Exercise 9.5

Answer 1E.

Given equation is

$$x - y' = xy$$

$$\Rightarrow y' + xy = x$$

Which is in the form $y' + p(x)y = q(x)$

Hence the given equation is linear differential equation.

Answer 2E.

Given equation is

$$y' + xy^2 = \sqrt{x}$$

Which is not in the form $y' + p(x)y = q(x)$

Hence the given equation is not a linear differential equation.

Answer 3E.

Given equation is

$$y' = \frac{1}{x} + \frac{1}{y}$$

Which is not in the form $y' + p(x)y = q(x)$

Hence the given equation is not a linear differential equation

Answer 4E.

Given equation is

$$y \sin x = x^2 y' - x$$

$$\Rightarrow x^2 y' - (\sin x)\, y = x$$

$$\Rightarrow y' - \frac{\sin x}{x^2} y = \frac{1}{x}$$

Which is in the form $y' + p(x)y = q(x)$

Hence the given equation is a linear differential equation

Answer 5E.

Given equation is $y' + y = 1$

Compare this equation with $y' + p(x)y = q(x)$ we get $p(x) = 1, q(x) = 1$

Integrating factor $IF = e^{\int p(x)\,dx} = e^{\int 1\,dx} = e^{x}$

Solution is

$$y(IF) = \int q(x)\ IF\ dx$$

$$\Rightarrow ye^{x} = \int 1e^{x}dx = \int e^{x}dx = e^{x} + c$$

$$\Rightarrow \boxed{y = 1 + ce^{-x}}$$

Answer 6E.

Given equation is $y' - y = e^{x}$

Compare this equation with $y' + p(x)y = q(x)$

we get $p(x) = -1, q(x) = e^{x}$

Integrating factor $IF = e^{\int p(x)\,dx} = e^{\int -1\,dx} = e^{-x}$

Solution is

$$y(IF) = \int q(x)\ IF\ dx$$

$$\Rightarrow ye^{-x} = \int e^{x}e^{-x}dx$$

$$= \int 1\,dx$$

$$= x + c$$

$$\Rightarrow \boxed{y = e^{x}(x + c)}$$

Answer 7E.

Given equation is

$y' = x - y$

$\Rightarrow y' + y = x$

Compare this equation with $y' + p(x)y = q(x)$ we get $p(x) = 1, q(x) = x$

Integrating factor $IF = e^{\int p(x)\,dx} = e^{\int 1\,dx} = e^{x}$

Solution is

$$y(IF) = \int q(x)\ IF\ dx$$

$$\Rightarrow ye^{x} = \int e^{x}x\,dx = e^{x}(x-1) + c$$

$$\Rightarrow \boxed{y = (x-1) + ce^{-x}}$$

Answer 8E.

Given equation is

$$4x^3y+x^4y' = \sin^3 x$$

$$\Rightarrow y' + \frac{4}{x}y = \frac{\sin^3 x}{x^4}$$

Compare this equation with $y' + p(x)y = q(x)$ we get $p(x) = \frac{4}{x}, q(x) = \frac{\sin^3 x}{x^4}$

Integrating factor $IF = e^{\int p(x)\,dx} = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$

Solution is

$$y(IF) = \int q(x)\ IF\ dx$$

$$\Rightarrow yx^4 = \int x^4 \frac{\sin^3 x}{x^4}dx = \int \sin^3 x\,dx = \int \frac{3\sin x - \sin 3x}{4}dx$$

$$\Rightarrow yx^4 = \frac{3}{4}(-\cos x) - \frac{1}{4}\frac{(-\cos 3x)}{3} + c$$

$$\Rightarrow \boxed{yx^4 = \frac{\cos 3x}{12} - \frac{3\cos x}{4} + c}$$

Answer 9EE.

We have to solve $xy' + y = \sqrt{x}$

$$\Rightarrow x\frac{dy}{dx} + y = \sqrt{x}$$

Dividing by x

$$\frac{dy}{dx} + \frac{1}{x}y = x^{-1/2} \qquad ---(1)$$

This is a linear differential equation

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

We have $P(x) = \frac{1}{x}$ and $Q(x) = x^{-1/2}$

Since $\int P(x)dx = \int \frac{1}{x}dx$

$$= \ln x$$

So the integrating factor $I = e^{\int P(x)dx}$

$$= e^{\ln x}$$

$$= x$$

Multiplying by x, both sides of the equation (1)

We have $x\frac{dy}{dx} + y = x^{1/2}$

$$\Rightarrow \frac{d}{dx}(xy) = x^{1/2}$$

Integrating both sides

$$xy = \int x^{1/2}dx$$

$$\Rightarrow xy = \frac{x^{3/2}}{3/2} + C$$

$$\Rightarrow \boxed{y = \frac{2}{3}\sqrt{x} + C/x}$$

Answer 11E.

Consider the following differential equation:

$$\sin x\frac{dy}{dx}+(\cos x)y=\sin\left(x^2\right)$$

Rewrite the equation as follows:

$$\frac{dy}{dx}+\left(\frac{\cos x}{\sin x}\right)y=\frac{\sin\left(x^2\right)}{\sin x}$$

The equation is in the form of a linear equation, with $P(x)=\frac{\cos x}{\sin x}$ and $Q(x)=\frac{(\sin x)^2}{\sin x}$.

The integrating factor is calculated as follows:

$$\begin{aligned}I(x)&=e^{\int P(x)dx}\\&=e^{\int\left(\frac{\cos x}{\sin x}\right)dx}\\&=e^{\ln(\sin x)}\qquad\qquad \int\left(\frac{\cos x}{\sin x}\right)dx=\sin x\\&=\sin x\end{aligned}$$

Multiply both sides of the differential equation by $\sin x$, to get the following:

$$\begin{aligned}\sin x\frac{dy}{dx}+\sin x\left(\frac{\cos x}{\sin x}\right)y&=\sin x\frac{\sin\left(x^2\right)}{\sin x}\\ \sin x\frac{dy}{dx}+\cos x\cdot y&=\sin\left(x^2\right)\\ \sin x\cdot\frac{dy}{dx}+\frac{d}{dx}(\sin x)\cdot y&=\sin\left(x^2\right)\\ \frac{d}{dx}(y\sin x)&=\sin\left(x^2\right)\qquad\qquad \text{Use the formula } (uv)'=uv'+vu'\end{aligned}$$

Integrate on both sides of the differential equation $\frac{d}{dx}(y\sin x)=\sin\left(x^2\right)$ with respect to x, to get the following:

$$\begin{aligned}y\sin x&=\int\sin\left(x^2\right)dx+C\qquad\qquad \text{Here } C \text{ is the arbitary constant}\\ y&=\frac{\int\sin\left(x^2\right)dx+C}{\sin x}\end{aligned}$$

Therefore, the solution is $\boxed{y=\frac{\int\sin\left(x^2\right)dx+C}{\sin x}}$

Answer 12E.

Consider the differential equation

$$x\frac{dy}{dx}-4y=x^4e^x$$

Rewrite the equation as

$$\frac{dy}{dx}-\frac{4}{x}y=\frac{x^4e^x}{x}$$
$$\frac{dy}{dx}-\frac{4}{x}y=x^3e^x$$

The equation in the form Liner equation, with $P(x)=-\frac{4}{x}$ and $Q(x)=x^3e^x$

The integrating factor is

$$I(x)=e^{\int P(x)dx}$$
$$=e^{\int\left(-\frac{4}{x}\right)dx}$$
$$=e^{-4\ln(x)} \qquad \int\left(\frac{1}{x}\right)dx=\ln x$$
$$=\frac{1}{e^{4\ln(x)}}$$
$$=\frac{1}{x^4} \qquad e^{n\ln(a)}=a^n$$

Multiply both side of the differential equation by $\frac{1}{x^4}$, we get

$$\frac{1}{x^4}\left[x\frac{dy}{dx}-4y\right]=\frac{x^4e^x}{x^4}$$
$$\frac{1}{x^3}\frac{dy}{dx}-\frac{4}{x^3}y=e^x$$
$$\frac{1}{x^3}\frac{dy}{dx}+\frac{d}{dx}\left(\frac{1}{x^3}\right)y=e^x \qquad \frac{d}{dx}\left(\frac{1}{x^n}\right)=-\frac{n+1}{x^n}$$
$$\frac{d}{dx}\left(\frac{1}{x^3}\cdot y\right)=e^x \qquad \frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$$

Integrate on both side of the differential equation $\frac{d}{dx}\left(\frac{1}{x^3}\cdot y\right)=e^x$ with respect to x, we get

$$\frac{y}{x^3}=\int e^xdx+C \qquad \text{Where } C \text{ is the arbitary constant}$$
$$\frac{y}{x^3}=e^x+C$$
$$y=x^3e^x+C_1 \qquad Cx^3=C_1 \text{ is the arbitary constant}$$

Therefore, the solution is $\boxed{y=x^3e^x+C_1}$

Answer 13E.

We have to solve $(1+t)\frac{du}{dt}+u=(1+t), \qquad t>0$

Dividing by (1+t)

$$\frac{du}{dt}+\frac{u}{(1+t)}=1 \qquad \text{--- (1)}$$

Comparing with $\frac{du}{dt}+P(t)u=Q(t)$

We get $P(t)=\frac{1}{(1+t)}$ *and* $Q(t)=1$

Since $\int P(t)dt=\int\frac{1}{(1+t)}dt=\ln|1+t|$

$=\ln(1+t)$ [$1+t>0$ since $t>0$

So integrating factor is

$$I=e^{\int P(t)dt}=e^{\ln(1+t)}=(1+t)$$

Then multiplying both sides of the equation (1) by (1+t)

$$(1+t)\frac{du}{dt}+u=(1+t)$$

$$\Rightarrow\frac{d}{dt}\left[(1+t)u\right]=(1+t)$$

Integrating both sides

$$(1+t)u=\int(1+t)dt$$

$$=t+\frac{t^2}{2}+C$$

$$=\frac{1}{2}\left(t^2+2t+2C\right)$$

$$\Rightarrow\boxed{u=\left(t^2+2t+2C\right)/\left[2(1+t)\right]}$$

Answer 14E.

We have to solve

$$t\ln t\frac{dr}{dt}+r=te^t$$

Dividing by $t\ln t$, we get

$$\frac{dr}{dt}+\frac{1}{t\ln t}r=\frac{e^t}{\ln t} \qquad ....(1)$$

Comparing with $\frac{dt}{dt}+P(t)=Q(t)$

We have $P(t)=\frac{1}{t\ln t}$ and $Q(t)=e^t/\ln t$

Since $\int P(t)dt=\int\frac{1}{t\ln t}dt$

Let $\ln t=u$

$$\Rightarrow\frac{1}{t}dt=du$$

So $\int P(t)dt=\int\frac{1}{u}du$

$$=\ln u$$

$$=\ln(\ln t)$$

Integrating factor $I=e^{\int P(t)dt}$

$$=e^{\ln(\ln t)}$$

$$=\ln t$$

Multiplying by $\ln t$, both sides of the equation (1)

$$\ln t\frac{dr}{dt}+\frac{1}{t}r=e^t$$

Or $$\frac{d}{dt}(r\ln t)=e^t$$

Integrating both sides

$$r\ln t=\int e^t dt$$

$$\Rightarrow r\ln t=e^t+C$$

$$\Rightarrow \boxed{r=\frac{e^t+C}{\ln t}}$$

Answer 15E.

Given equation is

$$x^2y'+2xy=\ln x$$

$$\Rightarrow y'+\frac{2}{x}y=\frac{\ln x}{x^2}$$

Compare this equation with $y'+p(x)y=q(x)$

we get $p(x)=\frac{2}{x}, q(x)=\frac{\ln x}{x^2}$

Integrating factor $IF=e^{\int p(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$

Solution is

$$y(IF)=\int q(x)\, IF\, dx$$

$$\Rightarrow yx^2=\int x^2\frac{\ln x}{x^2}dx=\int \ln x\, dx$$

$$\Rightarrow yx^2=x\ln x-x+c$$

We have $y(1)=2$

$$\Rightarrow 2=-1+c$$

$$\Rightarrow c=3$$

$$\therefore\ y=\frac{x\ln x-x+c}{x^2}=\boxed{\frac{\ln x-1}{x}+\frac{3}{x^2}}$$

Answer 16E.

Given equation is

$$t^3\frac{dy}{dt}+3t^2y=\cos t,\ \ y(\pi)=0$$

$$\Rightarrow y'+\frac{3}{t}y=\frac{\cos t}{t^3}$$

Compare this equation with $y'+p(t)y=q(t)$

we get $p(t)=\frac{3}{t}, q(t)=\frac{\cos t}{t^3}$

Integrating factor $IF=e^{\int p(t)dt}=e^{\int \frac{3}{t}dt}=e^{3\ln t}=t^3$

Solution is

$$y(IF)=\int q(t)\, IF\, dt$$

$$\Rightarrow yt^3=\int t^3\frac{\cos t}{t^3}dt=\int \cos t\, dt=\sin t+c$$

$$\Rightarrow yt^3=\sin t+c$$

We have $y(\pi)=0$

$$\Rightarrow 0=0+c$$

$$\Rightarrow c=0$$

$$\therefore\ yt^3=\sin t$$

$$\Rightarrow \boxed{y=\frac{\sin t}{t^3}}$$

Answer 17E.

Given equation is

$t\frac{du}{dt} = t^2 + 3u,\ \ u(2) = 4$

$\Rightarrow \frac{du}{dt} - \frac{3}{t}u = t$

Compare this equation with $u' + p(t)u = q(t)$

We get $p(t) = -\frac{3}{t}, q(t) = t$

Integrating factor $IF = e^{\int p(t)dt} = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3} = \frac{1}{t^3}$

Solution is

$y(IF) = \int q(t)\ IF\ dt$

$\Rightarrow u\frac{1}{t^3} = \int t\frac{1}{t^3}dt = \int \frac{1}{t^2}dt = -\frac{1}{t} + c$

$\Rightarrow u = t^3\left(-\frac{1}{t} + c\right) = -t^2 + ct^3$

We have $u(2) = 4$

$\Rightarrow 4 = -4 + 8c$

$\Rightarrow c = 1$

$\boxed{\therefore u = t^3 - t^2}$

Answer 18E.

We have to solve initial value problem

$2xy' + y = 6x, \quad x > 0, y(4) = 20$

Dividing by 2x

$$\frac{dy}{dx} + \frac{1}{2x}y = 3 \qquad \text{........(1)}$$

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

We have $P(x) = \frac{1}{2x}$

Then $\int P(x)dx = \int \frac{1}{2x}dx = \frac{1}{2}\ln|x|$

So integrating factor is $I = e^{\int P(x)dx} = e^{\ln\sqrt{x}} = \sqrt{x}$ $\qquad$ x>0

Multiplying by $\sqrt{x}$, both sides of the equation (1), we get

$\sqrt{x}\frac{dy}{dx} + \frac{\sqrt{x}}{2x}y = 3\sqrt{x}$

$\Rightarrow \sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}}y = 3\sqrt{x}$

$\Rightarrow \frac{d}{dx}\left(\sqrt{x}.y\right) = 3\sqrt{x}$

Integrating both sides $\sqrt{x}.y = 3\int \sqrt{x}dx$

$\Rightarrow \sqrt{x}y = 3\times 2\frac{x^{3/2}}{3} + C$

$\Rightarrow y = 2x + \frac{C}{\sqrt{x}}$

When x = 4 y = 20

So $20 = 8 + \frac{C}{\sqrt{4}} \Rightarrow \frac{C}{2} = 12$

$\Rightarrow \boxed{C = 24}$

Thus the solution is $\boxed{y = 2x + \frac{24}{\sqrt{x}}}$

Answer 19E.

We have to solve initial value problem

$$xy' = y + x^2 \sin x, \qquad y(\pi) = 0$$

$$\Rightarrow y' - \frac{1}{x} y = x \sin x$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} y = x \sin x \qquad ---(1)$$

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

We have $P(x) = -\frac{1}{x}$

Then $\int P(x)dx = \int -\frac{1}{x}dx$

$= -\ln x$

$= \ln(1/x)$

Integrating factor

$I = e^{\int P(x)dx}$

$= e^{\ln(1/x)}$

$= \frac{1}{x}$

Multiplying both sides of the equation (1) by $\frac{1}{x}$

$$\Rightarrow \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \sin x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}.y\right) = \sin x$$

Integrating both sides we get

$$\frac{1}{x} y = \int \sin x \, dx$$

Or $\frac{1}{x} y = -\cos x + C$

$$\Rightarrow y = Cx - x\cos x$$

We have $y(\pi) = 0$

So $\Rightarrow 0 = \pi C - \pi \cos \pi$

$\Rightarrow 0 = \pi C + \pi \Rightarrow C + 1 = 0$

$\Rightarrow \boxed{C = -1}$

Thus solution is $\boxed{y = -x(1 + \cos x)}$

Answer 20E.

Consider the differential equation

$$(x^2+1)\frac{dy}{dx}+3x(y-1)=0, \qquad y(0)=2$$

Rewrite the equation in standard form of linear equation

$$(x^2+1)\frac{dy}{dx}+3x(y-1)=0$$

$$\frac{dy}{dx}+\left(\frac{3x}{(x^2+1)}y-\frac{3x}{(x^2+1)}\right)=0$$

$$\frac{dy}{dx}+\frac{3x}{(x^2+1)}y=\frac{3x}{(x^2+1)}$$

The equation in the form Liner equation, with $P(x)=\frac{3x}{(x^2+1)}$ and $Q(x)=\frac{3x}{(x^2+1)}$

The integrating factor is

$$\begin{aligned}I(x)&=e^{\int P(x)dx}\\&=e^{\int\frac{3x}{(x^2+1)}dx}\\&=e^{\frac{3}{2}\int\frac{2x}{(x^2+1)}dx}\\&=e^{\frac{3}{2}\ln(x^2+1)} && \int\left(\frac{f'(x)}{f(x)}\right)dx=\ln(f(x))\\&=e^{\ln(x^2+1)^{\frac{3}{2}}} && n\ln(a)=\ln(a^n)\\&=(x^2+1)^{\frac{3}{2}} && e^{\ln(x)}=x\end{aligned}$$

Multiply both side of the differential equation by $(x^2+1)^{\frac{3}{2}}$, we get

$$(x^2+1)^{\frac{3}{2}}\left[\frac{dy}{dx}+\frac{3x}{(x^2+1)}y\right]=(x^2+1)^{\frac{3}{2}}\cdot\frac{3x}{(x^2+1)}$$

$$(x^2+1)^{\frac{3}{2}}\cdot\frac{dy}{dx}+(x^2+1)^{\frac{3}{2}}\cdot\frac{3x}{(x^2+1)}y=(x^2+1)^{\frac{3}{2}}\cdot\frac{3x}{(x^2+1)}$$

$$(x^2+1)^{\frac{3}{2}}\cdot\frac{dy}{dx}+(x^2+1)^{\frac{1}{2}}(3x)y=(x^2+1)^{\frac{1}{2}}(3x)$$

$$(x^2+1)^{\frac{3}{2}}\cdot\frac{dy}{dx}+\frac{d}{dx}\left((x^2+1)^{\frac{3}{2}}\right)y=\frac{d}{dx}\left((x^2+1)^{\frac{3}{2}}\right) \qquad \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{d}{dx}\left(y(x^2+1)^{\frac{3}{2}}\right)=\frac{d}{dx}\left((x^2+1)^{\frac{3}{2}}\right)$$

Integrate on both side of the above differential equation with respect to x , we get

$$y=(x^2+1)^{\frac{3}{2}}+C \qquad \text{Where C is the arbitary constant}$$

$$y=\frac{(x^2+1)^{\frac{3}{2}}+C}{(x^2+1)^{\frac{3}{2}}}$$

$$y=1+\frac{C}{(x^2+1)^{\frac{3}{2}}}$$

Therefore, the solution is $\boxed{y=1+\frac{C}{(x^2+1)^{\frac{3}{2}}}}$

Since $y(0)=2$

Plug in $x=0$ and $y=2$ in the general solution $y=1+\frac{C}{(x^2+1)^{\frac{3}{2}}}$, we get

$$2=1+\frac{C}{(0+1)^{\frac{3}{2}}}$$

$$C=1$$

Substitute $C=1$ in the solution $y=1+\frac{C}{(x^2+1)^{\frac{3}{2}}}$, we get

$$y=1+\frac{1}{(x^2+1)^{\frac{3}{2}}}$$

Therefore, the particular solution is $\boxed{y=1+\frac{1}{(x^2+1)^{\frac{3}{2}}}}$

Answer 21E.

We must solve the given differential equation show below:

$$xy'+2y=e^x$$

Two cases are considered.

Case 1: When $x=0$, $y=\frac{1}{2}$.

Case 2: $x \neq 0$

$xy'+2y=e^x$

The above equation is a first order linear differential equation. We can put it in the form $y'+P(x)y=Q(x)$.

$$xy'+2y=e^x$$

$$y'+\frac{2y}{x}=\frac{e^x}{x}$$

Where $P(x)=\frac{2}{x}$ and $Q(x)=\frac{e^x}{x}$.

We must find the integrating factor and multiply both sides of the equation by it in order to integrate both sides.

Find the integrating factor.

$$I(x) = e^{\int P(x)dx}$$
$$= e^{\int \frac{2}{x}dx}$$
$$= e^{2\ln x}$$
$$= e^{\ln x^2}$$
$$= x^2$$

Now multiply both sides of the equation by the integrating factor.

$$y' + \frac{2y}{x} = \frac{e^x}{x}$$
$$x^2 y' + 2xy = xe^x$$
$$\frac{d}{dx}\left(x^2 y\right) = xe^x \qquad \text{Reverse the Product Rule.}$$

Integrate.

$$\int \frac{d}{dx}\left(x^2 y\right)dx = \int xe^x dx$$
$$x^2 y = e^x(x-1) + C$$
$$\boxed{y = \frac{e^x(x-1)+C}{x^2}}$$

Where C is some constant.

Let

$u = x$, $dv = e^x dx \Rightarrow du = dx$, $v = e^x$

$$\int xe^x dx = uv - \int v du$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x$$
$$= e^x(x-1)$$

The following is a graph of several members of the family of solutions:

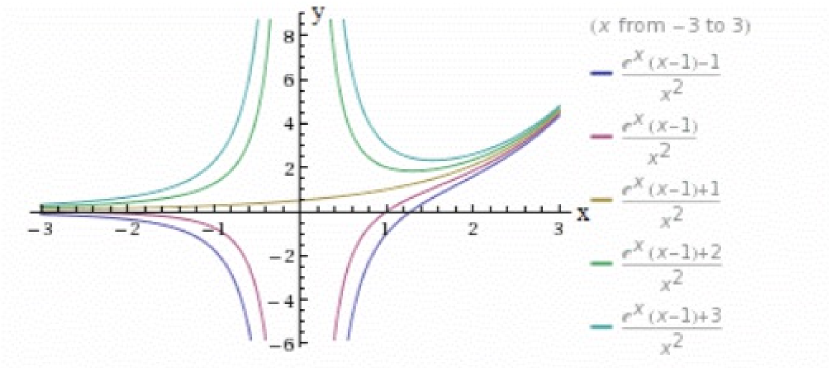

Figure 1: [-3, 3] by [-6, 8]

As C increases for $C > 0$ the hyberbolas move outward. As C decreases for $C \le 0$ the hyberbolas move outward.

Answer 23E.

Bernoulli differential equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

When $n = 0$, equation becomes

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{This is a linear equation}$$

When $n = 1$ equation becomes

$$\frac{dy}{dx} + P(x)y = Q(x)y$$

$$\Rightarrow \frac{dy}{dx} + \left(P(x) - Q(x)\right)y = 0$$

This is also a linear form

We have $$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad ---(1)$$

Let $y^{1-n} = u$ then $(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

Dividing by y^n, both sides of the equation (1) we get

$$y^{-n}\frac{dy}{dx} + y^{1-n}P(x) = Q(x)$$

$$\Rightarrow \frac{1}{(1-n)}\frac{du}{dx} + P(x)u = Q(x)$$

$$\Rightarrow \boxed{\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)}$$

This is a linear differential equation.

Answer 24E.

We have to solve $xy' + y = -xy^2$

Dividing by x

$$\frac{dy}{dx} + \frac{1}{x}y = -y^2 \qquad ---(1)$$

This is the form of Bernoulli equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Dividing by y^2, both sides of the equation (1)

We have $$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = -1$$

Let $$y^{-1} = u \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

Then we have $$-\frac{du}{dx} + \frac{1}{x}u = -1 \qquad .....(2)$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = 1$$ (This is a linear form)

Comparing with $\frac{du}{dx}+P(x)u=Q(x)$

We have $P(x)=-\frac{1}{x}$,

Then $\int P(x)dx=\int -\frac{1}{x}dx=-\ln|x|$

$$=-\ln x, \; for \; x>0$$

Integrating factor is $I=e^{\int P(x)dx}=e^{-\ln x}=1/x$

Multiplying both sides of the equation (2) by 1/x, we get

$$\frac{1}{x}\frac{du}{dx}-\frac{1}{x^2}u=\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}u\right)=\frac{1}{x}$$

Integrating both sides, we have

$$\Rightarrow \frac{1}{x}u=\int \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{x}u=\ln|x|+C$$

$$\Rightarrow u=x\ln|x|+Cx$$

$$\Rightarrow y^{-1}=x\ln|x|+Cx$$

$$\Rightarrow \boxed{y=\frac{1}{x(\ln|x|+C)}}$$

Answer 25E.

We have to solve $y'+\frac{2}{x}y=\frac{y^3}{x^2}$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y=\frac{y^3}{x^2} \qquad ---(1)$$

This is the form of Bernoulli equation $\frac{dy}{dx}+P(x)\,y=Q(x)\,y^n$

Dividing both sides of the equation (1) by y^3, we have

$$\Rightarrow y^{-3}\frac{dy}{dx}+\frac{2}{x}y^{-2}=\frac{1}{x^2}$$

Let $y^{-2}=t \Rightarrow -2y^{-3}\frac{dy}{dx}=\frac{dt}{dx}$

Then $-\frac{1}{2}\frac{dt}{dx}+\frac{2}{x}t=\frac{1}{x^2}$

$$\Rightarrow \frac{dt}{dx}-\frac{4}{x}t=-\frac{-2}{x^2} \qquad ---(2)$$

Comparing with the standard form of the linear equation we get $P(x) = -\frac{4}{x}$

Then $\int P(x)\,dx = -4\int \frac{1}{x}\,dx$

$$= -4\ln|x|$$

$$= \ln\frac{1}{x^4}$$

Integrating factor is $e^{\int P(x)dx} = e^{\ln(1/x^4)}$

$$= 1/x^4$$

Multiplying by $\frac{1}{x^4}$, both sides of the equation (2)

$$\Rightarrow \frac{1}{x^4}\frac{dt}{dx} - \frac{4}{x^5}t = -\frac{2}{x^6}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{t}{x^4}\right) = -\frac{2}{x^6}$$

Integrating both sides $\frac{t}{x^4} = -2\int x^{-6}dx$

$$\Rightarrow \frac{t}{x^4} = -2\frac{x^{-5}}{-5} + C$$

$$\Rightarrow t = \frac{2}{5}x^{-1} + Cx^4$$

$$\Rightarrow y^{-2} = \frac{2}{5x} + Cx^4$$

$$\Rightarrow y^{-1} = \pm\sqrt{\frac{2}{5x} + Cx^4}$$

$$\Rightarrow \boxed{y = \pm\left(\frac{2}{5x} + Cx^4\right)^{-1/2}}$$

Answer 26E.

Consider the following second-ordered differential equation:

$xy'' + 2y' = 12x^2$.

Substitute $u = y'$ in the equation $xy'' + 2y' = 12x^2$, to obtain the following:

$$xu' + 2u = 12x^2$$

$$u' + \frac{2}{x}u = 12x$$

The linear differential equation is of the following form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

So, the differential equation $u' + \frac{2}{x}u = 12x$ in the form of a linear equation with $P(x) = \frac{2}{x}$,

$Q(x) = 12x$.

The Integration factor is calculated as follows:

$$I(x) = e^{\int P(x)dx}$$
$$= e^{\int \frac{2}{x}dx}$$
$$= e^{2\int \frac{1}{x}dx}$$
$$= e^{2\ln(x)}$$
$$= e^{\ln(x^2)}$$
$$= x^2$$

Multiply both sides of the differential equation $u' + \frac{2}{x}u = 12x$ by x^2, and solve as follows:

$$x^2\left[u' + \frac{2}{x}u\right] = x^2 \cdot 12x$$
$$x^2u' + 2xu = x^2 \cdot 12x$$
$$x^2u' + (x^2)'u = 12x^3 \qquad (uv)' = uv' + vu'$$
$$(x^2u)' = 12x^3$$

Integrate on both sides of the above differential equation, and solve as follows:

$$\int (x^2u)' = \int 12x^3 dx$$
$$x^2u = 12 \cdot \frac{x^4}{4} + C \qquad \text{Here, } C \text{ is an arbitary constant.}$$
$$x^2u = 3x^4 + C$$

Substitute back $u = y'$ in the equation $x^2u = 3x^4 + C$, and solve as follows:

$$x^2y' = 3x^4 + C$$
$$y' = \frac{3x^4 + C}{x^2}$$
$$y' = 3x^2 + C_1 \qquad C_1 = \frac{C}{x^2}$$
$$\frac{dy}{dx} = 3x^2 + C_1$$

Since the equation is separable, so separate the variables on both side of the equation, to get as follows:

$$dy = (3x^2 + C_1)dx$$

Integrate on both side of the equation, and solve as follows:

$$\int dy = \int (3x^2 + C_1)dx$$
$$y = 3\frac{x^3}{3} + C_1x$$
$$y = x^3 + C_2 \qquad C_2 = C_1x \text{ is the arbitary constant.}$$

Therefore, $\boxed{y = x^3 + C_2}$.

Answer 27E.

(A) We have inductance $L = 2H$

Constant voltage $E(t) = 40v$

Resistance $R = 10\Omega$

We have the initial value problem

$$L\frac{dI}{dt} + RI = E(t)$$

$$\Rightarrow 2\frac{dI}{dt} + 10I = 40, \quad I(0) = 0$$

$$\Rightarrow \frac{dI}{dt} + 5I = 20 \qquad ......(1), \quad \text{with } I(0) = 0$$

Here $P(t) = 5$ so integrating factor $e^{\int P(t)dt} = e^{\int 5dt}$

$$= e^{5t}$$

Multiplying both sides of the equation (1) by e^{5t}

We have $$e^{5t}\frac{dI}{dt} + 5e^{5t}I = 20e^{5t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{5t}.I\right) = 20e^{5t}$$

Integrating both sides

$$e^{5t}.I = 20\int e^{5t}dt$$

$$\Rightarrow e^{5t}.I = 20\frac{e^{5t}}{5} + C$$

$$\Rightarrow \boxed{I(t) = 4 + Ce^{-5t}}$$

We have $I(0) = 0$

$$\Rightarrow 0 = 4 + C$$

$$\Rightarrow C = -4$$

so $\boxed{I(t) = 4\left(1 - e^{-5t}\right)}$

(B) Current after 0.1 second is

$$I(0.1) = 4\left(1 - e^{-5\times 0.1}\right) \approx \boxed{1.57\,A}$$

Answer 28E.

(A) We have been given $E(t) = 40\sin 60t$ volts

And $R = 20\Omega$

L = 1H

And I (0) =1A

We have the initial value problem

$$L\frac{dI}{dt}+RI=E(t) \qquad I(0)=1$$

$$\Rightarrow \frac{dI}{dt}+20I=40\sin 60t \qquad ---(1)$$

Integrating factor is $e^{\int 20\,dt}=e^{20t}$

Multiplying both sides of the equation (1) by e^{20t}

$$e^{20t}\frac{dI}{dt}+20e^{20t}I=40e^{20t}\sin 60t$$

$$\Rightarrow \frac{d}{dt}\left(e^{20t}I\right)=40e^{20t}.\sin 60t$$

Integrating both sides

$$\Rightarrow e^{20t}.I=40\int e^{20t}\sin 60t\,dt$$

Using the integral

$$\int e^{au}\sin bu\,du=\frac{e^{au}}{a^2+b^2}\left(a\sin bu-b\cos bu\right)+C$$

We have

$$e^{20t}I=40\left[\frac{e^{20t}}{400+3600}\left(20\sin 60t-60\cos 60t\right)\right]+C$$

$$=\frac{e^{20t}}{100}\left(20\sin 60t-60\cos 60t\right)+C$$

$$\Rightarrow I(t)=\frac{1}{100}\left(20\sin 60t-60\cos 60t\right)+Ce^{-20t}$$

We have I (0) =1

So $\quad 1=\frac{1}{100}\left(20\sin 0-60\cos 0\right)+Ce^{0}$

$$\Rightarrow 1=\frac{1}{100}(0-60)+C$$

$$\Rightarrow 1=-\frac{6}{10}+C$$

$$\Rightarrow C=1+\frac{6}{10}=\frac{16}{10}=\frac{8}{5}$$

Then $\boxed{I(t)=\frac{1}{100}\left(20\sin 60t-60\cos 60t\right)+\frac{8}{5}e^{-20t}}$

(B) Current after 0.1 s, is

$$I(0.1)=\frac{1}{100}\left(20\sin 6-60\cos 6\right)+\frac{8}{5}e^{-2}$$

$$\approx \boxed{-0.42A}$$

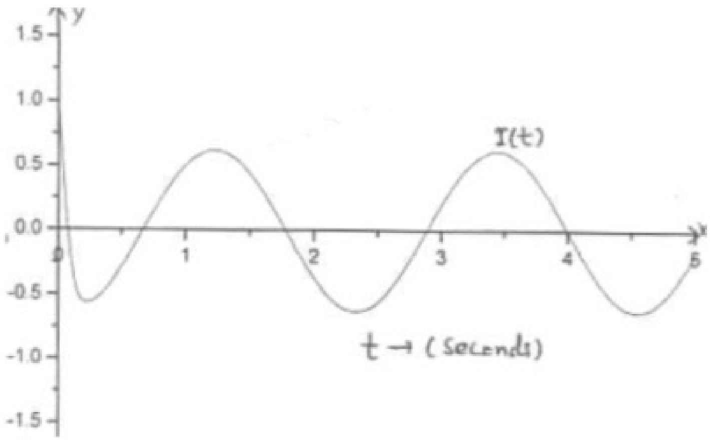

Fig. 1

Answer 29E.

We have been given, a differential equation

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E(t) \qquad \text{.......(1)}$$

Resistance $R=5\,\Omega$

Capacitance $C=0.05\text{F}$

Constant voltage $E\,(t)=60V$

And $Q\,(0)=0$ is the initial charge

Then form (1) we have initial value problem.

$$5\frac{dQ}{dt}+\frac{1}{0.05}Q=60\ , \qquad Q(0)=0$$

$$\Rightarrow \frac{dQ}{dt}+4Q=12\ , \qquad Q(0)=0 \qquad \text{..............(2)}$$

Integrating factor is $e^{\int 4\,dt}=e^{4t}$

Multiplying both sides of the equation (2) by e^{4t}

$$e^{4t}\frac{dQ}{dt}+4e^{4t}Q=12e^{4t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{4t}Q\right)=12e^{4t}$$

$$\Rightarrow e^{4t}Q=12\int e^{4t}dt \qquad \text{[Integrating both sides}$$

$$\Rightarrow e^{4t}Q=3e^{4t}+C$$

$$\Rightarrow Q(t)=3+Ce^{-4t}$$

We have $Q(0)=0$ thus $\Rightarrow 0=3+C \Rightarrow \boxed{C=-3}$

So we have

$$\boxed{Q(t)=3\left(1-e^{-4t}\right)} \text{ This is charge at time } t$$

Current at time t is $I(t) = \frac{dQ}{dt}$

$$= \frac{d}{dt}\left(3\left(1-e^{-4t}\right)\right)$$

$$\Rightarrow I(t) = 3\left(4e^{-4t}\right)$$

$$\Rightarrow \boxed{I(t) = 12e^{-4t}}$$

Answer 30E.

We have been given

Resistance R=2 Ω

Capacitance C=0.01F

E (t) = 10 Sin 60t

And Q(0) = 0

So we have the initial value problem

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \quad Q(0) = 0$$

$$\Rightarrow 2\frac{dQ}{dt} + \frac{1}{0.01}Q = 10\sin 60t$$

$$\Rightarrow \frac{dQ}{dt} + 50Q = 5\sin 60t \qquad \text{......(1)}, \ \ Q(0) = 0$$

Integrating factor is $e^{\int 50t} = e^{50t}$

Multiplying by e^{50t}, both sides of the equation (1),

$$e^{50t}\frac{dQ}{dt} + 50e^{50t}Q = 5e^{50t}\sin 60t$$

$$\Rightarrow \frac{d}{dt}\left(e^{50t}Q\right) = 5e^{50t}\sin 60t$$

Integrating both sides

$$e^{50t}Q = 5\int e^{50t}\sin 60t\,dt$$

Using the formula $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

$$\Rightarrow e^{50t}Q = 5\left[\frac{e^{50t}}{2500+3600}(50\sin 60t - 60\cos 60t)\right] + C$$

$$\Rightarrow e^{50t}.Q(t) = \frac{e^{50t}}{1220}(50\sin 60t - 60\cos 60t) + C$$

$$\Rightarrow Q(t) = \frac{1}{1220}(50\sin 60t - 60\cos 60t) + Ce^{-50t}$$

We have Q (0) = 0

So $\Rightarrow 0 = \frac{1}{1220}(0-60)+C$

$$\Rightarrow C = \frac{60}{1220} = \frac{6}{122} = \frac{3}{61}$$

So charge at time t is $\boxed{Q(t) = \frac{1}{122}(5\sin 60t - 6\cos 60t) + \frac{3}{61}e^{-50t}}$

Current at time t is $I(t) = \frac{dQ}{dt}$

$$\Rightarrow I = \frac{dQ}{dt} = \frac{d}{dt}\left[\frac{1}{1220}(50\sin 60t - 60\cos 60t) + \frac{3}{61}e^{-50t}\right]$$

$$\Rightarrow I(t) = \frac{1}{1220}(50\times 60\cos 60t + 60\times 60\sin 60t) - \frac{3\times 50}{61}e^{50t}$$

$$\Rightarrow I(t) = \frac{1}{1220}(3000\cos 60t + 3600\sin 60t) - \frac{150}{61}e^{-50t}$$

$$\Rightarrow I(t) = \frac{1}{122}(300\cos 60t + 360\sin 60t) - \frac{150}{61}e^{-50t}$$

$$\Rightarrow \boxed{I(t) = \frac{1}{61}\left[150\cos 60t + 180\sin 60t - 150e^{-50t}\right]}$$

Answer 31E.

A model for learning is given by the differential equation

$$\frac{dP}{dt} = k\left[M - P(t)\right]$$

Where $P(t)$ is the performance level at time t, M is the maximum performance level and k is the positive constant

We can rewrite the equation as

$$\frac{dP}{dt} = kM - kP(t)$$

$$\Rightarrow \frac{dP}{dt} + kP(t) = kM \quad \text{............(1)}$$ This is the form of linear equation

Integrating factor is $e^{\int k dt} = e^{kt}$

Multiplying both sides of the equation (1) by e^{kt}

$$e^{kt}\frac{dP}{dt} + ke^{kt}P = ke^{kt}M$$

$$\Rightarrow \frac{d}{dt}\left(e^{kt}P\right) = kMe^{kt}$$

Integrating both sides

$$e^{kt}P = M\int ke^{kt}\,dt$$

$$\Rightarrow e^{kt}P = Me^{kt} + C$$

$$\Rightarrow \boxed{P(t) = M + Ce^{-kt}}$$ Where C is a constant

We assume that at time $t = 0$, $P(0) = P_0$

So we have $P_0 = M + C\,e^0$

$$\Rightarrow C = P_0 - M$$

Then $P(t) = M + (P_0 - M)e^{-kt}$

$$\Rightarrow \boxed{P(t) = M - (M - P_0)e^{-kt}}$$

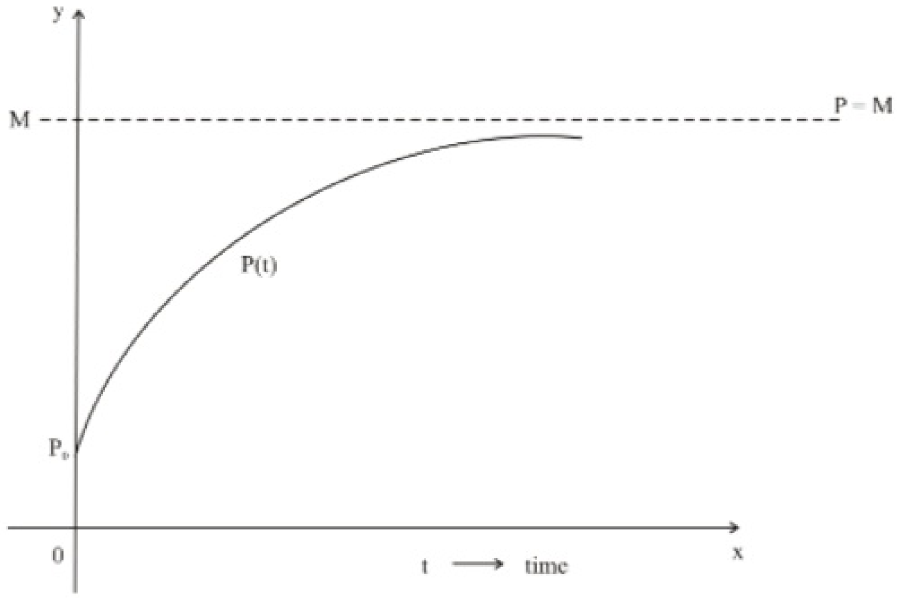

Fig. 1

Answer 32E.

A model for learning is given by the differential equation

$$\frac{dP}{dt} = k\left[M - P(t)\right]$$

Where P (t) is the performance level at time t, M is the maximum performance level and k is the positive constant

We can rewrite the equation as

$$\frac{dP}{dt} = kM - k\,P(t)$$

$$\Rightarrow \frac{dP}{dt} + kP(t) = kM \qquad \text{............(1)} \quad \text{This is the form of linear equation}$$

Integrating factor is $e^{\int k dt} = e^{kt}$

Multiplying both sides of the equation (1) by e^{kt}

$$e^{kt}\frac{dP}{dt} + ke^{kt}P = ke^{kt}M$$

$$\Rightarrow \frac{d}{dt}\left(e^{kt}P\right) = kM\,e^{kt}$$

Integrating both sides

$$e^{kt}P = M\int ke^{kt}\,dt$$

$$\Rightarrow e^{kt}P = Me^{kt} + C$$

$$\Rightarrow \boxed{P(t) = M + Ce^{-kt}} \qquad \text{Where C is a constant ....(2)}$$

We have been given, for the first worker.

$P_1(1) = 25$, $P_1(2) = 45$, $P_1(0) = 0$

Let the maximum performance level for the first worker be M_1

Then $P_1(t) = M_1 + Ce^{-kt}$ $\qquad$ *from*(2)

$$\Rightarrow 0 = M_1 + C \quad \Rightarrow C = -M_1$$

Then $P_1(t) = M_1\left(1 - e^{-kt}\right)$

By the given conditions we have

$$25 = M_1\ (1\text{-}e^{-k})$$

$$\Rightarrow \frac{25}{M_1} = 1 - e^{-k}$$

$$\Rightarrow e^{-k} = 1 - \frac{25}{M_1} \qquad ---(3)$$

And $45 = M_1\left(1 - e^{-2k}\right)$

$$\Rightarrow e^{-2k} = 1 - \frac{45}{M_1} \qquad ---(4)$$

$$\Rightarrow \left(e^{-k}\right)^2 = 1 - \frac{45}{M_1}$$

Therefore $\Rightarrow \left(1 - \frac{25}{M_1}\right)^2 = 1 - \frac{45}{M_1}$ from (3)

$$\Rightarrow 1 + \frac{625}{M_1^{\ 2}} - \frac{50}{M_1} = 1 - \frac{45}{M_1}$$

$$\Rightarrow \frac{625}{M_1^{\ 2}} - \frac{5}{M_1} = 0$$

$$\Rightarrow 625 - 5\ M_1 = 0$$

$$\Rightarrow M_1 = \frac{625}{5} = 125$$

So maximum number of units per hour that first worker is capable of processing is

$\boxed{125}$

We have been given, for the second worker

$$P_2(1) = 35,\ \ P_2(2) = 50,\ P_2(0) = 0$$

Let the maximum performance level for the second worker second worker be M_2

Then $P_2(t) = M_2 + Ce^{-kt}$ $\qquad$ *from*(2)

$\Rightarrow 0 = M_2 + C \ \Rightarrow C = -M_2$

Then $P_2(t) = M_2\left(1 - e^{-kt}\right)$

By the given conditions we have

$$35 = M_2\left(1 - e^{-k}\right)$$

$$\Rightarrow \frac{35}{M_2} = 1 - e^{-k}$$

$$\Rightarrow e^{-k} = 1 - \frac{35}{M_2} \qquad ---(5)$$

And $50 = M_2\left(1 - e^{-k2}\right)$

$$\Rightarrow e^{-2k} = 1 - \frac{50}{M_2} \qquad ---(6)$$

$$\Rightarrow \left(e^{-k}\right)^2 = 1 - \frac{50}{M_2}$$

Therefore

$$\Rightarrow\left(1-\frac{35}{M_2}\right)^2=1-\frac{50}{M_2} \qquad from\,(5)$$

$$\Rightarrow 1+\frac{1225}{M_2^2}-\frac{70}{M_2}=1-\frac{50}{M_2}$$

$$\Rightarrow \frac{1225}{M_2^2}-\frac{20}{M_2}=0$$

$$\Rightarrow 1225-20M_2=0 \Rightarrow M_2={}^{1225}\!/_{20}=61.25$$

So $\boxed{M_2\approx 61 \ \textit{units per hour}}$

Answer 33E.

Let y (t) be the amount of salt (in kg) after t minutes
Given that the tank contains 100L of water, so at time $t=0, y(0)=0$
Since at the beginning, amount of salt in the tank is 0g.
The rate of change of amount of salt is given by the equation

$$\frac{dy}{dt}=(\text{rate in})-(\text{rate out})$$

Rate in $=0.4\frac{\text{kg}}{\text{L}}\times 5\frac{\text{L}}{\text{min}}=2\ \text{kg/min}$

Since solution is being added with rate 5L / min and drained with the rate 3L /min
So amount of the solution in the tank at time t is =100+ (5-3) .t = (100+2t) L

Then rate out $=\frac{y(t)\ \text{kg}}{(100+2t)\text{L}}\times 3\frac{\text{L}}{\text{min}}=\frac{3y}{100+2t}\ \text{kg/min}$

Then we have $\boxed{\frac{dy}{dt}=2-\frac{3y}{100+2t}}$

We rewrite the equation as $\frac{dy}{dt}+\frac{3y}{(100+2t)}=2 \qquad ---(1)$

This is the form of Linear equation.

Integrating factor is $=e^{\int\frac{3}{(100+2t)}dt}$

$$=e^{(3/2)\ln(100+2t)}$$

$$=e^{\ln(100+2t)^{3/2}}$$

$$=(100+2t)^{3/2}$$

Multiplying both sides of the equation (1) by $(100+2t)^{3/2}$

$$\Rightarrow (100+2t)^{3/2}\frac{dy}{dt}+3(100+2t)^{1/2}y=2(100+2t)^{3/2}$$

$$\Rightarrow \frac{d}{dt}\left((100+2t)^{3/2}y\right)=2(100+2t)^{3/2}$$

Integrating both sides, we have

$$(100+2t)^{3/2}y=2\int(100+2t)^{3/2}dt$$

$$=2\frac{(100+2t)^{5/2}}{(5/2).2}+C$$

$$=\frac{2}{5}(100+2t)^{5/2}+C$$

$$\Rightarrow y(t)=\frac{2}{5}(100+2t)+C(100+2t)^{-3/2}$$

We have $y(0)=0$

So $0=\frac{2}{5}(100)+C(100)^{-3/2}$

$\Rightarrow -40=C(100)^{-3/2}$

$\Rightarrow C=-40\times(100)^{3/2}=-40000$

Then the solution is $\boxed{y(t)=\frac{2}{5}(100+2t)-40000(100+2t)^{-3/2}}$

Amount of salt after 20 minutes is

$$y(20)=\frac{2}{5}(100+2\times 20)-40000(100+2\times 20)^{-3/2}$$

$$=56-\frac{40000}{140\sqrt{140}}$$

$$=56-\frac{4000}{14\sqrt{140}}\approx 31.85\text{kg}$$

And amount of solution in the tank after 20 minutes

$=100+2\times 20 \quad =140L$

Then concentration $=\frac{y(20)\text{ kg}}{140\text{L}}\approx\frac{31.85\text{kg}}{140\text{L}}$

$\approx \boxed{0.2275\text{ kg/L}}$

Answer 34E.

Let y (t) be the amount of salt (in grams) after t seconds so initial amount of salt

y (0) = Concentration of salt $\times$amount of solution

$=400\times 0.05\text{ g}$

y (0) = 20 grams

The rate of change of amount of salt is given by the equation

$$\frac{dy}{dt}=(rate\,in)-(rate\,out)$$

Rate in =0 g/s because only pure water is pumped in to the tank

Since pumping rate is 4 L/s and draining rate is 10 L/s then amount of solution after t seconds $=400-(10-4)\,t$

$=400-6t$

Then rate out $=\frac{y(t)}{400-6t}\times 10$ g/s

Therefore $\frac{dy}{dt}=-\frac{10y}{400-6t}$

$$\Rightarrow \frac{1}{y}dy=-\frac{10}{400-6t}dt$$

Integrating both sides

$$\int \frac{1}{y} dy = -10 \int \frac{1}{400-6t} dt$$

$$\Rightarrow \ln y = \frac{-10}{-6} \ln(400-6t) + C$$

$$\Rightarrow \ln y = \frac{5}{3} \ln(400-6t) + C$$

$$\Rightarrow \ln y = \ln(400-6t)^{5/3} + \ln e^C$$

$$\Rightarrow y(t) = e^C (400-6t)^{5/3}$$

Then $y(t) = k(400-6t)^{5/3}$ grams {Let $e^C = k$}

We have y (0) =20 grams, therefore

$$20 = k(400-0)^{5/3}$$

$$\Rightarrow k = \frac{20}{(400)^{5/3}} \approx 9.21 \times 10^{-4}$$

So $y(t) = \frac{20}{(400)^{5/3}} (400-6t)^{5/3}$ grams

Then $\boxed{y(t) = 20(1-0.015t)^{5/3}}$ g

Answer 35E.

(A) We have to solve the differential equation $m\frac{dv}{dt} = mg - cv$

Rewrite the equation as $m\frac{dv}{dt} + cv = mg$

$$\Rightarrow \frac{dv}{dt} + \frac{c}{m} v = g \qquad ---(1)$$ This is a linear equation

Integrating factor is $I = e^{\int \frac{c}{m} dt} = e^{ct/m}$

Multiplying both sides of the equation (1) by $e^{ct/m}$

We have $e^{ct/m} \frac{dv}{dt} + \frac{c}{m} e^{ct/m} v = g e^{ct/m}$

Or $\frac{d}{dt}\left(e^{ct/m}.v\right) = g e^{ct/m}$

Integrating both sides, we have

$$e^{ct/m}.v = g \int e^{ct/m} dt$$

$$\Rightarrow e^{ct/m}.v = g\frac{m}{c} e^{ct/m} + k$$ k is a constant

$$\Rightarrow v = \frac{gm}{c} + k e^{-ct/m}$$

Integrating both sides, we have

$$e^{ct/m}.v = g \int e^{ct/m} dt$$

$$\Rightarrow e^{ct/m}.v = g\frac{m}{c} e^{ct/m} + k$$ k is a constant

$$\Rightarrow v = \frac{gm}{c} + k e^{-ct/m}$$

(B) For finding limiting velocity

We have to find $\lim_{t\to\infty} v(t)$

From (1) we have $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{c}\left(1-e^{-ct/m}\right)$

$$= \frac{mg}{c} - 0$$

$$= \frac{mg}{c}$$

So limiting velocity is $\boxed{mg/c}$

(C) Distance $s(t) = \int v(t)\,dt$

$$= \int \frac{mg}{c}\left(1-e^{-ct/m}\right)dt$$

$$= \frac{mg}{c}\int \left(1-e^{-ct/m}\right)dt$$

$$= \frac{mg}{c}\left[t+\frac{m}{c}e^{-ct/m}\right]+A \qquad A \text{ is a constant}$$

When $t = 0$, $s(0) = 0$ because object starts from the rest

So $\quad 0 = \frac{mg}{c}\left(\frac{m}{c}\right)+A$

$$\Rightarrow A = -\frac{m^2 g}{c^2}$$

Then distance after t seconds is $\boxed{s(t) = \frac{mg}{c}\left[t+\frac{m}{c}e^{-ct/m}\right]-\frac{m^2 g}{c^2}}$

Answer 36E.

The velocity of a falling object of mass m is given by

$$v = \frac{mg}{c}\left(1-e^{-ct/m}\right).$$

Differentiation of both sides with respect to m gives

$$\frac{dv}{dm} = \frac{g}{c}\left(1-e^{-ct/m}\right)+\frac{mg}{c}\left(-e^{-ct/m}\cdot\frac{ct}{m^2}\right)$$

$$\Rightarrow \boxed{\frac{dv}{dm} = \frac{g}{c}\left[1-e^{-ct/m}-\frac{ct}{m}e^{-ct/m}\right]}$$

This is the expression of a mass m.

Now suppose m_1 and m_2 be two masses such that $m_2 > m_1$ then for the heavier object to fall faster we must have $\frac{dv}{dm_2}-\frac{dv}{dm_1} > 0$.

$$\frac{dv}{dm_2}-\frac{dv}{dm_1} = \frac{g}{c}\left[1-e^{-ct/m_2}-\frac{ct}{m_2}e^{-ct/m_2}-1+e^{-ct/m_1}+\frac{ct}{m_1}e^{-ct/m_1}\right]$$

$$= \frac{g}{c}\left[e^{-ct/m_1}\left(1+\frac{ct}{m_1}\right)-e^{-ct/m_2}\left(1+\frac{ct}{m_2}\right)\right]$$

Now since $m_2 > m_1, \Rightarrow \frac{1}{m_1} > \frac{1}{m_2} \Rightarrow \frac{ct}{m_1} > \frac{ct}{m_2} \Rightarrow 1+\frac{ct}{m_1} > 1+\frac{ct}{m_2}$

And since $\frac{ct}{m_1} > \frac{ct}{m_2} \Rightarrow e^{-ct/m_1} > e^{-ct/m_2}$

There fore $e^{-ct/m_1}\left(1+\frac{ct}{m_1}\right) > e^{-ct/m_2}\left(1+\frac{ct}{m_2}\right)$

$\Rightarrow \frac{dv}{dm_2} - \frac{dv}{dm_1} > 0 \Rightarrow \frac{dv}{dm_2} > \frac{dv}{dm_1}$

Thus acceleration of heavier mass is more that the acceleration of lighter mass.

So heavier objects do fall faster than lighter objects

Answer 37E.

(a). Given equation is

$p' = kp\left(1-\frac{p}{M}\right)$ ------(1)

$Put\ \frac{1}{p} = z$

$\Rightarrow p = \frac{1}{z}$

$\Rightarrow p' = -\frac{1}{z^2}dz$

(1) $becomes -\frac{1}{z^2}dz = \frac{k}{z}\left(1-\frac{1}{zM}\right)$

$\Rightarrow \boxed{z' + kz = \frac{k}{M}}$

(b). Here $p(t) = k, q(t) = \frac{k}{M}$

Integrating factor $IF = e^{\int p(t)dt} = e^{\int k dt} = e^{kt}$

Solution is

$z(IF) = \int q(t) IF\ dt$

$\Rightarrow ze^{kt} = \int \frac{k}{M}e^{kt}dt = \frac{k}{M}\frac{e^{kt}}{k} + c$

$\Rightarrow ze^{kt} = \frac{e^{kt}}{M} + c$

$\Rightarrow z = \frac{1}{M} + ce^{-kt}$

$\Rightarrow \frac{1}{p(t)} = \frac{1}{M} + ce^{-kt}$

$= \frac{1+Mce^{-kt}}{M}$

$\Rightarrow \boxed{p(t) = \frac{M}{1+Mce^{-kt}}}$

Answer 38E.

(a) Given the logistic differential equation,

$$\frac{dP}{dt}=k(t)P\left(1-\frac{P}{M(t)}\right)$$

Substituting $z=\frac{1}{P}$ we get

$$\frac{d\left(\frac{1}{z}\right)}{dt}=\frac{k(t)\left(1-\frac{1}{zM(t)}\right)}{z}$$

$$\Rightarrow\frac{d\left(\frac{1}{z}\right)}{dt}=\frac{k(t)\left(\frac{zM(t)}{zM(t)}-\frac{1}{zM(t)}\right)}{z}$$

$$\Rightarrow-\frac{1}{z^2}\frac{dz}{dt}=\frac{k(t)(zM(t)-1)}{z^2M(t)}$$

$$\Rightarrow\frac{dz}{dt}=-\frac{k(t)zM(t)-k(t)}{M(t)}$$

$$\Rightarrow\boxed{\frac{dz}{dt}+k(t)z=\frac{k(t)}{M(t)}}$$

(b) We need to find the solution of the differential equation

$$\frac{dz}{dt}+k(t)z=\frac{k(t)}{M(t)}$$

This equation is a first order linear differential equation in standard form:

$$y'+P(x)y=Q(x)$$

Where $P(x)=k(t)$, and $Q(x)=\frac{k(t)}{M(t)}$

We must find the integrating factor and multiply both sides of the equation by it in order to integrate both sides.

Find the integrating factor.

$$I(x)=e^{\int P(x)dx}$$
$$=e^{\int k(t)dt}$$

Now multiply both sides of the equation by the integrating factor.

$$\frac{dz}{dt}+k(t)z=\frac{k(t)}{M(t)}$$

$$e^{\int k(t)dt}\frac{dz}{dt}+e^{\int k(t)dt}k(t)z=e^{\int k(t)dt}\frac{k(t)}{M(t)}$$

This can be written as

$$\frac{d}{dt}\left(e^{\int k(t)dt}z\right)=e^{\int k(t)dt}\frac{k(t)}{M(t)}$$

On integrating we get

$$\int \frac{d}{dt}\left(e^{\int k(t)dt} z\right)dt = \int e^{\int k(t)dt} \frac{k(t)}{M(t)}dt$$

$$\Rightarrow e^{\int k(t)dt} z = C + \int e^{\int k(t)dt} \frac{k(t)}{M(t)}dt$$

$$\Rightarrow \boxed{z = \frac{C + \int e^{\int k(t)dt} \frac{k(t)}{M(t)}dt}{e^{\int k(t)dt}}}$$

C, is the constant of integration.

If $M(t)$ is constant:

$$z = \frac{C + \int e^{\int k(t)dt} \frac{k(t)}{M}dt}{e^{\int k(t)dt}}$$

$$z = \frac{C + \frac{1}{M}\int e^{\int k(t)dt} k(t)dt}{e^{\int k(t)dt}}$$

Recall $z = \frac{1}{P}$

So we get

$$P(t) = \frac{e^{\int k(t)dt}}{C + \frac{1}{M}\int e^{\int k(t)dt} k(t)dt}$$

$$\Rightarrow P(t) = \frac{Me^{\int k(t)dt}}{MC + \int e^{\int k(t)dt} k(t)dt}$$

$$\Rightarrow P(t) = \frac{M}{MCe^{-\int k(t)dt} + e^{-\int k(t)dt}\int e^{\int k(t)dt} k(t)dt}$$

If we let

$$u = \int k(t)dt$$

$$\Rightarrow du = k(t)dt$$

So

$$\int e^{\int k(t)dt} k(t)dt = \int e^{u}du$$

$$= e^{u}$$

$$= e^{\int k(t)dt}$$

Then the function simplifies to

$$P(t) = \frac{M}{\left(MCe^{-\int k(t)dt} + e^{-\int k(t)dt} e^{\int k(t)dt}\right)}$$

So we get

$$\boxed{P(t) = \frac{M}{\left(CMe^{-\int k(t)dt} + 1\right)}}$$

Now

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{\left(CMe^{-\int k(t)dt}+1\right)}$$

$$=\lim_{t\to\infty} \frac{M}{\left(\frac{CM}{e^{\int k(t)dt}}+1\right)}$$

$$=\frac{M}{\left(\frac{CM}{\infty}+1\right)} \qquad \left(\text{since} \int_0^\infty k(t)\,dt = \infty\right)$$

$$=\frac{M}{(0+1)} = M$$

Thus

If $\int_0^\infty k(t)\,dt = \infty$ then $\boxed{\lim_{t\to\infty} P(t) = M}$

(c) Assuming k is constant but M varies:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{1+CMe^{-\int k(t)dt}}$$

$$=\frac{M}{1+CM\left(\lim_{t\to\infty} e^{-\int k(t)dt}\right)}$$

$$=\frac{M}{1+CM\left(e^{-\lim_{t\to\infty}\int k(t)dt}\right)}$$

$$=\frac{M}{1+CM\left(e^{-\infty}\right)}$$

$$=\frac{M}{1+CM(0)}$$

$$=\frac{M}{(1+0)}$$

$$=M$$

So we have $\boxed{\lim_{t\to\infty} P(t) = \lim_{t\to\infty} M(t)}$

We found earlier that:

$$z(t) = \frac{C+\int e^{\int k(t)dt}\frac{k(t)}{M(t)}dt}{e^{\int k(t)dt}}$$

When $k(t)$ is constant but $M(t)$ varies the above expression simplifies to:

$$z(t) = \frac{C+\int_0^t e^{ks}\frac{k}{M(s)}ds}{e^{kt}} = Ce^{-kt} + e^{-kt}\int_0^t e^{ks}\frac{k}{M(s)}ds$$

So we have

$$\boxed{z(t) = Ce^{-kt} + e^{-kt}\int_0^t e^{ks}\frac{k}{M(s)}ds}$$

Taking the limit as $t \to \infty$ gives:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left[\frac{1}{Ce^{-kt} + e^{-kt}\int_0^t e^{ks}\frac{k}{M(s)}ds}\right]$$

$$= \lim_{t\to\infty}\left[\frac{e^{kt}}{C + \int_0^t e^{ks}\frac{k}{M(s)}ds}\right]$$

$$= \frac{\lim_{t\to\infty} e^{kt}}{\lim_{t\to\infty}\left[C + \int_0^t e^{ks}\frac{k}{M(s)}ds\right]}$$

$$= \frac{\lim_{t\to\infty} e^{kt}}{\left[C + \int_0^\infty e^{ks}\frac{k}{M(s)}ds\right]}$$

If $\lim_{t\to\infty} M(t)$ exists, then the denominator integral blows to ∞ since $\lim_{s\to\infty} e^{ks}$ tends to ∞ and the factors in the integrand are either constant k or constant in the limit $M(s)$.

Then, the above limit becomes $\left(\frac{\infty}{\infty}\right)$, and l'Hospital's rule applies.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\frac{\frac{d}{dt}\left[e^{kt}\right]}{\frac{d}{dt}\left[C + \int_0^t e^{ks}\frac{k}{M(s)}ds\right]}$$

$$= \lim_{t\to\infty}\frac{\left[ke^{kt}\right]}{\left[e^{kt}\frac{k}{M(t)}\right]}$$

$$= \lim_{t\to\infty}\frac{1}{\left[\frac{1}{M(t)}\right]}$$

$$= \lim_{t\to\infty} M(t)$$