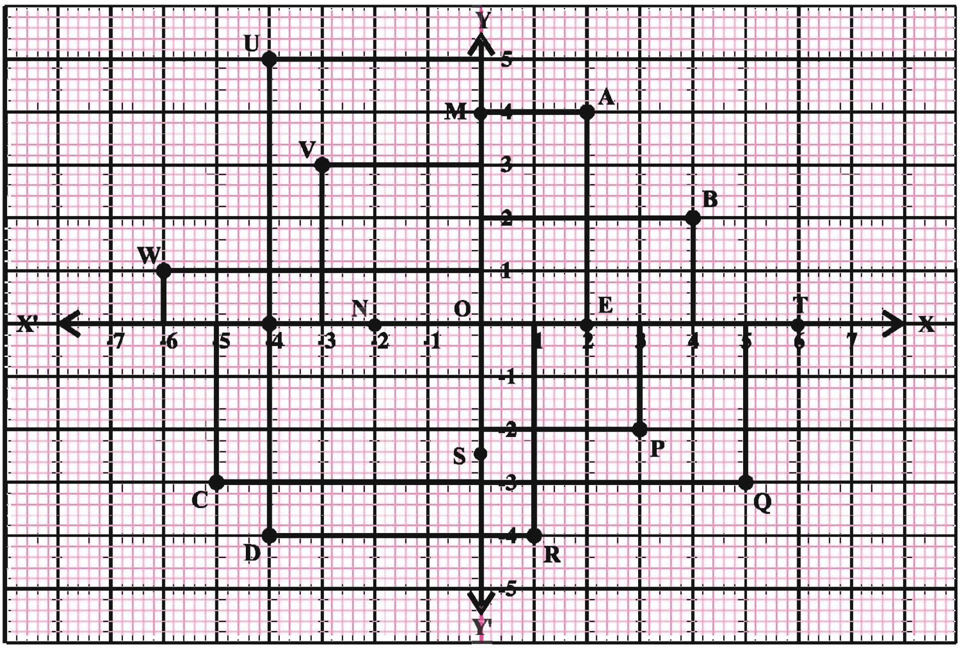

Figure 4.6

Solution : (a) The feet of perpendiculars from the point A to the Y-axis and X-axis are respectively M and E. The unique real numbers 2 and 4 are associated with E and M respectively.

∴ The x-coordinate of the point A is 2 and the y-coordinate of the point A is 4. So we write A(2,4). For the point B; the foot of perpendicular from the point B to the Y-axis is associated with the unique real number 2 and the foot of perpendicular from the point B to the X-axis is associated with the unique real number 4.

∴ 4 is the x-coordinate of the point B and 2 is the y-coordinate of the point B i.e. B(4,2). Now from the point U the feet of perpendiculars to the X-axis and to the

Y-axis are associated with the unique real numbers −4 and 5 respectively. Hence, for the point U, the x-coordinate is −4 and the y-coordinate is 5.

For the point D, we get unique numbers −4 on the X-axis and −4 on the Y-axis associated with the feet of perpendiculars from the point D. So the coordinates of D are (−4, −4), i.e. D(−4, −4). Similarly for the point P the *x*-coordinate of P is 3 and the *y*-coordinate of P is −2. i.e. P(3, −2). Similarly W(−6, 1), C(−5, −3), Q(5, −3), R(1, −4) and V(−3, 3).

(b) Similarly for the given points the coordinates are T(6, 0), M(0, 4), N(−2, 0) and $S\left(0, \frac{-5}{2}\right)$

EXERCISE 4.1

1. Answer as directed :
   (1) Write the name of the axes in the cartesian plane.
   (2) Give the name of each subset (part) of the plane partitioned by the axes.
   (3) Do the axes intersect ? If yes, give the name of the point of intersection and also write its coordinates.
2. See the figure 4.7 and answer the questions that follows :

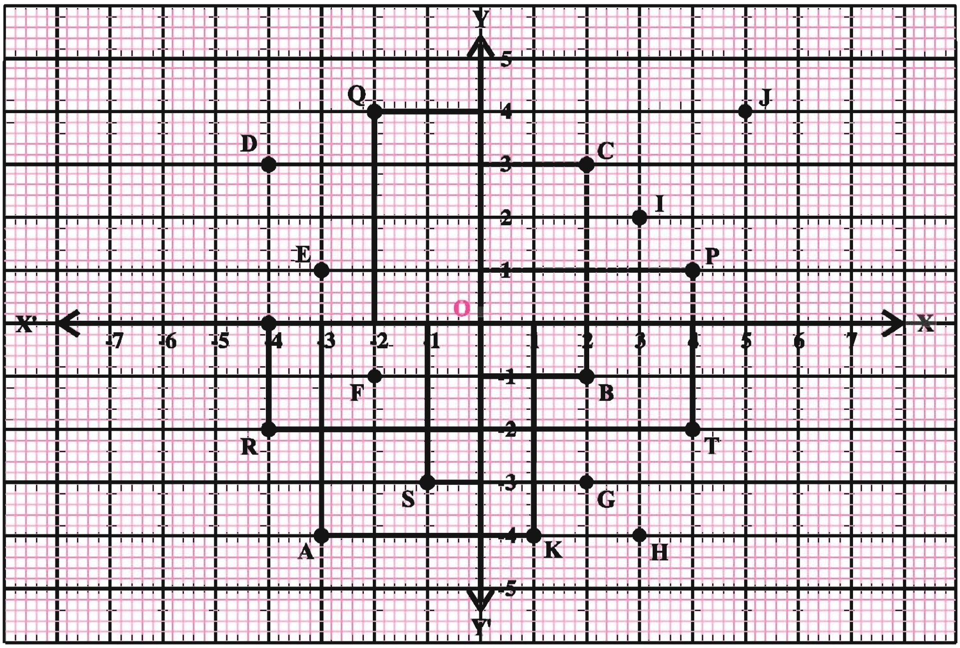

Figure. 4.7

(1) The coordinates of P and Q
(2) Point identified as (−2, 4).
(3) The abscissa of the point A.
(4) The ordinate of the point R.
(5) Write the coordinates of the points A, B, C, D, E, F, G, H, I, J, R, S and T.

*

4.3 Plotting a Point in the Plane if its Coordinates are Given

Let us obtain a point in the plane corresponding to the ordered pair (2, 3). The x-coordinate and y-coordinate are positive. On X-axis on the right side of O, there is a unique point M corresponding to 2. On Y-axis, in the upper half-plane there will be a unique point N corresponding to 3. Draw lines from M and N, perpendicular to X-axis and to Y-axis respectively. The unique point P of their intersection is the point in the plane corresponding to (2, 3).

Now let us represent graphically the point corresponding to the ordered pair (−2, −3) in the plane. Both the coordinates of (−2, −3) are negative. On X-axis, on the left hand side of O, there is a unique point A corresponding to −2 and on Y-axis, in the lower half plane of the X-axis, there is a unique point B corresponding to −3. Draw lines perpendicular to X-axis from A and to Y-axis from B respectively. Their point of intersection, the unique point Q, is the point in the plane corresponding to (−2, −3). Similarly (−2, 3) and (2, −3) are represented as points R and S respectively (See figure 4.8).

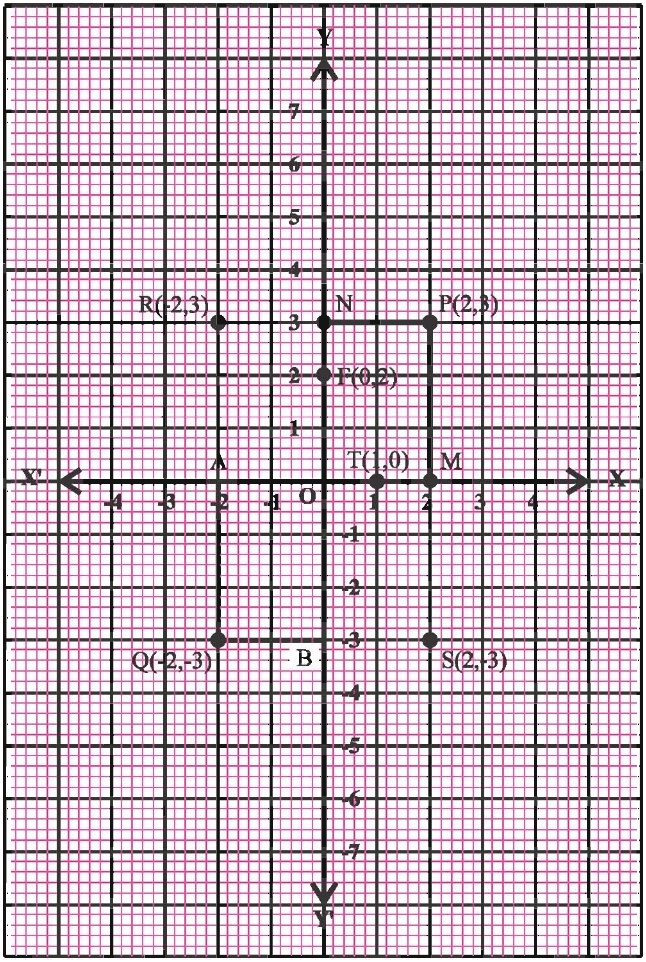

Figure. 4.8

We have seen that, a point x-coordinate of which is zero lies on the Y-axis and a point y-coordinate of which is zero lies on the X-axis. T represents (1, 0) and F represents (0, 2)

From this illustration, we can say that to each ordered pair of real numbers, a unique point of the plane is associated. **(i)**

We have also seen that corresponding to each point in the plane there is a unique ordered pair of real numbers. **(ii)**

From (i) and (ii) we can say that there is a one-one correspondence between the plane and R × R and if a point P of the plane and the ordered pair (x, y) correspond to each other, then we write P (x, y).

P is called the representation of (x, y) in the plane and x and y are called cartesian co-ordinates of P. x is called the x-coordinate and y is called the y-coordinate of P. In fact, we identify P and (x, y) and say that (x, y) (like P) is a point of the plane.

By drawing graph of a set A × B we mean plotting of points of A × B in the Cartesian plane.

Example 2 : Locate the point corresponding to ordered pairs.

(−3, 4), (−3,−1), (4, 0), (0, 5), (1, −2) and (2, 3) in the Cartesian plane.

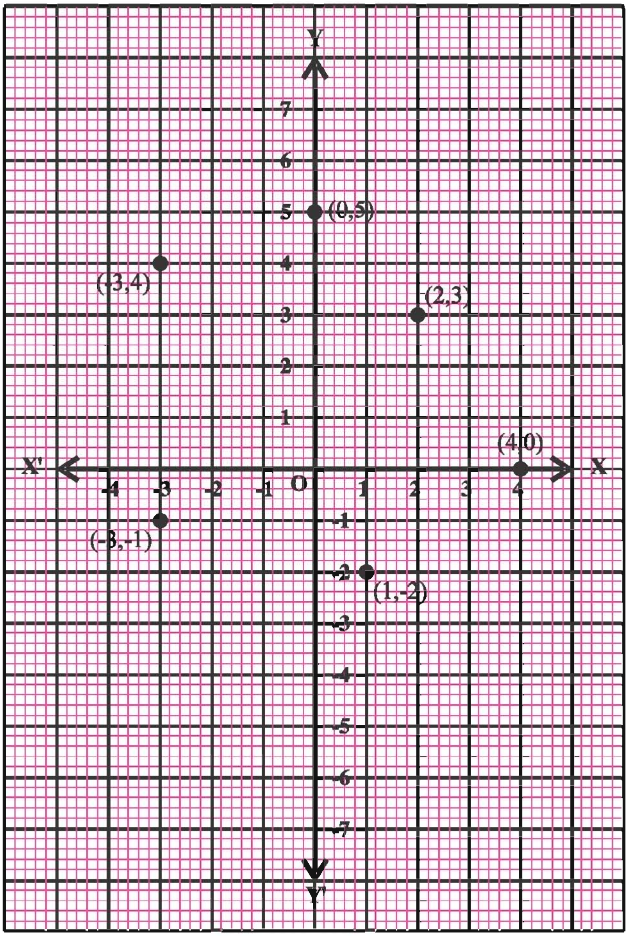

Figure. 4.9

Solution : Taking the scale 1 cm = 1 unit on the axes draw the X-axis and Y-axis on the graph paper. The positions of the points are shown by dots in the Fig. 4.9

Note : For the ordered pairs (a, b) and (p, q), $(a, b) = (p, q)$ if and only if $a = p$ and $b = q$. For example, let us find x and y, if $(5, 4y - 1) = (3x - 4, 7)$

Here, $(3x - 4, 7) = (5, 4y - 1)$

$\therefore$ $3x - 4 = 5$ and $4y - 1 = 7$

$\therefore$ $3x = 5 + 4$ and $4y = 7 + 1$

$\therefore$ $3x = 9$ and $4y = 8$

$\therefore$ $x = \frac{9}{3}$ and $y = \frac{8}{4}$

$\therefore$ $x = 3$ and $y = 2$

EXERCISE 4.2

1. Plot the following ordered pairs (x, y) in the plane :

$(-4, -3)$, $(-3, 5)$, $(-2, -4)$, $(-1, 6)$, $(0, 2)$, $(1, -3.5)$, $(2, 3)$, $(4, -2)$.

2. Plot the points (x, y) in the cartesian plane obtained by taking values of x in the polynomial $y = 3x - 2$, $x = -3, -2, -1, 0, 1, 2, 3, 4$.

3. If P = $\{0, 1, -1\}$ and Q = $\{-3, 2\}$, then draw the graph of P $\times$ Q and Q $\times$ P.

4. If A = $\{-2, 3\}$ and B = $\{-1, 1, 4\}$, then draw the graphs of

(1) A $\times$ B (2) B $\times$ A (3) A $\times$ A (4) B $\times$ B

5. Plot the points A(4, 5), B(−2, −1), C(−3, 6) and D(5, −2). From the graph, find the midpoints of $\overline{AB}$ and $\overline{CD}$.

6. Represent the points M(3, 4), N(−3, −2), P(−2, 5) and Q(4, −1) in the plane. Draw $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. From the graph, find their point of intersection.

7. Examine the validity of the following statements :

(1) Point (4, 0) lies on the X-axis.

(2) P(−2, 3) is a point in the third quadrant.

(3) For the point A, if the abscissa is 4 and the ordinate is −3, then A lies in the fourth quadrant.

(4) The point of intersection of the axes has co-ordinates (0, 0).

(5) In the plane the position of (y, x) is the same as the position of (x, y), where $x \neq y$.

(6) B(0, −9) is a point on $\overrightarrow{OY}$.

(7) For $x = 3$, $y = 2$, $u = -7$, $v = 11$ the point $(x - u, y - v)$ lies in the 1st quadrant.

(8) Point (4, −5) lies in the lower half-plane of the X-axis and to the right hand side of Y-axis.

(9) If $(2x - 1, 1)$ and $(-3, 3y - 2)$ are two equal ordered pairs of $R \times R$, then $x = -1$ and $y = 1$.

EXERCISE 4

1. In which quadrant do the following points lie ?

(1) (2, 4) (2) (3, −5) (3) $\left(-5, -\frac{3}{2}\right)$ (4) (−5, 2) (5) $\left(\frac{1}{2}, \frac{-1}{2}\right)$ (6) (1, 3)

2. Plot the following points on a graph paper :

(1) (3, −2) (2) (−2, 3) (3) (−5, 4) (4) (−4, −3) (5) (0, 4) (6) (3, 0)

3. If A = {−3, 2, 4} and B = {−1, 1}, then draw the graphs of

(1) $A \times B$ (2) $A \times A$ (3) $B \times B$ (4) $B \times A$

4. Write the coordinates of each of the following points marked in the graph paper (figure 4.10).

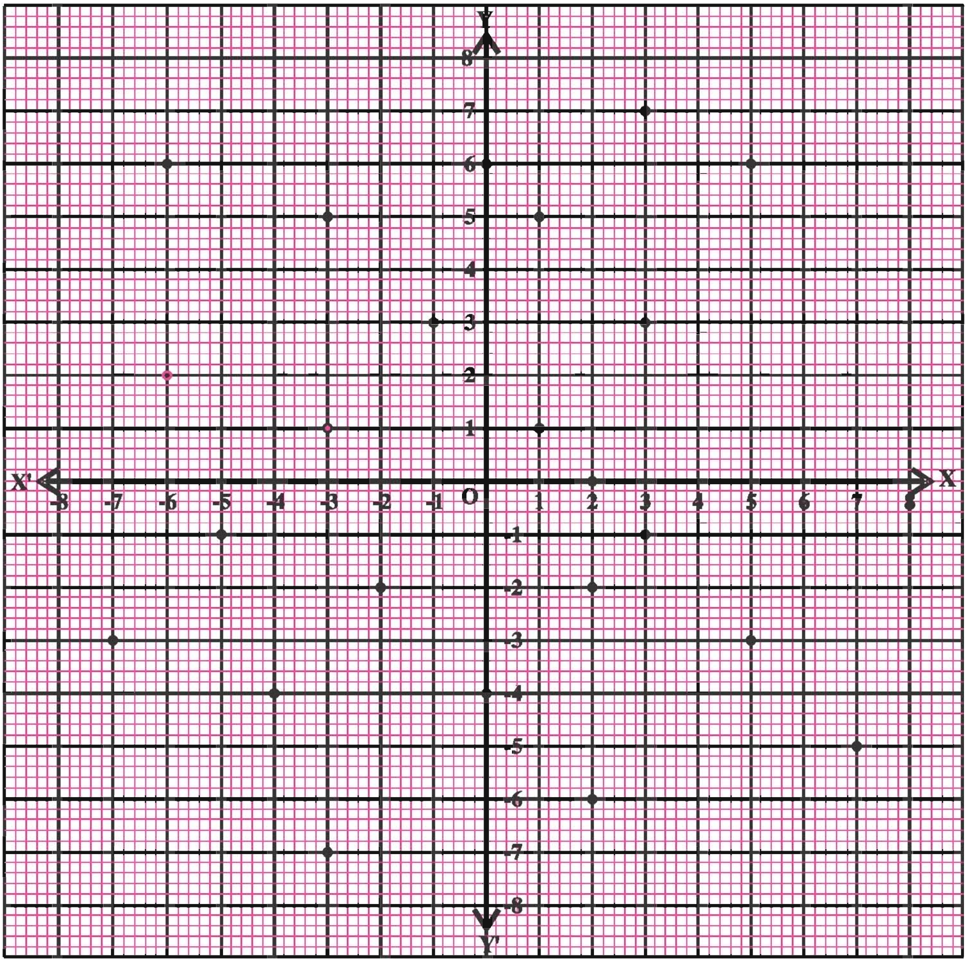

Figure 4.10

5. If $x = -1$, $y = 5$, $z = 3$, $w = -4$, then in which quadrants do the points $(x + y, z + w)$, $(y - z, w + x)$ and $(x - w, y + z)$ lie ?

6. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Point (4, 0) lies on ...... ☐

(a) $\overrightarrow{OX'}$ (b) $\overrightarrow{OY}$ (c) $\overrightarrow{OX}$ (d) $\overrightarrow{OY'}$

(2) For a point, if the abscissa is –3 and the ordinate is 5, then it lies in the ...... quadrant. ☐

(a) I (b) II (c) III (d) IV

(3) The point of intersection of the axes has co-ordinates .... ☐

(a) (0, 1) (b) (1, 0) (c) (0, 0) (d) (0, –1)

(4) The point (–2, 0) lies on ...... ☐

(a) $\overrightarrow{OY}$ (b) $\overrightarrow{OX'}$ (c) 1^{st} quadrant (d) $\overrightarrow{OX}$

(5) Point (5, –2) lies in the ...... quadrant. ☐

(a) I (b) II (c) III (d) IV

(6) For the point (7, –4), the abscissa is ...... ☐

(a) –4 (b) –7 (c) 4 (d) 7

(7) For the point (3, –5), the ordinate is ...... ☐

(a) 3 (d) 5 (c) –3 (d) –5

(8) For the origin O, abscissa and ordinate are both ...... ☐

(a) 1 (b) –1 (c) 0 (d) 0.5

(9) The 3rd quadrant is the interior of ...... ☐

(a) ∠YOX' (b) ∠X'OY' (c) ∠Y'OX (d) ∠XOY

(10) The coordinates of any point on the Y-axis are of the form (0, *b*), where | *b* | is the distance of the point from the ...... ☐

(a) Y-axis (b) X-axis (c) (0, 1) (d) (1, 0)

(11) The measure of the angle between the $\overleftrightarrow{X'X}$ and $\overleftrightarrow{Y'Y}$ is ...... ☐

(a) 90 (b) 0 (c) 180 (d) 60

(12) For $x = 3$, $y = 2$, $u = -9$, $v = 13$ the point $(x + y, u + v)$ lies in the ...... quadrant. ☐

(a) III (b) II (c) IV (d) I

(13) In the plane, $(x, y) = (y, x)$ if ...... ☐

(a) $x = 3, y = 3$ (b) $x = 3, y = 2$ (c) $x = 2, y = 3$ (d) $x = 1, y = 0$

(14) If the co-ordinates of the points are of the same sign (both positive or both negative), then points lie in the ...... quadrants. ☐

(a) I and II (b) I and III (c) I and IV (d) II and IV

(15) The point having coordinates of the opposite signs lies in ...... ☐

(a) I and II (b) I and III (c) I and IV (d) II and IV

(16) Any point on the X-axis is of the type ...... ☐

(a) (0, x) (b) (0, y) (c) (0, 1) (d) (a, 0)

(17) The coordinate axes divide plane into ...... parts called quadrants. ☐

(a) two (b) five (c) four (d) six

(18) X-axis is a horizontal line passing through ...... ☐

(a) Point (0, 1) (b) origin (c) Point (0, −1) (d) quadrant I

(19) The vertical line through the origin is called the ...... ☐

(a) X-axis (b) XY-plane (c) Y-axis (d) $\overleftrightarrow{Y'Y}$

(20) The ...... quadrant is bounded by the $\overrightarrow{OX'}$ and the $\overrightarrow{OY}$. ☐

(a) 1st (b) 3rd (c) 2nd (d) 4th

(21) In the plane origin O (0,0) lies on the ...... ☐

(a) X-axis only (b) Y-axis only

(c) 1st quadrant (d) X-axis and Y-axis both

(22) The point (0, 3) lies on the ...... ☐

(a) X-axis (b) $\overleftrightarrow{Y'Y}$ (c) 1st quadrant (d) 2nd quadrant

(23) The point (−4, 0) lies on the ...... ☐

(a) 2nd quadrant (b) $\overrightarrow{OX}$ (c) 3rd quadrant (d) $\overrightarrow{OX'}$

(24) The point (0, −2) lies on the ...... ☐

(a) Y-axis (b) X-axis

(c) 1st and 4th quadrant (d) 3rd quadrant

(25) The point (−3, 4) lies in the ...... ☐

(a) 1st quadrant (b) 3rd quadrant

(c) interior of ∠YOX' (d) interior of ∠Y'OX

*

Summary

In this chapter, you have studied the following points :

1. To locate the position of an ordered pair or a point in a plane, we require coordinate axes, namely, X-axis (the horizontal line) and Y-axis (the vertical line).
2. The plane is called the cartesian plane or coordinate plane or cartesian coordinate plane.
3. The point of intersection of the axes is called the origin O (0, 0).
4. If the x-coordinate (the abscissa) is a and the y-coordinate (the ordinate) is b, then a and b are called the coordinates of the point.
5. The coordinate axes divide the plane into four parts called quadrants.
6. On the X-axis every point is of the form $(x, 0)$ and on the Y-axis every point is of the form $(0, y)$. x and y are real numbers.
7. If $x \neq y$, then $(x, y) \neq (y, x)$ and $(x, y) = (y, x)$, if $x = y$

●

Birth Place	:	La Haye en Touraine, Touraine (present-day Descartes, Indre-et-Loire), France
Era	:	17th-century philosophy
Region	:	Western Philosophy
School	:	Cartesianism, Rationalism, Foundationalism
Main interests	:	Metaphysics, Epistemology, Mathematics
Notable ideas	:	Cogito ergo sum, method of doubt, Cartesian coordinate system, Cartesian dualism, ontological argument for the existence of Christian God; Folium of Descartes
Signature	:	

René Déscartes

CHAPTER 5

LINEAR EQUATIONS IN TWO VARIABLES

5.1 Introduction

In earlier classes, we have studied linear equations in one variable of the form $ax + b = c$ (where a, b, c are constants and $a \neq 0$). Such equations have a unique (i.e one and only one) solution.

Here, $ax + b = c$

So, $ax = c - b$

$\therefore$ Since $a \neq 0$, $x = \dfrac{c-b}{a}$ is the solution of the equation $ax + b = c$.

For example $x - 2 = 0$, $x + \sqrt{3} = 0$ and $\sqrt{5}x - \sqrt{7} = 0$ are linear equations in one variable. These equations have a unique solution.

Linear equations in one variable of the form $ax + b = cx + d$

(where $a, b, c, d \in R$, $a \neq c$) have a unique solution.

Linear equations in one variable of the form $\dfrac{ax+b}{cx+d} = k$; k is constant (where, $cx + d \neq 0$, $a, b, c, d \in R$, $a \neq kc$) have a unique solution.

Now, let us consider a practical problem related to the linear equation given by the statement. "The sum of the ages of two friends is 27 years and the difference of their ages is 3 years." Find their ages.

It can be translated into an equation form as follows. Suppose the older friend has age x years, then the other has age $(x - 3)$ years. Since sum of ages of both is 27.

$\therefore$ $x + (x - 3) = 27$

$\therefore$ $2x - 3 = 27$. This is a linear equation in one variable.

In this chapter, we shall recall the knowledge of linear equations in one variable and will extend it to linear equations in two variables. We shall also discuss whether the solution of a linear equation in two variables is unique or not and how the solution can be represented in the cartesian coordinate plane.

5.2 Linear Equations

Consider the linear equation $2x - 7 = 0$ in one variable.

Its solution (i.e. root of the equation) is $x = \frac{7}{2}$ and it can be represented on the number line as shown below :

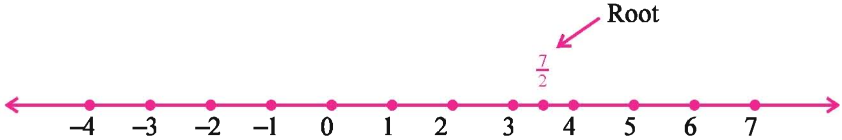

Figure 5.1

Linear Equations in Two Variables : "The sum of two numbers is 6", we shall express this statement in the form of an equation. Let these numbers be x and y.

Then the statement is translated in symbols as $x + y = 6$.

This is an equation having two variables x and y. Since, **the exponent of each variable is 1 and there is no product term xy, such an equation is called a linear equation in two variables.** It is customary to denote the variables in such equations by x and y, but other letters like p and q, r and s, u and v etc. may also be used. Some examples of linear equations in two variables are :

$3p + 2q = 12$, $2.5r + 4s = 7$, $6u + \pi v = 9$ and $\sqrt{3}x - 5y = 4$

The equations $x + y = 6$, $3x + 2y + 4 = 0$, $x - 2y + 3 = 0$ and $5x + \frac{3}{2}y = 4$ are linear equations in two variables x and y.

In these equations there are two variables and each variable occurs to index 1. The linear equation $3x - 2 = 5$ is an equation in one variable. It can also be expressed as an equation $3x + 0y - 7 = 0$ in two variables x and y.

Similarly, $4y - 3 = 0$ can be expressed as $0x + 4y - 3 = 0$; $5x = 0$ as $5x + 0y = 0$ and $3y = 0$ as $0x + 3y = 0$.

Thus, **each linear equation in one variable can be expressed in the form of a linear equation in two variables. The standard form of a linear equation in two variables is $ax + by + c = 0$, where $a, b, c \in$ R, a and b are not zero simultaneously, i.e. $a^2 + b^2 \neq 0$.**

Note : $a^2 \geq 0$ and $b^2 \geq 0$. So $a^2 + b^2 \geq 0$. If $a^2 + b^2 = 0$, then $a = b = 0$. Hence $a^2 + b^2 \neq 0$ means a and b are not simultaneously zero.

Now onwards whenever we consider a linear equation $ax + by + c = 0$ in two variables, we shall accept that $a, b, c \in R$ and a, b are not simultaneously zero, even if it is not explicitly mentioned.

Example 1 : State which of the following equations are linear equations in two variables and indicate the values of a, b and c in each case.

(1) $4x + 7y = 5$ (2) $3x = 6$ (3) $2 = 3x - 5y$ (4) $2x^2 = 3y$

(5) $6y = 0$ (6) $xy = 3$ (7) $x^2 + 3x + 2 = 0$

(8) $y - 3 = \sqrt{2}x$ (9) $7x = 8y$ (10) $4x + \frac{7}{3}y = \frac{11}{3}$

Solution : The standard of a linear equation in two variables is $ax + by + c = 0$.

(1) $4x + 7y = 5$ is a linear equation in two variables.
The standard form is $4x + 7y - 5 = 0$ where, $a = 4$, $b = 7$, $c = -5$.

(2) $3x = 6$ is a linear equation in two variables.
It can be written as $3x + 0y - 6 = 0$ in the standard form.
Here, $a = 3$, $b = 0$, $c = -6$.

(3) $2 = 3x - 5y$ is a linear equation in two variables.
It can be written as $3x - 5y - 2 = 0$ in the standard form.
Here, $a = 3$, $b = -5$, $c = -2$.

(4) $2x^2 = 3y$ is not a linear equation, as exponent of variable x is 2.

(5) $6y = 0$ is a linear equation in two variables.
It can be written as $0x + 6y + 0 = 0$ in the standard form.
Here, $a = 0$, $b = 6$, $c = 0$

(6) $xy = 3$ is not a linear equation in two variables as it is not in standard form $ax + by + c = 0$. Note that indices of x and y are 1 and xy has index $1 + 1 = 2$.

(7) $x^2 + 3x + 2 = 0$ is not a linear equation because in x^2, the index of variable x is 2.

(8) $y - 3 = \sqrt{2}x$ is a linear equation in two variables.
It can be written as $\sqrt{2}x - y + 3 = 0$ in the standard form.
Here, $a = \sqrt{2}$, $b = -1$, $c = 3$.

(9) $7x = 8y$ is a linear equation in two variables.
It can be written as $7x - 8y + 0 = 0$ in the standard form.
Here, $a = 7$, $b = -8$, $c = 0$

(10) $4x + \frac{7}{3}y = \frac{11}{3}$ is a linear equation in two variables.
It can be written as $12x + 7y - 11 = 0$ in the standard form.
Here, $a = 12$, $b = 7$, $c = -11$.
or $4x + \frac{7}{3}y - \frac{11}{3} = 0$, where $a = 4$, $b = \frac{7}{3}$, $c = -\frac{11}{3}$.

EXERCISE 5.1

1. "The cost of a notebook is twice the cost of a pen." Represent this statement as a linear equation in two variables.
2. State which of the following equations are linear equations in two variables and express them in the standard form $ax + by + c = 0$ and indicate the values of a, b and c in each case :

(1) $5x = 6y$	(2) $y^2 = 3x$	(3) $7x = 0$	(4) $6y - 4x = 3$
(5) $3x + 4.5y = 8.2$	(6) $y - \frac{x}{4} - 3 = 0$	(7) $9x = 3$	(8) $3x = 2y - 4$
(9) $y = 2x + 5$	(10) $\frac{3}{2}x + \frac{7}{2}y = 1$	(11) $3y^2 + 2x = 2$	(12) $\frac{x}{3} - \frac{4}{y} = 2$

*

5.3 Solution of a Linear Equation in Two Variables

We have seen that every linear equation in one variable has a unique solution. What can be said about the solution of a linear equation in two variables ? As there are two variables in the equation, a solution means a pair of values, one for x and one for y which will satisfy the given equation. Let us consider the equation $x + y = 6$.

Taking $x = 4$ and $y = 2$, we get

$\therefore \quad x + y = 6$

Thus, $x = 4$ and $y = 2$ is a solution of the equation $x + y = 6$ or (4, 2) is a solution of this equation. Similarly, ordered pairs (3, 3), (1, 5), (2, 4), (5, 1), (−1, 7), (−2, 8) etc. also satisfy the equation $x + y = 6$. All these ordered pairs are solutions of this equation. It is not necessary that the solutions are in integers.

If we take $x = \frac{3}{2}$, $y = \frac{9}{2}$, then $x + y = \frac{3}{2} + \frac{9}{2} = \frac{3+9}{2} = \frac{12}{2} = 6$.

Similarly, the ordered pairs $\left(\frac{5}{3}, \frac{13}{3}\right)$, $\left(\frac{1}{2}, \frac{11}{2}\right)$, $\left(\frac{1}{3}, \frac{17}{3}\right)$, $\left(\frac{2}{3}, \frac{16}{3}\right)$, $\left(\frac{1}{4}, \frac{23}{4}\right)$ are also solutions of $x + y = 6$.

Further, taking $x = \frac{\sqrt{3}}{2}$, $y = \frac{12-\sqrt{3}}{2}$ we get

$$x + y = \frac{\sqrt{3}}{2} + \frac{12-\sqrt{3}}{2} = \frac{\sqrt{3}+12-\sqrt{3}}{2} = \frac{12}{2} = 6$$

So, $\left(\frac{\sqrt{3}}{2}, \frac{12-\sqrt{3}}{2}\right)$ is also a solution.

Further $(\sqrt{3}, 6 - \sqrt{3})$, $(\sqrt{5}, 6 - \sqrt{5})$, $(\pi, 6 - \pi)$ are also solutions of the equation $x + y = 6$

Thus, ordered pairs (6, 0), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (0, 6), (−1, 7), $\left(\frac{9}{2}, \frac{3}{2}\right)$, $\left(\frac{17}{3}, \frac{1}{3}\right)$, (7, −1), $(\sqrt{5}, 6 - \sqrt{5})$, $(\sqrt{3}, 6 - \sqrt{3})$, $(\pi, 6 - \pi)$ etc. are solutions of $x + y = 6$. So, there is no end to different solutions of a linear equation in two variables.

Hence, we can say that a linear equation in two variables has infinite number of solutions. (or infinitely many solutions).

We have studied in the chapter of coordinate geometry how to plot a point on a graph paper. Let us plot the solutions (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) of $x + y = 6$ on a graph paper.

These points are collinear. If we join them by a straight edge, we get the graph of $x + y = 6$. Thus we see that the graph of $x + y = 6$ is a line (see figure 5.2).

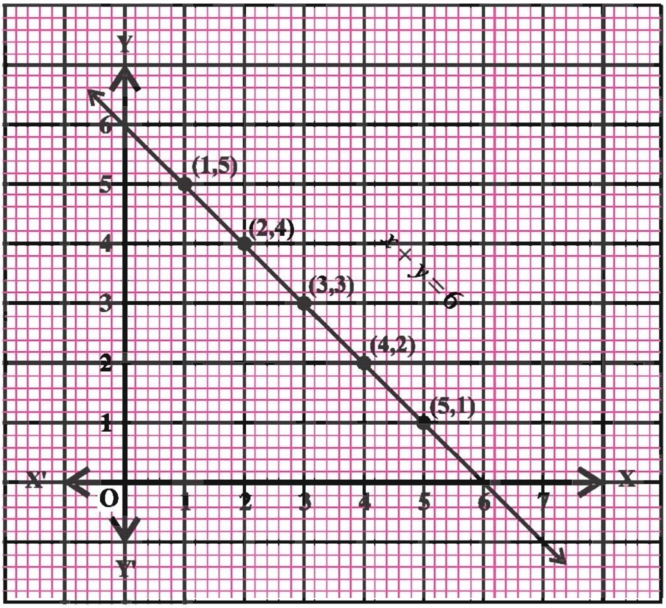

Figure 5.2

So, that is why the equation $x + y = 6$ is called a linear equation.

For real numbers x and y, if (x, y) satisfies the equation $ax + by + c = 0$, then (x, y) is called a solution of $ax + by + c = 0$. $a, b, c \in R$ and $a^2 + b^2 \neq 0$.

Therefore, **the set $\{(x, y) \mid ax + by + c = 0;\ x, y \in R\}$ is a solution set of the linear equation $ax + by + c = 0$ in two variables.**

Example 2 : Find any three elements of the solution set of the equation $4x + 3y = 12$.

Solution : The equation $4x + 3y = 12$ gives,

$\therefore \quad 3y = 12 - 4x$

$\therefore \quad y = \frac{12 - 4x}{3}$

For $x = 0$, $y = \frac{12 - 4(0)}{3} = \frac{12 - 0}{3} = \frac{12}{3} = 4$. So, (0, 4) is a solution of the equation.

For $x = 1$, $y = \frac{12 - 4(1)}{3} = \frac{8}{3}$. So, $\left(1, \frac{8}{3}\right)$ is a solution of the equation.

For $x = 3$: $y = \frac{12 - 4(3)}{3} = \frac{12 - 12}{3} = \frac{0}{3} = 0$. So, another solution is (3, 0).

Thus, (0, 4), $(1, \frac{8}{3})$ and (3, 0) are three elements of the solution set of the equation $4x + 3y = 12$.

Remark : Note that an easy way of getting a solution is to take $x = 0$ and get the corresponding value of y. Similarly, we can put $y = 0$ and obtain the corresponding value of x.

Example 3 : Find four different solutions of the equation $2x + y = 6$.

Solution : The given equation is $2x + y = 6$

$2x + y = 6$

$\therefore \quad y = 6 - 2x$

For $x = 0, y = 6 - 2\,(0) = 6 - 0 = 6$

For $x = -1,\ y = 6 - 2\,(-1) = 6 + 2 = 8$

For $x = \frac{1}{2},\ y = 6 - 2\left(\frac{1}{2}\right) = 6 - 1 = 5$

For $x = 3, y = 6 - 2\,(3) = 6 - 6 = 0$

Thus , (0, 6), (–1, 8), $\left(\frac{1}{2}, 5\right)$ and (3, 0) are four different solutions of the equation $2x + y = 6$.

Example 4 : Show that $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}-5}{3}\right)$ are solutions of the equation $2x - 3y - 5 = 0$.

Solution : For the point $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$, taking $x = \sqrt{3},\ y = \frac{2\sqrt{3}-5}{3}$ we have

$$2x - 3y - 5 = 2\sqrt{3} - 3\left(\frac{2\sqrt{3}-5}{3}\right) - 5$$

$$= 2\sqrt{3} - (2\sqrt{3} - 5) - 5$$

$$= 2\sqrt{3} - 2\sqrt{3} + 5 - 5 = 0$$

Thus, the equation $2x - 3y - 5 = 0$ is verified for $x = \sqrt{3},\ y = \frac{2\sqrt{3}-5}{3}$

So, $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$ is a solution of the linear equation $2x - 3y - 5 = 0$.

Substituting $x = \frac{\sqrt{3}}{2}$ and $y = \frac{\sqrt{3}-5}{3}$, we have

$$2x - 3y - 5 = 2 \cdot \left(\frac{\sqrt{3}}{2}\right) - 3 \cdot \left(\frac{\sqrt{3}-5}{3}\right) - 5$$

$$= \sqrt{3} - (\sqrt{3} - 5) - 5$$

$$= \sqrt{3} - \sqrt{3} + 5 - 5$$

$$= 0$$

So, $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}-5}{3}\right)$ is a solution of the linear equation $2x - 3y - 5 = 0$

EXERCISE 5.2

1. Find five different solutions of each of the equations.

(1) $2x = 3y + 5$ (2) $6y = 9$

2. Find two solutions for each of the equations.

(1) $3x + 4y = 12$ (2) $5x - 2y = 0$ (3) $3x + 5 = 0$ (4) $\pi x + y = 4$

3. Find three elements of the solution set of the following equations.

(1) $3x - 2y = 3$ (2) $2x = 4$ (3) $6y = 15$ (4) $5x + 3y = 0$

(5) $3x + 4y = 6$ (6) $x + y = 0$ (7) $x - y = 0$ (8) $6x + 3y = 9$

4. Which one of the following options is true and why ?

$3y = 2x + 7$ has...

(1) a unique solution (2) only two solutions (3) infinitely many solutions

5. Examine which of the following points are solutions of the equation $2x - y = 5$ and which are not :

(1) (3, 1) (2) (−2, −9) (3) (0, 5) (4) (5, 0) (5) (0, −5) (6) (4, 2)

(7) (2, 1) (8) $\left(\frac{-1}{2}, \frac{-11}{2}\right)$ (9) $(1 + \sqrt{2}, -3 + 2\sqrt{2})$ (10) (1, −6)

6. Find the value of k in each of the following questions :

(1) $x = 1, y = 2$ is a solution of the equation $3x - 2y = k$

(2) $x = 1, y = 3$ is a solution of the equation $3x + ky = 9$

(3) $kx + 5y = 11$ has a solution (4, −1)

(4) (2, 5) is a solution of the equation $4x + ky = 13k$

*

5.4 Graph of a Linear Equation in Two Variables

We know that a solution (x, y) of the linear equation $ax + by + c = 0$ in two variables is a point of the coordinate plane. If all the solutions are plotted in the plane, then they are collinear. Joining them by a straight edge, we get a line. This line is the graph of linear equation in two variables.

Since there are infinitely many solutions of a linear equation in two variables, it is not possible to plot all the solutions. As the graph of a linear equation in two variables is a line, it is enough to plot two ordered pairs of the solutions, then by joining them with a straight edge we can get the graph. We know that two points determine a line, however we shall plot at least three elements and prepare a graph using these points.

Remark : $ax + by + c = 0$ is a polynomial equation of degree one in two variables x and y. The equation $ax + by + c = 0$ is called a linear equation, simply because, its geometrical representation is a straight line.

Example 5 : Draw the graph of $2x - 3y = 0$

Solution : Here, $2x - 3y = 0$

$\therefore \quad 3y = 2x$

$\therefore \quad y = \frac{2x}{3}$

For $x = 0$, $y = \frac{2(0)}{3} = \frac{0}{3} = 0$

For $x = 3$, $y = \frac{2(3)}{3} = \frac{6}{3} = 2$

For $x = -3$, $y = \frac{2(-3)}{3} = \frac{-6}{3} = -2$

Three elements of the solution set of $2x - 3y = 0$ are

x	-3	0	3
y	-2	0	2

Plot the points $(-3, -2)$, $(0, 0)$, $(3, 2)$ on a graph paper.

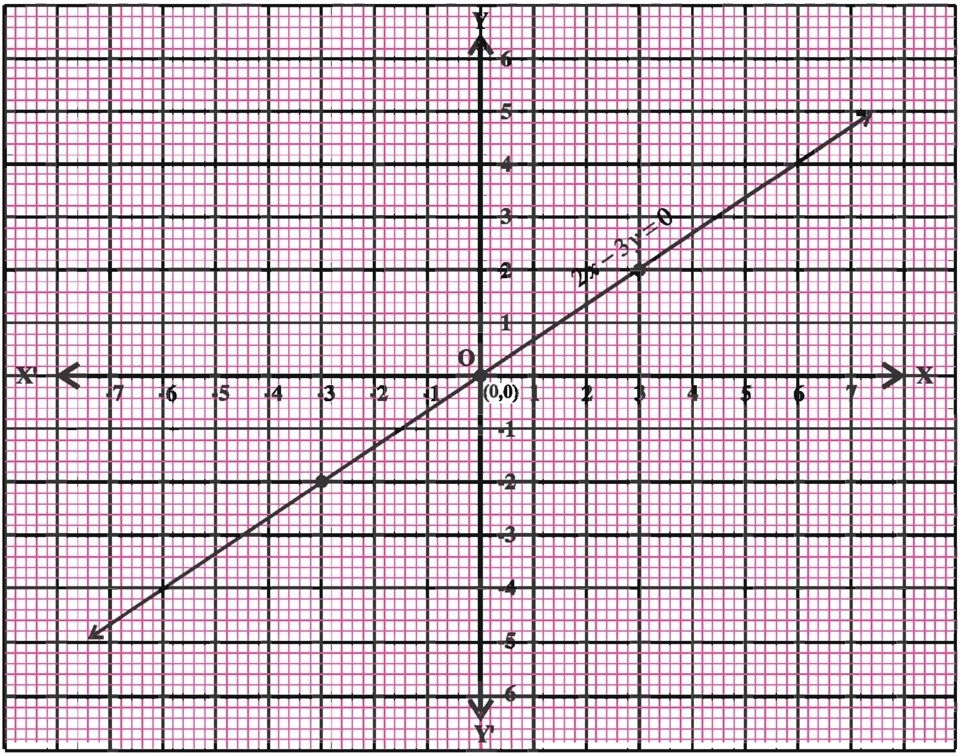

Figure 5.3

These points are collinear. Joining them by a straight edge, we get a line (see figure 5.3).

Observe that the graph of $2x - 3y = 0$ is a line passing through the origin O. If the constant term c is zero then the graph of the equation $ax + by + c = 0$ is always a line passing through the origin.

Example 6 : Draw the graph of $x = 0$

Solution : The equation $x = 0$ can be written as $x + 0y = 0$. In this equation coefficient of y is zero.

So, for any value of y we get $x = 0$

$\therefore$ (0, 1), (0, 4) and (0, −2) are three elements of the solution set. Plotting these points on the graph paper, it can be seen that all the three points are on the Y-axis.

i.e. the graph of $x = 0$ is the Y-axis (see Fig. 5.4)

The line $x = 0$ is the Y-axis.

Similarly, the equation $y = 0$

i.e. $0x + y = 0$ has solutions (1, 0), (2, 0), (3, 0), (−1, 0), (−3, 0), $\left(\frac{3}{2}, 0\right)$, $\left(\frac{5}{3}, 0\right)$,...

Plotting these points on a graph paper, we can see that the graph of the equation $y = 0$ is the X-axis (see figure 5.4). **Thus, the graph of the equation $x = 0$ is the Y-axis and the graph of the equation $y = 0$ is the X-axis.**

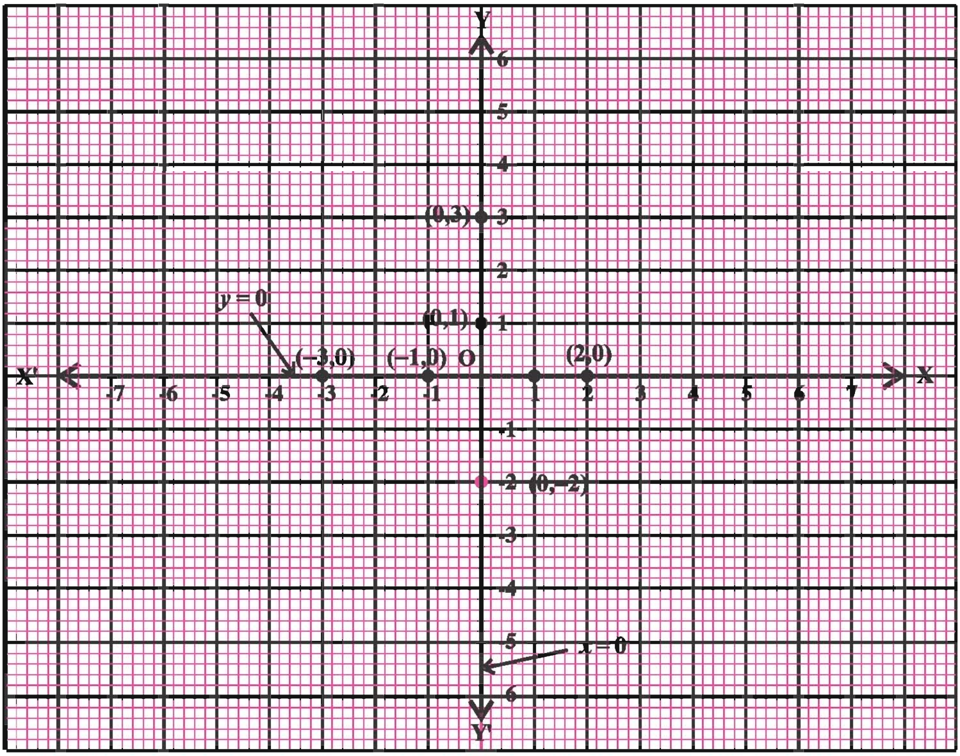

Figure 5.4

Example 7 : Draw the graph of $3x + 3y = 12$

Solution : Here the equation $3x + 3y = 12$ is given.

$\therefore \quad x + y = 4$ (Dividing by 3)

$\therefore \quad y = 4 - x$

For $x = 0$, $y = 4 - 0 = 4$

For $x = 4$, $y = 4 - 4 = 0$

For $x = 2$, $y = 4 - 2 = 2$

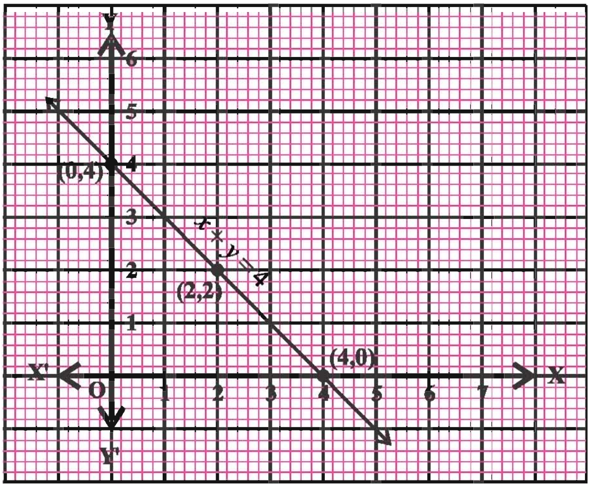

Figure 5.5

Three elements of the solution set of the given equation are (0, 4), (4, 0) and (2, 2). Plotting these points on the graph paper and joining them by a straight edge, we get the graph of $x + y = 4$, which is a line, intersecting both X-axis and Y-axis (see figure 5.5)

Note : If $a \neq 0$, $b \neq 0$, $c \neq 0$ then the graph of the equation $ax + by + c = 0$ is a line, intersecting both X-axis and Y-axis in distinct points.

Here, the graph of $x + y = 4$ intersects X-axis in (4, 0) and Y-axis in (0, 4)

Example 8 : For each of the graph in figure 5.6, 5.7, 5.8 select the correct equation graph of which is from the choices given below.

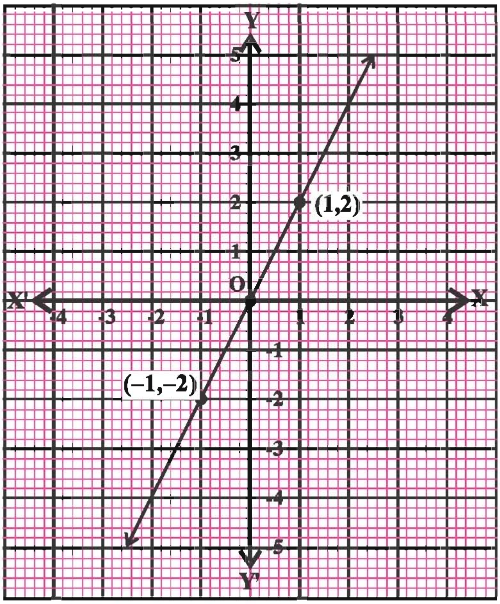

Figure 5.6

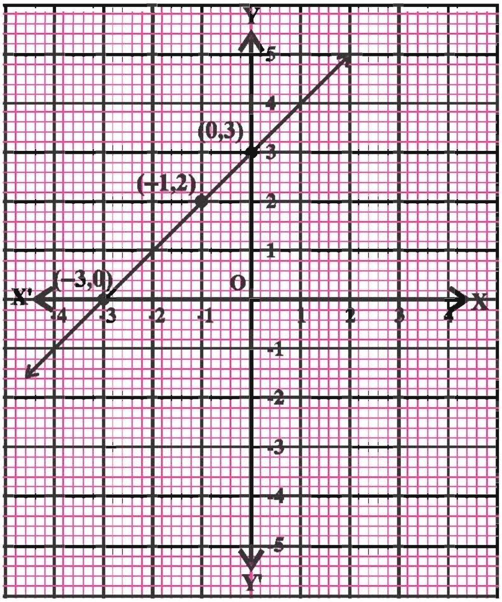

Figure 5.7

(a) For figure 5.6

(1) $x - y = 0$ (2) $y = -x$

(3) $y = 2x$ (4) $y = 3x + 1$

(b) For figure 5.7

(1) $x + 2y = 0$ (2) $y = -3x$

(3) $x - y + 3 = 0$ (4) $2y = 4x + 5$

(c) For figure 5.8

(1) $y = 2x + 1$ (2) $3x - 4y = 12$

(3) $2y = 3x + 2$ (4) $4x - 3y = 12$

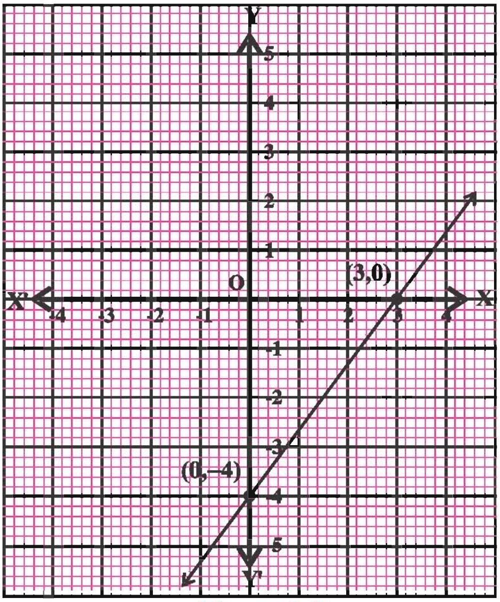

Figure 5.8

Solution :

(a) In figure 5.6, points (–1, –2), (0, 0), (1, 2) are on the line, these points satisfy the equation $y = 2x$.

$\therefore$ $y = 2x$ is the equation corresponding to this graph.

(b) In figure 5.7, the points (0, 3), (–3, 0), (–1, 2) lie on the line and satisfy the equation $x - y + 3 = 0$.

(c) In figure 5.8, obviously (3, 0) and (0, –4) satisfy $4x - 3y = 12$. The equation is $4x - 3y - 12 = 0$.

EXERCISE 5.3

1. Draw the graph of each of the following linear equations in two variables :

(1) $x + y = 6$ (2) $x - y = 2$ (3) $x - 2y = 6$

(4) $y = 3x$ (5) $y = x + 1$ (6) $3x + y = 2$

2. If the point (2, 3) lies on the graph of the equation $2y = ax + 10$, find the value of a.

3. Given the point (2, 3), find the equations of four distinct lines on which it lies. Draw the graph of each line. Also write the point of intersection of each line with the coordinate axes.

4. Consider the linear equation that converts Fahrenheit (F) to Celsius (C) :

$$F = \left(\frac{9}{5}\right)C + 32$$

(1) Draw the graph of this linear equation using Celsius for X-axis and Fahrenheit for Y-axis.

(2) If the temperature is 30° C, what is the corresponding temperature in Fahrenheit ?

(3) If the temperature is 95° F, what is the corresponding temperature in Celsius ?

(4) If the temperature is 0° C, what is the corresponding temperature in Fahrenheit and if the temperature is 0° F, what is the corresponding temperature in Celsius ?

(5) Is there a temperature which is numerically the same in both Fahrenheit and Celsius ? If yes, find it.

*

5.5 Equation of Lines Perpendicular to the X-axis and Y-axis

Consider the equation $3x = 6$, i.e. $x = \frac{6}{3}$. So, $x = 2$ or $x - 2 = 0$

Here, $x - 2 = 0$, if it is treated as an equation in one variable x only, then it has the unique solution $x = 2$, which is a point on the number line (see figure 5.9).

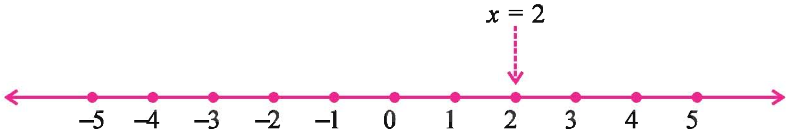

Figure 5.9

Now, in two variables the equation $x - 2 = 0$ can be expressed as $x + 0y - 2 = 0$

Since, the coefficient of y is zero, i.e. for any value of y we get $x = 2$. i.e. an equation $x + 0y - 2 = 0$ has infinitely many ordered pairs as solution. Thus all the solutions of this equation are of the form.

(2, r), where r is any real number (2, 0), (2, 1), (2, 2) are three of the solutions of the equation $x + 0y = 2$

Drawing the graph by plotting these points, we get a line parallel to Y-axis.

Thus the graph of $x = 2$ is a line perpendicular to X-axis (see figure 5.10)

If the coefficient of y is 0 in a linear equation in two variables, then its graph is a line perpendicular to X-axis.

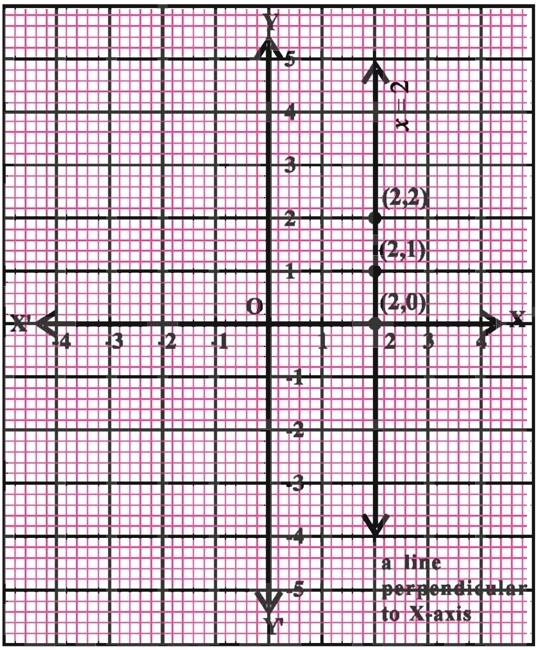

Figure 5.10

Similarly the equation $y = 3$, in two variables can be written as $0x + y = 3$

As discussed above, it has infinitely many solutions of the type (r, 3), where r is any real number. (–2, 3), (0, 3), (2, 3) are three of the solutions of the equation $y = 3$. Plotting these points, we can see that the graph of $y = 3$ is a line perpendicular to Y-axis. (See figure 5.11)

If the coefficient of x is 0 in a linear equation in two variables, then its graph is a line perpendicular to Y-axis.

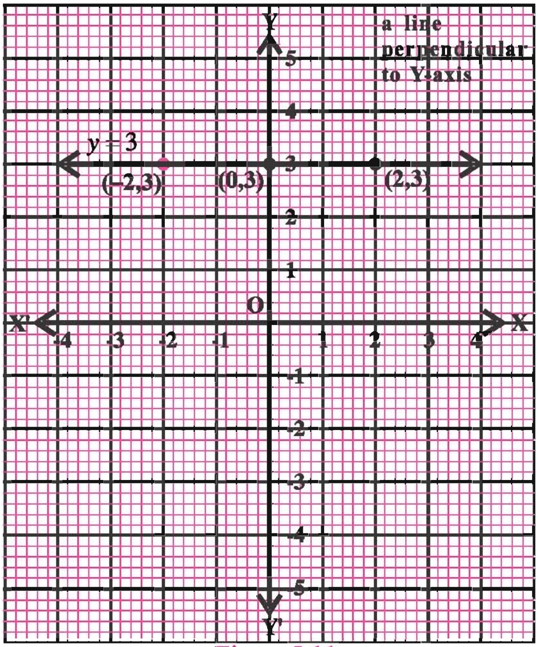

Figure 5.11

Conclusions :

From the above illustrations we have following facts :

For the graph of a linear equation $ax + by + c = 0$ in two variables,

(1) If $a = 0$, $c = 0$, i.e. the graph of the equation $y = 0$ is the X-axis.

(2) If $b = 0$, $c = 0$, i.e. the graph of the equation $x = 0$ is the Y-axis.

(3) If $a = 0$, $c \neq 0$, the equation is $by = -c$ or $y = p$; $p = \left(\frac{-c}{b}\right)$ then the graph of the equation is a line perpendicular to the Y-axis.

(4) If $b = 0$, $c \neq 0$, the equation is $ax = -c$ or $x = q$; $q = \frac{-c}{a}$ then the graph of the equation is a line perpendicular to the X-axis.

(5) If $a \neq 0$, $b \neq 0$, $c = 0$, then the graph is a line passing through the origin O.

EXERCISE 5.4

1. Give the geometric representations of the equations (1) $y = -4$ (2) $2x + 9 = 0$ in one variable and in two variables.
2. Draw the graph of the following linear equations in two variables
   (1) $3y = 6$ (2) $x = 4$ (3) $2y = 10$ (4) $5x + 10 = 0$
3. Solve the equations (1) $2x + 1 = x - 2$ (2) $y - 1 = 2y - 5$, and represent their solutions on the (i) number line (ii) Cartesian plane.
4. Draw the graphs of $y = x + 1$ and $x + y - 3 = 0$ on the same graph paper and observe that these lines intersect at the point (......, ...).

EXERCISE 5

1. Examine whether the following expressions are linear equations in two variables or not.

   (1) $\frac{7}{3}x + \frac{5}{2}y + \frac{1}{2} = 0$ (2) $\frac{3}{x} + 2y - 1 = 0$

   (3) $\frac{x}{2} + \frac{y}{2} = 3$ (4) $\frac{2}{y} + \frac{2}{x} = \frac{1}{3}$

   (5) $y + 3 = 0$ (6) $2x - 5 = 0$

2. Find three distinct solutions of each of the following equations :

(1) $\frac{x}{2} + y = 6$ (2) $x + \frac{y}{3} = 9$

(3) $x + y - 1 = 0$ (4) $x - y + 1 = 0$

(5) $2x + 3y = 6$ (6) $3x - 5y - 15 = 0$

3. If $a = 2k$, $b = 5k$, $c = 7k$; $k \neq 0$ and $k \in$ R, then find the value of k in the following cases : If

(1) $(b - a, c - b)$ is a solution of the linear equation $2x + 3y = 10$.

(2) $(c - 3a, 3b - 2c)$ is a solution of the equation $x + y - 3 = 0$.

(3) $(c + b - 5a, c - b - a)$ is a solution of the equation $2x + y - 8 = 0$.

(4) $(a + b, c + 1)$ is a solution of the equation $y = 2x$.

(5) $(2a + b - c - 2, 3b + 2a - 3c + 2)$ is the point of intersection of the coordinate axes.

4. Draw the graph of each of the following linear equations in two variables; also find their points of intersection with the axes :

(1) $x + y = 0$ (2) $x - y = 0$

(3) $x + y = 2$ (4) $x - y = 3$

(5) $3x + 4y + 12 = 0$ (6) $3x - 2y - 6 = 0$

(7) $3x + 2y - 6 = 0$ (8) $3x - 4y + 12 = 0$

(9) $2x + 5 = 0$ (10) $4y - 8 = 0$

5. Represent geometrically the solutions of the following equations :

(1) $3x + 2 = -x + 10$ (2) $4y - 3 = y + 6$

(3) $2x + 3 = x - 1$ (4) $3y + 2 = 2y - 3$

on the (i) same number line, (ii) same cartesian plane.

6. Draw the graphs of following in R^2.

(1) $x = 4$ (2) $y = 4$ (3) $x = -4$

(4) $y = -4$ (5) $y = x$ (6) $y = -x$

on the same graph paper and write the points where these lines intersect each other.

7, Draw the graphs of (1) $x + 3y - 6 = 0$ and (2) $2x - y - 5 = 0$, on the same graph paper and write the point, where these two lines intersect each other.

8. Draw the graphs of (1) $3x + 2y = 9$ and (2) $x + 4y = 8$, on the same graph paper and observe that the graphs intersect each other at the point $\left(2, \frac{3}{2}\right)$.

9. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Graph of the equation $y = x$ passes through the ...... quadrants and origin. ☐

(a) I and II (b) II and III (c) I and III (d) III and IV

(2) Line $x + y = 2$ passes through the ...... quadrants. ☐

(a) 1st and 3rd both (b) 2nd and 3rd

(c) 3rd and 4th both (d) 1st, 2nd and 4th all

(3) $x + y = 0$ passes through ...... quadrants. ☐

(a) I and II (b) I and II (c) II and IV (d) III and IV

(4) $ax + by = c$; $a^2 + b^2 \neq 0$, passes through origin, if ☐

(a) $a = 0, c \neq 0$ (b) $b = 0, c \neq 0$ (c) $c = 0$ (d) $a \neq 0, c \neq 0$

(5) The linear equation $4x - y + 8 = 0$ has ...... ☐

(a) no solution (b) unique solution

(c) only two solutions (d) infinitely many solutions

(6) If $x = 2$, $y = 5$ is a solution of the equation $5x + 7y - k = 0$, then the value of k is ☐

(a) 12 (b) 35 (c) 45 (d) −45

(7) If the equation is $F = \left(\frac{9}{5}\right)C + 32$, then C = ... ☐

(a) 5F − 160 (b) $\frac{1}{9}(5F - 160)$ (c) $\frac{5}{9}F - 32$ (d) $\frac{5}{9}(F - 32)$

(8) In the equation $F = \left(\frac{9}{5}\right)C + 32$, F = C ...... ☐

(a) is impossible (b) if C = 40 (c) if C = −40 (d) if F = 32

(9) If $F = \left(\frac{9}{5}\right)C + 32$, and F = −274, then C = ... ☐

(a) −338 (b) −274 (c) −170 (d) 170

(10) In the plane, the equation $y = mx$ represents ...... for different values of m. ☐

(a) perpendicular lines (b) parallel lines

(c) lines through origin (d) lines through the point other than origin.

(11) Line $y = 4$ is ...... ☐

(a) parallel to Y-axis (b) intersects both the axis

(c) parallel to X-axis (d) passing through the origin.

(12) Line $x = -2$ is ...... ☐

(a) parallel to X-axis (b) parallel to Y-axis

(c) passing through the origin. (d) intersecting both the axis

(13) One of the solutions of the linear equation $2x + 3y = 7$ is ......

(a) (1, 2) (b) (–1, 3) (c) (–2, 5) (d) (–2, 4)

(14) The graph of the equation ...... is a line parallel to Y-axis

(a) $x - 3 = 0$ (b) $x - y = 1$ (c) $y = 1$ (d) $x + y = 1$

(15) The graph of the equation ...... is a line passing through the origin.

(a) $x + y = 0$ (b) $x + y = 1$ (c) $2y - 3 = 0$ (d) $2x - 2y = 1$

*

Summary

In this chapter, you have studied the following points :

1. $ax + by + c = 0$ is a linear equation where a, b, c are real numbers; $a^2 + b^2 \neq 0$.
2. A linear equation in two variables has infinite number of solutions.
3. The graph of every linear equation in two variables is a straight line.
4. An equation of the type $y = mx$ represents a line passing through the origin.
5. The equation of Y-axis is $x = 0$ and the equation of X-axis is $y = 0$
6. The graph of $x = a$ is a straight line perpendicular to X-axis, i.e. $x = a$ is a vertical line.
7. The graph of $y = b$ is a straight line perpendicular to Y-axis i.e. $y = b$ is a horizontal line

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CHAPTER 6

STRUCTURE OF GEOMETRY

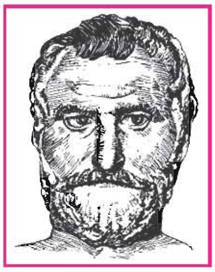
'A multitude of words is no proof of a prudent mind.' - ***Thales***

'Hope is the poor man's bread.' - ***Gary Herbert***

'The past is certain the future obscure.' - ***Thales***

6.1 Introduction

The word 'geometry' comes from the combination of two Greek words 'geo' meaning the 'earth' and 'metrein' meaning to 'measure'. Geometry appears to have originated from the need for measuring land. This branch was studied in various forms in every ancient civilization, like India, Greece etc.



Thales
(Born : 624-625 BC
Died : 546-547 BC)

It is believed that the knowledge of geometry passed on from Egyptians to Greeks. A Greek mathematician Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected by its diameter. Thales is considered to be the pioneer of geometry, because he used the word geometry for the first time. Pythagoras was one of most popular student of Thales. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent.

In the Indian subcontinent the excavations of Harappa and Mohen-Jo-Daro show that Indus Valley civilization made extensive use of geometry about 300 BC. The cities were well planned and organised. The roads were parallel to each other and the drainage system was underground. A house had rooms of different shapes and the bricks used had the ratio of 4:2:1 for length, breadth and height respectively.

Some of the Theorems of Thales :

(1) A circle is bisected by its diameter.

(2) Angles at the base of any isosceles triangle are equal.

(3) If two straight lines intersect, the opposite angles formed are equal.

(4) If one triangle has two angles and one side equal to those of another triangle the two triangles are equal in all respects.

(5) 'Any angle inscribed in semicircle is a right angle' is known as Thales theorem.

Sulbasutras provided literature for constructions using geometry. *Sulbasutras* were created between 800 BC to 500 BC. *Bodhayan Sulbasutra* is the oldest. Implicitly their construction implies knowledge of proof of Pythagoras' principle. Altars for public worship contained combinations of rectangles, triangles and trapaziums. Thus Indians knew Pythagoras' theorem before his birth. Aryabhatt, Brahmgupta and Bhaskaracharya contributed to the development of geometry.

Sacred fire found locations according to definite instructions about shapes and sizes. Squares and circular altars were used for residential rituals. *Sriyantra* consists of nine interwoven isosceles triangles arranged, so as to produce 43 subsidiary triangles.

Geometry was being developed in unorganized manner. Egyptians gave only statements of results. Egyptians and Babylonians used geometry solely for application and practical utility. But Greeks laid the basis of deductive reasoning.

6.2 Euclid's Approach

Euclid was a teacher of mathematics at Alexandria. He collected all the known results and compiled them in a series of a thirteen chapters each called a **'Book'**.

Euclid
(325 BC to 265 BC)

The compilation was named **'Elements'**. This **'Elements'** greatly influenced world's notion of geometry for years to come. The notions of point, line, plane, surface were derived from surrounding objects. An abstract geometrical concept of a solid object was created and developed from studies of space. Boundaries of a solid are surfaces partitioning space. 'Surfaces' have no thickness and their boundaries are curves or straight lines.

The lines end in points. Gradually proceeding from solids to point we lose 'dimension'.

Geometric quantity	Dimension
Solid	3
Surface	2
Line	1
Point	None