

Permutations and Combinations

- **Fundamental Principle of Counting:** If an event occurs in m different ways, following which another event occurs in n different ways, then the total number of occurrence of the events in the given order is $m \times n$. This is called the fundamental principle of counting.

Example:

Find the number of 5-letter words, with or without meaning, which can be formed out of the letters of the word MATHS, where the repetition of digits is not allowed.

Solution:

There are as many words as there are ways of filling 5 vacant places $\square\square\square\square\square$ by the 5 letters.

The first place can be filled with any of the 5 letters in 5 different ways, following which the second place can be filled with any of the remaining 4 letters in 4 different ways, following which the third place can be filled in 3 different ways, following which the fourth place can be in 2 different ways, following which the fifth place can be filled in 1 way.

Thus, the number of ways in which the 5 places can be filled, by the multiplication principle, is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Note: If repetition of letters had been allowed, then the required number of words would be $5 \times 5 \times 5 \times 5 \times 5 = 3125$.

- **Factorial notation:** The notation $n!$ represents the product of the first n natural numbers, i.e.,

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 5 \times 4 \times 3 \times 2 \times 1$$

$$0! = 1$$

Example:

- **Permutation when all objects are distinct:** A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

$$\frac{12!}{8!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

The number of permutations of n different things taken r at a time, when

$$= 12 \times 11 \times 10 \times 9$$

$$= 11880$$

◦ repetition is not allowed,

is ${}^n P_r = \frac{n!}{(n-r)!}$, where $0 \leq r \leq n$.

◦ repetition is allowed, is n^r , where $0 \leq r \leq n$.

Example 1: Twenty five students are participating in a competition. In how many ways, can the first three prizes be won in such a way that a prize cannot be shared by more than one student?

Solution: The total number of ways in which first three prizes can be won is the number of arrangements of 25 different things taken 3 at a time.

So, required number of ways = ${}^{25} P_3$

$$= \frac{25!}{(25-3)!}$$

$$= \frac{25!}{22!}$$

$$= \frac{25 \times 24 \times 23 \times 22!}{22!}$$

$$= 25 \times 24 \times 23 = 13800$$

Example 2: Find the total number of four digit numbers that can be formed by using the digits 0, 2, 5, and 6?

Solution: A four digit number has four places i.e., units, tens, hundreds and thousands. Units, tens and hundreds place can be filled with either 0, 2, 5, or 6 where as thousands place can be filled with 2, 5 or 6 only.

Number of ways to fill the units place = 4

Number of ways to fill the tens place = 4

Number of ways to fill the hundreds place = 4

Number of ways to fill the thousands place = 3 ways.

∴ Total number of four digit numbers = $4 \times 4 \times 4 \times 3 = 192$

- **Concept of permutations when all objects are not distinct**

- The number of permutations of n objects, when p objects are of the same kind and the rest are all different, is $\frac{n!}{p!}$.
- In general, the number of permutations of n objects, when p_1 objects are of one kind, p_2 are of the second kind, ..., p_k are of the k^{th} kind and the rest, if any, are of different kinds, is $\frac{n!}{p_1! p_2! \dots p_k!}$.

Example: Find the number of permutations of the letters of the word ARRANGEMENT.

Solution: Here, there are 11 objects (letters) of which there are 2A's, 2R's, 2N's, 2E's and the rest are all different.

∴ Required number of arrangements

$$\begin{aligned}
 &= \frac{11!}{2!2!2!2!} \\
 &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 2}{2 \times 2 \times 2 \times 2} \\
 &= 2494800
 \end{aligned}$$

- **Combinations:** The number of combinations of n different things taken r at a time is denoted by ${}^n C_r$, which is given by

$${}^n C_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq n.$$

In particular, ${}^n C_0 = {}^n C_n = 1$

Example 1: A box contains 8 red bulbs and 5 blue bulbs. Determine the number of ways in which 4 red and 2 blue bulbs can be selected.

Solution:

It is given that a box contains 8 red bulbs and 5 blue bulbs.

Now, 4 red bulbs can be selected from 8 red bulbs in ${}^8 C_4$ number of ways, and 2 blue bulbs can be selected from 5 blue bulbs in ${}^5 C_2$ number of ways.

Hence, 4 red bulbs and 2 blue bulbs can be selected from a box containing 8 red bulbs and 5 blue bulbs in ${}^8 C_4 \times {}^5 C_2$ number of ways.

Now,

$${}^8 C_4 \times {}^5 C_2$$

$$\begin{aligned}
&= \frac{8!}{4! \times (8-4)!} \times \frac{4!}{2!(4-2)!} \\
&= \frac{8!}{4! \times 4!} \times \frac{4!}{2!2!} \\
&= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} \\
&= 70 \times 6
\end{aligned}$$

$$= 420$$

Thus, the number of ways of selecting the bulbs is 420.

$$\bullet \quad {}^n C_{n-r} = \frac{n!}{(n-r)! [n-(n-r)]!} = \frac{n!}{(n-r)! r!} = {}^n C_r$$

In other words, selecting r objects out of n objects is the same as rejecting $(n-r)$ objects.

$$\bullet \quad {}^n C_a = {}^n C_b \Rightarrow a = b \text{ or } a = n, \text{ i.e., } n = a + b$$

Example 2: If ${}^{19} C_{3r} = {}^{19} C_{2r+4}$, then find the value of r .

Solution:

$${}^{19} C_{3r} = {}^{19} C_{2r+4}$$

$$\Rightarrow 3r + (2r + 4) = 19 \quad \text{or} \quad 3r = 2r + 4$$

$$\Rightarrow 3r + (2r + 4) = 19 \quad \Rightarrow r = 4$$

$$\Rightarrow 5r + 4 = 19$$

$$\Rightarrow 5r = 19 - 4 = 15$$

$$\Rightarrow r = 3$$

\therefore The value of r is either 3 or 4.

$$\bullet \quad {}^n C_{r-1} + {}^n C_r = {}^{n+1} C_r$$

Example 3: If ${}^n C_{r-1} + {}^n C_r + {}^{n+1} C_{r+1} + {}^{n+2} C_{r+2} = {}^{n+a} C_{r+(a-1)}$, then find the value of a .

Solution:

$${}^n C_{r-1} + {}^n C_r + {}^{n+1} C_{r+1} + {}^{n+2} C_{r+2}$$

$$\begin{aligned}
&= {}^{n+1}C_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} \\
&= {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} \\
&= {}^{n+3}C_{r+2} \\
&\therefore {}^{n+3}C_{r+2} = {}^{n+a}C_{r+(a-1)} \\
&\Rightarrow a = 3
\end{aligned}$$

$$(\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r)$$