CHAPTER 10

Matrix Methods of Analysis

10.1 Algebra of Matrices

Let us consider a set of simultaneous equations

$$x + 3y + 5z = 0$$
$$4x + 2y + 3z = 0$$
$$5x + 3y + 4z = 0$$

Now we write down the coefficient of x, y and z of the above equations and enclose them within brackets and then, we get

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and column and bounded by brackets is called a matrix.

$[A]_{m \times n}$ means matrix of 'm' rows and 'n' columns.

$$[A] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ & \ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots \\ a_{m_1} & a_{m_2} \ldots\ldots a_{m_n} \end{bmatrix}$$

A matrix is denoted by $[A]$ and its determinant is denoted by $|A|$.

10.2 Types of Matrices

(a) **Row matrix:** If a matrix has only one row and any number of column then it is called a row matrix e.g

$$[A] = [1 \quad 3 \quad 5]$$

(b) **Column matrix:** If a matrix has only one column and any number of row then it is called a column matrix e.g.

$$[A]=\begin{bmatrix}4\\1\\5\end{bmatrix}$$

(c) **Square matrix:** A matrix which has equal number of rows and column is called a square matrix e.g.

$$[A]_{n\times n}=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$$

(d) **Diagonal matrix:** A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero. e.g.

$$[A]=\begin{bmatrix}1&0&0\\0&2&0\\0&0&4\end{bmatrix}$$

(e) **Symmetric matrix:** A square matrix will be called symmetric matrix if element have symmetry about it diagonal i.e., $a_{ij}=a_{ji}$, e.g.

$$[A]=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}$$

(f) **Transpose of a matrix:** If in a matrix, we interchange the rows and columns with each other, the new matrix obtained is called transpose of a matrix

$$[A]=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$$

$$[A^T]=\begin{bmatrix}1&4&5\\3&2&3\\5&3&4\end{bmatrix}$$

10.3 Matrices Operations

(a) Addition of matrices: Addition of matrices can be done only when they are of same order $[A]_{m\times n}+[B]_{m\times n}=[C]_{m\times n}$ i.e.,

If $[A]=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$ and $[B]=\begin{bmatrix}1&2&4\\-2&5&3\\1&4&3\end{bmatrix}$.

Then,

$$A+B=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}+\begin{bmatrix}1&2&4\\-2&5&3\\1&4&3\end{bmatrix}$$

$$=\begin{bmatrix}(1+1)&(3+2)&(5+4)\\(4-2)&(2+5)&(3+3)\\(5+1)&(3+4)&(4+3)\end{bmatrix}=\begin{bmatrix}2&5&9\\2&7&6\\6&7&7\end{bmatrix}_{3\times 3}$$

(b) **Subtraction of matrices:** Subtraction of matrices can be done only if they have the same order.

$[A]_{m\times n}-[B]_{m\times n}=[D]_{m\times n}$ i.e.,

If $[A]=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$ and $[B]=\begin{bmatrix}1&2&4\\-2&5&3\\1&4&3\end{bmatrix}$.

Then,

$$[B-A]=\begin{bmatrix}1&2&4\\-2&5&3\\1&4&3\end{bmatrix}-\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$$

$$=\begin{bmatrix}(1-1)&(2+3)&(4-5)\\(-2-4)&(5-2)&(3+3)\\(1-5)&(4-3)&(3-4)\end{bmatrix}=\begin{bmatrix}0&-1&-1\\-6&3&0\\-4&1&-1\end{bmatrix}_{3\times 3}$$

(c) **Scalar multiplication of matrix:** If $[A]$ is any matrix and k is any scalar number, if matrix $[A]$ is multiplied by scalar quantity k, then each element of $[A]$ is multiplied by k. i.e.,

$$[A]=\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}$$

Then,

$$2A=2\begin{bmatrix}1&3&5\\4&2&3\\5&3&4\end{bmatrix}=\begin{bmatrix}2\times1&2\times3&2\times5\\2\times4&2\times2&2\times3\\2\times5&2\times3&2\times4\end{bmatrix}=\begin{bmatrix}2&6&10\\8&4&6\\10&6&8\end{bmatrix}$$

(d) **Multiplication of matrices:** The multiplication of two matrices $[A]$ and $[B]$ is only possible if the number of columns in $[A]$ is equal to the number of rows in $[B]$.

$$[A]_{m\times p}\times[B]_{p\times n}=[C]_{m\times n}$$

Where,

The elements of row of matrix $[A]$ is multiplied to the corresponding elements of column of matrix $[B]$.

Properties of matrix multiplication:

1. Multiplication of matrices is not cumulative,

$$AB\neq BA$$

2. Matrix multiplication is associative,

$$A(BC)=(AB)C$$

3. Matrix multiplication is distributive i.e.,

$$A(B+C)=AB+AC$$

Example 10.1

If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1 & 2\\ 0 & 1 & 3\\ 1 & 3 & 2\end{bmatrix}$, find the product AB?

Solution:

$$AB=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 2\\ 0 & 1 & 3\\ 1 & 3 & 2\end{bmatrix}$$

$$\begin{aligned}
C_{11} &= a_{11}\times b_{11}+a_{12}\times b_{21}+a_{13}\times b_{31}\\
&= (1\times 0)+(-2\times 0)+(3\times 1)\\
&= 0-0+3=3\\
C_{12} &= a_{11}\times b_{12}+a_{12}\times b_{22}+a_{13}\times b_{32}\\
&= (1\times 1)+(-2\times 1)+(3\times 3)\\
&= 1-2+9=8\\
C_{13} &= a_{11}\times b_{13}+a_{12}\times b_{23}+a_{13}\times b_{33}\\
&= 1\times 2+(-2\times 3)+(3\times 2)\\
&= 2-6+6=2\\
C_{21} &= a_{21}\times b_{11}+a_{22}\times b_{21}+a_{23}\times b_{31}\\
&= (2\times 0)+(3\times 0)+(-1\times 1)\\
&= 0+0-1=-1\\
C_{22} &= a_{21}\times b_{12}+a_{22}\times b_{22}+a_{23}\times b_{32}\\
&= (2\times 1)+(3\times 1)+(-1\times 3)\\
&= 2+3-3=2\\
C_{23} &= a_{21}\times b_{13}+a_{22}\times b_{23}+a_{23}\times b_{33}\\
&= (2\times 2)+(3\times 3)+(-1\times 2)\\
&= 4+9-2=11\\
C_{31} &= a_{31}\times b_{11}+a_{32}\times b_{21}+a_{33}\times b_{31}\\
&= (-3\times 0)+(1\times 0)+(2\times 1)\\
&= 0+0+2=2\\
C_{32} &= a_{31}\times b_{12}+a_{32}\times b_{22}+a_{33}\times b_{32}\\
&= (-3\times 1)+(1\times 1)+(2\times 3)\\
&= -3+1+6=4\\
C_{33} &= a_{31}\times b_{13}+a_{32}\times b_{23}+a_{33}\times b_{33}\\
&= (-3\times 2)+(1\times 3)+(2\times 2)\\
&= -6+3+4=1
\end{aligned}$$

$$\therefore \quad [C]=\begin{bmatrix}3 & 8 & 2\\ -1 & 2 & 11\\ 2 & 4 & 1\end{bmatrix}$$

(e) Adjoint of a square matrix: The adjoint of $[A]$ is the transpose matrix of the cofactors of matrix A.

Let, $A=\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$

Cofactor of matrix A are

$$A_1=\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}=b_2c_3-b_3c_2 \qquad A_2=-\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}=-b_1c_3+b_3c_1$$

$$A_3=\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}=b_1c_2-b_2c_1 \qquad B_1=-\begin{vmatrix}a_2 & a_3\\ c_2 & c_3\end{vmatrix}=-a_2c_3+a_3c_2$$

$$B_2=\begin{vmatrix}a_1 & a_3\\ c_1 & c_3\end{vmatrix}=a_1c_3-a_3c_1 \qquad B_3=-\begin{vmatrix}a_1 & a_2\\ c_1 & c_2\end{vmatrix}=-a_1c_2+a_2c_1$$

$$C_1=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}=a_2b_3-a_3b_2 \qquad C_2=-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}=-a_1b_3+a_3b_1$$

$$C_3=\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=a_1b_2-a_2b_1$$

The matrix formed by cofactors.

$$[A]'=\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}$$

$$Adj\,[A]=\text{Transpose of }[A]'$$

$$Adj\,[A]=\begin{bmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{bmatrix}$$

(f) Inverse of a matrix: Inverse of A matrix is represented by A^{-1}

$$A^{-1}=\frac{Adj[A]}{|A|}$$

Example 10.2

If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}.

Solution:

We know, $$A^{-1} = \frac{Adj[A]}{|A|}$$

$$[A] = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|A| = 3(-3+4) + 3(2-0) + 4(-2-0)$$
$$= 3 + 6 - 8 = 1$$

Cofactors of $|A|$ are,

$$A_1 = -3 + 4 = 1,\ B_1 = 3 - 4 = -1,\ C_1 = -12 + 12 = 0$$
$$A_2 = -2 - 0 = -2,\ B_2 = 3 - 0 = 3,\ C_2 = -12 + 8 = -4$$
$$A_3 = -2 - 0 = -2,\ B_3 = 3 - 0 = 3,\ C_3 = -9 + 6 = -3$$

Matrix formed by cofactors is $$[A]' = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$Adj[A]$ = Transpose of the matrix of cofactors

$$Adj[A] = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore \quad A^1 = \frac{Adj[A]}{|A|} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

NOTE

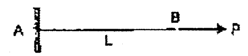
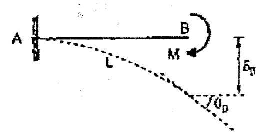
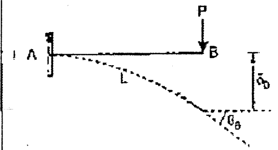
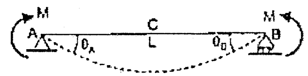
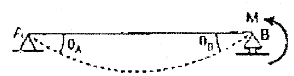
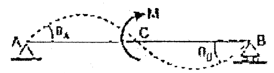
- In structural problems, flexibility and stiffness matrices are always square matrix.
- In flexibility and stiffness matrices, the diagonals are non-zero and non-negative.
- Flexibility and stiffness matrices are always symmetrical material. It means the transpose of $[A]$ (flexibility or stiffness matrix) is same as matrix $[A]$.

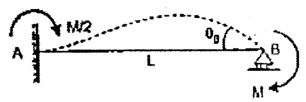
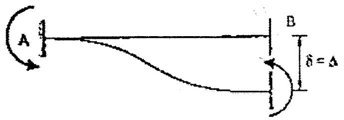
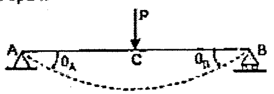
(g) **Determinant of matrix:** A determinant is a real number associated with every square matrix. It is denoted by "det A" or $|A|$.

Properties of Determinant

- A determinant will have zero value if any two rows or columns are identical.
- If element of two rows or columns are interchanged, the sign of determinant changes
- The elements in a row or column may be multiplied by a constant or added to another row or column without changing the value of determinant.
- The adjoint of square matrix $[A]$ is obtained by the transpose of the matrix of cofactors of A.

10.4 Standard Results of Slope and Deflection

S.No.	Loading	Slope	Deflection
1.	Axial load at free end of cantilever: 	$\theta_B = 0$	$\delta_B = \frac{PL}{AE}$ (Axial)
2.	Moment at free end of cantilever: 	$\theta_B = \frac{ML}{EI}$	$\delta_B = \frac{ML^2}{2EI}$
3.	Point load at free end of cantilever: 	$\theta_B = \frac{PL^2}{2EI}$	$\delta_B = \frac{PL^3}{3EI}$
4.	Simply supported beam with moment at both end: 	$\theta_A = \theta_B = \frac{ML}{2EI}$	$\delta_C = \frac{ML^2}{8EI}$
5.	Simply supported beam with moment at one end: 	$\theta_A = \frac{ML}{6EI}$ $\theta_B = \frac{ML}{3EI}$	$\delta_A = 0$ $\delta_B = 0$
6.	Simply supported beam with moment at mid span: 	$\theta_A = \theta_B = \frac{ML}{24EI}$ $\theta_C = \frac{ML}{12EI}$	$\delta_A = 0$ $\delta_C = 0$ $\delta_B = 0$

7.	Propped cantilever subjected to moment at propped support: 	$\theta_B = \frac{ML}{4EI}$	$\delta_A = 0$ $\delta_B = 0$
8.		$\bar{M}_{AB} = \bar{M}_{BA} = \frac{6EI\Delta}{L^2}$	$\delta_A = 0$ $\delta_B = \Delta$
9.	Simply supported beam subjected to point load at mid span: 	$\theta_A = \theta_B = \frac{PL^2}{16EI}$	$\delta_A = \delta_A = 0$ $\delta_C = \frac{PL^3}{48EI}$

10.5 Flexibility and Stiffness

$$\Delta = \frac{P}{K}$$

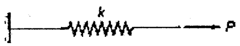


Fig.10.1

The displacement caused by unit force is known as flexibility.

$\therefore \quad f = \frac{\Delta}{P}$

The force required to produce unit displacement is known as stiffness.

$\therefore \quad K = \frac{P}{\Delta}$

Also note that, $\quad K = \frac{1}{f}$ or $K \times f = 1$

10.6 Flexibility Matrix

10.6.1 Properties

1. The flexibility matrix will always be a square matrix ($n \times n$) in which diagonal elements will be non-negative and non-zero.
2. Order of flexibility matrix will be equal to degree of static indeterminacy. (i.e., no. of redundants)

10.6.2 Procedure to Develop Flexibility Matrix

If there are N coordinate then flexibility matrix will be $N \times N$ size square matrix. The element of flexibility matrix represents displacement of any point produced by unit force in the direction of chosen coordinate. Consider a cantilever beam, the chosen coordinates are shown in figure 10.2.

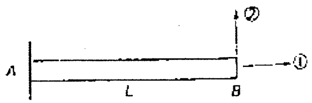


Fig. 10.2

There are two coordinates, therefore flexibility matrix will be of square size (2 × 2).

$$[f] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

Note that in typical element of flexibility matrix i.e f_{ij}

j - represents direction of applied unit force.

i - represents direction of displacement measured.

Here,

f_{11} = Displacement in direction of (1) when unit force is applied in the direction of (1) alone.

f_{12} = Displacement in direction of (1) when unit force is applied in the direction of (2) alone.

f_{21} = Displacement in direction of (2) when unit force is applied in the direction of (1) alone.

f_{22} = Displacement in direction of (2) when unit force is applied in the direction of (2) alone.

According to Maxwell's reciprocal theorem,

$$f_{ij} = f_{ji}$$

Hence, $\quad f_{12} = f_{21}$

Step-1. To generate first column of flexibility matrix apply unit force in the direction of coordinate (1) alone. Now measure the displacements produce in the directions of coordinate 1, 2, ...N.

Step-2. To generate second column of flexibility matrix now apply unit force in the direction of coordinate (2) alone and measure the displacements produce in the directions of coordinate 1, 2,..... N.

For given cantilever, apply unit load in direction of coordinate (1).

f_{11} = Displacement of point B in the direction of coordinate (1) due to unit force in the direction of (1) alone.

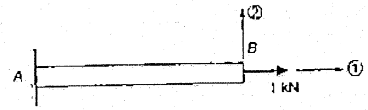


Fig. 10.3

$\therefore \quad f_{11} = \frac{1 \times L}{AE} = \frac{L}{AE}$

f_{21} = Displacement of point B in the direction of coordinate (2) due to unit force in the direction (1) alone.

$\therefore \quad f_{21} = 0$

Also from Maxwell's reciprocal theorem.

$f_{12} = f_{21} = 0$

Now apply unit force in the direction of coordinate (2) alone.

f_{22} = Displacement of point B in the direction of coordinate (2) due to unit force in the direction of (2) alone.

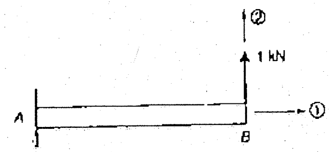


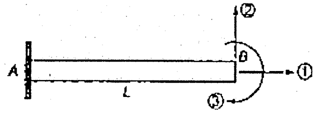
Fig. 10.4

$\therefore \quad f_{22} = \frac{1 \times L^3}{3EI} = \frac{L^3}{3EI}$

Therefore flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \frac{L}{AE} & 0 \\ 0 & \frac{L^3}{3EI} \end{bmatrix}$$

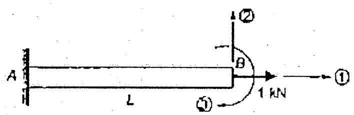
Example 10.3 For the coordinate marked in figure, develop flexibility matrix.



Solution:

First Column:

Apply unit force in the direction of coordinate (1) alone and measure displacements in the direction of (1), (2) and (3).



$$f_{11} = \frac{1 \times L}{AE} = \frac{L}{AE}$$

$$f_{21} = 0$$

$$f_{31} = 0$$

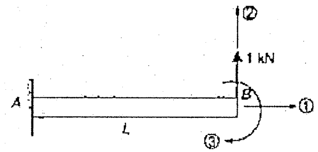
Also from the Maxwell's reciprocal theorem,

$$f_{12} = f_{21} = 0$$

and $$f_{13} = f_{31} = 0$$

Second Column:

Apply unit force in the direction of coordinate (2) alone and measure displacements in direction of (1), (2) and (3).

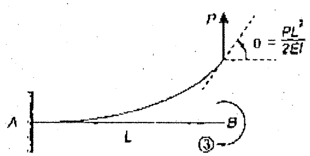


$f_{12} = 0$ (already known)

$$f_{22} = \frac{1 \times L^3}{3EI} = \frac{L^3}{3EI}$$

$$\therefore \quad f_{32} = -\frac{1 \times L^2}{2EI} = -\frac{L^2}{2EI}$$

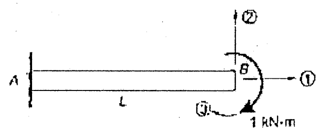
Also, from the Maxwell's reciprocal theorem,



$$f_{23} = f_{32} = -\frac{L^2}{2EI}$$

Third Column:

Apply unit moment in the direction of coordinate (3) alone and measure displacements in the directions of (1), (2) and (3).



$f_{13} = 0$ (Already known)

$f_{23} = -\frac{L^2}{2EI}$ (Already known)

$$f_{33} = \frac{1 \times L}{EI} = \frac{L}{EI}$$

Therefore the flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \frac{L}{AE} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & -\frac{L^2}{2EI} \\ 0 & -\frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

Example 10.4 Flexibility matrix for a beam element is written in the form:

$$[A] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

What is the corresponding stiffness matrix?

(a) $\frac{6EI}{L^3}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$ (b) $\frac{6EI}{7L^3}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$

(c) $\frac{6EI}{L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$ (d) $\frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$

Ans. (d)

Product of flexibility and stiffness matrix is an identity matrix i.e. flexibility matrix and stiffness matrix are inverse of each other.

$$[f][k] = [I]$$

$$\therefore \quad [k] = [f]^{-1}$$

We know, for square matrix, $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of $[A]$ is given by, $$[A]^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

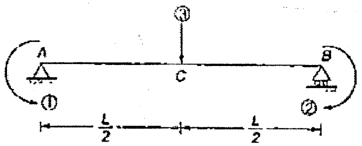
Similarly, $$[f]^{-1} = \frac{1}{|f|}\frac{6EI}{L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$\therefore \quad [k] = [f]^{-1} = \frac{6EI}{|f|L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$|f| = 2 \times 16 - 5 \times 5 = 7$$

$$\therefore \quad [k] = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

Example 10.5 For the beam with coordinate shown in figure. Develop the flexibility matrix. EI is constant.



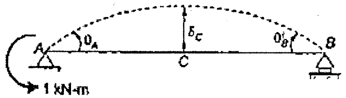
Solution:

First column: Apply unit moment at A in direction of (1).

$$f_{11} = \frac{L}{3EI}$$

$$f_{21} = \frac{L}{6EI}$$

$$f_{31} = f_{13} \quad \text{(By Reciprocal theorem)}$$

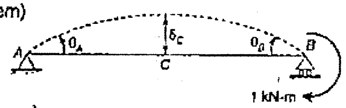


Second column: Apply unit moment at B in direction of coordinate (2).

$$f_{12} = f_{21} = \frac{L}{6EI} \quad \text{(By Reciprocal theorem)}$$

$$f_{22} = \frac{L}{3EI}$$

$$f_{32} = f_{23} \quad \text{(By Reciprocal theorem)}$$

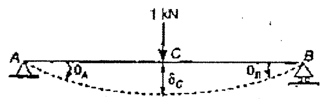


Third column: Apply unit load at C in direction of coordinate (3).

$$f_{13} = -\frac{L^2}{16EI}$$

$$f_{23} = -\frac{L^2}{16EI}$$

$$f_{33} = \frac{L^3}{48EI}$$



Hence,

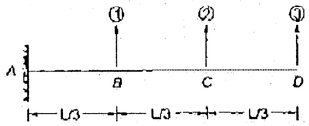
$$f_{31} = f_{13} = -\frac{L^2}{16EI}$$

$$f_{32} = f_{23} = -\frac{L^2}{16EI}$$

Therefore the flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \frac{L}{3EI} & \frac{L}{6EI} & \frac{-L^2}{16EI} \\ \frac{L}{6EI} & \frac{L}{3EI} & \frac{-L^2}{16EI} \\ \frac{-L^2}{16EI} & \frac{-L^2}{16EI} & \frac{L^3}{48EI} \end{bmatrix}$$

Example 10.6 Generate flexibility matrix for given coordinate system.



Solution:

First column: Apply unit load at B in the direction of coordinate (1) and measure displacements in the direction of 1, 2 and 3.

$$f_{11} = \frac{\left(\frac{L}{3}\right)^3}{3EI} = \frac{L^3}{81EI}$$

$$f_{21} = \delta_B + \theta_B \cdot L_{BC}$$

$$f_{21} = \frac{L^3}{81EI} + \frac{(L/3)^2}{2EI} \times \frac{L}{3} = \frac{5}{162}\frac{L^3}{EI}$$

$$f_{31} = \delta_D$$

$$\therefore \quad f_{31} = \delta_B + \theta_B \cdot L_{BD}$$

$$\therefore \quad f_{31} = \frac{L^3}{81EI} + \frac{(L/3)^2}{2EI} \times \frac{2L}{3} = \frac{4}{81}\frac{L^3}{EI}$$

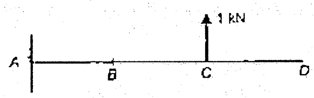
Second column: Apply unit load at C in the direction of coordinate (2) and measure displacements in the direction of 1, 2 and 3.

$$f_{12} = f_{21} = \frac{5}{162}\frac{L^3}{EI} \quad \text{(By reciprocal theorem)}$$

$$f_{22} = \frac{1 \cdot \left(\frac{2L}{3}\right)^3}{3EI} = \frac{8}{81}\frac{L^3}{EI}$$

$$f_{32} = \delta_D = \delta_C + \theta_C \times L_{CD}$$

$$\therefore \quad f_{32} = \frac{8}{81}\frac{L^3}{EI} + \frac{\left(\frac{2L}{3}\right)^2}{2EI} \times \frac{L}{3} = \frac{14}{81}\frac{L^3}{EI}$$

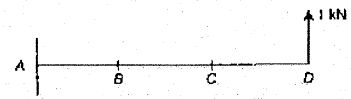


Third column: Apply unit load at D in the direction of coordinate (3) and measure displacements in the directions of 1, 2 and 3.

$$f_{13} = f_{31} = \frac{4}{81}\frac{L^3}{EI} \quad \text{(By reciprocal theorem)}$$

$$f_{23} = f_{32} = \frac{14}{81}\frac{L^3}{EI} \quad \text{(By reciprocal theorem)}$$

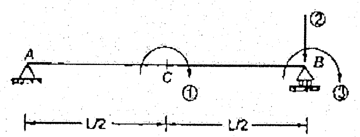
$$f_{33} = \frac{L^3}{3EI}$$



Hence for the given coordinate system the flexibility matrix is,

$$[f] = \begin{bmatrix} \frac{L^3}{81EI} & \frac{5}{162}\frac{L^3}{EI} & \frac{4}{81}\frac{L^3}{EI} \\ \frac{5}{162}\frac{L^3}{EI} & \frac{8}{81}\frac{L^3}{EI} & \frac{14}{81}\frac{L^3}{EI} \\ \frac{4}{81}\frac{L^3}{EI} & \frac{14}{81}\frac{L^3}{EI} & \frac{L^3}{3EI} \end{bmatrix}$$

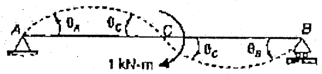
Example 10.7 Develop the flexibility matrix for the simply supported beam AB with coordinate system shown in figure.



Solution:

First column: Apply unit moment in the direction of coordinate (1) and measure displacements in the direction of 1, 2 and 3.

$$f_{11} = \frac{1 \times L}{12EI} = \frac{L}{12EI}$$



$$f_{21} = 0$$

$$f_{31} = -\frac{1 \times L}{24EI} = \frac{-L}{24EI}$$

Also, from the Maxwell's reciprocal theorem,

$$f_{12} = f_{21} = 0$$

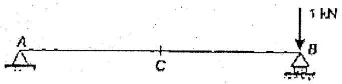
$$f_{13} = f_{31} = -\frac{L}{24EI}$$

Second column: Apply unit load in the direction of coordinate and measure displacements in the direction of 1, 2 and 3.

$f_{12} = 0$ (Already known)

$f_{22} = 0$

$f_{32} = 0$



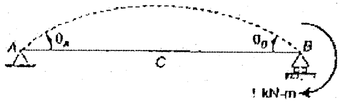
Also, from Maxwell's reciprocal theorem,

$$f_{23} = f_{32} = 0$$

Third column: Apply unit moment in the direction of coordinate (3) and measure displacements in the direction of 1, 2 and 3.

$f_{13} = -\frac{L}{24EI}$ (Already known)

$f_{23} = 0$ (Already known)

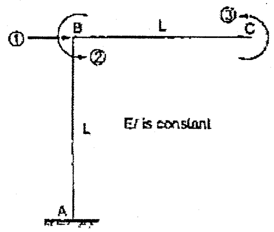


$$f_{33} = \frac{1 \times L}{3EI} = \frac{L}{3EI}$$

Hence the flexibility matrix for the coordinate system is

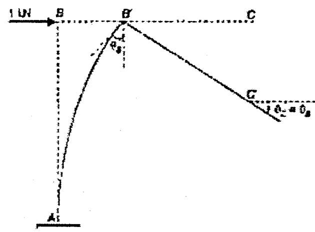
$$[f] = \begin{bmatrix} \frac{L}{12EI} & 0 & -\frac{L}{24EI} \\ 0 & 0 & 0 \\ -\frac{L}{24EI} & 0 & \frac{L}{3EI} \end{bmatrix}$$

Example 10.8 Generate flexibility matrix for cantilever frame shown in figure.



Solution:

First column:



$$f_{11} = \frac{L^3}{3EI}$$

$$f_{21} = -\frac{L^2}{2EI}$$

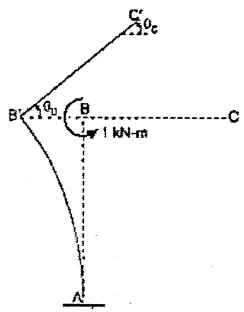
$$f_{31} = -\frac{L^2}{2EI}$$

Also, from Maxwell's reciprocal theorem,

$$f_{12} = f_{21} = \frac{-L^2}{2EI}$$

$$f_{13} = f_{31} = \frac{-L^2}{2EI}$$

Second column:



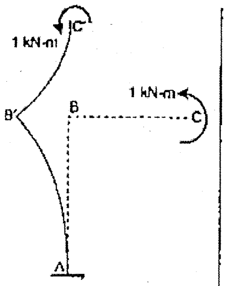
$f_{12} = \frac{-L^2}{2EI}$ (Already known)

$$f_{22} = \frac{L}{EI}$$

$$f_{32} = \frac{L}{EI}$$

Also, from Maxwell's reciprocal theorem,

$$f_{23} = f_{32} = \frac{L}{EI}$$



Third column:

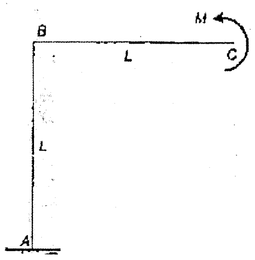
$$f_{13} = \frac{-L^2}{2EI} \text{ (Already known)}$$

$$f_{23} = \frac{L}{EI} \text{ (Already known)}$$

For f_{33}:

Using strain-energy method

Strain-energy stored in frame



$$U = U_{AB} + U_{BC}$$

$$U = \frac{M^2L}{2EI} + \frac{M^2L}{2EI} = \frac{M^2L}{EI}$$

$$\theta_C = \frac{\partial U}{\partial M} = \frac{2ML}{EI}$$

If $M = 1$ kN-m, then, θ_c becomes

$$f_{33} = \frac{2L}{EI}$$

Hence the flexibility matrix for given cantilever frame is

$$[f] = \begin{bmatrix} \frac{L^3}{3EI} & \frac{-L^2}{2EI} & \frac{-L^2}{2EI} \\ -\frac{L^2}{2EI} & \frac{L}{EI} & \frac{L}{EI} \\ -\frac{L^2}{2EI} & \frac{L}{EI} & \frac{2L}{EI} \end{bmatrix}$$

10.7 Stiffness Matrix

10.7.1 Properties

1. The stiffness matrix is always square matrix having non-zero and non-negative diagonal elements.
2. The order of matrix = Degrees of freedom (D_k)

NOTE: (a) If load is vertical in beams, then axial displacement should be neglected.
(b) If members are axially rigid, then also axial displacement should be ignored.

10.7.2 Procedure to Develop Stiffness Matrix

If there are N coordinate then stiffness matrix will be $N \times N$ size square matrix. The element of stiffness matrix represents force produced by unit displacement in the direction of chosen coordinate

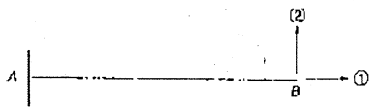


Fig. 10.5

k_{xy} = Force/moment produced in the direction of x when unit displacement (Δ or θ) is applied in y-direction alone.

It is also noticed that,

$k_{xy} = k_{yx}$ (According to Maxwell's reciprocal theorem)

Step-1. To generate first column of stiffness matrix, give unit displacement in the direction of coordinate (1) alone without any displacement in other coordinate directions (i.e. No Δ or θ at other coordinate) and measure forces developed in all coordinate directions.

Step-2. To generate second column of stiffness matrix, give unit displacement in the direction of coordinate (2) alone without any displacement in other coordinate directions (i.e. No Δ or θ at other coordinates) and measure forces developed in all coordinate directions.

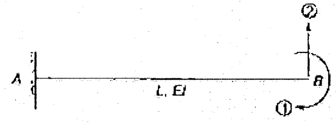


Fig. 10.6

Consider a cantilever beam with coordinate as shown in figure.

First column: To generate first column of stiffness matrix, give unit displacement in the direction of coordinate (1) alone without any displacement in the direction of other coordinate i.e. no Δ or θ at other coordinates and measure force produced in the directions of all coordinates.

$\therefore$ Give $\theta_B = 1$ ↻ and ensure $\Delta_B = 0$

So provide hinge support at B.

k_{11} = Force developed in the direction of coordinate (1) when unit displacement is provided in the direction of coordinate (1).

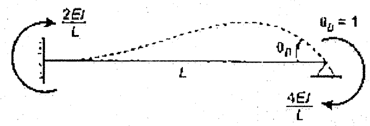


Fig. 10.7

k_{21} = Force developed in the direction of coordinate (2) when unit displacement is provided in the direction of coordinate (1).

Take, $\Sigma M_A = 0$

$$R_B \times L - \frac{4EI}{L} - \frac{2EI}{L} = 0$$

$$R_B = \frac{6EI}{L^2}$$

$$\therefore \quad k_{11} = \frac{4EI}{L}$$

$$\therefore \quad k_{21} = R_B = \frac{6EI}{L^2}$$

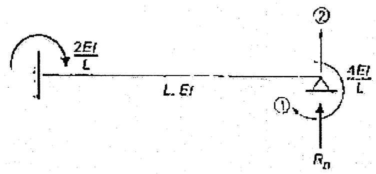


Fig. 10.8

Second column: To generate second column of stiffness matrix, give unit displacement in the direction coordinate (2) and measure force developed in the direction of coordinates (1) and (2)

$\therefore$ Give $\Delta_B = 1$ (↑) and ensure $\theta_B = 0$

So, fix the end B at B' so that

BB' = 1 unit

Take, $\Sigma M_A = 0$

$$\frac{6EI}{L^2} + \frac{6EI}{L^2} - R_B \times L = 0$$

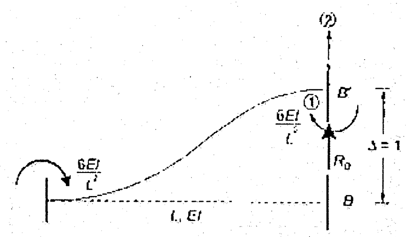


Fig. 10.9

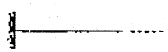
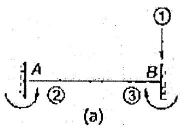
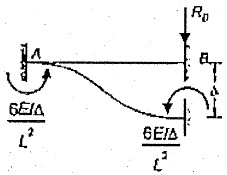
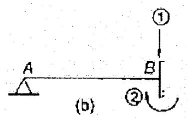
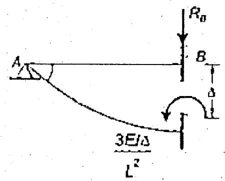
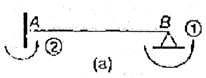
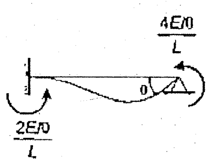
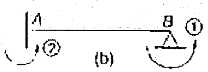
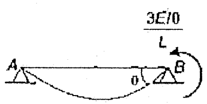
$$R_B = \frac{12EI}{L^3}(\uparrow)$$

$$\therefore \quad k_{12} = \frac{6EI}{L^2} \quad \text{and} \quad k_{22} = R_B = \frac{12EI}{L^3}$$

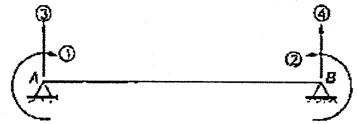
Hence the stiffness matrix for given beam is

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{12EI}{L^3} \end{bmatrix}$$

10.7.3 Standard Results for Stiffness

Type of Displacement	Co-ordinate system	Displacement diagram	Stiffness
1. Axial			$k_{11} = \frac{AE}{L}$
2. Transverse displacement (a) with far end fixed			$k_{11} = \frac{12EI}{L^3}$, $k_{21} = \frac{6EI}{L^2}$, $k_{31} = -\frac{6EI}{L^2}$
(b) with far end hinged			$k_{11} = \frac{3EI\Delta}{L^3}$, $k_{21} = -\frac{3EI}{L^2}$
3. Flexural displacement (a) with far end fixed			$k_{11} = \frac{4EI}{L}$, $k_{21} = \frac{2EI}{L}$
(b) with far end hinged			$k_{11} = \frac{3EI}{L}$, $k_{21} = 0$

Example 10.9 Generate stiffness matrix for the coordinate shown in figure.

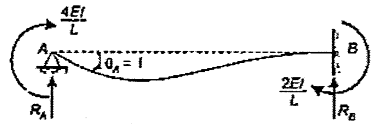


Solution:

First column: Give unit displacement in the direction of coordinate (1).

$\therefore \quad \theta_A = 1$ ↻ and ensure $\Delta_A = 0$, $\Delta_B = 0$ and $\theta_B = 0$

So, replace support B by fixed support.



$\Sigma F_y = 0; \quad R_A + R_B = 0 \quad \ldots(i)$

$\Sigma M_B = 0; \quad R_A \times L + \frac{4EI}{L} + \frac{2EI}{L} = 0$

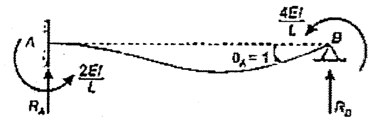
$$R_A = -\frac{6EI}{L^2} \text{ and } R_B = \frac{6EI}{L^2}$$

$$k_{11} = \frac{4EI}{L} \qquad k_{21} = \frac{2EI}{L}$$

$$k_{31} = -\frac{6EI}{L^2} \qquad k_{41} = \frac{6EI}{L^2}$$

Second column: Give unit displacement in the direction of coordinate (2).

$\therefore \quad \theta_B = 1$ ↻ and ensure $\Delta_B = \Delta_A = 0$ and $\theta_A = 0$



$\Sigma F_y = 0; \quad R_A + R_B = 0$

$\Sigma M_A = 0; \quad R_B \times L + \frac{4EI}{L} + \frac{2EI}{L} = 0$

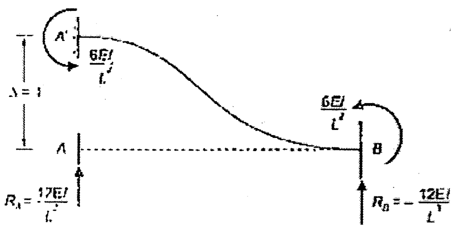
$$R_B = \frac{-6EI}{L^2} \text{ and } R_A = \frac{6EI}{L^2}$$

$$k_{12} = -\frac{2EI}{L} \qquad k_{22} = \frac{4EI}{L}$$

$$k_{32} = \frac{6EI}{L^2} \qquad k_{42} = \frac{-6EI}{L^2}$$

Third column: Give unit displacement in the direction of coordinate (3).

$\therefore \quad \Delta_A = 1\ (\uparrow)$ and ensure $\theta_A = \theta_B = 0$ and $\Delta_B = 0$



$$k_{13} = \frac{-6EI}{L^2}$$

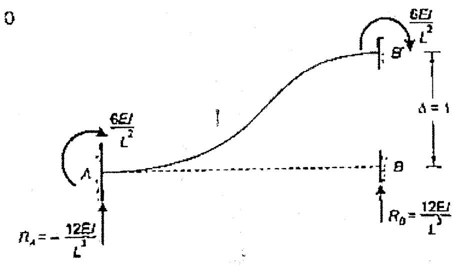
$$k_{23} = \frac{6EI}{L^2}$$

$$k_{33} = \frac{12EI}{L^3}$$

$$k_{43} = \frac{-12EI}{L^3}$$

Fourth column: Give unit displacement in the direction of coordinate (4).

$\therefore$ $\Delta_B = 1$ and ensure $\theta_A = \theta_B = 0$ and $\Delta_A = 0$



$$k_{14} = \frac{6EI}{L^2}$$

$$k_{24} = -\frac{6EI}{L^2}$$

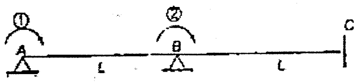
$$k_{34} = -\frac{12EI}{L^3}$$

$$k_{44} = \frac{12EI}{L^3}$$

Hence the stiffness matrix for given coordinate system is

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{-2EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\ \frac{-2EI}{L} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{-12EI}{L^3} \\ \frac{6EI}{L^2} & \frac{-6EI}{L^2} & \frac{-12EI}{L^3} & \frac{12EI}{L^3} \end{bmatrix}$$

Example 10.10 Generate stiffness matrix for coordinate shown in figure.

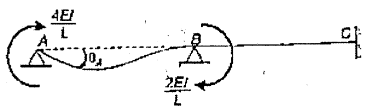


EI is constant

Solution:

First column: Give unit displacement in the direction of coordinate (1).

$\therefore$ $\theta_A = 1$ and ensure $\theta_B = 0$

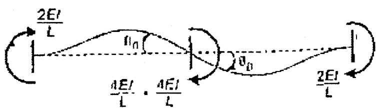


$$k_{11} = \frac{4EI}{L}$$

$$k_{21} = \frac{2EI}{L}$$

Second column: Give unit displacement in the direction of coordinate (2).

$\therefore$ $\theta_B = 1$ and ensure $\theta_A = \theta_C = 0$



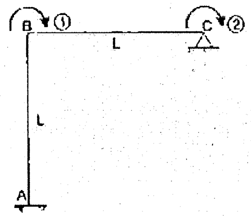
$$k_{12} = \frac{2EI}{L}$$

$$k_{22} = \frac{8EI}{L}$$

Hence the stiffness matrix for given beam is,

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{8EI}{L} \end{bmatrix}$$

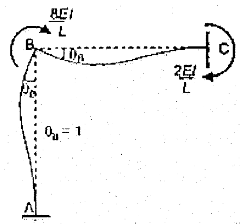
Example 10.11 For the frame shown in figure generate stiffness matrix.



Solution:

First column: Give unit displacement in the direction of coordinate (1).

$\therefore$ $\theta_B = 1$ and ensure $\theta_C = 0$

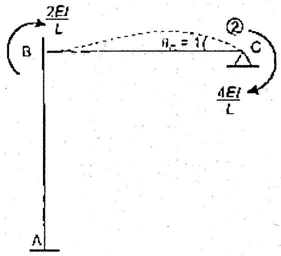


$$k_{11} = \frac{8EI}{L}$$

$$k_{21} = \frac{2EI}{L}$$

Second column: Give unit displacement in the direction of coordinate (2).

$\therefore$ $\theta_C = 1$ and ensure $\theta_B = 0$



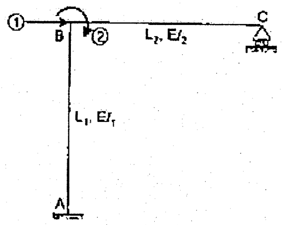
$$k_{12} = \frac{2EI}{L}$$

$$k_{22} = \frac{4EI}{L}$$

Hence stiffness matrix for given frame is

$$[k] = \begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Example 10.12 Draw stiffness matrix for the frame shown below.



Solution:

First column: Give unit displacement in the direction of coordinate (1).

$\therefore\quad \Delta_B = 1\ (\rightarrow)$ and ensure $\theta_B = 0$.

Take, $\quad \Sigma M_B' = 0$

$$H_A \times L_1 - \frac{6EI_1}{L_1^2} - \frac{6EI_1}{L_1^2} = 0$$

$$H_A = \frac{12EI_1}{L_1^3}$$

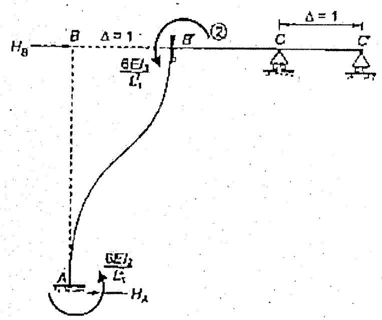
Also, $\quad \Sigma F_x = 0$

$\therefore \quad H_A = H_B$

$\therefore \quad H_B = \frac{12EI_1}{L_1^3}$

$$k_{11} = \frac{12EI_1}{L_1^3}$$

$$k_{21} = \frac{-6EI_1}{L_1^3}$$



Second column: Give unit displacement in the direction of coordinate (2).

$\therefore\quad \theta_B = 1$ and ensure $\Delta_B = 0\ (\rightarrow)$

$\Sigma F_x = 0$

$H_A = H_B$

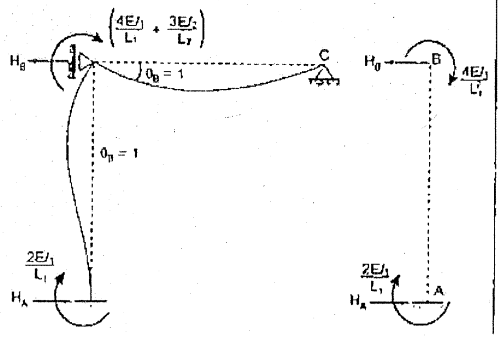
Also, $\quad \Sigma M_B = 0;$

$$-H_A \times L_1 + \frac{2EI_1}{L_1} + \frac{4EI_1}{L_1} = 0$$

$\therefore \quad H_A = \frac{6EI_1}{L_1^2}$

Hence, $\quad H_B = \frac{6EI_1}{L_1^2}$

$$k_{12} = -\frac{6EI_1}{L_1^2}$$



$$k_{22} = \frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}$$

Hence, stiffness matrix for given frame is

$$[k] = \begin{bmatrix} \frac{12EI_1}{L_1^3} & -\frac{6EI_1}{L_1^2} \\ \frac{-6EI_1}{L_1^2} & \left(\frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}\right) \end{bmatrix}$$

10.8 Analysis of Beam and Frame Using Flexibility Matrix Method

10.8.1 Degree of Static Indeterminacy (D_S)

(a) **2D-Beam:** As the beam has open configuration, the degree of internal indeterminacy is zero. Hence the degree of static indeterminacy is given by,

$$D_S = r_e - 3$$

If beam has internal hinges, then the degree of static indeterminacy,

$$D_S = r_e - 3 - n$$

where, r_e = No. of independent external reactions

n = No. of Internal hinges

(b) **2-D rigid frame:** For 2-D rigid frame, the degree of static indeterminacy is given by

$$D_S = 3m - r_e - 3j - r_r$$

where, m = number of members

r_e = number of independent external reactions

j = number of joints

r_r = number of reactions released

10.8.2 Basic Released Structure

It is statically determinate and stable structure which is obtained by releasing a sufficient number of internal forces or external reaction component in order to obtain determinate structure from corresponding statically indeterminate structure.

Consider a continuous beam *ABCD* as shown in figure.

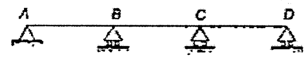


Fig 10.10

Degree of static indeterminacy for above beam is

$$D_S = r_e - 3$$

Here, $\quad r_e = 5$

$\therefore \quad D_S = 5 - 3 = 2$

$\therefore$ Beam is statically indeterminate to second degree. Thus to make the beam statically determinate, two reactions component either internal or external have to be released.

Case-1: Consider external reactions at *B* and *C* are redundant. The released structure can be obtained by removing restraint offered by reactions. For above beam released structure is simply supported beam.

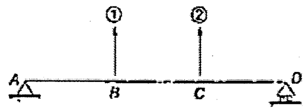


Fig 10.11 Released structure when R_B and R_C are released

Case-2: If bending moment at B and external support reaction at C is released. Then the released structure comprises two simply supported beam AB and BD as shown in figure.

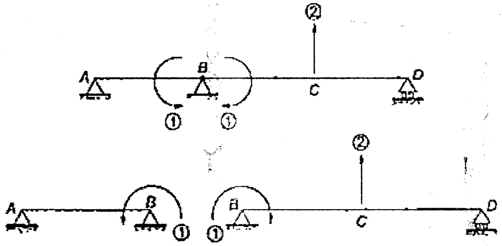


Fig. 10.12 Released structure when M_B and R_C are released

Case-3: If bending moment at B and C is released. Then the released structure comprises three simply supported beams AB, BC and CD as shown in figure.

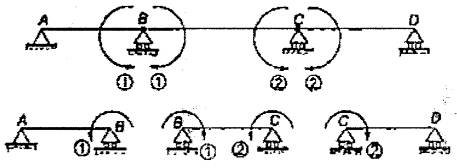


Fig. 10.13 Released structure when M_B and M_C are released

Procedure of Analysis

In flexibility method, unknown forces are taken as redundant and number of redundant are equal to degree of static indeterminacy. If there are N redundants, then flexibility matrix will be a square matrix of size $N \times N$.

Consider a beam with coordinates as shown in figure.

f_{11} = Deflection in the direction of coordinate (1) when unit load is applied in the direction of coordinate (1).

f_{12} = Deflection in the direction of coordinate (1) when unit load is applied in the direction coordinate (2).

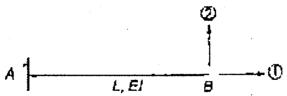


Fig. 10.14

If load P_1 is acting in the direction of coordinate (1) and load P_2 is acting in the direction of coordinate (2).

Thus the total deflection in the direction of coordinate (1) is,

$$\Delta_1 = f_{11}P_1 + f_{12}P_2 \quad \ldots(i)$$

Similarly, the total deflection in the direction of coordinate (2) is,

$$\Delta_2 = f_{21}P_1 + f_{22}P_2 \quad \ldots(ii)$$

The equations (*i*) and (*ii*) can be represented in matrix form as

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$[\Delta] = [f][P]$$

$$\therefore \quad P = [f]^{-1}[\Delta]$$

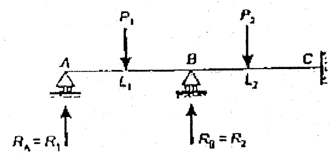


Fig. 10.15

Step-1. Find degree of static indeterminacy (D_S). Now identify the redundants in such a way that released structure remain stable and determinate. Neglect all axial effect in beams. Example

In above case there are two redundant say $R_A = R_1$ and $R_B = R_2$.

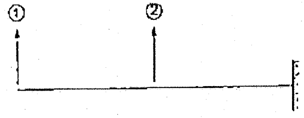


Fig. 10.16

Step-2. Remove the redundants and assign one coordinate in the direction of each redundant.

Step-3. Develop flexibility matrix for above coordinate system and find inverse of flexibility matrix i.e. $[f]^{-1}$.

Step-4. Remove redundant and obtain basic released structure with given loading which is statically determinate and stable.

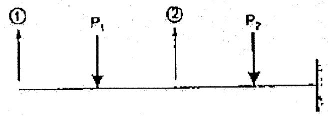


Fig. 10.17 Basic released structure

Step-5. For basic released structure, find deflection due to given loading in the direction of assign coordinate. Let Δ_{1L} and Δ_{2L} are deflections in the direction of coordinate (1) and (2) due to given loading in basic released structure.

Step-6. Remove loading and apply redundant forces in the direction of assign coordinate and find deflection at coordinate (1) and (2). Let Δ_{1R} and Δ_{2R} are the displacements due to redundant reactions in the direction of coordinates (1) and (2) respectively.

$$\therefore \quad \Delta_{1R} = f_{11}R_1 + f_{12}R_2$$

$$\Delta_{2R} = f_{21}R_1 + f_{22}R_2$$

Hence,

$$\begin{bmatrix} \Delta_{1R} \\ \Delta_{2R} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \ldots(i)$$

Find final deflection due to given loading and redundant reaction at coordinate (1) and (2).

$$\Delta_1 = \Delta_{1L} + \Delta_{1R}$$

$$\Delta_2 = \Delta_{2L} + \Delta_{2R}$$

Since in the direction of coordinate (1) and (2) redundant reactions are present. Hence final deflections will be zero.

$$\therefore \quad \Delta_{1R} = -\Delta_{1L}$$

$$\Delta_{2R} = -\Delta_{2L}$$

Also,

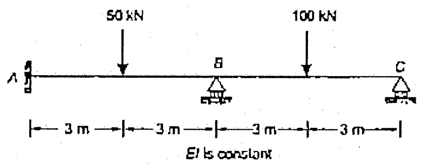
$$\begin{bmatrix} \Delta_{1R} \\ \Delta_{2R} \end{bmatrix} = \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix}$$

From equation (*i*), we get

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1} \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix} \quad \ldots(ii)$$

Example 10.13 Analyse the beam shown in figure using flexibility matrix method.



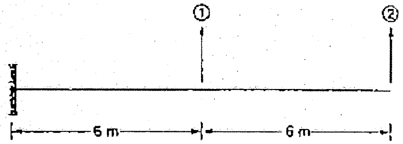
Solution:

$D_S = r_a - 3$

$D_S = 5 - 3 = 2$

Thus given continuous beam is redundant to second degree.

Let us consider reactions R_B and R_C as redundant. Also let coordinate (1) in the direction of R_B and coordinate (2) in the direction of R_C.



Co-ordinate System

Flexibility Matrix

Column 1st: Apply unit load in the direction of coordinate (1) and measure displacements in the direction of coordinate (1) and (2).

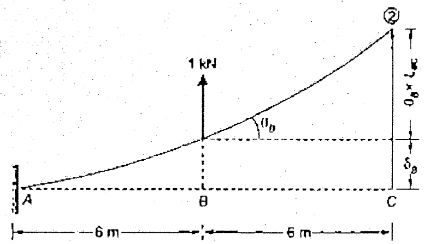
$$f_{11} = \frac{1\times 6^3}{3EI} = \frac{72}{EI}$$

$$f_{21} = \delta_B + \theta_B \times L_{BC}$$

$$f_{21} = \frac{72}{EI} + \frac{1\times(6)^2}{2EI}\times 6$$

$$= \frac{72}{EI} + \frac{108}{EI}$$

$$f_{21} = \frac{180}{EI}$$



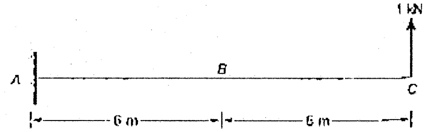
Also, from reciprocal theorem,

$$f_{12} = f_{21} = \frac{180}{EI}$$

Column 2nd: Apply unit load in the direction of coordinate (2) and measure displacement is the direction of coordinate (1) and (2).

$$f_{12} = \frac{180}{EI}$$

$$f_{22} = \frac{1\times(12)^3}{3EI} = \frac{576}{EI}$$



Hence, the flexibility matrix for basic released structure is

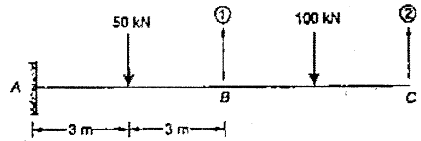
$$[f] = \begin{bmatrix} \frac{72}{EI} & \frac{180}{EI} \\ \frac{180}{EI} & \frac{576}{EI} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 72 & 180 \\ 180 & 576 \end{bmatrix}$$

Inverse of flexibility matrix,

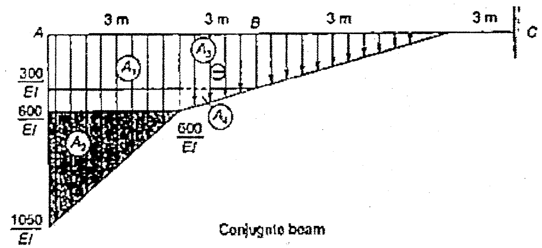
$$[f]^{-1} = \frac{Adj[f]}{|f|} = \frac{EI\begin{bmatrix} 576 & -180 \\ -180 & 72 \end{bmatrix}}{(72\times 576 - 180\times 180)}$$

$$\therefore \quad [f]^{-1} = EI\begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:



Using conjugate beam method,



Conjugate beam

$$\Delta_{1L} = \text{B.M at } B$$

$$= A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 + A_4\bar{x}_4$$

$$= -\frac{600}{EI}\times 3\times 4.5 - \frac{1}{2}\times\frac{450}{EI}\times 3\times\left(3+\frac{2}{3}\times 3\right) - \frac{300}{EI}\times 3\times 1.5 - \frac{1}{2}\times\frac{300}{EI}\times 3\times\frac{2}{3}\times 3$$

$$= -\frac{1}{EI}\left[600\times 3\times 4.5 + \frac{450\times 3\times 5}{2} + 300\times 3\times 1.5 + 300\times 3\times 1\right]$$

$$= -\frac{13725}{EI}(\downarrow)$$

$$\Delta_{2L} = \text{BM at } C$$

$$\Delta_{2L} = -\frac{600}{EI}\times 3\times 10.5-\frac{1}{2}\times\frac{450}{EI}\times 3\times\left(4+\frac{2}{3}\times 3\right)-\frac{1}{2}\times\frac{300}{EI}\times 3\times\left(6+\frac{2}{3}\times 3\right)$$

$$-\frac{300}{EI}\times 3\times(6+1.5)-\frac{1}{2}\times\frac{300}{EI}\times 3\left(3+\frac{2}{3}\times 3\right)$$

$$= -\frac{1}{EI}\left[600\times 3\times 10.5+\frac{450\times 3\times 11}{2}+\frac{300\times 3\times 8}{2}+300\times 3\times 7.5+\frac{300\times 3\times 5}{2}\right]$$

$$= -\frac{38925}{EI}\ (\downarrow)$$

$\therefore \quad R = [f]^{-1}\{-\Delta_L\}$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}\times\frac{1}{EI}\begin{bmatrix} -(-13725) \\ -(-38925) \end{bmatrix}$$

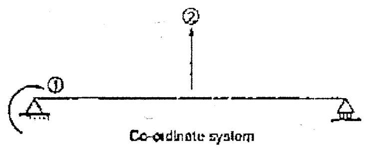
$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}\begin{bmatrix} 13725 \\ 38925 \end{bmatrix}$$

$\therefore \quad R_B = R_1 = 0.0634 \times 13725 - 0.0198 \times 38925 = +99.45$ kN

$R_C = R_2 = -0.0198 \times 13725 + 0.0079 \times 38925 = +35.75$ kN

Alternate Solution

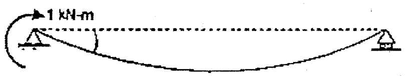
Now let us consider M_A and R_B as redundant. Also let coordinate (1) in the direction of M_A and coordinate (2) in the direction of R_B.



Co-ordinate system

Flexibility Matrix

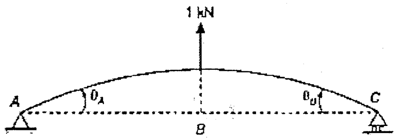
Column 1st:



$$f_{11} = \frac{1\times L}{3EI} = \frac{4}{EI}$$

$f_{21} = f_{12}$ (By Reciprocal theorem)

Column 2nd:



$$f_{12} = -\frac{1\times L^2}{16EI} = -\frac{9}{EI}$$

$$f_{22} = \frac{1\times L^3}{48EI} = \frac{36}{EI}$$

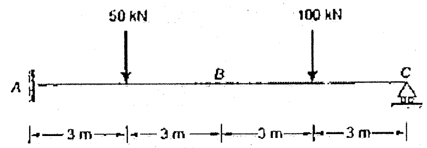
Hence flexibility matrix for selected coordinate system is

$$[f] = \frac{1}{EI}\begin{bmatrix} 4 & -9 \\ -9 & 36 \end{bmatrix}$$

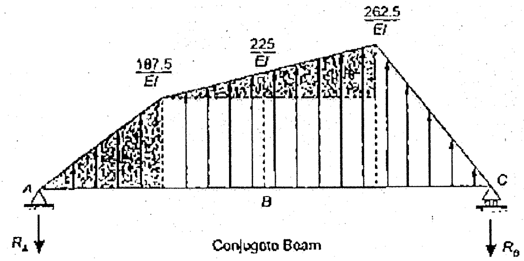
The inverse of flexibility matrix,

$$[f]^{-1} = \frac{Adj|f|}{|f|} = EI\begin{bmatrix} 0.5714 & +0.1428 \\ +0.1428 & 0.0634 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:



Using conjugate beam method,



$$R_A + R_B = \frac{1}{2}\times\frac{187.5}{EI}\times 3+\frac{187.5}{EI}\times 6+\frac{1}{2}\times\left(\frac{262.5}{EI}-\frac{187.5}{EI}\right)\times 6+\frac{1}{2}\times\frac{262.5}{EI}\times 3-\frac{2025}{EI} \quad \ldots(i)$$

Applying, $\quad \Sigma M_C = 0$

$$\Rightarrow -R_A\times 12+\frac{1}{2}\times 3\times\frac{187.5}{EI}\times 10+\frac{187.5}{EI}\times 6\times 6+\frac{1}{2}\times 6\times\frac{75}{EI}\times 5+\frac{1}{2}\times\frac{262.5}{EI}\times 3\times\frac{2}{3}\times 3 = 0$$

$$\Rightarrow -12R_A+\frac{(2812.5+6750+1125+787.5)}{EI} = 0$$

$$\Rightarrow \quad R_A = \frac{956.25}{EI}$$

$$\therefore \quad R_B = \frac{1068.75}{EI} \qquad \left(R_B = \frac{2025}{EI}-R_A\right)$$

$$\therefore \quad \Delta_{1L} = \theta_A = \text{SF at A in C.B} = -\frac{956.25}{EI} \circlearrowright = +\frac{956.25}{EI}$$

(in the direction of (2)

Δ_{2L} = B.M at B in C.B

$$= -R_A\times 6+\left(\frac{1}{2}\times 3\times\frac{187.5}{EI}\times 4\right)+\left(\frac{187.5}{EI}\times 3\times 1.5\right)+\left(\frac{1}{2}\times 3\times\frac{37.5}{EI}\times 1\right)$$

$$= -\frac{956.25\times 6}{EI}+\frac{3\times 187.5\times 4}{2EI}+\frac{187.5\times 3\times 1.5}{EI}+\frac{3\times 37.5}{2EI} = -\frac{7425}{EI}\ (\downarrow)$$

$\therefore \quad [R] = [f]^{-1}[-\Delta_L]$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.5714 & 0.1428 \\ 0.1428 & 0.0634 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -956.25 \\ +3712.5 \end{bmatrix}$$

$\therefore \quad R_1 = 0.5714 \times -956.25 + 0.1428 \times 3712.5$

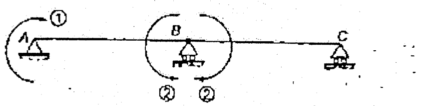
$M_A = R_1 = -16.25$ kN

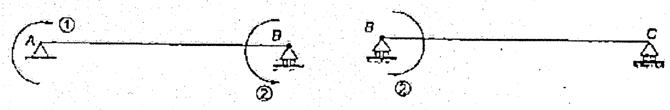
and $\quad 0.1428 \times -956.25 + 0.0634 \times 3712.5$

$R_B = R_2 = 98.82$ kN (↑)

Alternate Solution

Let us consider M_A and M_B as redundant. Also let coordinate (1) in the direction of M_A and coordinate (2) in the direction of M_B.



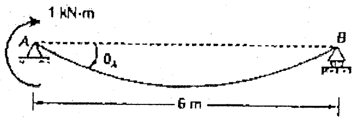


Flexibility Matrix

Column 1st:

$$f_{11} = \frac{1 \times L}{3EI} = \frac{6}{3EI} = \frac{2}{EI}$$

$$f_{21} = \frac{L}{6EI} = \frac{1}{EI}$$



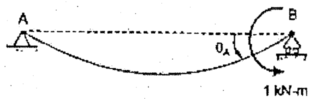
Column 2nd:

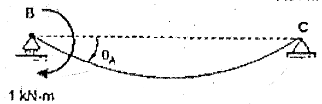
$$f_{12} = f_{21} = \frac{1}{EI}$$

$$f_{22} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$f_{22} = \frac{L}{3EI} + \frac{L}{3EI} = \frac{2L}{3EI} = \frac{2 \times 6}{3EI}$$

$\therefore \quad f_{22} = \frac{4}{EI}$





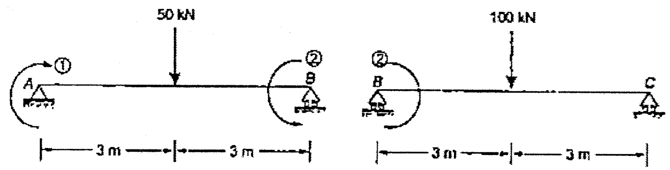
Hence flexibility matrix for selected coordinate is

$$[f] = \frac{1}{EI}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj\,[f]}{|f|} = EI\begin{bmatrix} 0.5714 & -0.1428 \\ -0.1428 & 0.2857 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:



$$\Delta_{1L} = \theta_A = \frac{50 \times 6^2}{16EI} = \frac{112.5}{EI}$$

$$\Delta_{2L} = \frac{50 \times 6^2}{16EI} + \frac{100 \times 6^2}{16EI} = \frac{337.5}{EI}$$

$\therefore \quad R = [f]^{-1}[-\Delta_L]$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.5714 & -0.1428 \\ -0.1428 & 0.2857 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -112.5 \\ -337.5 \end{bmatrix}$$

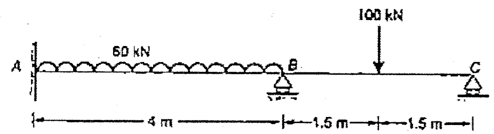
$\therefore \quad M_A = R_1 = 0.5714 \times (-112.5) - 0.1428 \times (-337.5)$

$M_A = -16.08$ kN-m

and $\quad M_B = R_1 = -0.1428 \times (-112.5) + 0.2857 \times (-337.5)$

$M_B = -80.35$ kN-m

Example 10.14 Analyses the continuous beam shown in figure. Use flexibility matrix method.



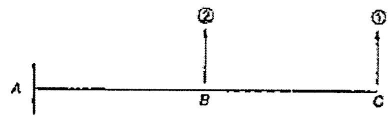
EI is constant

Solution:

$$D_S = 5 - 3 = 2$$

Thus the given beam is indeterminate to second degree.

Let us consider R_C and R_B be the redundants. Also let coordinate (1) in the direction of R_C and coordinate (2) in the direction of R_B.

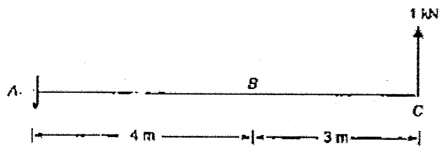


Co-ordinate System

Flexibility Matrix

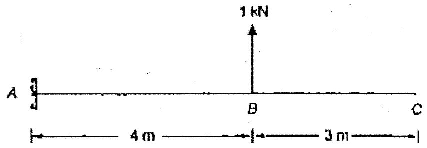
Column 1st:

$$f_{11} = \frac{1 \times (7)^3}{3EI} = \frac{343}{3EI}$$



$f_{21} = f_{12}$ (From Maxwell's reciprocal theorem)

Column 2nd:



$$f_{12} = \frac{1\times4^3}{3EI} + \frac{1\times4^2}{2EI}\times3$$

$$f_{12} = \frac{136}{3EI}$$

$$f_{22} = \frac{1\times4^3}{3EI} = \frac{64}{3EI}$$

Hence the flexibility matrix for selected coordinate is

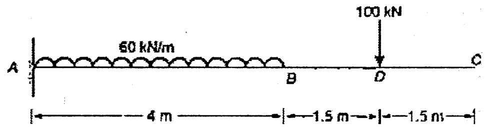
$$[f] = \frac{1}{EI}\begin{bmatrix} \frac{343}{3} & \frac{136}{3} \\ \frac{136}{3} & \frac{64}{3} \end{bmatrix}$$

The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj[f]}{|f|} = \frac{EI}{|f|}\begin{bmatrix} \frac{64}{3} & \frac{-136}{3} \\ \frac{-136}{3} & \frac{343}{3} \end{bmatrix}$$

$\therefore$ $$[f]^{-1} = EI\begin{bmatrix} 0.055 & -0.118 \\ -0.118 & 0.297 \end{bmatrix}$$

This displacement in the direction of coordinates due to external loading:



$$\Delta_{1L} = \Delta_C = -\left[\frac{100\times(AD)^3}{3EI} + \frac{100\times(AD)^2}{2EI}\times CD + \frac{60\times(AB)^4}{8EI} + \frac{60\times(AB)^3}{6EI}\times BC\right]$$

$$= -\left[\frac{100\times5.5^3}{3EI} + \frac{100\times5.5^2}{2EI}\times1.5 + \frac{60\times4^4}{8EI} + \frac{60\times4^3}{6EI}\times3\right] = -\frac{11654.58}{EI}$$

$$\Delta_{2L} = \Delta_B = -\left[\frac{100\times(AB)^3}{3EI} + \frac{100\times(AB)^2}{2EI}\times BD + \frac{60\times(AB)^4}{8EI}\right]$$

$$\Delta_{2L} = -\frac{5253.33}{EI}$$

$\therefore$ $$[R] = [f]^{-1}[-\Delta_L]$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.055 & -0.118 \\ -0.118 & 0.297 \end{bmatrix}\times\frac{1}{EI}\begin{bmatrix} -(-11654.58) \\ -(-5253.33) \end{bmatrix}$$

$\therefore$ $$R_C = R_1 = 0.055\times11654.58 - 0.118\times5253.33$$

$$R_C = 21.10 \text{ kN}$$

and $$R_B = R_2 = -0.118\times11654.58 + 0.297\times5253.33$$

$\therefore$ $$R_B = 185 \text{ kN}$$

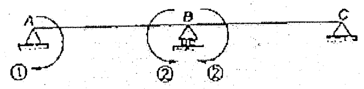
$\Sigma F_y = 0;$

$$R_A + R_B + R_C = 100 + 60\times4$$

$$R_A = 340 - 21.10 - 185 = 133.9 \text{ kN}$$

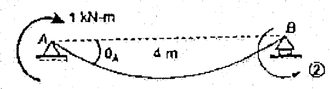
Alternate Solution

Now, let us select M_A and M_B as redundant.



Flexibility Matrix

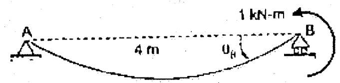
Column 1st:



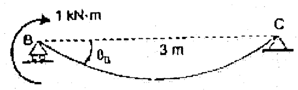
$$f_{11} = \frac{4}{3EI}$$

$$f_{21} = \frac{4}{6EI}$$

Column 2nd:



$$f_{12} = \frac{4}{6EI}$$



$$f_{22} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$= \frac{4}{3EI} + \frac{3}{3EI} = \frac{7}{3EI}$$

Hence the flexibility matrix for selected coordinate is

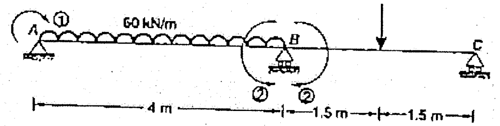
$$[f] = \frac{1}{EI}\begin{bmatrix} \frac{4}{3} & \frac{4}{6} \\ \frac{4}{6} & \frac{7}{3} \end{bmatrix}$$

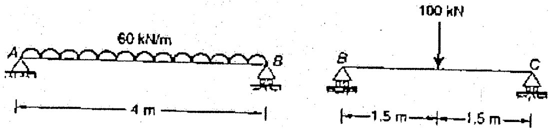
The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj[f]}{|f|} = \frac{EI}{2.654}\begin{bmatrix} 2.33 & -0.667 \\ -0.667 & 1.33 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.877 & -0.251 \\ -0.251 & 0.501 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:





$$\Delta_{1L} = \theta_A = \frac{60\times4^3}{24EI} = \frac{160}{EI}$$

$$\Delta_{2L} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$= \frac{60\times4^3}{24EI} + \frac{100\times3^2}{16EI} = \frac{216.25}{EI}$$

$$[R] = [f]^{-1}[-\Delta_L]$$

$$\therefore \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.877 & -0.251 \\ -0.251 & 0.501 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -160 \\ -216.25 \end{bmatrix}$$

$$\therefore \quad R_1 = 0.877\times(-160) + (-0.251)\times(-216.25)$$

$$M_A = R_1 = -86.04 \text{ kN-m}$$

$$M_B = R_2 = -0.251\times(-160) + 0.501\times(-216.25) = -68.18 \text{ kN-m}$$

10.9 Analysis of Beam and Frame Using Stiffness Matrix Method

Procedure of Analysis using Stiffness Matrix Method

Step-1. Determine degree of kinematic indeterminacy neglecting axial deformations. Identify independent joint displacement $\Delta_1, \Delta_2, \ldots\ldots\Delta_n$.

Step-2. Assign one coordinate in the direction of each unknown displacement of joint. Develop stiffness matrix and find inverse of matrix.

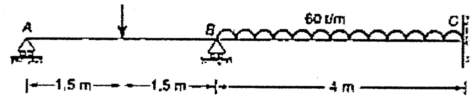
Step-3. Remove displacements in the direction of coordinates and obtain locked structure. Find the forces developed in the direction of assign coordinate due to given loading in the locked structure.

Let $P_{1L}, P_{2L} \ldots\ldots P_{nL}$ are forces developed in the direction of coordinate. Then displacement $\Delta_1, \Delta_2, \Delta_3 \ldots\ldots \Delta_n$ will be

$$\therefore \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12}\ldots\ldots k_{1n} \\ k_{21} & k_{22}\ldots\ldots k_{2n} \\ \vdots & \vdots \ldots\ldots k_{nn} \\ k_{n1} & \end{bmatrix} \begin{bmatrix} -P_{1L} \\ -P_{2L} \\ -P_{3L} \end{bmatrix}$$

$$\therefore \quad \Delta = [k]^{-1}[-P_L]$$

Example 10.15 Using stiffness matrix method analyses the beam shown in figure. Find final moment and also draw bending moment diagram.



EI is constant

Solution:

$$D_K = 3j - r_e - m''$$

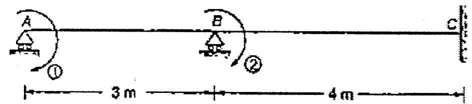
Here, $j = 3$, $r_e = 5$ and m'' = axially rigid member = 2

$$\therefore \quad D_K = 3\times3-5-2$$

$$D_K = 2$$

Take θ_A and θ_B as unknown displacement.

Assign coordinate (1) in the direction of θ_A and coordinate (2) in the direction θ_B



Stiffness Matrix

Column 1st:

Give unit displacement in the direction of coordinate (1).

$\therefore \quad \theta_A = 1$ and ensure $\theta_B = 0$

$$k_{11} = \frac{4EI}{3}$$

$$k_{21} = \frac{2EI}{3}$$

Column 2nd:

Give unit displacement in the direction of coordinate (2).

$\therefore \quad \theta_B = 1$ and ensure $\theta_A = 0$

$$k_{12} = \frac{2EI}{3}$$

$$k_{22} = \frac{4EI}{3} + \frac{4EI}{4} = \frac{7}{3}EI$$

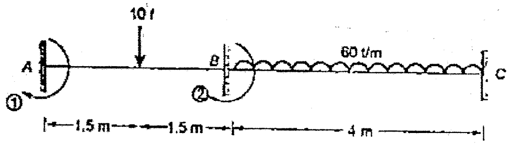
The stiffness matrix for selected coordinate is

$$[k] = \begin{bmatrix} \frac{4EI}{3} & \frac{2EI}{3} \\ \frac{2EI}{3} & \frac{7EI}{3} \end{bmatrix} = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

The inverse of stiffness matrix is

$$[k]^{-1} = \frac{1}{8EI}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

Let P_{1L} and P_{2L} are forces (Moments) developed in the coordinate directions due to the given load in locked structure.



$$\overline{M}_{AB} = -\frac{10\times3}{8} = -3.75 \text{ t-m}$$

$$\overline{M}_{BA} = 3.75 \text{ t-m}$$

$$\overline{M}_{BC} = -\frac{6\times4^2}{12} = -8 \text{ t-m}$$

$$\overline{M}_{CB} = 8 \text{ t-m}$$

$$P_{1L} = \overline{M}_{AB} = -3.75 \text{ t-m}$$

$$P_{2L} = \overline{M}_{AB} - \overline{M}_{BC}$$

$$= 3.75 - 8 = -4.25 \text{ t-m}$$

$\therefore$ $$\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix} = \frac{1}{8EI}\begin{bmatrix}7 & -2\\ -2 & 4\end{bmatrix}\begin{bmatrix}-(-3.75)\\ -(-4.25)\end{bmatrix}$$

$\therefore$ $$\Delta_1 = \frac{1}{8EI}[7\times3.75 - 2\times4.25]$$

$\therefore$ $$\Delta_1 = \theta_A = \frac{2.22}{EI}$$

and $$\Delta_2 = \frac{1}{8EI}[-2\times3.75 + 4\times4.25]$$

$\therefore$ $$\Delta_2 = \theta_B = \frac{1.1875}{EI}$$

Final end moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$M_{AB} = -3.75 + \frac{2EI}{3}\left(2\times\frac{2.22}{EI} + \frac{1.1875}{EI} - 0\right)$$

$$M_{AB} = 0 \text{ t-m}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

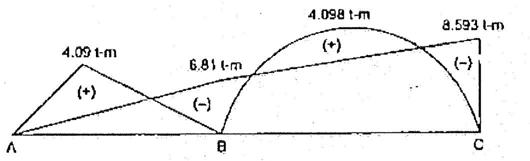
$$= 3.75 + \frac{2EI}{3}\left(\frac{2\times1.1875}{EI} + \frac{2.22}{EI} - 0\right) = 6.81 \text{ t-m}$$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

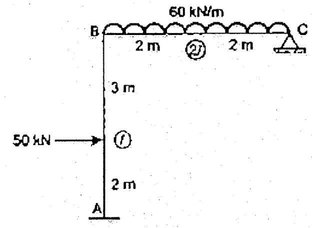
$$= -8 + \frac{2EI}{4}\left(\frac{2\times1.1875}{EI} + 0 - 0\right) = -6.81 \text{ t-m}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(\theta_B + 2\theta_C - \frac{3\Delta}{L}\right)$$

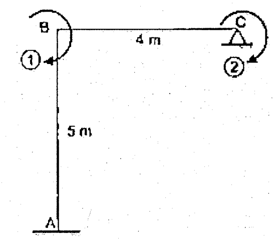
$$= 8 + \frac{2EI}{4}\left(\frac{1.1875}{EI} + 0 - 0\right) = 8.593 \text{ t-m}$$



Example 10.16 Analysis the frame shown in figure using stiffness matrix method.



Solution:



$$D_K = 3j - r_e - m''$$

Here, $j = 3$, $r_e = 5$, $m'' = 2$

$\therefore$ $$D_K = 3\times3 - 5 - 2 = 2$$

Take θ_B and θ_C as unknown displacement. Assign coordinate (1) in the direction of θ_A and coordinate (2) in the direction of θ_B.

Stiffness Matrix

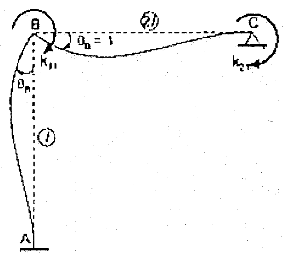
Column 1st:

Give unit displacement in the direction of coordinate (1).

$\therefore$ $\theta_B = 1$ and ensure $\theta_C = 0$

$$k_{11} = \frac{4EI}{5} + \frac{4E(2I)}{4} = \frac{14EI}{5}$$

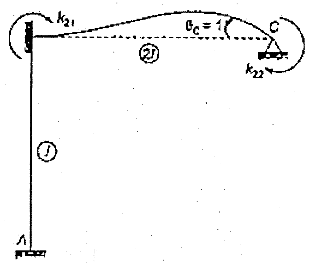
$$k_{21} = \frac{2E(2I)}{4} = EI$$



Column 2nd:

Give unit displacement in the direction of coordinate (2).

$\therefore \quad \theta_C = 1$ and ensure $\theta_B = 0$



$$k_{12} = k_{21} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$

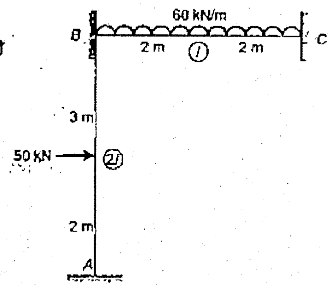
Hence, the stiffness matrix for selected coordinate is

$$[k] = \begin{bmatrix} \frac{14}{5}EI & EI \\ EI & 2EI \end{bmatrix}$$

The inverse of stiffness matrix is

$$[k]^{-1} = \frac{1}{23EI}\begin{bmatrix} 10 & -5 \\ -5 & 14 \end{bmatrix} \text{ or } \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}$$

Let P_{1L} and P_{2L} are forces (moments) developed in the coordinate directions due to the given loading in locked structure.



$$\overline{M}_{AB} = -\frac{Pab^2}{L^2} = -\frac{50 \times 2 \times 3^2}{5^2} = -36 \text{ kN-m}$$

$$\overline{M}_{BA} = \frac{Pba^2}{L^2} = \frac{50 \times 3 \times 2^2}{5^2} = 24 \text{ kN-m}$$

$$\overline{M}_{BC} = -\frac{wl^2}{12} = -\frac{60 \times 4^2}{12} = -80 \text{ kN-m}$$

$$\overline{M}_{CB} = \frac{wl^2}{12} = \frac{60 \times 4^2}{12} = 80 \text{ kN-m}$$

$\therefore \quad P_{1L} = \overline{M}_{BA} - \overline{M}_{BC}$

$\Rightarrow \quad P_{1L} = 24 - 80 = -56$ kN-m

and $\quad P_{2L} = \overline{M}_{CB}$

$\therefore \quad P_{2L} = 80$ kN

Hence,
$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = [k]^{-1}\begin{bmatrix} -P_{1L} \\ -P_{2L} \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}\begin{bmatrix} -(-56) \\ -(80) \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}\begin{bmatrix} 56 \\ -80 \end{bmatrix}$$

$$\therefore \quad \Delta_1 = \frac{1}{EI}[0.434 \times 56 + (-0.217) \times (-80)]$$

$$\Rightarrow \quad \theta_B = \Delta_1 = \frac{41.664}{EI}$$

and
$$\Delta_2 = \frac{1}{EI}[-0.217 \times 56 + 0.608 \times (-80)]$$

$$\Rightarrow \quad \theta_C = \Delta_2 = -\frac{60.792}{EI}$$

Final end moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L_{AB}}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -36 + \frac{2EI}{5}\left(0 + \frac{41.664}{EI} - 0\right) = -19.33 \text{ kN-m}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L_{AB}}\left(\theta_A + 2\theta_B - \frac{3\Delta}{L}\right)$$

$$= 24 + \frac{2EI}{5}\left(0 + \frac{2 \times 41.664}{EI} - 0\right) = 57.46 \text{ kN-m}$$

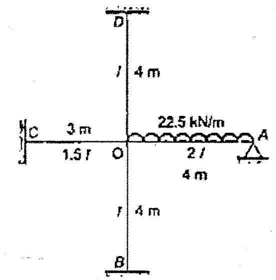
$$M_{BC} = \overline{M}_{BC} + \frac{2E(2I)}{L_{BC}}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -80 + \frac{4EI}{4}\left(2 \times \frac{41.664}{EI} - \frac{60.792}{EI}\right) = -57.46 \text{ kN-m}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2E(2I)}{L_{BC}}\left(\theta_B + 2\theta_C - \frac{3\Delta}{L}\right)$$

$$M_{CB} = +80 + \frac{4EI}{4}\left(\frac{41.664}{EI} - \frac{2 \times 60.792}{EI} - 0\right) = 0$$

Example 10.17 Analyse the frame shown in figure using stiffness matrix method.



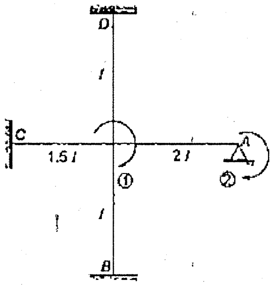
Solution:

$$D_K = 3j - r_e - m''$$

Here, $j = 5$, $r_e = 11$

$m'' = 2$ (Axial displacement of *OD* and *OB* already prevented i.e. fixed at both end)

$$\therefore \quad D_K = 3 \times 5 - 11 - 2 = 2$$



Take θ_0 and θ_A as unknown displacement. Assign coordinate (1) in the direction of θ_0 and coordinate (2) in the direction of θ_A.

Stiffness Matrix:

Column 1st:

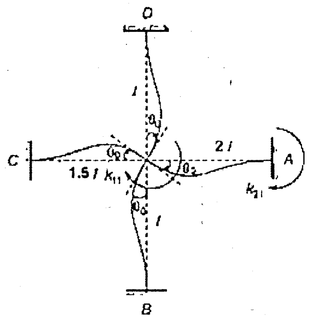
Give unit displacement in the direction of coordinate (1).

$\therefore \quad \theta_0 = 1$ and ensure $\theta_A = 0$.

$$k_{11} = \frac{4E(2I)}{4} + \frac{4EI}{4} + \frac{4E(1.5I)}{3} + \frac{4EI}{4}$$

$$k_{11} = 6EI$$

$$k_{21} = \frac{2E(2I)}{4} = EI$$



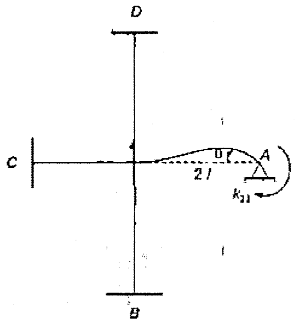
Column 2nd:

Give unit displacement in the direction of coordinate (2).

$\therefore \quad \theta_A = 1$ and ensure $\theta_0 = 0$

$$k_{12} = k_{21} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$



The stiffness matrix for selected coordinate is,

$$[k] = \begin{bmatrix} 6EI & EI \\ EI & 2EI \end{bmatrix}$$

The inverse of stiffness matrix is

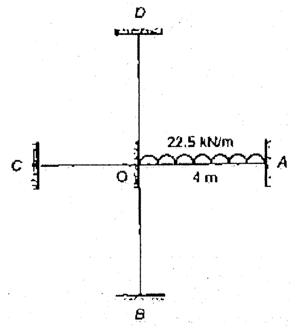
$$[k]^{-1} = \frac{Adj\,[k]}{|k|} = \frac{1}{11EI}\begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}$$

Let P_{1L} and P_{2L} are forces (moments) developed in the coordinate directions due to the given loading in locked structure.

$$P_{1L} = \bar{M}_{OA} = -\frac{22.5 \times 4^2}{12}$$

$$\therefore \quad P_{1L} = -30 \text{ kN-m}$$

$$P_{2L} = \bar{M}_{AO} = +30 \text{ kN-m}$$



Hence,
$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{11EI}\begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} -(-30) \\ -(30) \end{bmatrix}$$

$$\therefore \quad \Delta_1 = \frac{1}{11EI}(2 \times 30 - 1 \times (-30))$$

$$\Rightarrow \quad \theta_0 = \Delta_1 = \frac{8.18}{EI}$$

and
$$\Delta_2 = -1 \times 30 + 6 \times -30$$

$$\Rightarrow \quad \theta_A = \Delta_2 = \frac{-19.09}{EI}$$

Final end moments:

$$M_{AO} = \bar{M}_{AO} + \frac{2E(2I)}{L}\left(2\theta_A + \theta_0 - \frac{3\Delta}{L}\right)$$

$$= 30 + \frac{4EI}{4}\left(\frac{2 \times -19.09}{EI} + \frac{8.18}{EI} - 0\right) = 0$$

$$M_{OA} = \bar{M}_{OA} + \frac{2E(2I)}{L}\left(\theta_A + 2\theta_0 - \frac{3\Delta}{L}\right)$$

$$= -30 + \frac{4EI}{4}\left(-\frac{19.09}{EI} + \frac{2 \times 8.18}{EI} - 0\right) = -32.73 \text{ kN-m}$$

$$M_{BO} = \bar{M}_{BO} + \frac{2EI}{4}\left(2\theta_B + \theta_0 - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{8.18}{EI}\right) = 4.09 \text{ kN-m}$$

$$M_{OB} = \bar{M}_{OB} + \frac{2EI}{4}\left(\theta_B + 2\theta_0 - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{2 \times 8.18}{EI} - 0\right) = 8.18 \text{ kN-m}$$

$$M_{CO} = \bar{M}_{CO} + \frac{2E(1.5EI)}{4}\left(2\theta_C + \theta_0 - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{3EI}{4}\left(0 + \frac{8.18}{EI} - 0\right) = 6.135 \text{ kN-m}$$

$$M_{OC} = \bar{M}_{OC} + \frac{2E(1.5I)}{4}\left(\theta_C + 2\theta_0 - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{3EI}{4}\left(0 + \frac{2 \times 8.18}{EI} - 0\right) = 12.27 \text{ kN-m}$$

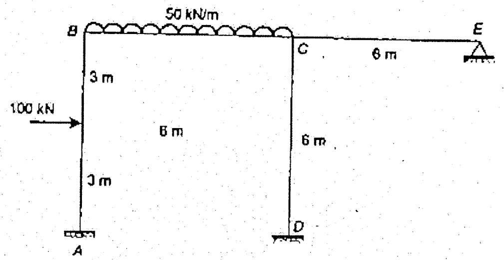
$$M_{DO} = \bar{M}_{DO} + \frac{2EI}{4}\left(2\theta_D + \theta_O - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{8.18}{EI} - 0\right) = 4.09 \text{ kN-m}$$

$$M_{OD} = \bar{M}_{OD} + \frac{2EI}{4}\left(\theta_D + 2\theta_O - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{2\times 8.18}{EI} - 0\right) = 8.18 \text{ kN-m}$$

Example 10.18 Using stiffness matrix method, analyse the frame shown in figure.



EI is constant

Solution:

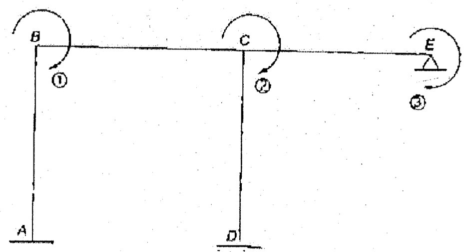
$$D_K = 3j - r_e - m''$$

Here, $j = 5$, $r_e = 8$, $m'' = 4$

$\therefore$

$$D_K = 3 \times 5 - 8 - 4$$
$$= 3$$

Take θ_B, θ_C and θ_E as unknown displacement. Assign coordinate (1) in the direction of θ_B, coordinate (2) in the direction of θ_C and coordinate (3) in the direction of θ_E.



Stiffness Matrix

Column 1st:

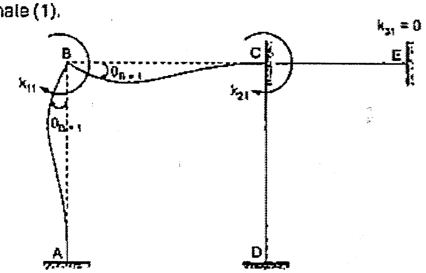
Give unit displacement in the direction of coordinate (1).

$\therefore$ $\theta_B = 1$ and ensure $\theta_C = \theta_E = 0$

$$k_{11} = \frac{4EI}{6} + \frac{4EI}{6} = \frac{4}{3}EI$$

$$k_{21} = \frac{2EI}{6} = \frac{EI}{3}$$

$$k_{31} = 0$$



Column 2nd:

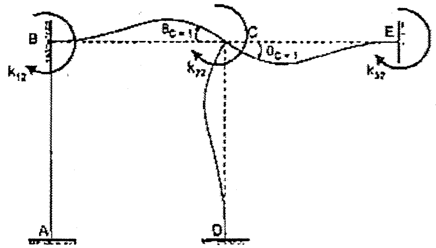
Give unit displacement in the direction of coordinate (2).

$\therefore$ $\theta_C = 1$ and ensure $\theta_B = \theta_E = 0$

$$k_{12} = k_{21} = \frac{EI}{3}$$

$$k_{22} = \frac{4EI}{6} + \frac{4EI}{6} + \frac{4EI}{6} = 2EI$$

$$k_{32} = \frac{2EI}{6} = \frac{EI}{3}$$



Consider 3rd:

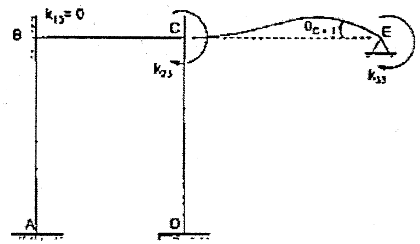
Give unit displacement in the direction of coordinate (3).

$\therefore$ $\theta_E = 1$ and ensure $\theta_B = \theta_C = 0$

$$k_{13} = k_{13} = 0$$

$$k_{23} = k_{32} = \frac{EI}{3}$$

$$k_{33} = \frac{4EI}{6} = \frac{2EI}{3}$$

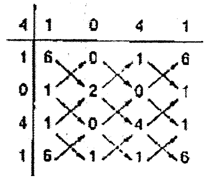


The stiffness matrix for selected coordinate is,

$$[k] = \begin{bmatrix} \frac{4EI}{3} & \frac{EI}{3} & 0 \\ \frac{EI}{3} & 2EI & \frac{EI}{3} \\ 0 & \frac{EI}{3} & \frac{2EI}{3} \end{bmatrix}$$

$$[k] = \frac{EI}{3}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The inverse of stiffness matrix [k]:



$$Adj[k] = \frac{3}{EI}\begin{bmatrix} 11 & -2 & 1 \\ -2 & 8 & -4 \\ 1 & -4 & 23 \end{bmatrix}$$

$$|k| = 4(12-1) - 1(2-0) + 0$$

$$|k| = 42$$

$$\therefore \quad [k]^{-1} = \frac{Adj[k]}{|k|} = \frac{3}{EI}\begin{bmatrix} 0.261 & -0.00476 & 0.0238 \\ -0.00476 & 0.1904 & -0.0952 \\ 0.0238 & -0.0952 & 0.5476 \end{bmatrix}$$

$$[k]^{-1} = \frac{1}{EI}\begin{bmatrix} 0.783 & -0.142 & 0.0714 \\ -0.142 & 0.5712 & -0.2856 \\ 0.0714 & -0.2856 & 1.6428 \end{bmatrix}$$

Let P_{1L}, P_{2L} and P_{3L} are forces developed in coordinate direction due to the given load in locked structure

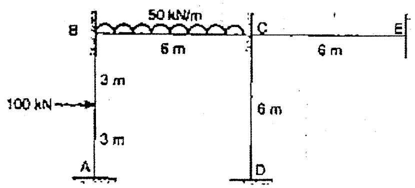
$$P_{1L} = \overline{M}_{BA} - \overline{M}_{BC}$$

$$= \frac{100\times 6}{8} - \frac{50\times 6^2}{12}$$

$$= -75 \text{ kN-m}$$

$$P_{2L} = 150 \text{ kN-m}$$

$$P_{3L} = 0$$



$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.783 & -0.142 & 0.0714 \\ -0.142 & 0.5712 & -0.2856 \\ 0.0714 & -0.2856 & 1.6428 \end{bmatrix}\begin{bmatrix} -(-75) \\ -(150) \\ -0 \end{bmatrix}$$

$$\therefore \quad \Delta_1 = \frac{1}{EI}[0.783\times 75 + 0.142\times 50 + 0]$$

$$\Rightarrow \quad \theta_B = \Delta_1 = \frac{80.02}{EI}$$

and $$\Delta_2 = \frac{1}{EI}[-0.142\times 75 - 0.5712\times 150 + 0]$$

$$\Rightarrow \quad \theta_C = \Delta_2 = -\frac{96.33}{EI}$$

and $$\Delta_3 = \frac{1}{EI}[0.0714\times 75 + 0.2856\times 150 + 0]$$

$$\theta_E = \Delta_3 = \frac{48.19}{EI}$$

Final end moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{6}\left[2\theta_A + \theta_B - \frac{3\Delta}{L}\right]$$

$$= -75 + \frac{2EI}{6}\left[0 + \frac{80.02}{EI} - 0\right] = -48.33 \text{ kNm}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{6}\left[\theta_A + 2\theta_B - \frac{3\Delta}{L}\right]$$

$$= 75 + \frac{2EI}{6}\left[0 + 2\times\frac{80.02}{EI} - 0\right] = 128.35 \text{ kNm}$$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{6}\left[2\theta_B + \theta_C - \frac{3\Delta}{L}\right]$$

$$= -150 + \frac{2EI}{6}\left[\frac{80.02\times 2}{EI} - \frac{96.33}{EI} - 0\right] = -128.76 \text{ kNm}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{6}\left[\theta_B + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 150 + \frac{2EI}{6}\left[\frac{80.02}{EI} - \frac{2\times 96.33}{EI} - 0\right] = 112.47 \text{ kNm}$$

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{6}\left[2\theta_D + \theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[0 - \frac{96.33}{EI} - 0\right] = -32.11 \text{ kNm}$$

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{6}\left[\theta_D + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[0 + \frac{2\times -96.33}{EI} - 0\right] = -64.22 \text{ kNm}$$

$$M_{EC} = \overline{M}_{EC} + \frac{2EI}{6}\left[2\theta_E + \theta_C - \frac{3\Delta}{L}\right]$$

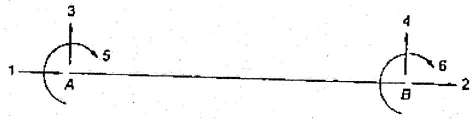
$$= 0 + \frac{2EI}{6}\left[\frac{2\times 48.19}{EI} - \frac{96.33}{EI} - 0\right] = 0 \text{ kNm}$$

$$M_{CE} = \overline{M}_{CE} + \frac{2EI}{6}\left[\theta_E + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[\frac{48.19}{EI} - \frac{2\times 96.33}{EI}\right] = -48.14 \text{ kNm}$$

Illustrative Examples

Example 10.19 Develop the stiffness matrix for the end loaded prismatic beam element AB with reference to the coordinates shown in figure.

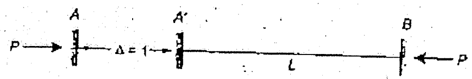


Solution:

First column: To develop the first column of stiffness matrix, give unit displacement in the direction of coordinate 1 and find force induce in other coordinate directions.

$$\Delta = \frac{PL}{AE}$$

if $\Delta = 1$ then $P = k$



$\therefore$ $$k_{11} = \frac{AE}{L}$$

and $$k_{21} = -\frac{AE}{L}$$

By Maxwell's reciprocal theorem,

$$k_{12} = k_{21} = -\frac{AE}{L}$$

$k_{31} = 0,\ k_{41} = 0,\ k_{51} = 0,\ k_{61} = 0$

Second Column:

$$k_{22} = \frac{AE}{L}$$

$$k_{12} = -\frac{AE}{L}$$

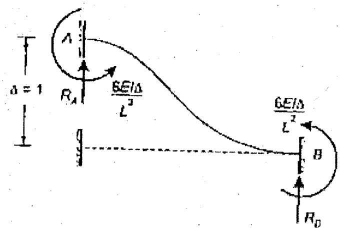
$k_{32} = 0,\ k_{42} = 0,\ k_{52} = 0,\ k_{62} = 0$

Third Column:

$k_{13} = 0,\ k_{23} = 0$

$$k_{53} = -\frac{6EI}{L^2}$$

$$k_{63} = -\frac{6EI}{L^2}$$



$\Sigma M_B = 0;$ $$R_A \times L - \frac{12EI}{L^2} = 0$$

$$R_A = \frac{12EI}{L^3}$$

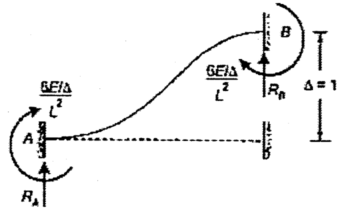
and $$R_B = -\frac{12EI}{L^3}$$

$\therefore$ $$k_{33} = \frac{12EI}{L^3}$$

$$k_{43} = -\frac{12EI}{L^3}$$

Fourth Column:

$\Sigma M_B = 0;$ $$R_A \times L + \frac{12EI}{L^2} = 0$$



$$R_A = -\frac{12EI}{L^3}$$

and $$R_B = \frac{12EI}{L^3}$$

$$k_{44} = \frac{12EI}{L^3}$$

$$k_{34} = -\frac{12EI}{L^3}$$

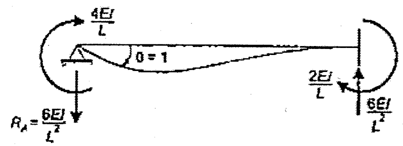
$k_{14} = 0,\ k_{24} = 0$

$$k_{54} = \frac{6EI}{L^2}$$

$$k_{64} = \frac{6EI}{L^2}$$

Fifth Column:

$$k_{55} = \frac{4EI}{L}$$



$k_{15} = 0,\ k_{25} = 0$

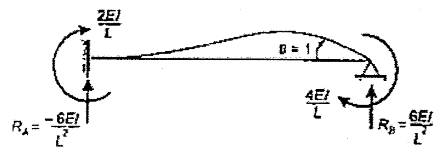
$$k_{35} = -\frac{6EI}{L^2}$$

$$k_{45} = \frac{6EI}{L^2}$$

$$k_{65} = \frac{2EI}{L}$$

Sixth Column:

$$k_{66} = \frac{4EI}{L}$$



$$k_{56} = \frac{2EI}{L}$$

$k_{16} = 0,\ k_{26} = 0$

$$k_{36} = -\frac{6EI}{L^2}$$

$$k_{46} = \frac{6EI}{L^2}$$

Hence, the stiffness matrix for given coordinate system is

$$[k] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI}{L^3} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & 0 & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ 0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Example 10.20 Analyse the portal frame shown in figure by flexibility matrix method. EI is constant.

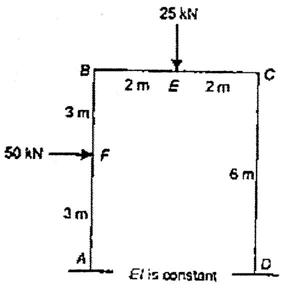


Fig. Portal frame

Solution:

Here $r_e = 3 + 3 = 6$

$D_S = r_e - 3$

∴ $D_S = 6 - 3 = 3$

The redundants selected are

$$[P] = \begin{bmatrix} H_D \\ R_D \\ M_D \end{bmatrix}$$

Coordinate 1, 2 and 3 selected in the direction of redundant shown in figure. The basic released structure is a cantilever frame. The required deflection may be calculated from unit load method.

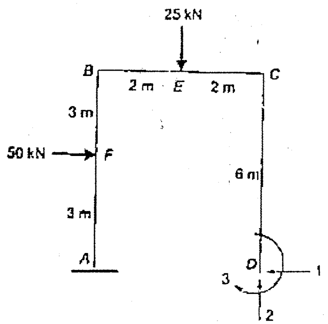


Fig. Basic released structure

Portion	DC	CE	EB	BF	FA
Origin	D	C	E	B	F
Limit	0 − 6	0 − 2	0 − 2	0-3	0 − 3
M	0	0	$-25x$	−50	$-50(1 + x)$
m_1	0	x	$x + 2$	4	4
m_2	$-x$	−6	−6	$x - 6$	$x - 3$
m_3	−1	−1	−1	−1	−1

Where M = BM at any section due to given loading

m_1 = BM at any section when unit load is in the direction of coordinate 1

m_2 = BM at any section when unit load is in the direction of coordinate 2

m_3 = BM at any section when unit moment is in the direction of coordinate 3

$$\Delta_{1L} = \int \frac{Mm_1 dx}{EI}$$

$$EI(\Delta_{1L}) = \int^{DC} Mm_1dx + \int^{CE} Mm_1dx + \int^{EB} Mm_1dx + \int^{BF} Mm_1dx + \int^{FA} Mm_1dx$$

$$EI(\Delta_{1L}) = 0 + 0 + \int_0^2 -25x(x+2)dx + \int_0^3 -50 \times 4dx + \int_0^3 -50(1+x)4dx$$

$$= \int_0^2 (-25x^2 - 50x)dx - \int_0^3 200dx - \int_0^3 200(1+x)dx$$

$$= \left[-25\frac{x^3}{3} - \frac{50x^2}{2}\right]_0^2 - 200\left[x + x + \frac{x^2}{2}\right]_0^3$$

$$= \left[-\frac{25 \times 2^3}{3} - 50 \times 2\right] - 200\left[2 \times 3 + \frac{3^2}{2}\right]$$

$$\therefore \quad \Delta_{1L} = -\frac{2266.67}{EI}$$

$$\Delta_{2L} = \int \frac{Mm_2}{EI} dx$$

$$EI(\Delta_{2L}) = 0 + 0 + \int_0^2 (-25x)(-6)dx + \int_0^3 -50(x-6)dx + \int_0^3 -50(1+x)(x-3)dx$$

$$= \int_0^2 150x\,dx - \int_0^3 50(x-6)dx - 50\int_0^3 (1+x)(x-3)dx$$

$$= \left[150\frac{x^2}{2}\right]_0^2 - 50\left[\frac{x^2}{2} - 6x\right]_0^3 - 50\left[\frac{x^2}{2} - 3x + \frac{x^3}{3} - 3\frac{x^2}{2}\right]_0^3$$

= 300 - 675 + 450)

$$\Delta_{2L} = \frac{75}{EI}$$

$$\Delta_{3L} = \int \frac{Mm_3}{EI} dx$$

$$EI(\Delta_{3L}) = 0 + 0 + \int_0^2 25x\,dx + \int_0^3 50\,dx + \int_0^3 50(1+x)\,dx$$

$$= 50 + 150 + 375$$

$$\Delta_{3L} = \frac{575}{EI}$$

$$\delta_{11} = \int \frac{m_1^2}{EI} dx$$

$$EI(\delta_{11}) = 0 + \int_0^2 x^2 dx + \int_0^2 (x+2)^2 + \int_0^3 16\,dx + \int_0^3 16\,dx$$

$$= 2.67 + 18.67 + 96$$

$$\therefore \quad \delta_{11} = \frac{117.34}{EI}$$

$$\delta_{21} = \delta_{12} = \int \frac{m_1 m_2\, dx}{EI} \qquad \text{(By Reciprocal theorem)}$$

$$EI(\delta_{12}) = 0 + \int_0^2 -6x\,dx + \int_0^2 -6(x+2)\,dx + \int_0^3 4(x-6)\,dx \int_0^3 4(x-3)\,dx$$

$$= -12 - 36 - 54 - 18$$

$$\therefore \quad \delta_{12} = -\frac{120}{EI}$$

$$\delta_{31} = \delta_{13} = \int \frac{m_1 m_3\, dx}{EI} \qquad \text{(By Reciprocal theorem)}$$

$$EI(\delta_{31}) = 0 + \int_0^2 -x\,dx + \int_0^2 -(x+2)\,dx + \int_0^3 -4\,dx + \int_0^3 -4\,dx$$

$$= -\int_0^2 x\,dx - \int_0^2 (x+2)\,dx - 8\int_0^3 dx$$

$$= -2 - 6 - 24$$

$$\therefore \quad \delta_{31} = -\frac{32}{EI}$$

$$\delta_{22} = \int \frac{(m_2)^2\, dx}{EI}$$

$$EI(\delta_{22}) = \int_0^6 x^2 dx + \int_0^2 36\,dx + \int_0^2 36\,dx + \int_0^3 (x-6)^2 dx + \int_0^3 (x-3)^2 dx$$

$$= 72 + 72 + 72 + 63 + 9$$

$$\therefore \quad \delta_{22} = \frac{208}{EI}$$

$$\delta_{32} = \delta_{23} = \int \frac{m_2 m_3}{EI} dx \qquad \text{(By Reciprocal theorem)}$$

$$EI(\delta_{32}) = \int_0^6 x\,dx + \int_0^2 6\,dx + \int_0^2 6\,dx + \int_0^3 -(x-6)\,dx + \int_0^3 -(x-3)\,dx$$

$$= 18 + 24 + 13.5 + 4.5$$

$$\therefore \quad \delta_{32} = \frac{60}{EI}$$

$$\delta_{33} = \int \frac{(m_3)^2\, dx}{EI}$$

$$EI(\delta_{33}) = \int_0^6 1.dx + \int_0^2 1.dx + \int_0^2 1.dx + \int_0^3 1.dx + \int_0^3 1.dx$$

$$= 6 + 2 + 2 + 3 + 3$$

$$\therefore \quad \delta_{33} = \frac{16}{EI}$$

The compatibility equation is

$$[\delta]\{P\} = [\Delta] - [\Delta_L]$$

$$\frac{1}{EI} = \begin{bmatrix} 117.34 & -120 & -32 \\ -120 & 288 & 60 \\ -32 & 60 & 16 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{EI} \begin{bmatrix} -2266.67 \\ 75 \\ 575 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 117.34 & -120 & -32 \\ -120 & 280 & 60 \\ -32 & 60 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 2266.67 \\ -75 \\ -575 \end{bmatrix} \qquad \text{...(i)}$$

On solving (i), we get

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 20.93 \\ 33.03 \\ -117.95 \end{bmatrix}$$

$R_D = P_1 = 9.48$ kN (↑)

$H_D = P_2 = 20.60$ kN (←)

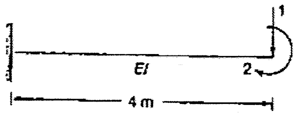
$M_D = P_3 = -117.95$ kN-m (↻) or 117.95 kN-m (↺)

Summary

- The flexibility of a structure is defined as the displacement caused by unit force and the stiffness is defined as the force required to produce unit displacement.
- Flexibility and stiffness matrix are always square matrix.
- Elements of flexibility and stiffness matrix are always non-zero and non-negative along the diagonal.
- Flexibility of stiffness matrix is always symmetrical about it leading diagonal or it forms a mirror image about the diagonal.

Objective Brain Teasers

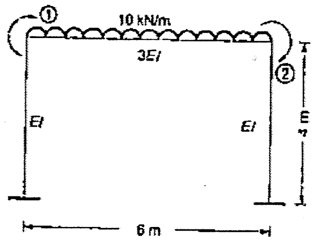
Q.1 The flexibility matrix of the beam shown in the given figure is



(a) $\begin{bmatrix}\frac{64}{3EI} & -\frac{8}{EI}\\ -\frac{8}{EI} & \frac{64}{3EI}\end{bmatrix}$ (b) $\begin{bmatrix}\frac{64}{3EI} & \frac{8}{EI}\\ \frac{8}{EI} & -\frac{64}{3EI}\end{bmatrix}$

(c) $\begin{bmatrix}\frac{64}{3EI} & \frac{8}{EI}\\ \frac{8}{EI} & \frac{4}{EI}\end{bmatrix}$ (d) $\begin{bmatrix}\frac{64}{3EI} & \frac{8}{EI}\\ \frac{4}{EI} & \frac{8}{EI}\end{bmatrix}$

Q.2 Considering only flexural deformations, which is the stiffness matrix for the plane frame shown in the figure given below?

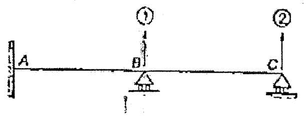


(a) $\begin{bmatrix}4 & 3\\ 3 & 4\end{bmatrix}EI$ (b) $\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}EI$

(c) $\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}EI$ (d) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}EI$

Q.3 Flexibility matrix of the beam shown below is

$f = \frac{1}{3EI}\begin{bmatrix}1 & 2\\ 2 & 8\end{bmatrix}$.



If support B settles by $\frac{\Delta}{EI}$ units, what is the reaction at B?

(a) 0.75 Δ (b) 3.0 Δ
(c) 6.0 Δ (d) 24.0 Δ

Q.4 The stiffness matrix of a beam element is $\left(\frac{2EI}{L}\right)\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$. Which one of the following is its flexibility matrix?

(a) $\left(\frac{L}{2EI}\right)\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ (b) $\left(\frac{L}{6EI}\right)\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$

(c) $\left(\frac{L}{5EI}\right)\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$ (d) $\left(\frac{L}{6EI}\right)\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$

Q.5 Flexibility matrix for a beam element is

$$[F] = \frac{1}{EI}\begin{bmatrix}36 & 9\\ 9 & 4\end{bmatrix}$$

What is the corresponding stiffness matrix $[S]$?

(a) $[S] = \frac{EI}{63}\begin{bmatrix}36 & -9\\ -9 & 4\end{bmatrix}$ (b) $[S] = \frac{EI}{63}\begin{bmatrix}36 & 9\\ 9 & 4\end{bmatrix}$

(c) $[S] = \frac{EI}{63}\begin{bmatrix}4 & -9\\ -9 & 36\end{bmatrix}$ (d) $[S] = \frac{EI}{63}\begin{bmatrix}4 & 9\\ 9 & 36\end{bmatrix}$

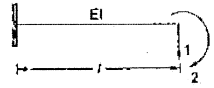
Q.6 Consider the following statements relating to structural analysis:

1. Flexibility matrix and its transpose are equal.
2. Elements of main diagonal of stiffness matrix are always positive.
3. For unstable structures, coefficients in leading diagonal matrix can be negative.

Which of these statements is/are correct?

(a) 1, 2 an 3 (b) 1 and 2 only
(c) 2 and 3 only (d) 3 only

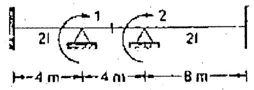
Q.7 A cantilever beam is shown in the figure below with degree of freedom, then k_{11} will be



(a) $\frac{L^3}{3EI}$ (b) $-\frac{L^3}{3EI}$

(c) $-\frac{12EI}{L^3}$ (d) $\frac{12EI}{L^3}$

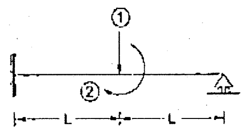
Q.8 Displacement coordinates for a beam are shown in the given figure. The stiffness matrix is given by



(a) $\begin{bmatrix}3EI & EI\\ EI & 2EI\end{bmatrix}$ (b) $\begin{bmatrix}3EI & -0.5EI\\ -0.5EI & 2EI\end{bmatrix}$

(c) $\begin{bmatrix}3EI & 0\\ 0 & 2EI\end{bmatrix}$ (d) $\begin{bmatrix}3EI & 0.5EI\\ 0.5EI & 2EI\end{bmatrix}$

Q.9 The stiffness matrix for the given coordinates of the beam as shown in the figure is



(a) $\begin{bmatrix}\frac{15EI}{L^3} & \frac{-3EI}{L^2}\\ \frac{-3EI}{L^2} & \frac{7EI}{L}\end{bmatrix}$ (b) $\begin{bmatrix}\frac{15EI}{L^3} & \frac{3EI}{L^2}\\ \frac{3EI}{L^2} & \frac{7EI}{L}\end{bmatrix}$

(c) $\begin{bmatrix}\frac{15EI}{L^3} & \frac{-9EI}{L^2}\\ \frac{-9EI}{L^2} & \frac{7EI}{L}\end{bmatrix}$ (d) $\begin{bmatrix}\frac{15EI}{L^3} & \frac{9EI}{L^2}\\ \frac{9EI}{L^2} & \frac{7EI}{L}\end{bmatrix}$

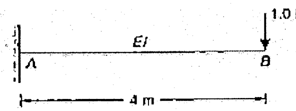
Answers

1. (c) 2. (b) 3. (c) 4. (b) 5. (c)
6. (b) 7. (d) 8. (d) 9. (a)

Hints and Explanations:

1. (c)

Elements of flexibility matrix can be obtained by applying unit force in the direction of any one coordinate and calculating displacement in the direction of both coordinates.



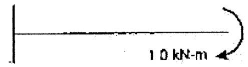
Applying unit load in the direction of 1.

Deflection at B, $f_{11} = \frac{1\times 4^3}{3EI} = \frac{64}{3EI}$

Rotation at B, $f_{21} = \frac{1\times 4^2}{2EI} = \frac{8}{EI}$

The positive sign is used when the displacement is in the direction of coordinate, otherwise negative sign is used.

Applying unit moment in the direction of 2
Deflection at B,



$f_{12} = \frac{1\times 4^2}{2EI} = \frac{8}{EI}$

Rotation at B,

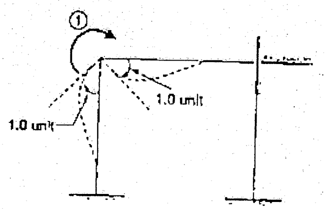
$f_{22} = \frac{1\times 4}{EI} = \frac{4}{EI}$

So flexibility matrix

$$= \begin{bmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix} = \begin{bmatrix}\frac{64}{3EI} & \frac{8}{EI}\\ \frac{8}{EI} & \frac{4}{EI}\end{bmatrix}$$

2. (b)

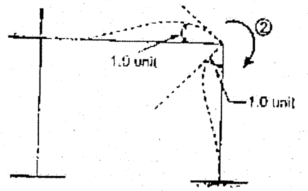
Keeping $D_1 = 1.0$ and $D_2 = 0$, the frame will be



$$s_{11} = \frac{4\times(3EI)}{6} + \frac{4EI}{4} = 3EI$$

$$s_{21} = \frac{2\times(3EI)}{6} = EI$$

Keeping $D_1 = 0$ and $D_2 = 1.0$, the frame will be



$$s_{12} = \frac{2\times 3EI}{6} = EI$$

and $$s_{22} = 3\,EI$$

3. (c)

We know that,

$$D_Q = D_{QS} + FQ$$

D_Q is the final displacement matrix corresponding to the redundants in the actual structure.

$$D_{QS} = D_{QL} + D_{QT} + D_{QP} + D_{QR}$$

D_{QS} includes the effects of external loads (D_{QL}), temperature (D_{QT}), prestrain effects (D_{QP}), support settlements, etc.

F is the Flexibility matrix and Q is the unknown reactions matrix.

Assuming upward displacements as positive

$$D_{QL} = \begin{bmatrix}0\\0\end{bmatrix};\ D_{QP} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$D_{QT} = \begin{bmatrix}0\\0\end{bmatrix};\ D_{QR} = \begin{bmatrix}-\frac{\Delta}{EI}\\0\end{bmatrix}$$

$$\therefore \quad D_{QS} = \begin{bmatrix}-\frac{\Delta}{EI}\\0\end{bmatrix} \text{ and } Q = \begin{bmatrix}V_B\\V_C\end{bmatrix}$$

Final displacement,

$$D_Q = 0$$

$$\therefore \quad D_{QS} + FQ = 0$$

$$\Rightarrow \quad FQ = -D_{QS}$$

$$\Rightarrow \quad Q = -F^{-1} D_{QS}$$

$$\Rightarrow \quad \begin{bmatrix}V_B\\V_C\end{bmatrix} = -\frac{3EI}{4}\begin{bmatrix}8 & -2\\-2 & 1\end{bmatrix}_{2\times2}\begin{bmatrix}-\frac{\Delta}{EI}\\0\end{bmatrix}$$

$$\begin{bmatrix}V_B\\V_C\end{bmatrix} = -\frac{3EI}{4}\times\left(-\frac{\Delta}{EI}\right)\begin{bmatrix}8 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}V_B\\V_C\end{bmatrix} = \frac{3\Delta}{4}\begin{bmatrix}8+(-2)\times 0\\(-2)\times 1 + 1\times 0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}V_B\\V_C\end{bmatrix} = \frac{3\Delta}{4}\begin{bmatrix}8\\-2\end{bmatrix}$$

$$\therefore \quad V_B = \frac{3\Delta}{4}\times 8 = 6\,\Delta$$

and $$V_C = \frac{3\Delta}{4}\times(-2) = -1.5\,\Delta$$

4. (b)

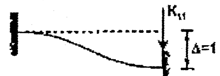
The flexibility and stiffness matrices are inverse of each other and both are symmetrical.

The value of determinant of stiffness matrix is $\frac{2EI}{L}\times 3 = \frac{6EI}{L}$.

The flexibility matrix will be $= \frac{L}{6EI}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$

7. (d)

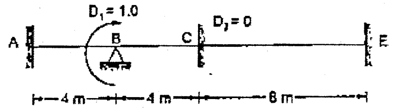
K_{11} = Force required along degree of freedom '1' to produce unit displacement in the direction of degree of freedom '1'. And degree of freedom '2' must be locked.



$$K_{11} = \frac{12EI}{L^3} \text{ (standard result)}$$

8. (d)

Stiffness matrix can be obtained by making second coordinate i.e. rotation in the direction of 2 as zero and considering unit rotation in the direction of coordinate 1.

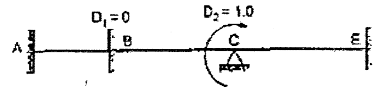


The beam AB has rotational stiffness, $\frac{4E(2I)}{4} = 2EI$ and beam BC has rotational stiffness, $\frac{4E(I)}{4} = EI$.

So moment required in the direction of 1 to produce unit rotation will be $2EI + EI = 3EI$. Thus $K_{11} = 3EI$.

The moment generated at point C in the direction of coordinate 2 is 1/2 × EI = 0.5 EI as the carry over factor is half. So K_{21} = 0.5 EI.

Similarly make $D_1 = 0$ and $D_2 = 1.0$.



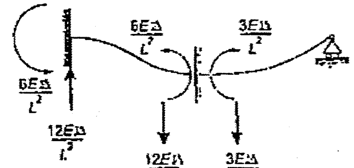
$$K_{22} = \frac{4EI}{4} + \frac{4E(2I)}{8} = 2EI$$

$$K_{12} = 0.5\,EI$$

$$\text{Stiffness matrix} = \begin{bmatrix}K_{11} & K_{12}\\K_{21} & K_{22}\end{bmatrix} = \begin{bmatrix}3EI & 0.5EI\\0.5EI & 2EI\end{bmatrix}$$

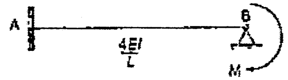
9. (a)

For coordinate 1,



$$K_{11} = \frac{15EI}{L^3}$$

$$K_{21} = \frac{-3EI}{L^3} = K_{12}$$

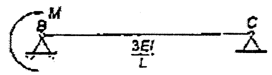


$$\frac{4EI}{L}$$

$$M = K\theta$$

$$M = \frac{4EI}{L}\theta$$

$$K = \frac{4EI}{L}$$



$$\frac{3EI}{L}$$

$$\theta = \frac{ML}{3EI}$$

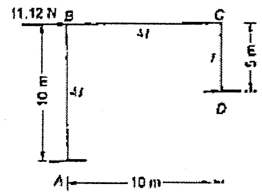
$$M = \frac{3EI\theta}{L}$$

$$K = \frac{3EI}{L}$$

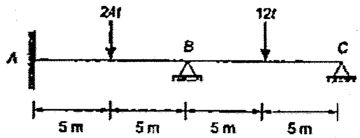
$$K_{22} = \frac{4EI}{L} + \frac{3EI}{L} = \frac{7EI}{L}$$

Conventional Practice Questions

Q.1 Analyse the portal frame $ABCD$ loaded as shown in figure below. The members are made of same material. Length and moment of inertia of each member are mentioned in the figure.

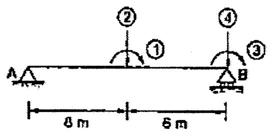


Q.2 Analyse the continuous beam shown below using stiffness matrix method. (Take EI = Constant)



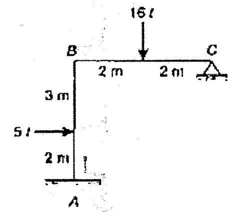
Ans. (R_A = 12.64 t, R_B = 19.93 t and R_C = 3.43 t)

Q.3 Develop the flexibility matrix for the simply supported beam AB with reference to the coordinates shown in figure. (Take EI = Constant)

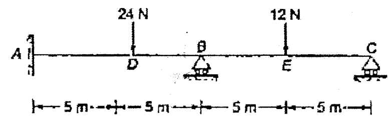


$$\text{Ans. } [\delta] = \frac{1}{EI}\begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 36 & -9 & 0 \\ -0.5 & -9 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q.4 Analyse the frame shown using stiffness matrix.

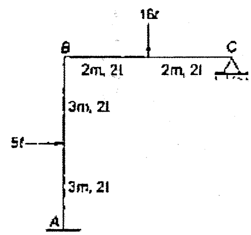


Q.5 Analyse the continuous beam ABC loaded as shown assuming EI as constant.



Ans. 34.375 Nm

Q.6 Analyse the frame shown using stiffness matrix.



■■■■