3 Trigonometric Functions

NASA scientists utilise the knowledge of trigonometry to build and launch rockets and space shuttles. Without trigonometry, humans would not have been able to travel to the moon.

Topic Notes

- *Basic Concepts of Angles*
- *Trigonometric Transformation Formulae*

BASIC CONCEPTS OF ANGLES 1

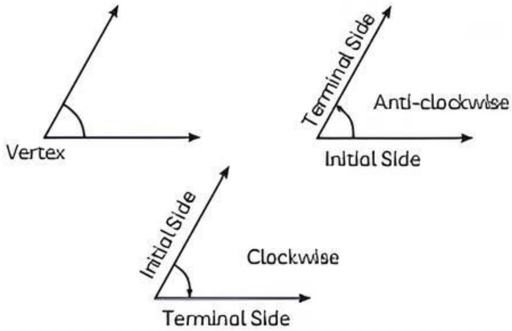
TOPIC 1

ANGLES AND THEIR MEASUREMENT

Angle

An angle is the figure formed by two rays sharing a common end-point. The two rays are called sides of the angle, and the common end-point is called vertex of the angle.

The word angle is also used to designate the measure of an angle of a rotation. Also, the sides of an angle are called initial sides and terminal sides.



We have the following conventions:

Positive Angle: If the direction of rotation is anti-clockwise, then the angle is taken as positive.

Negative Angle: If the direction of rotation is clockwise, then the angle is taken as negative.

Units for Measurement of Angles

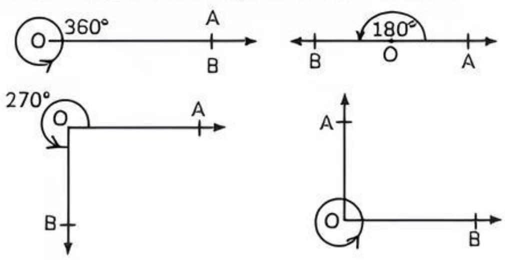
The two most commonly used units for measurement of angles are the following:

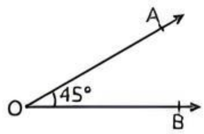
Degree Measure

An angle is said to be of 1 degree (denoted by $1°$) if it is $\left(\frac{1}{360}\right)^{th}$ part of a revolution.

So, one revolution is of 360°. One advantage of this unit is that many angles common in simple geometry are measured as a whole number of degrees. Fractions of a degree may be written in normal decimal notation (e.g. 7.5° for seven and a half degrees) but the 'degree-minute-second' system is also in use.

Some of the common angles are shown below:





Minutes

Each degree is divided into 60 equal parts called minutes. A measure of an angle in minutes is denoted by a single prime ().

$$1° = 60'$$

So, 7.5° can be called 7 degrees and 30 minutes, written as 7° 30′.

Second

Each minute is further divided into 60 equal parts called seconds. A measure of an angle in seconds is denoted by a double prime (″) .

$$1' = 60''$$

So, an angle of 2 degree 5 minutes 30 seconds is written as 2° 5′ 30″.

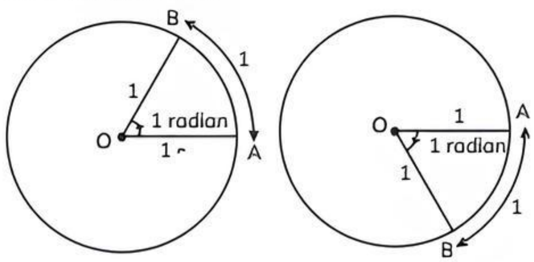
The division of degrees into minutes and seconds of angle is analogous to the division of hours into minutes and seconds of time.

Radian Measure

An angle is said to be 1 radian (denoted by 1 rad) if the length of arc is equal to the radius of the circle.

But the radian notation is frequently omitted. So, any measure of angle without units means that the angle is in radian.

The angles that measures 1 radian (1^c) and –1 radian (-1^c) are shown below:



Relation between Degree and Radian

Consider a circle of radius r. Then, the angle (in radian) subtended by the circle at the centre is given by

$$\theta = \frac{1}{r} = \frac{2\pi r}{r} = 2\pi\theta$$

Also, the angle (in degrees) subtended by the circle at the centre is 360°, which implies.

$$360° = 2\pi \text{ radian}$$

We now list some frequently used angles in degrees and radians.

Degrees	0°	30°	45°	60°	90°	120°	135°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$

Degrees	150°	180°	225°	270°	315°
Radians	$\frac{5\pi}{6}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$

Important

- *Radian measure* = $\frac{\pi}{180}$ × *Degree measure*
- *Degree measure* = $\frac{180}{\pi}$ × *Radian measure*

Conversion from Degree Measure to Radians

Step I. Convert the seconds (if given) into minutes by using the relation

$$1 \text{ second} = \frac{1}{60} \text{ minutes or } 1'' = \left(\frac{1}{60}\right)'$$

Step II. Convert the total minutes (given minutes + minues obained in step I) into degrees by using the relation

$$1 \text{ second} = \frac{1}{60} \text{ degree or } 1' = \left(\frac{1}{60}\right)^{\circ}$$

Step III. Convert the total degrees (given degree + degree obtained in step II) into radians by using the relation

$$\text{Redian measure} = \frac{\pi}{180} \times \text{ Degree measure}$$

Illustration: Convert 240° into radians.

$$240^{\circ} = \frac{\pi}{180} \times 240 \text{ rad}$$

$$= \frac{4\pi}{3} \text{ rad.}$$

Conversion from Radian Measure to Degree

Step I. Convert given radian into degree by using the relation $\left(\text{Use } \pi = \frac{22}{7}\right)$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{radian measure}$$

Step II. Convert the fraction part (obained in Step I) into minutes by using the relation

$$1^{\circ} = 60' \text{ or } 1 \text{ degree} = 60 \text{ min.}$$

Step III. If fraction is again obtained in step II then convert it into second by using the relation

$$1' = 60'' \text{ and } 1 \text{ min} = 60 \text{ seconds}$$

Example 1.1: Find the radian measures corresponding to the following degree measures −47° 30′. [NCERT]

Ans. As we know that

$$180^{\circ} = \pi \text{ radian}$$

So, $1^{\circ} = \frac{\pi}{180^{\circ}}$

and $60' = 1^{\circ}$

$$30' = \left(\frac{30^{\circ}}{60}\right)$$

Given, $-47^{\circ} 30' = -(47^{\circ} + 30')$

$$= -\left(47^{\circ} + \frac{30^{\circ}}{60}\right)$$

$$= -\left(47^{\circ} + \frac{1^{\circ}}{2}\right)$$

$$= -\left(\frac{94^{\circ} + 1^{\circ}}{2}\right)$$

$$= -\left(\frac{95^{\circ}}{2}\right)$$

$$\text{Radian measure} = \frac{\pi}{180} \times \text{degree measure}$$

$$= \frac{\pi}{180} \times \left(\frac{-95^{\circ}}{2}\right)$$

$$= \pi \times \frac{-19}{72}$$

$$= \frac{-19\pi}{72} \text{ radian}$$

Example 1.2: Find the radian measures corresponding to the following degree measures:

(A) 25°

(B) 240°

(C) 520° [NCERT]

Ans. (A) We know that 180° = π radian

$$1^{\circ} = \frac{\pi}{180^{\circ}} \text{ radian}$$

$$25^{\circ} = \frac{\pi}{180} \times 25 \text{ radian}$$

$$= \frac{5\pi}{36} \text{ radians}$$

(B) We know that 180° = π radian

$$1^{\circ} = \frac{\pi}{180^{\circ}} \text{ radian}$$

$$\therefore \quad 240° = \frac{\pi}{180°} \times 240 \text{ radian}$$

$$= \frac{4}{3}\pi \text{ radians}$$

(C) We know that $180° = \pi$ radian

$$1° = \frac{\pi}{180°} \text{ radian}$$

$$\therefore \quad 520° = \frac{\pi}{180} \times 520$$

$$= \frac{26\pi}{9} \text{ radians}$$

Example 1.3: If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii. [NCERT]

Ans. We know that, $l = r\,\theta$

There are 2 circles of different radius.

So, the radius be denoted by r_1 and r_2.

Length of an arc of 1st circle	Length of an arc of 2nd circle
$l = r_1\,\theta$	$l = r_2\,\theta$
$= r_1 \times 60°$	$= r_2 \times 75°$
$= r_1 \times 60° \times \frac{\pi}{180°}$	$= r_2 \times 75° \times \frac{\pi}{180°}$
$= r_1 \times \frac{\pi}{3}$	$= r_2 \times \frac{5\pi}{12}$

It is given that arcs are of same length.

Hence,

Length of 1st arc = length of 2nd arc

$$r_1 \times \frac{\pi}{3} = r_2 \times \frac{5\pi}{12}$$

$$\frac{r_1}{r_2} = \frac{5\pi}{12} \times \frac{3}{\pi}$$

$$\frac{r_1}{r_2} = \frac{5}{4}$$

Hence, $r_1 : r_2 = 5 : 4$

So, ratio of their radii = 5 : 4

Example 1.4: Find the degree measure of the angle subtended at the center of a circle of radius 100 cm by an arc of length 22 cm. (Use $\pi = \frac{22}{7}$) [NCERT]

Ans. We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ radian at the center, then

$$\theta = \frac{l}{r}$$

Therefore, $r = 100$ cm, $l = 22$ cm, we have

$$\theta = \frac{22}{100} \text{ degree}$$

As we know that

$$1° = \frac{\pi}{180°} \text{ radian}$$

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$\theta = \frac{22}{100} \text{ degree}$$

$$\text{radian} = \frac{180}{\pi} \times \frac{22}{100} \text{ degree}$$

$$= \frac{180 \times 7 \times 22}{22 \times 100} \text{degree}$$

$$= \frac{126}{10} \text{degree}$$

$$= 12\frac{3}{5} \text{ degree} = 12° \; 36'$$

$[1° = 60']$

Thus, the required angle is 12° 36′.

Example 1.5: Find the angle in radian through which a pendulum swings if its radius is 75 cm and the tip describe an arc of length.

(A) 10 cm

(B) 15 cm

(C) 21 cm [NCERT]

Ans. We know that in a circle of radius r unit, if an arc of length l unit subtends and angle θ radian at the center, then

$$\theta = \frac{l}{r}$$

It is given that, $r = 75$cm

(A) Here, $l = 10$cm

$$\therefore \quad \theta = \frac{10}{75} \text{radian} = \frac{2}{15} \text{ radians}$$

(B) Here, $l = 15$cm

$$\therefore \quad \theta = \frac{15}{75} \text{ radian} = \frac{1}{5} \text{ radians}$$

(C) Here, $l = 21$cm

$$\therefore \quad \theta = \frac{21}{75} \text{ radian} = \frac{7}{25} \text{ radians}$$

Example 1.6: A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second? [NCERT]

Ans. Number of revolutions made by the wheel in 1 minute = 360

$\therefore$ Number of revolutions made by the wheel in

1 second = $\frac{360}{60}$ = 6 revolutions per second

In one complete revolution, the wheel turns an angle of 2π radian.

Hence, in 6 complete revolutions, it will turn an angle of $6 \times 2\pi$ radian, i.e., 12π radian

Thus, in one second, the wheel turns an angle of 12π radian.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. A wheel rotates, making 18 revolutions per second. If the radius of the wheel is 49 cm, what linear distance does a point of its rim travel in three minutes? (Take π = 22/7)

(a) 9.97 km (b) 9.90 km

(c) 9.80 km (d) 9.85 km

Ans. *(a) 9.97 km*

Explanation: Radius of the wheel = 49 cm

$\therefore$ Circumference of the wheel

$= 2\pi \times 49$ cm = 308 cm

Hence, the linear distance travelled by a point of the rim in one revolution = 308 cm.

Number of revolutions made by the wheel in 3 minutes i.e. 180 seconds = $18 \times 3 \times 60 = 3240$

$\therefore$ The linear distance travelled by a point of the rim in 3 minutes

$= 308 \times 3240$

= 997920 cm = 9.97 km.

2. The angle of a triangle are in A.P. and the ratio of angle in degree of the least to the angle in radians of the greatest is 60 : π, find the angles in degrees.

(a) 30°, 60°, 90° (b) 40°, 60°, 90°

(c) 30°, 30°, 120° (d) 20°, 130°, 30°

Ans. *(a) 30°, 60°, 90°*

Explanation: Let, the angles of the triangle are, $(a - d)°$, $a°$ and $(a + d)°$.

Then, $a - d + a + a + d = 180°$

$\Rightarrow \quad a = 60°$

So, the angles are $(60 - d)°$, $(60°$, $(60 + d)°$.

Here, $(60 - d)°$ is the least angle and $(60 + d)°$ is the greatest angle.

Now, greater angle = $(60 + d)°$

$$= \left\{(60+d)\frac{\pi}{180}\right\}^c$$

Also, $\frac{\text{number of degrees in the least angle}}{\text{number of radians in the greatest angle}}$

$$= \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{60-d}{\left\{(60+d)\frac{\pi}{180}\right\}^c} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{180(60-d)}{\pi(60+d)} = \frac{60}{\pi}$$

$$\Rightarrow \quad 4d = 120$$

$$\Rightarrow \quad d = 30$$

Hence, the angles are $(60 - 30)°$, 60°, $(60 + 30)°$ i.e., 30°, 60°, 90°.

3. The large hand of a clock is 49 cm long. How much distance does its extremity move in 30 minutes?

(a) 154 cm (b) 80 cm

(c) 75 cm (d) 77 cm

Ans. *(a) 154 cm*

Explanation:

The large hand of the clock makes a complete revolution in 60 minutes.

Angle rotated in 60 minutes = 360°

$\therefore$ Angle traced out by the large hand in 30 minutes (of time)

$$\theta = \frac{360 \times 30}{60}$$

$$= 180°$$

$$= \frac{180°}{180}\pi \text{ radian}$$

$$= \pi \text{ radian}$$

Hence, the distance moved by the extremity of the large hand

$$l = r \times \theta$$

$$49 \times \pi = 49 \times \frac{22}{7} = 154 \text{ cm.}$$

4. The radius of the circle whose arc length 15π cm makes an angle of $\frac{3\pi}{4}$ radian at the centre is:

(a) 10 cm (b) 20 cm

(c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm [Diksha]

Ans. *(b)* *20 cm*

Explanation: Given, arc of length = 15π cm and angle

$$\theta = \frac{3\pi}{4}$$

Length of an arc of a circle is given by

$l = r \times \theta$

Where, l = length of arc

r = radius of a circle

θ = angle

$$\therefore \quad 15\pi = r \times \frac{3\pi}{4}$$

$$r = \frac{15\pi \times 4}{3\pi}$$

$r = 20$ cm

Hence, the radius of a circle is 20 cm.

5. **The radius of the circle in which a central angle of 30° intercepts an arc of length 22 cm is: (Use $\pi = \frac{22}{7}$)**

(a) 43 cm **(b) 41 cm**

(c) 42 cm **(d) 40 cm**

Ans. *(c)* *42 cm*

Explanation: Here, $l = 22$ cm and $\theta = 30° = \frac{\pi}{6}$

We know that length of an arc of a circle is given by

$$r = \frac{l}{\theta},$$

$$r = \frac{22 \times 6}{\pi}$$

$$= \frac{22 \times 6 \times 7}{22}$$

= 42 cm

6. **The angle in the radian through which a pendulum swings its length is 80 cm and tip describes an arc of length 20 cm is:**

(a) $\frac{1}{4}$ **(b) $\frac{2}{25}$**

(c) $\frac{3}{25}$ **(d) $\frac{4}{25}$**

Ans. *(a)* $\frac{1}{4}$

Explanation: Given, length of pendulum = 20 cm

Radius (r) = length of pendulum = 80 cm

Length of arc (l) = 20 cm

As we know that, $\theta = \frac{l}{r}$

$$\theta = \frac{20}{80} = \frac{1}{4} \text{ radian}$$

7. **The angle between the minute and hour hands of a clock at 8 : 30 is:**

(a) 80° **(b) 75°**

(c) 60° **(d) 105°** **[Diksha]**

Ans. *(b) 75°*

Explanation: We know that the hour hand of a clock completes one rotation in 12 hours.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

Now,

Angle traced by the hour hand in 8 hours 30 minutes, i.e., $\left(\frac{360}{60} \times 30°\right) = 180° = 255°$

We also know that the minute hand of a clock completes one rotation in 60 minutes.

$\therefore$ Angle traced by the minute hand in 60 minutes = 360°

Now,

Angle traced by the minute hand in 30 minutes

$= \left(\frac{360}{60} \times 30°\right) = 180°$

$\therefore$ Required angle between two hands of the clock = 255° – 180° = 75°.

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

8. **Assertion (A):** The radius of the circle in which a central angle of 60° intercepts an arc of length 44 cm is 42 cm.

Reason (R): Length of an arc of a circle is $l = r\theta$, where r is non-italic angle.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Here, $l = 44$ cm and $\theta = 60° = \frac{\pi}{3}$

As we know that

$l = r\theta$

$$r = \frac{l}{\theta}$$

$$= \frac{44 \times 3}{\pi}$$

$$= \frac{44 \times 3 \times 7}{22} = 42 \text{ cm}$$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

9. *Nitish is playing with a Pinwheel toy which he bought from a village fair. He noticed that the pinwheel toy revolves as fast as he blows it. Consider the Pinwheel toy that makes 360 revolutions per minute.*

(A) Find the number of revolutions made by Pinwheel toy in 120 second.

(B) Find the number of revolutions made by Pinwheel toy in 1 sec and angle made by Pinwheel toy (in degree) in 6 revolutions.

(C) Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm. (Use $\pi = \frac{22}{7}$).

Ans. (A) Since the number of revolutions made by Pinwheel toy in 1 minute = 360

And 1 min = 60 seconds

So, the number of revolution made by Pinwheel toy in 60 seconds = 360

The number of revolution made by Pinwheel toy in 1 second = $\frac{360}{60}$

$\therefore$ Number of revolutions made by Pinwheel toy in 120 seconds = $\frac{360 \times 120}{60} = 720$

(B) The number of revolution made by Pinwheel toy in 1 seconds = $\frac{360}{60} = 6$

Since, angle made by Pinwheel toy in 1 revolutions = 360°.

Thus, angle made by Pinwheel toy in 6 revolutions = 360° × 6 = 2160°

(C) Given,

Length of the arc = l = 37.4 cm

Central angle = $\theta = 60° = \frac{60 \times \pi}{180}$ radian

$= \frac{\pi}{3}$ radians

We know that,

$$r = \frac{l}{\theta}$$

$$= (37.4) \times \left(\frac{\pi}{3}\right)$$

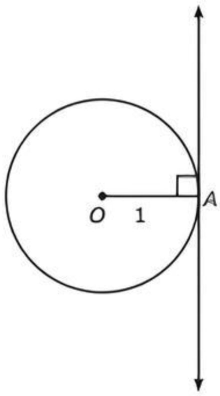
$$= \frac{(37.4)}{\left(\frac{22}{7 \times 3}\right)}$$

$$= 35.7 \text{ cm}$$

Hence, the radius of the circle is 35.7 cm.

10. *Consider a unit circle with centre O. Let A be any point on the circle. Consider O A as the initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle.*

A circle subtends an angle at the centre whose radian measure is 2 π and its degree measure is 360°.



(A) The radian measure of 240° is:

(a) $\frac{4\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

(B) A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

(a) 6π (b) 4π

(c) 3π (d) 12π

(C) The degree measure of 1.2 radian is:

(a) 68° (b) 68°43′37.8″

(c) 68°45′36″ (d) 58°46′27″

(D) The radius of the circle in which a central angle of 45° intercepts an arc of 132 cm, is:

(Use π) = $\frac{22}{7}$

(a) 168 cm (b) 50 cm

(c) 160 cm (d) 148 cm

(E) The minute hand of a watch is 35 cm long. How far does it move in 9 minutes?

(a) 15 cm (b) 30 cm

(c) 46 cm (d) 33 cm

Ans. (A) *(a)* $\frac{4\pi}{3}$

Explanation: As we know that

$$180° = \pi \text{ radian}$$

$$1° = \frac{\pi}{180°} \text{ radian}$$

Radian measure of 240° $= 240 \times \frac{\pi}{180°}$

$$= \frac{4\pi}{3}$$

(B) *(d)* 12π

Explanation: Given that a wheel makes 360 revolutions in 1 minute.

i.e., a wheel makes 360 revolutions in 60 seconds.

$\therefore$ In 1 second, no. of revolutions $= \frac{360}{60}$

= 6 revolutions

In 1 revolution, the angle made by the wheel = 360°

$\therefore$ Angle made by the wheel in 6 revolutions

$$= 6 \times 360°$$

$$= 2160°$$

Radian made in 6 revolutions

$$= 2160° \times \frac{\pi}{180°}$$

$$= 12\pi$$

(C) *(b) 68°43′37.8″*

Explanation: As we know that,

$$180° = \pi \text{ radian}$$

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$(1.2) = \left(1.2 \times \frac{180}{\pi}\right)$$

$$= 1.2 \times \frac{180}{22} \times 7$$

$$= 68.7272°$$

$$= 68° (0.7272 \times 60)'$$

$$= 68° 43'(0.63 \times 60)''$$

$$= 68° 43' 37.8''$$

(D) *(a) 168 cm*

Explanation: We have,

$l = 132$ cm and $\theta - 45° = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$

Now, $\theta = \frac{l}{r} = \frac{132}{r}$

$$\frac{\pi}{4} = \frac{132}{r}$$

$$\Rightarrow \quad r = \frac{132 \times 4}{\pi}$$

$$\Rightarrow \quad r = \frac{132 \times 4}{\frac{22}{7}} = 168 \text{ cm}$$

(E) *(d) 33 cm*

Explanation: The angle made by minute hand in 9 minutes = (9 × 6)°

$$= 54° = 54° \times \frac{\pi}{180} = \frac{3\pi}{10}$$

$$\therefore \quad \theta = \frac{l}{r}$$

$$\Rightarrow \quad \frac{3\pi}{10} = \frac{l}{35}$$

$$\Rightarrow \quad l = \frac{35 \times 3\pi}{10} = \frac{21\pi}{2}$$

$$\Rightarrow \quad l = \frac{21}{2} \times \frac{22}{7} = 33 \text{ cm}$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

11. **Convert the following decimal-degree to degree minute second measures: 15.5757°**

Ans.

$$15.5757° = 15° + 0.5757°$$

$$= 15° + (0.5757 \times 60)'$$

$$= 15° + 34.542'$$

$$= 15° + 34' + 0.542'$$

$$= 15° + 34' + (0.542 \times 60)''$$

$$= 15° + 34' + 32.52''$$

$$= 15° + 34' + 33'$$

$$= 15° 34' 33''$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

12. **Convert following radian measure into degree measures $\frac{-2}{9}$.**

Ans. $\frac{-2}{9}$ radian $= \frac{-2}{9} \times \frac{180}{\pi}$

$$= -\left(2 \times \frac{20}{\pi}\right)$$

$$= -\left(2 \times \frac{20 \times 7}{22}\right)$$

$$= -\left(\frac{20 \times 7}{11}\right)$$

$$= -\left(\frac{140}{11}\right)$$

$$= -12.7272°$$

13. **Find the radius of the circle in which a central angle of 30° intercepts an arc of length 66 cm. (Use $\pi = \frac{22}{7}$)**

Ans. Given that, length of arc, l = 66 cm and central angle, $\theta = 30°$

$$\text{Angle } (\theta) = \left(30 \times \frac{\pi}{180}\right) \text{ radian}$$

$$\left[\because 1° = \frac{\pi}{180} \text{ radian}\right]$$

$$= \frac{\pi}{6} \text{ radian}$$

We know that

$$r = \frac{l}{\theta} \quad \text{(where } \theta \text{ is in radian)}$$

$$= \frac{66 \times 6}{\pi} = \frac{66 \times 6 \times 7}{22} = 126 \text{ cm}$$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

14. **The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$, find the third angle in the degree measure.**

Ans. As two angles are given as $\frac{1}{2}$ and $\frac{1}{3}$ radian so,

$$\frac{1}{3} \text{ radian} = \left(\frac{1}{2} \times \frac{180}{\pi}\right)^{\circ} = \left(\frac{1}{\alpha} \times \frac{180}{32} \times 7\right)^{\circ} = 28.66$$

$$\frac{1}{3} \text{ radian} = \left(\frac{1}{3} \times \frac{180}{\pi}\right)^{\circ} = \left(\frac{1}{3} \times \frac{180}{22} \times 7\right) = 19.10°$$

Now as we know that sum of all the angle can as

$28.66 + 19.10 + x = 180$

(Let the third angles be x)

$$x = 132.23° = 132°\left(\frac{23 \times 60}{90}\right)^{\circ} = 132°15.33'$$

$$x = 132°15'\,12.6''$$

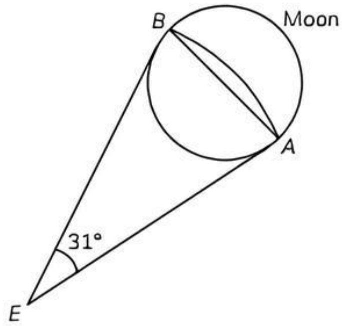
LONG ANSWER Type Questions (LA)

[4 & 5 marks]

15. **The moon's distance from the earth is 360,000 km and its diameter subtends an angle of 31' at the eye of the observer. Find the diameter of the moon.**

Ans. Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB are arc AB. Let d be the diameter of the moon. Then, d = arc AB.

We have,



$$\theta = 31' = \left(\frac{31}{60}\right)^{\circ} = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^{c}$$

and $r = 360000$ kms

$$\theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$$

$$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right) \text{km}$$

$$= 3247.62 \text{ kms}$$

Hence, the diameter of the moon is 3247.62 km.

TRIGONOMETRIC TRANSFORMATION FORMULAE 2

TOPIC 1

TRIGONOMETRIC FUNCTIONS

In earlier classes, we have studied trigonometric ratios (for acute angles) as the ratio of sides of a right-angled triangle. To recall, there are six trigonometric ratios defined as follows:

(i) $\sin\theta = \frac{P}{H}$

(ii) $\text{cosec}\,\theta = \frac{H}{P}$

(iii) $\cos\theta = \frac{B}{H}$

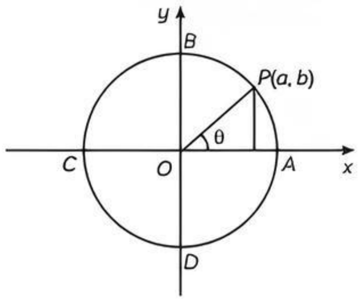
(iv) $\sec\theta = \frac{H}{B}$

(v) $\tan\theta = \frac{P}{B}$

(vi) $\cot\theta = \frac{B}{P}$

Where θ is the acute angle, P is the perpendicular, B is the base and H is the hypotenuse of the right-angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a system of coordinate axes with origin at O. Also, consider a unit circle with centre at O. For each real number θ, let P (a, b) be the point on the circle such that OP makes an angle θ (measured anticlockwise) with the positive direction of x – axis, as shown in figure.



We define,

Sine Function

(denoted by sin)as

$\sin\theta = y$ – coordinate of point P.

Cosine Function

(denoted by cos) as

$\cos\theta = x$ – coordinates of point P.

From figure, it is clear that the coordinates of the point A, B, C and D are (1, 0), (0, 1), (– 1, 0) and (0, –1).

We observe the following:

$\sin 0 = y$ – coordinate of point $A = 0$

$\cos 0 = x$ – coordinate of point $A = 1$

$\sin\frac{\pi}{2} = y$ – coordinate of point $B = 1$

$\cos\frac{\pi}{2} = x$ – coordinate of point $B = 0$

$\sin\pi = y$ – coordinates of point $C = 0$

$\cos\pi = x$ – coordinate of point $c = -1$

$\sin\frac{3\pi}{2} = y$ – coordinate of point $D = -1$

$\cos\frac{3\pi}{2} = x$ – coordinated of point $D = 0$

We observe that, y – coordinates of the points A and C are 0.

So, Sin $\theta = 0$. for $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

i.e., $\sin\theta = 0$, when θ is an integral multiple of π

Also, we observe that, x – coordinates of the points B and D are 0.

So, $\cos\theta = 0$, for $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{3}, \pm\frac{5\pi}{2}, \ldots$

i.e., $\cos\theta = 0$, when θ is an odd multiple of $\frac{\pi}{2}$

$\sin\theta = 0$

$\Rightarrow$ $\theta = n\pi$, where $n \in Z$

$\cos\theta = 0$

$\Rightarrow$ $\theta = (2n+1)\frac{\pi}{2}$, where $n \in Z$

We now define other trigonometric functions in terms of sine and cosine functions:

Cosecant Function

(denoted by cosec) defined as

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{y\text{ - coordinates of point } P'}$$

Where, $\theta \neq n\pi$ (form $\in Z$)

Secant Function

(denoted by sec) is defined as

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x\text{-coordinate of point } P'}$$

Where, $\theta \neq (2n+1)\frac{\pi}{2}$ (form $\in Z$).

Tangent Function

(denoted by tan) is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y\text{-coordinates of point } P}{x\text{-coordinates of point } P'}$$

Where, $\theta \neq (2n+1)\frac{\pi}{2}$ (form $\in Z$)

Cotangent Function

(denoted by cot) is defined as

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x\text{-coordinates of point}}{y\text{-coordinates of point}}$$

Where, $\theta \neq n\pi$ (form $\in Z$).

From above definitions, we can have the following table:

	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n.d.	0	n.d.	0
cosec	n.d.	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	n.d.	– 1	n.d.
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	n.d.	– 1	n.d.	1
cot	n.d.	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	n.d.	0	n.d.

Here, 'n.d.' means that the trigonometric function is not defined at that value.

We know that, one complete revolution subtends an angle of 2π radian at the centre of the circle. If we take one complete revolution from the point P, we again come back to the same point P. Thus, we observe that if θ increases (or decreases) by an integral multiple of 2π, the value of sine and cosine functions do not change.

Hence, $\sin(2n\pi + \theta) = \sin\theta$, for all $n \in Z$

And $\cos(2n\pi + \theta) = \cos\theta$, for all $n \in Z$

By definition of other four trigonometric functions, we have

$$\text{cosec}\,(2n\pi+\theta) = \frac{1}{\sin(2n\pi+\theta)} = \frac{1}{\sin\theta} = \text{cosec}\,\theta$$

$$\sec(2n\pi+\theta) = \frac{1}{\cos(2n\pi+\theta)} = \frac{1}{\cos\theta} = \cot\theta$$

$$\tan(2n\pi+\theta) = \frac{\sin(2n\pi+\theta)}{\cos(2n\pi+\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\cot(2n\pi+\theta) = \frac{\cos(2n\pi+\theta)}{\sin(2n\pi+\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Hence, we can summarise the results as follows:

$\sin(n \times 360° + \theta) = \sin\theta$ for all $n \in Z$

$\cos(n \times 360° + \theta) = \cos\theta$ for all $n \in Z$

$\text{cosec}(n \times 360° + \theta) = \text{cosec}\,\theta$ for all $n \in Z$

$\sec(n \times 360° + \theta) = \sec\theta$ for all $n \in Z$

$\tan(n \times 360° + \theta) = \tan\theta$ for all $n \in Z$

$\cot(n \times 360° + \theta) = \cot\theta$ for all $n \in Z$

Periods of Trigonometric Functions

We shall now understand the meaning of a periodic function. In simple words, a periodic function is a function that repeats its values in regular intervals or periods.

Periodic Function

A function $f : D \to R$ is said to be periodic if there exists a non-zero real number a such that $f(x + a) = f(x)$ for all $x \in D$.

Period of a Periodic Function

Let $f : D \to R$ be a periodic function. The least positive real number p such that $f(x + a) = f(x)$ for all $x \in D$ is called period of f.

From the discussion in the previous subsection, it is clear that all trigonometric functions are periodic functions with period 2π.

Signs of Trigonometric Functions

Let us now find the signs of the trigonometric functions for different values of θ in their respective domains. We observe that the signs of these functions depend on the quadrant in which θ lies.

For example, in I quadrant (i .e .0.< $\frac{\pi}{2}$) and II quadrant (i.e . $\frac{\pi}{2} < \theta < \pi$), the y-coordinate of point P is positive.

So, by definition, sin x is positive.

But, in III quadrant (i. e .$\pi < \theta < \frac{3\pi}{2}$) and IV quadrant (i.e., $\frac{3\pi}{2} < \theta < 2\pi$), the y-coordinate of point P is negative.

So, by definition, sin x is negative.

Similarly, we can find the signs of other trigonometric functions in different quadrants.

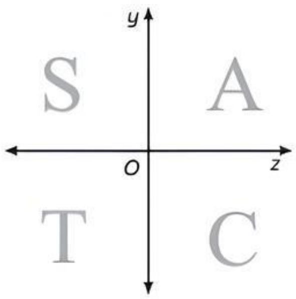
The final result has been summarised in the following table:

	I Quadrant	II Quadrant	III Quadrant	IV Quadrant
sin θ, cosec θ	+	+	–	–
tan θ, cot θ	+	+	+	–
cos θ, sec θ	+	–	–	+

One way to remember which functions are positive and which are negative in the various quadrants is to remember a simple four-letter acronym, ASTC.

This acronym can remind you that All are positive in the I quadrant, Since is positive in II quadrant, Tangent is positive in III quadrant and cosine is positive in IV quadrant.

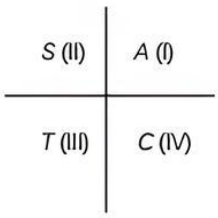
This acronym could stand for After School To College, or(Add Sugar To Coffee) some other four-word expression that will help you remember the relationships.



Example 2.1: **Find the values of other five trigonometric functions if sin $x = \frac{3}{5}$, x lies in the second quadrant.** **[NCERT]**

Ans. Since, x is in IInd quadrant.

So, sin will be positive but cos and tan will be negative.



Here, $\sin x = \frac{3}{5}$

We know that,

$$\sin^2 x + \cos^2 x = 1$$

$$\left(\frac{3}{5}\right)^2 + \cos^2 x = 1$$

$$\frac{9}{25} + \cos^2 x = 1$$

$$\cos^2 x = 1 - \frac{9}{25}$$

$$\cos^2 x = \frac{25-9}{25}$$

$$\cos^2 x = \frac{16}{25}$$

$$\cos x = \pm\sqrt{\frac{16}{25}}$$

$$\cos x = \pm\frac{4}{5}$$

As, x lies in the IInd quadrant and cos x is negative in IInd quadrant.

$$\therefore \quad \cos x = \frac{-4}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{\frac{-4}{5}}$$

$$= \frac{3}{5} \times \frac{5}{-4} = \frac{-3}{4}$$

$$\text{cosec } x = \frac{1}{\sin x} = \frac{1}{\frac{3}{5}}$$

$$= \frac{1 \times 5}{3} = \frac{5}{3}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\frac{-4}{5}}$$

$$= \frac{1 \times 5}{-4} = \frac{-5}{4}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{-3}{4}}$$

$$= \frac{-4}{3}$$

Example 2.2: **Find the value of the trigonometric function cosec (– 1410°).** **[NCERT]**

Ans. Given, cosec(– 1410°)

As, $\sin(-x) = -\sin x$,

So, $\text{cosec}(-x) = -\text{cosec } x$

$$\text{cosec}(-1410°) = -\text{cosec}(1410°)$$

$$= -\text{cosec}\left(1410 \times \frac{\pi}{180}\right)$$

$$= -\text{cosec } \frac{47\pi}{6}$$

$$= -\text{cosec}\left(7\frac{5}{6}\pi\right)$$

$$= -\text{cosec}\left(8\pi - \frac{1}{6}\pi\right)$$

$$= -\text{cosec}\left(4\times 2\pi - \frac{1}{6}\pi\right)$$

Values of cosec x repeats after an interval of 2π.

Hence, ignoring $4 \times (2\pi)$.

$$= -\text{cosec}\left(-\frac{1}{6}\pi\right)$$

$$= -\text{cosec}\left(-\frac{1}{6}\times 180^\circ\right)$$

$$= -\text{cosec}(-30^\circ)$$

$$= -(-\text{cosec } 30^\circ)$$

$$= \text{cosec } 30^\circ$$

$$= \frac{1}{\sin 30^\circ}$$

$$= \frac{1}{\left(\frac{1}{2}\right)}$$

$$= \frac{2}{1}$$

$$= 2$$

Example 2.3: Find the value of the trigonometric function $\tan \frac{19\pi}{3}$. **[NCERT]**

Ans. Given, trigonometric function is $\tan\frac{19\pi}{3}$

It can be rewritten as

$$= \tan 6\frac{1}{3}\pi$$

$$= \tan\left(6\pi + \frac{1}{3}\pi\right)$$

$$= \tan\left(3(2\pi) + \frac{1}{3}\pi\right)$$

Values of tan x repeats after an interval of 2π.

Hence, ignoring $3 \times (2\pi)$

$$= \tan\left(\frac{1}{3}\pi\right)$$

$$= \tan\left(\frac{1}{3}\times 180^\circ\right)$$

$$= \tan 60^\circ$$

$$= \sqrt{3} \qquad [\because \tan 60^\circ = \sqrt{3}]$$

Example 2.4: Find the value of the trigonometric function $\sin\left(\frac{11\pi}{3}\right)$. **[NCERT]**

Ans. Given, $\sin\left(-\frac{11\pi}{3}\right)$

It can be rewritten as

$$= -\sin\frac{11\pi}{3} \qquad [\text{As } \sin(-x) = -\sin x]$$

$$= -\sin\left(3\frac{2}{3}\pi\right)$$

$$= -\sin\left(4\pi - \frac{1}{3}\pi\right)$$

Values of $\sin x$ repeats after an interval of 2π.

Hence, ignoring 4π i.e., $2 \times (2\pi)$

$$= -\sin\left(\frac{-1}{3}\pi\right)$$

$$= -\left(-\sin\left(\frac{1}{3}\pi\right)\right)$$

$$[\text{As } \sin(-x) = -\sin x]$$

$$= \sin\left(\frac{1}{3}\pi\right)$$

$$= \sin\left(\frac{180^\circ}{3}\right)$$

$$= \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2}$$

Example 2.5: Find the values of other five trigonometric functions if $\sec x = \frac{13}{5}$, x lies in fourth quadrant. **[NCERT]**

Ans. Since, x lies in the IV[th] quadrant, cos will be positive. But sin and tan will be negative

We know that

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \tan^2 x = \left(\frac{13}{5}\right)^2$$

$$\tan^2 x = \left(\frac{13}{5}\right)^2 - 1$$

$$\tan^2 x = \frac{169}{25} - 1$$

$$\tan^2 x = \frac{169-25}{25}$$

$$\tan^2 x = \frac{144}{25}$$

$$\tan x = \pm\sqrt{\frac{144}{25}}$$

$$\tan x = \pm\frac{12}{5}$$

Since, x is in IVth quadrant.

tan x is negative in IVth quadrant.

$$\therefore \quad \tan x = \frac{-12}{5}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{-12}{5}}$$

$$= \frac{-5}{12}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\frac{13}{5}}$$

$$= \frac{5}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sin x = (\tan x) \times (\cos x)$$

$$= \frac{-12}{5} \times \frac{5}{13}$$

$$= \frac{-12}{13}$$

$$\therefore \quad \text{cosec } x = \frac{1}{\sin x} = \frac{-13}{12}$$

Example 2.6: Case Based:

In the school project Pankaj was asked to construct a triangle and name it as ABC.

Consider $\sin A = \frac{4}{5}$ *and* $\cos B = \frac{5}{13}$, *where* $0 < A$ *and* $B < \frac{\pi}{2}$.

Based on the above information answer the following questions.

(A) The value of $\cos A + \sin B$ is:

(a) $\frac{5}{13}$ (b) $\frac{99}{65}$

(c) $\frac{20}{65}$ (d) $\frac{9}{13}$

(B) The value of $\sin(A + B)$ is:

(a) $\frac{17}{13}$ (b) $\frac{29}{65}$

(c) $\frac{56}{65}$ (d) $\frac{-13}{17}$

(C) The value of $\cos(A + B)$ is:

(a) $-\frac{11}{19}$ (b) $-\frac{33}{65}$

(c) $-\frac{19}{11}$ (d) $-\frac{65}{33}$

(D) The value of $\sin(A - B)$ is:

(a) $\frac{-16}{65}$ (b) $\frac{-39}{40}$

(c) $\frac{-65}{16}$ (d) $\frac{-40}{39}$

(E) The value of $\cos(A - B)$ is:

(a) $\frac{65}{73}$ (b) $\frac{63}{65}$

(c) $\frac{73}{65}$ (d) $\frac{72}{65}$

Ans. Given, $\sin A = \frac{4}{5}, 0 < A < \frac{\pi}{2}$

$$\therefore \quad \cos A = \sqrt{1 - \sin^2 A}$$

[$\because$ A lies in 1st quadrant]

$$= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}}$$

$$\Rightarrow \quad \cos A = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

and $\cos B = \frac{5}{13}, 0 < B < \frac{\pi}{2}$

$$\therefore \quad \sin B = \sqrt{1 - \cos^2 B}$$

[$\because$ B lies in 1st quadrant]

$$= \sqrt{1 - \left(\frac{5}{13}\right)^2} = \sqrt{1 - \frac{25}{169}}$$

$$= \sqrt{\frac{144}{169}} = \frac{12}{13}$$

(A) *(b)* $\frac{99}{65}$

Explanation:

$$\cos A + \sin B = \frac{3}{5} + \frac{12}{13}$$

$$= \frac{39+60}{65} = \frac{99}{65}$$

(B) *(c)* $\frac{56}{65}$

Explanation:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$= \frac{4}{5} \times \frac{5}{13} \times \frac{12}{13} \times \frac{3}{5}$$

$$= \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

(C) *(b)* $\frac{-33}{65}$

Explanation:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{3}{5} \times \frac{5}{13} - \frac{4}{5} \times \frac{12}{13}$$

$$= \frac{15}{65} - \frac{48}{65} = \frac{-33}{65}$$

(D) *(a)* $\frac{-16}{55}$

Explanation:

$$\sin(A + B) = \sin A \cos B - \cos A \sin B$$

$$= \frac{4}{5} \times \frac{5}{13} - \frac{3}{5} \times \frac{12}{13}$$

$$= \frac{20}{65} - \frac{36}{65}$$

$$= \frac{-16}{65}$$

(E) *(b)* $\frac{63}{65}$

Explanation:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13}$$

$$= \frac{15}{65} + \frac{48}{65} = \frac{63}{64}$$

TOPIC 2

GRAPH OF TRIGONOMETRIC FUNCTION

Trigonometric Functions of Sum and Difference of Two Angles

Trigonometric Function of Some Allied Angles In Terms of θ

T-Ratios / Allied Angles	sin	cos	tan	cot	sec	cosec
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\text{cosec}\,\theta$
$\left(\frac{\pi}{2} \pm \theta\right)$	$\cos\theta$	$\pm\sin\theta$	$\pm\cot\theta$	$\pm\tan\theta$	$\pm\text{cosec}\,\theta$	$\sec\theta$
$(\pi \pm \theta)$	$\pm\sin\theta$	$-\cos\theta$	$\pm\tan\theta$	$\pm\cot\theta$	$\pm\sec\theta$	$\pm\text{cosec}\,\theta$
$\left(\frac{3\pi}{2} \pm \theta\right)$	$-\cos\theta$	$\pm\sin\theta$	$\mp\cot\theta$	$\mp\tan\theta$	$\mp\text{cosec}\,\theta$	$\mp\sec\theta$
$(2\pi \pm \theta)$	$\pm\sin\theta$	$\cos\theta$	$\pm\tan\theta$	$\pm\cot\theta$	$\sec\theta$	$\pm\text{cosec}\,\theta$

Trigonometric function of Compound Angles

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

7. $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

8. $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

Transformation Formulae

In this topic, we deal with mainly two types of transformation.

1. Transformation of product into sum or difference.
2. Transformation of sum or difference into product.

Transformation of Product into Sum or Difference

- $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
- $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
- $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
- $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Transformation of Sum or Difference into Product

- $\sin A+\sin B=2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\sin A-\sin B=2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
- $\cos A+\cos B=2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\cos A-\cos B=-2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

Trigonometric Functions of Multiple Angles

- $\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A$

$$=2\cos^2 A-1=\frac{1-\tan^2 A}{1+\tan^2 A}$$

- $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$
- $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A=2\sin A-4\sin^3 A$
- $\cos 3A=4\cos^3 A-3\cos A$
- $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Trigonmetric Functions of Sub-Multiple Angles

1. $\cos x=\cos^2\left(\dfrac{x}{2}\right)-\sin^2\left(\dfrac{x}{2}\right)$

$$=2\cos^2\left(\frac{x}{2}\right)-1$$

$$=1-2\sin^2\left(\frac{x}{2}\right)$$

$$\cos x=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

2. $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$

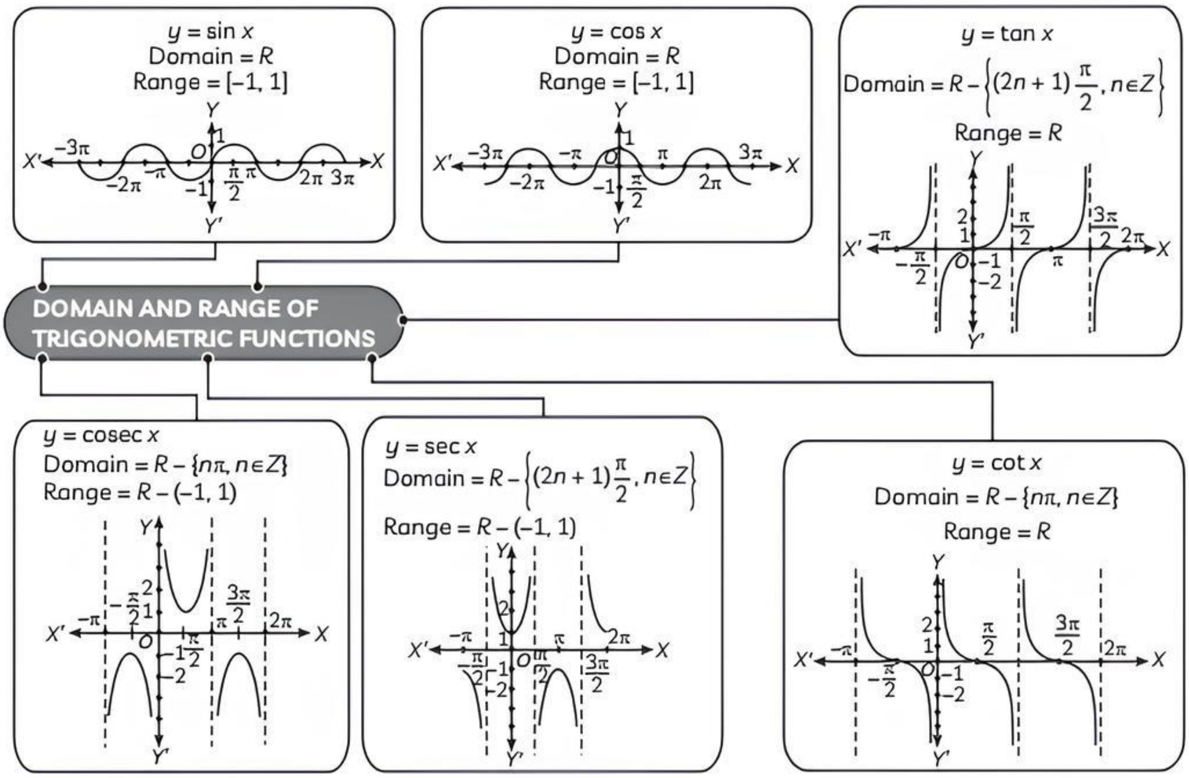
$$=\frac{2\tan\frac{x}{2}}{1+\tan^2\left(\frac{x}{2}\right)}$$

3. $\tan x=\dfrac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$

Trigonometric Ratios of Some Specific Angles

1. $\sin 18°=\cos 72°=\dfrac{\sqrt{5}-1}{4}$
2. $\cos 18°=\sin 72°=\dfrac{\sqrt{10+2\sqrt{5}}}{4}$
3. $\cos 36°=\sin 54°=\dfrac{\sqrt{5}+1}{4}$
4. $\sin 36°=\cos 54°=\dfrac{\sqrt{10-2\sqrt{5}}}{4}$

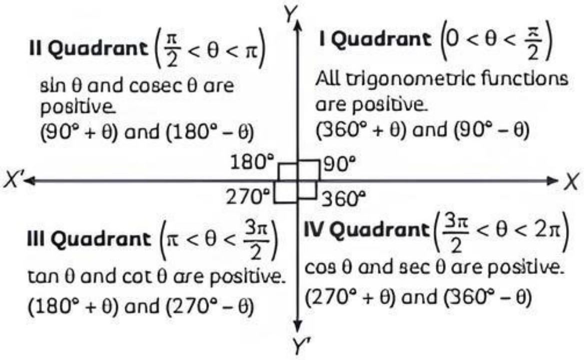
Domain and range of trigonometric functions



Value of T-Function for some Particular Angles

Angles \ T-Functions	sin	cos	tan	cot	sec	cosec
0	0	1	0	not defined	1	not defined
$\frac{\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{2}(\sqrt{3}-1)$	$\sqrt{2}(\sqrt{3}+1)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$\frac{\pi}{2}$	1	0	not defined	0	not defined	1
π	0	–1	0	not defined	–1	not defined
$\frac{3\pi}{2}$	–1	0	not defined	0	not defined	–1
2π	0	1`	0	not defined	1	not defined

Signs of Trigonometric Functions in Different Quadrants



Quadrant / T-Functions	I	II	III	IV
sin x	+	+	−	−
cos x	+	−	−	+
tan x	+	−	+	−
cot x	+	−	+	−
sec x	+	−	−	+
cosec x	+	+	−	−

Example 2.7: Prove that: $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$.

Ans. Given,

$$\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$$

$$\text{L.H.S} = \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$$

$$= \frac{2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}}{2\cos\frac{5x+3x}{2}\cos\frac{5x-3x}{2}}$$

$$\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \frac{\sin 4x \cos x}{\cos 4x \cos x} = \tan 4x = \text{R.H.S.}$$

Example 2.8: Prove that: $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan\dfrac{x-y}{2}$

Ans. $\text{L.H.S} = \dfrac{\sin x - \sin y}{\cos x + \cos y} = \dfrac{2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}}$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \quad \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \tan\frac{x-y}{2} = \text{R.H.S}$$

Hence, proved.

Example 2.9: Prove that: $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

Ans. $\text{L.H.S.} = \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \dfrac{\sin 3x + \sin x}{\cos 3x + \cos x}$

$$= \frac{2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}}{2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}}$$

$$\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \frac{\sin 2x \cos x}{\cos 2x \cos x} = \tan 2x = \text{R.H.S}$$

Hence, proved.

Example 2.10: Prove that: $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

Ans. $\text{L.H.S.} = \dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \dfrac{\sin 3x - \sin x}{\cos^2 - \sin^2 x}$

$$= \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{\cos 2x}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \text{ and } \cos^2 x - \sin^2 x = \cos 2x\right]$$

$$= \frac{2\cos 2x \sin x}{\cos 2x} = 2\sin x = \text{R.H.S}$$

Hence, proved.

Example 2.11: Prove that:

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x.$$

Ans. $\text{L.H.S.} = \dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$

$$= \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\cos 3x}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\sin 3x}$$

$$\left[\because \quad \cos C+\cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x}$$

$$= \frac{\cos 3x\,(2\cos x+1)}{\sin 3x(2\cos x+1)} = \cot 3x = \text{R.H.S.}$$

Hence, proved.

Example: 2.12: Prove that: $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

Ans. Take, $\cot 3x = \cot (2x + x)$

$$\frac{\cot 3x}{1} = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x}$$

$$\left[\because \quad \cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}\right]$$

$\Rightarrow$ $\cot 3x \cot 2x + \cot 3x \cot x = \cot 2x \cot x - 1$

$\Rightarrow$ $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

$\therefore$ L.H.S. = R.H.S

Example 2.13: Prove that:

$$\tan 4x = \frac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^2 x}$$

Ans. L.H.S. $= \tan 4x = \tan 2\,(2x) = \frac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\frac{2\tan x}{1-\tan^2 x}}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}$$

$$\left[\because \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}\right]$$

$$= \frac{4\tan x}{1-\tan^2 x} \times \frac{(1-\tan^2 x)^2}{(1-\tan^2 x)^2 - 4\tan^2 x}$$

$$= \frac{4\tan x\,(1-\tan^2 x)}{1+\tan^4 x - 2\tan^2 x - 4\tan^2 x}$$

$$= \frac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x} = \text{R.H.S.}$$

Hence, proved.

Example 2.14: Prove that:

$\cos 4x = 1 - 8 \sin^2 x \cos^2 x$.

Ans. L.H.S. $= \cos 4x = 1 - 2\sin^2 2x = 1 - 2(\sin 2x)^2$

$[\because \ \cos 2x = 1 - 2\sin^2 2x \cos^2]$

$= 1 - 2(2\sin x \cos x)^2$

$= 1 - 8\sin 2x \cos^2 x$ = R.H.S

$[\because \ \sin 2x = 2\sin x \cos x]$

Hence, proved.

Example 2.15: Prove that: $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Ans. L.H.S. $= \cos 6x = \cos 2\,(3x)$

$= 2\cos^2 3x - 1 = 2(\cos 3x)^2 - 1$

$(\because \ \cos 2\theta = 2\cos^2\theta - 1)$

$= 2(4\cos^3 x - 3\cos x)^2 - 1$

$(\because \ \cos 3\theta = 4\cos^3\theta - 3\cos\theta)$

$= 2(16\cos^6 x + 9\cos^2 x - 24\cos^4 x) - 1$

$= 32\cos^6 x - 48\cos^4 x + 18\cos 2x - 1$

= R.H.S.

Hence proved.

Example 2.16: If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta = y\cos\theta$ then find the value of $x^2 + y^2$:

Ans. Given, $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ ...(i)

And, $x\sin\theta = y\cos\theta$

$$\Rightarrow y = x\frac{\sin\theta}{\cos\theta} \quad \text{...(ii)}$$

Putting in (i), $x\sin^3\theta + x\frac{\sin\theta}{\cos\theta}\cdot\cos^3\theta = \sin\theta\cos\theta$

$\Rightarrow$ $x\sin^3\theta + x\sin\theta\cos^2\theta = \sin\theta\cos\theta$

$x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$x(\sin^2\theta + \cos^2\theta) = \cos\theta$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow$ $x = \cos\theta$

Putting the value of x in (ii), we get

$y = \sin\theta$

$\Rightarrow$ $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$

Example 2.17: Find the value of $\sec^2\theta + \text{cosec}^2\theta$.

Ans.

$$\sec^2\theta + \text{cosec}^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \text{cosec}^2\theta\sec^2\theta$$

Example 2.18: If $x \sin\frac{\pi}{4}\cos^2\frac{\pi}{3}$

$$= \frac{\tan^2\left(\frac{\pi}{3}\right)\text{cosec}\left(\frac{\pi}{6}\right)}{\sec\left(\frac{\pi}{4}\right)\cot^2\left(\frac{\pi}{6}\right)}$$ then, find the value of x.

Ans. $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \text{ cosec } 30°}{\sec 45° \cot^2 30°}$

$$x\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{4} = \frac{3.2}{\sqrt{2}.3}$$

$$\Rightarrow \quad x = 8$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. If sin θ + cosec θ = 2. then $\sin^2\theta + \text{cosec}^2\theta$ is equal to:

(a) 1 (b) 4
(c) 2 (d) None of these

[NCERT Exemplar]

Ans. *(c)* 2

Explanation: According to the question.

$\sin\theta + \text{cosec}\,\theta = 2$

On squaring both sides we get.

$\Rightarrow (\sin\theta + \text{cosec}\,\theta)^2 = 2^2$

$\Rightarrow \sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\,\text{cosec}\,\theta = 4$

$\Rightarrow \sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\left(\frac{1}{\sin\theta}\right) = 4$

$\Rightarrow \sin^2\theta + \text{cosec}^2\theta + 2 = 4$

$\Rightarrow \sin^2\theta + \text{cosec}^2\theta = 2$

2. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of θ + ϕ is:

(a) $\frac{\pi}{6}$ (b) π
(c) 0 (d) $\frac{\pi}{4}$

[NCERT Exemplar]

Ans. *(d)* $\frac{\pi}{4}$

Explanation: According to the question.

$\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$

We know that.

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

On putting the values, we get

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$\tan(\theta+\phi) = 1$

$\Rightarrow \tan(\theta+\phi) = \tan\frac{\pi}{4}$ $\left[\because \tan\frac{\pi}{4} = 1\right]$

$\Rightarrow \theta + \phi = \frac{\pi}{4}$

3. What is the value of tan 75° ?

(a) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$
(c) $\frac{\sqrt{3}}{\sqrt{3}+1}$ (d) $\frac{\sqrt{3}}{\sqrt{3}-1}$

[Delhi Gov. QB 2022]

Ans. *(b)* $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

Explanation: tan 75° = tan(45° + 30°)

Expand the above expression using identity $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$

So, tan 75° = tan(45° + 30°)

$$\tan 75° = \frac{\tan 45° + \tan 50°}{1-\tan 45°\tan 30°}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

4. What is the value of sin 75° ?

(a) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
(c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

[Delhi Gov. QB 2022]

Ans. *(c)* $\frac{\sqrt{3}+1}{2\sqrt{2}}$

Explanation:

$$\sin 75° = \sin(45° + 30°)$$

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Putting $x = 45°$ and $y = 30°$

$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

5. The value of $\tan\left(\frac{61\pi}{64}\right)$ is:

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$

(c) 1 (d) 0

Ans. *(a)* $\frac{1}{\sqrt{3}}$

Explanation: $\tan\left(\frac{61\pi}{64}\right) = \tan\left(10\pi + \frac{\pi}{6}\right)$

$$= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

6. If $\cos x = \frac{-1}{2}$ and $0 < x < 2\pi$, then the solutions are:

(a) $x = \frac{\pi}{3}, \frac{4\pi}{3}$ (b) $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

(c) $x = \frac{2\pi}{3}, \frac{7\pi}{3}$ (d) $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$ [Diksha]

Ans. *(b)* $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Explanation: Given, $\cos x = -\frac{1}{2}$

Then, $\cos x = -\cos\frac{\pi}{3}$

$$\cos x = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\cos x = \cos\frac{2\pi}{2}$$

$$x = \frac{2\pi}{3}$$

and $\cos x = \cos\left(\pi + \frac{\pi}{3}\right)$

$$\cos x = \cos\frac{4\pi}{3}$$

lie in $0 < x < 2\pi$.

$$x = \frac{4\pi}{3}$$

So, both lie in $0 < n < 2\pi$

Hence, $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

7. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is:

(a) 1 (b) $\sqrt{3}$

(c) $\frac{\sqrt{3}}{2}$ (d) 2

[Delhi Gov. QB 2022]

Ans. *(c)* $\frac{\sqrt{3}}{2}$

Explanation: Let $\theta = 15°$

$\Rightarrow 2\theta = 30°$

Now, since we know that,

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\Rightarrow \cos 30° = \frac{1-\tan^2 15°}{1+\tan^2 15°}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{1-\tan^2 15°}{1+\tan^2 15°}$$

8. Find the value of $\tan(\alpha + \beta)$, given that $\cot\alpha = \frac{1}{2}, \alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = \frac{-5}{3}, \beta \in \left(\frac{\pi}{2}, \pi\right)$.

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$

(c) $\frac{5}{11}$ (d) $\frac{3}{11}$

Ans. *(b)* $\frac{2}{11}$

Explanation: Given,

$\therefore$ $\tan\alpha = \frac{1}{\cot\alpha} = 2, \alpha$ is in III quadrant

and $\sec\beta = -\frac{5}{3}$

$\cos\beta = -\frac{3}{5}$, β is in III quadrant.

Let, $AB = \sqrt{(5)^2-(3)^2}$

$= \sqrt{25-9}$

$= 4$

Then, $\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{4/5}{-3/5} = \frac{-4}{3}$

$\Rightarrow \tan\beta = \pm\frac{4}{3}$

$\therefore$ β is in II quadrant.

But, $\tan\beta = \frac{-4}{3}$ [$\because$ tan β is –ve in II quadrant]

Now,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta} = \frac{2+\left(-\frac{4}{3}\right)}{1-(2)\left(\frac{-4}{3}\right)} = \frac{2}{11}$$

9. If sin θ + cos θ = 1, then the value of sin 2θ is equal to:

(a) 1 **(b) $\frac{1}{2}$**

(c) 0 **(d) 2**

[Delhi Gov. QB 2022]

Ans. *(c) 0*

Explanation: Given, sin θ + cos θ = 1

$\Rightarrow (\sin\theta + \cos\theta)^2 = (1)^2$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$\Rightarrow 1 + \sin 2\theta = 1$

$\Rightarrow \sin 2\theta = 1 - 1 = 0$

10. If $\cos x + \sqrt{3}\sin x = 2$, then the value of x is:

(a) $\frac{\pi}{3}$ **(b) $\frac{2\pi}{3}$**

(c) $\frac{4\pi}{3}$ **(d) $\frac{5\pi}{3}$** **[Diksha]**

Ans. *(a)* $\frac{\pi}{3}$

Explanation: Given, $\cos x + \sqrt{3}\sin x = 2$

$$\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = 1 \quad ...(i)$$

This equation is of the form $a\cos x + b\sin x = c$

Where $a = 1$, $b = \sqrt{3}$ and $c = 2$

Let $a = r\cos\alpha$ and $b = \sin\alpha$

Now,

$1 = r\cos\alpha$, $\sqrt{3} = r\sin\alpha$

$r = \sqrt{a^2+b^2} = \sqrt{1+3}$

$r = 2$

and, $\tan\alpha = \frac{b}{a}$

$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$

On putting $a = 1 = r\cos\alpha$

and $b = \sqrt{3} = r\sin\alpha$

In eq. (i) we get

$\Rightarrow r\cos x\cos\alpha + r\sin x\sin\alpha = 2$

$\Rightarrow r\cos(x-\alpha) = 2$

$2\cos\left(x-\frac{\pi}{3}\right) = 2$

$\cos\left(x-\frac{\pi}{3}\right) = 1$

$\cos\left(x-\frac{\pi}{3}\right) = \cos 0$

$x-\frac{\pi}{3} = 2n\pi \pm 0$

$x = 2n\pi \pm \frac{\pi}{3}$

for $n = 0$, $x = \frac{\pi}{3}$

$x = \frac{\pi}{3}$

11. The value of tan 75° – cot 75° is:

(a) $2\sqrt{3}$ **(b) $2+\sqrt{3}$**

(c) $2-\sqrt{3}$ **(d) 1** **[NCERT Exemplar]**

Ans. *(a)* $2\sqrt{3}$

Explanation: We have, tan 75° – cot 75°

$= \frac{\sin 75°}{\cos 75°} - \frac{\cos 75°}{\sin 75°}$

$= \frac{\sin^2 75° - \cos^2 75°}{\cos 75°\sin 75°}$

$= \frac{2(\sin^2 75° - \cos^2 75°)}{2\cos 75°\sin 75°}$

$= \frac{-2\cos 150°}{\sin 150°}$

$= 2\cot 150°$

$= -2\cot(180° - 30°)$

$= 2\cot 30°$

$= 2\sqrt{3}$

12. If $\alpha+\beta=\frac{\pi}{4}$, then value of (1 + tan α).(1 + tan β) is:

(a) 1 (b) 2

(c) –2 (d) Not defined

[Delhi Gov. QB 2022]

Ans. *(b) 2*

Explanation:

$$\tan(\alpha+\beta)=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1$$

$$\Rightarrow \tan\alpha+\tan\beta+\tan\alpha\tan\beta=1$$

$$\Rightarrow 1+\tan\alpha+\tan\beta+\tan\alpha\tan\beta=2$$

$$\Rightarrow (1+\tan\alpha)(1+\tan\beta)=2$$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

13. Assertion (A): Value of sin (– 270)° is 1.

Reason (R): sin (180° + θ) = – sin θ.

Ans. *(a) Both (A) and (R) are true and R is the correct explanation of (A).*

Explanation: sin(– 270°) = – sin (180° + 90°)

We know that

$$\sin(180°+\theta)=-\sin\theta$$

$$=(-\sin 90°)=1$$

14. Assertion (A): The value of $\theta=\frac{\pi}{3}$ or $\frac{2\pi}{3}$, when θ lies between (0, 2π) and $\sin^2\theta=\frac{3}{4}$.

Reason (R): sin θ is positive in the first and second quadrant.

Ans. *(d) (A) is false but (R) is true.*

Explanation: Given, $\sin^2\theta=\frac{3}{4}$

$$\Rightarrow \sin\theta=\frac{\sqrt{3}}{2} \text{ or } -\frac{\sqrt{3}}{2}.$$

Case I: When $\sin\theta=\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}$

$$\Rightarrow \sin\theta=\sin\frac{\pi}{3} \text{ or } \sin\left(\pi-\frac{\pi}{3}\right)$$

$$\Rightarrow \theta=\frac{\pi}{3} \text{ or } \pi-\frac{\pi}{3},$$

i.e., $\theta=\frac{\pi}{3}$ or $\frac{2\pi}{3}$

Case II: When $\sin\theta=-\frac{\sqrt{3}}{2}$, then θ lies either in the third or fourth quadrant.

Now, $\sin\theta=-\frac{\sqrt{3}}{2}=-\sin\frac{\pi}{3}$

$$=\sin\left(\pi+\frac{\pi}{3}\right) \text{ or } \sin\left(2\pi-\frac{\pi}{3}\right)$$

$$\theta=\pi+\frac{\pi}{3} \text{ or } 2\pi-\frac{\pi}{3}$$

$$\Rightarrow \theta=\frac{4\pi}{3} \text{ or } \frac{5\pi}{3}.$$

Hence, $\sin^2\theta=\frac{3}{4}$, $0<\theta<2\pi$

$$\Rightarrow \theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3},$$

15. Let sec θ + tan θ = m, where $0<m<1$.

Assertion (A): $\sec\theta=\frac{m^2+1}{2m}$ and $\sin\theta=\frac{m^2-1}{m^2+1}$

Reason (R): θ lies in the third quadrant.

Ans. *(c) (A) is true but (R) is false.*

Explanation: Given, sec θ + tan θ = m,

where, $0<m<1$ ...(i)

We know that, $\sec^2\theta-\tan^2\theta=1$ ...(ii)

dividing (ii) by (i), we get

$$\sec\theta-\tan\theta=\frac{1}{m} \quad ...(iii)$$

Note that $\frac{1}{m}>1$ ($\because$ $0<m<1$)

On adding (i) and (iii), we get

$$\sec\theta=\frac{m^2+1}{2m}>0$$

And subtracting (iii) from (i), we get

$$\tan\theta=\frac{m^2-1}{2m}<0$$

As sec θ > 0 and tan θ < 0

∴ θ lies in the fourth quadrant.

Also, $\sin\theta = \tan\theta\cos\theta = \dfrac{\tan\theta}{\sec\theta} = \dfrac{m^2 - 1/2m}{m^2 + 1/2m}$

$$= \frac{m^2 - 1}{m^2 + 1}$$

16. Let α be a real number lying between 0 and $\dfrac{\pi}{2}$ and n be a positive integer.

Assertion (A): $\tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Reason (R): $\cot\alpha - \tan\alpha = 2\cot 2\alpha$.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation:

Given, $\cot\alpha - \tan\alpha = \dfrac{1}{\tan\alpha} - \tan\alpha = \dfrac{1 - \tan^2\alpha}{\tan\alpha}$

$$= 2\left(\frac{1 - \tan^2\alpha}{2\tan\alpha}\right) = 2\cot 2\alpha$$

From here, we get $\tan\alpha = \cot\alpha - 2\cot 2\alpha$

Making repeated use of this identify, we shall obtain

$\tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha$

$= (\cot\alpha - 2\cot 2\alpha) + 2(\cot 2\alpha - 2\cot 2^2\alpha) + 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha) + \ldots + 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha) + 2^n\cot 2^n\alpha = \cot\alpha$

17. Assertion (A): The value of $\sin(-690°)\cos(-300°) + \cos(-750°)\sin(-240°) = 1$

Reason (R): The value of sin and cos is negative in the third and fourth quadrant respectively.

Ans. *(c) (A) is true but (R) is false.*

Explanation: $\sin(-690°) = -\sin 690°$

$$= -\sin(2 \times 360° - 30°)$$

$$= -(-\sin 30°) = \frac{1}{2}$$

$\cos(-300°) = \cos 300° = \cos(360° - 60°)$

$$= \cos 60° = \frac{1}{2}$$

$\cos(-750°) = \cos 750° = \cos(2 \times 360° + 30°)$

$$= \cos 30° = \frac{\sqrt{3}}{2}$$

$\sin(-240°) = -\sin 240° = -\sin(180° + 60°)$

$$= -(-\sin 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$\therefore$ $\sin(-690°)\cos(-300°) + \cos(-750°)\sin(-240°)$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1$$

18. If $A + B + C = 180°$, then

Assertion(A): $\cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} - \cos^2\dfrac{C}{2}$

$$= 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Reason (R): $\cos C + \cos D$

$$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Given, $\cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} - \cos^2\dfrac{C}{2}$

$$= \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} - \frac{1 + \cos C}{2}$$

$$= \frac{1 + (\cos A + \cos B - \cos C)}{2} \quad \text{...(i)}$$

Now, $\cos A + \cos B - \cos C$

$$= 2\cos\frac{A+B}{2}\cos\left(\frac{A-B}{2}\right) - \cos\left(2\cdot\frac{C}{2}\right)$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - \left(1 - 2\sin^2\frac{C}{2}\right)$$

$$\left[\because \cos\left(\frac{A+B}{2}\right) = \cos\left(90° - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)\right]$$

$$= 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2}\right)\right\} - 1$$

$$= -1 + 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right\}$$

$$= -1 + 4\sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) \quad \text{...(ii)}$$

From (i) and (ii), we get

L.H.S of the given identity

$$= \frac{1 + \left(-1 + 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)}{2}$$

$$= 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

19. *Sudhir who is a student of class XI got a Maths assignment from his class teacher.*
He did all the questions except a few. If the value of $\sin x = \frac{3}{5}$ and $\cos y = -\frac{12}{13}$, where x and y both lie in the second quadrant, then help Sudhir in solving these questions.

(A) What will be the value of $\cos x$?

(a) $\frac{4}{5}$ (b) $\frac{-3}{5}$

(c) $\frac{-4}{5}$ (d) $\frac{3}{5}$

(B) What will be the value of $\sin y$?

(a) $\frac{5}{12}$ (b) $\frac{-12}{13}$

(c) $\frac{-5}{13}$ (d) $\frac{5}{13}$

(C) Which of the following options is correct?

(a) $\sin(x - y) = \sin x \cos y + \cos x \sin y$

(b) $\sin(x + y) = \cos x \sin y - \sin x \cos y$

(c) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(d) $\sin(x - y) = \sin x \sin y - \cos x \cos y$

(D) The value of $\sin(x + y)$ is:

(a) $\frac{-56}{65}$ (b) $\frac{56}{65}$

(c) $\frac{55}{67}$ (d) $\frac{-55}{67}$

(E) The value of $\sin 75°$ is:

(a) $\frac{1-\sqrt{3}}{\sqrt{2}}$ (b) $\frac{1+\sqrt{3}}{2\sqrt{2}}$

(c) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (d) $\frac{1+\sqrt{3}}{2}$

Ans. (A) *(c)* $\frac{-4}{5}$

Explanation: Given, $\sin x = \frac{3}{5}$

As we know that

$$\cos^2 x = 1 - \sin^2 x$$

$$= 1 - \left(\frac{3}{5}\right)^2$$

$$= 1 - \frac{9}{25}$$

$$= \frac{25-9}{25} = \frac{16}{25}$$

Thus, $\cos x = \pm\frac{4}{5}$

Since x lies in second quadrant

∴ $\cos x$ is negative

∴ $\cos x = -\frac{4}{5}$

(B) *(d)* $\frac{5}{13}$

Explanation: Given,

$$\cos y = \frac{-12}{13}$$

As we know that

$$\sin^2 y = 1 - \cos^2 y$$

$$= 1 - \frac{144}{169} = \frac{25}{169} = \pm\frac{5}{13}$$

Since, y lies in second quadrant

∴ $\sin y$ is positive

∴ $\sin y = \frac{5}{13}$

(C) *(c) $\sin(x + y) = \sin x \cos y + \cos x \sin y$*

Explanation: Trigonometric function of compound angle

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

(D) (a) $\frac{-56}{65}$

Explanation:

As we know that

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\sin(x+y) = \left(\frac{3}{5}\right)\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\left(\frac{5}{13}\right)$$

$$= -\frac{36}{65}-\frac{20}{65} = -\frac{56}{65}$$

(E) (b) $\frac{1+\sqrt{3}}{2\sqrt{2}}$

Explanation: Given,

$\sin 75° = \sin(30° + 45°)$

$= \sin 30° \cos 45° + \cos 30° \sin 45°$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

20. **Consider $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$ where $0 < A, B < \frac{\pi}{2}$.**

(A) Find the value of $\cos A + \sin B$.

(B) Find value of $\cos(A + B)$.

(C) Find the value of $\sin(A - B)$.

Ans. (A) Given, $\sin A = \frac{4}{5}, 0 < A < \frac{\pi}{2}$

$\therefore \quad \cos A = \sqrt{1-\sin^2 A}$

[$\because$ A lies in 1st quadrant]

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$

$$\Rightarrow \quad \cos A = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

And $\quad \cos B = \frac{5}{13}, 0 < B < \frac{\pi}{2}$

$\therefore \quad \sin B = \sqrt{1-\cos^2 B}$

[$\because$ B lies in 1st quadrant]

$$= \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}}$$

$$\Rightarrow \quad \sin B = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

$$\cos A + \sin B = \frac{3}{5}+\frac{12}{13} = \frac{39+60}{65} = \frac{99}{65}$$

(B) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{3}{5}\times\frac{5}{13}-\frac{4}{5}\times\frac{12}{13}$$

$$= \frac{15-48}{65} = \frac{-33}{65}$$

(C) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{4}{5}\times\frac{5}{13}-\frac{3}{5}\times\frac{12}{13}$$

$$= \frac{20-36}{65} = \frac{-16}{65}$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

21. **If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$ then show that $a^2 + b^2 = m^2 + n^2$**

[NCERT Exemplar]

Ans. Given, $a\cos\theta + b\sin\theta = m$ (i)

and $\quad a\sin\theta - b\cos\theta = n$ (ii)

On squaring and adding of eqs (i) and (ii), we get

$m^2 + n^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$

$\Rightarrow m^2 + n^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$

$\Rightarrow m^2 + n^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$\Rightarrow m^2 + n^2 = a^2 + b^2$

Hence, proved.

22. **Find the value of $\sin(-240°)$.**

Ans. $\sin°(-240°) = -\sin 240°$

$= -\sin(180° + 60°)$

$= -(-\sin 60°)$

$= \sin 60°$

$$= \frac{\sqrt{3}}{2}$$

23. Find the value of tan 22° 30′.

[NCERT Exemplar]

Ans. Let, $22° 30' = \frac{\theta}{2}$

$\therefore \quad \theta = 45°$

$$\tan 22° 30' = \tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}$$

$$\therefore \quad \frac{\sin\theta}{1+\cos\theta} = \frac{\sin 45°}{1+\cos 45°}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}$$

$$= \frac{1\times(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

Hence, $\tan 22° 30' = \sqrt{2}-1$

24. Find the value of $\cos^2 75° - \cos^2 15°$.

Ans. Given, $\cos^2 75 - \cos^2 15 =$

$= (\cos 75° + \cos 15°) . (\cos 75° - \cos 15°)$

$= (2 \cos 90° . \cos 60°)(2 \sin 90° . \sin 60°)$

$$= \left(2.0.\frac{1}{2}\right)\left(2.1.\frac{\sqrt{3}}{2}\right)$$

$= 0$

25. Prove that $\sin (n + 1)x.\sin(n + 2)x + \cos(n + 1)x.\cos(n + 2)x = \cos x$.

[Delhi Gov. SQP 2022]

Ans. L.H.S. $= \sin (n + 1)x \sin(n + 2)x + \cos(n + 1)x \cos(n + 2)x = \cos x$

$= \cos[(n + 1)x - (n + 2)x]$

$[\because \cos (A - B) = \cos A \cos B + \sin A \sin B]$

$= \cos [nx + x - nx - 2x]$

$= \cos (-x) = \cos x =$ R.H.S.

$[\because \cos (-\theta) = \cos \theta]$

26. Find the value of 2 sin 45° . cos 45°.

Ans. We have,

$2 \sin 45° \cos 45°$

$$= 2\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}$$

$= 1$

27. If $\cos x + \sin x = \sqrt{2}$, then find the value of x.

Ans. $\cos x + \sin x = \sqrt{2}$

$$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = 1$$

$\sin 45° \cos x + \cos 45° \sin x = 1$

$\sin (45° + x) = 1$

$\sin (45° + x) = \sin 90°$

$45° + x = 90°$

$x = 45°$

$$x = \frac{\pi}{4}$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

28. Solve $\tan 4x = -\cot\left[x+\frac{\pi}{4}\right]$.

Ans. $\tan \theta = \cot [90° - \theta]$

$$\tan 4x = -\cot\left[x+\frac{n}{4}\right]$$

$$= \tan\left[\frac{\pi}{2}+x+\frac{\pi}{4}\right]$$

$$\tan 4x = \tan\left[x+\frac{3\pi}{4}\right]$$

$$4x = \left[x+\frac{3\pi}{4}\right]$$

$$4x = n\pi + x + \frac{3\pi}{4}$$

Where $n \in Z$.

$$3x = n\pi + \frac{3\pi}{4}$$

29. Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}$.

Ans. $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Substituting this in the above equation we get,

$$\frac{1+\tan^2\frac{\theta}{2}+2\tan\frac{\theta}{2}-1+\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}+2\tan\frac{\theta}{2}+1-\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \frac{2\tan\frac{\theta}{2}+2\tan^2\frac{\theta}{2}}{2+2\tan\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

30. If $m\sin\theta = n\sin(\theta+2a)$ then prove that $\tan(\theta+a)\cot a = \dfrac{m+n}{m-n}$ [NCERT Exemplar]

Ans. Given, $m\sin\theta = n\sin(\theta+2\alpha)$

$$\therefore \frac{\sin(\theta+2\alpha)}{\sin\theta}=\frac{m}{n}$$

Applying componendo and dividendo, we get

$$\frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta}=\frac{m+n}{m-n}$$

$$\Rightarrow \frac{2\sin\left(\frac{\theta+2\alpha+\theta}{2}\right)\cdot\cos\left(\frac{\theta+2\alpha-\theta}{2}\right)}{2\cos\left(\frac{\theta+2\alpha+\theta}{2}\right)\cdot\sin\left(\frac{\theta+2\alpha-\theta}{2}\right)}=\frac{m+n}{m-n}$$

$$\left[\begin{array}{l}\therefore\ \sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\ \textit{and}\\ \sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\end{array}\right]$$

$$\Rightarrow \frac{\sin(\theta+\alpha)\cdot\cos\alpha}{\cos(\theta+\alpha)\cdot\sin\alpha}=\frac{m+n}{m-n}$$

$$\Rightarrow \tan(\theta+\alpha)\cdot\cot\alpha=\frac{m+n}{m-n}$$

Hence, proved.

31. Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$.

[NCERT Exemplar]

Ans. L.H.S. $= \dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}$

$$= \frac{(\tan A+\sec A)-(\sec^2 A-\tan^2 A)}{(\tan A-\sec A+1)}$$

$[\therefore\ \sec^2 A-\tan^2 A=1]$

$$= \frac{\tan A+\sec A-\{(\sec A-\tan A)(\sec A+\tan A)\}}{(\tan A-\sec A+1)}$$

$[\therefore\ a^2-b^2=(a-b)(a+b)]$

$$= \frac{(\sec A+\tan A)(1-\sec A+\tan A)}{1-\sec A+\tan A}$$

$= \sec A+\tan A$

$$= \frac{1}{\cos A}+\frac{\sin A}{\cos A}=\frac{1+\sin A}{\cos A}=\text{R.H.S.}$$

Hence, proved.

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

32. Find the value of:

$$\sin\frac{\pi}{18}+\sin\frac{\pi}{9}+\sin\frac{2\pi}{9}+\sin\frac{5\pi}{18}.$$

Ans. $\sin\frac{\pi}{18}+\sin\frac{\pi}{9}+\sin\frac{2\pi}{9}+\sin\frac{5\pi}{18}$

$$=\sin\frac{\pi}{18}+\sin\frac{5\pi}{18}+\sin\frac{\pi}{9}+\sin\frac{2\pi}{9}$$

$$=2\sin\frac{\left(\frac{5\pi}{18}+\frac{\pi}{18}\right)}{2}\cos\frac{\left(\frac{5\pi}{18}-\frac{\pi}{18}\right)}{2}+2\sin\left[\frac{\left(\frac{2\pi}{9}+\frac{\pi}{9}\right)}{2}\right]\cos\left[\frac{\left(\frac{2\pi}{9}-\frac{\pi}{9}\right)}{2}\right]$$

[Using identity:

$\sin A+\sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$$=2\sin\frac{3\pi}{18}\cos\frac{2\pi}{18}+2\sin\frac{3\pi}{18}\cos\left(\frac{\pi}{18}\right)$$

$$=2\sin\frac{3\pi}{18}\left(\cos\frac{2\pi}{18}+\cos\frac{\pi}{18}\right)$$

$$=2\sin\frac{\pi}{6}\left(\cos\frac{\pi}{9}+\cos\frac{\pi}{18}\right)$$

$$=2\times\frac{1}{2}\left(\cos\frac{\pi}{9}+\cos\frac{\pi}{18}\right) \qquad \left[\because \sin\frac{\pi}{6}=\frac{1}{2}\right]$$

$$=\cos\frac{\pi}{9}+\cos\frac{\pi}{18}$$

$$=\sin\left(\frac{\pi}{2}-\frac{\pi}{9}\right)+\sin\left(\frac{\pi}{2}-\frac{\pi}{18}\right)$$

$$\left[\because \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\right]$$

$$=\sin\left(\frac{9\pi-2\pi}{18}\right)+\sin\left(\frac{9\pi-\pi}{18}\right)$$

$$=\sin\left(\frac{7\pi}{18}\right)+\sin\left(\frac{8\pi}{18}\right)$$

$$=\sin\left(\frac{7\pi}{18}\right)+\sin\left(\frac{4\pi}{9}\right)$$

33. If $2\sin^2\theta = 3\cos\theta$ where, $0 \le \theta \le 2\theta$, then find the value of θ. **[NCERT Exemplar]**

Ans. Given,
$$2\sin^2\theta = 3\cos\theta$$
$$2(1-\cos^2\theta) = 3\cos\theta$$
$$2-2\cos^2\theta-3\cos\theta = 0$$
$$2\cos^2\theta+3\cos\theta-2=0$$
$$2\cos^2\theta+4\cos\theta-\cos\theta-2=0$$
$$2\cos\theta(\cos\theta+2)-1(\cos\theta+2)=0$$
$$(\cos\theta+2)(2\cos\theta-1)=0$$

So, either $\cos\theta+2=0$ or $2\cos\theta-1=0$

But, $\cos\theta \ne -2$ $\quad [-1 \le \cos\theta \le 1]$

$2\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{2}$ and $\cos\theta = -2$ is not possible because $\cos\theta \ge -1$

$$\cos\theta = \cos\frac{\pi}{3} \text{ or } \cos\theta = \cos\left(2\pi-\frac{\pi}{3}\right)$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

34. **Prove that:**

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}.$$

Ans. L.H.S. $= \cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$

$$=\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\pi-\frac{3\pi}{8}\right)+\cos^4\left(\pi-\frac{\pi}{8}\right)$$

$$=\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$$

$$=2\left[\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)\right]$$

$$=2\left(\left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1+\cos\frac{3\pi}{4}}{2}\right)^2\right)$$

$$=\frac{2}{4}\left(\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right)=\frac{1}{2}(2+1)$$

$$=\frac{3}{2} = \text{R.H.S}$$

35. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$. **[NCERT Exemplar]**

Ans. Given, $\tan x = \frac{b}{a}$

$$\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}=\frac{\sqrt{(a+b)^2}+\sqrt{(a-b)^2}}{\sqrt{(a-b)(a+b)}}$$

$$=\frac{(a+b)+(a-b)}{\sqrt{a^2-b^2}}$$

$$=\frac{2a}{\sqrt{a^2-b^2}}$$

$$=\frac{2a}{a\sqrt{1-\left(\frac{b}{a}\right)^2}}$$

$$=\frac{2}{\sqrt{1-\tan^2 x}} \qquad \left[\because \frac{b}{a}=\tan x\right]$$

$$=\frac{2\cos x}{\sqrt{\cos^2 x-\sin^2 x}}$$

$$=\frac{2\cos x}{\sqrt{\cos 2x}} \qquad [\therefore \cos 2x = \cos^2 x - \sin^2 x]$$

36. Find the value of $m \sin x + n \cos x$, if $\tan \frac{x}{2} = \frac{m}{n}$.

Ans. We have, $\tan\frac{x}{2} = \frac{m}{n}$

and, $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2\frac{m}{n}}{1+\frac{m^2}{n^2}} = \dfrac{2mn}{m^2+n^2}$

$$\cos x = \sqrt{1-\left(\frac{2mn}{m^2+n^2}\right)^2} = \sqrt{\frac{(m^2+n^2)^2-4m^2n^2}{(m^2+n^2)^2}}$$

$$= \frac{m^2-n^2}{m^2+n^2}$$

$$\therefore\ m\sin x + n\cos x = m\left(\frac{2mn}{m^2+n^2}\right)+n\left(\frac{m^2-n^2}{m^2+n^2}\right)$$

$$= \frac{2m^2n}{m^2+n^2}+\frac{nm^2}{m^2+n^2}-\frac{n^3}{m^2+n^2}=\frac{3m^2n-n^3}{m^2+n^2}$$

37. Evaluate $\tan\left(\frac{13\pi}{12}\right)$. **[Delhi Gov. SQP 2022]**

Ans. $\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right)$

Since $(\pi + \theta)$ is in third quadrant. By ASTC rule $\tan\theta$ is positive in third quadrant.

$$\therefore\ \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12}$$

$$= \tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

Using compound angle formula,

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$= \frac{\tan\frac{\pi}{3}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$

We know that $\tan\frac{\pi}{4}=1$ and $\tan\frac{\pi}{3}=\sqrt{3}$ we have

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-2\sqrt{3}$$

38. If $\dfrac{2\sin a}{1+\cos a+\sin a} = y$, then prove that $\dfrac{1-\cos a+\sin a}{1+\sin a}$ is also equal to y.

[NCERT Exemplar]

Ans. Given, $\dfrac{2\sin a}{1+\cos a+\sin a} = y$

Now, $\dfrac{1-\cos a+\sin a}{1+\sin a}$

$$= \frac{(1-\cos a+\sin a)}{1+\sin a}\cdot\frac{(1+\cos a+\sin a)}{(1+\cos a+\sin a)}$$

$$= \frac{\{(1+\sin a)-\cos a\}}{1+\sin a}\cdot\frac{\{(1+\sin a)+\cos a\}}{1+\cos a+\sin a}$$

$$= \frac{(1+\sin a)^2-\cos^2 a}{(1+\sin a)(1+\cos a+\sin a)}$$

$$= \frac{(1+\sin^2 a+2\sin a)-\cos^2 a}{(1+\sin a)(1+\cos a+\sin a)}$$

$$= \frac{(1+\sin^2 a+2\sin a)-(1-\sin^2 a)}{(1+\sin a)(1+\cos a+\sin a)}$$

$$= \frac{1+\sin^2 a+2\sin a-1+\sin^2 a}{(1+\sin a)(1+\sin a+\cos a)}$$

$$= \frac{2\sin^2 a+2\sin a}{(1+\sin a)(1+\cos a+\sin a)}$$

$$= \frac{2\sin a(\sin a+1)}{(1+\sin a)(1+\cos a+\sin a)}$$

$$= \frac{2\sin a}{1+\cos a+\sin a} = y$$

Hence, proved.

39. $\cos 46° \cos 14° - \sin 46° \sin 14° + \sin 75°$.

Ans. We know, $\cos(A + B) = \cos A\cos B - \sin A\sin B$

So using this formula,

$\cos 46°\cos 14° - \sin 46°\sin 14°$

$= \cos(46° + 14°)$

$= \cos 60°$

And $\cos 60° = \frac{1}{2}$ ...(i)

Now, $\sin 75° = \sin (45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$[\because \sin (A + B) = \sin A \cos B + \cos A \sin B]$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$\left[\because \sin 30° = \frac{1}{2}, \cos 30° = \frac{\sqrt{3}}{2} \text{ and } \sin 45° = \cos 45° = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad ...(ii)$$

On adding eq. (i) and eq. (ii), we get

$\cos 47° \cos 13° - \sin 47° \sin 13° + \sin 75°$

$$= \frac{1}{2} + \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{1}{2} + \frac{(\sqrt{3}+1) \times \sqrt{2}}{2\sqrt{2} \times \sqrt{2}}$$

$$= \frac{1}{2} + \frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

40. **If $a \cos \theta + b \sin \theta = m$ and $a \sin \theta - b \cos \theta = n$, then show that $a^2 + b^2 = m^2 + n^2$.** **[NCERT Exemplar]**

Ans. Given, $a \cos \theta + b \sin \theta = m$ ...(i)

And $a \sin \theta - b \cos \theta = n$ ...(ii)

Squaring and adding of eqs (i) and (ii), we get

$m^2 + n^2 = (a \cos \theta + b \sin \theta)^2 + (a \sin \theta - b \cos \theta)^2$

$\Rightarrow m^2 + n^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta + 2\, ab \cos \theta \sin \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta - 2\, ab \sin \theta \cos \theta$

On solving

$\Rightarrow m^2 + n^2 = a^2 (\cos^2 \theta + \sin^2 \theta) + b^2 (\sin^2 \theta + \cos^2 \theta)$

$\Rightarrow m^2 + n^2 = a^2 + b^2$

Hence, proved.

41. **Prove that: $\sin a + \sin b + \sin c - \sin (a + b + c) = 4 \sin \frac{a+b}{2} \sin \frac{b+c}{2} \sin \frac{a+c}{2}$**

Ans. L.H.S = $(\sin a + \sin b) + (\sin c - \sin (a + b + c))$

$\sin a + \sin b + \sin c - \sin (a + b + c)$

$$= \left(2 \sin \frac{(a+b)}{2} \cos \frac{(a-b)}{2}\right) + \left(2 \cos \frac{(c+a+b+c)}{2} \sin \frac{(c-a-b-c)}{2}\right)$$

$$= 2 \sin \frac{(a+b)}{2} \cos \frac{(a-b)}{2} - 2 \cos \frac{(a+b+2c)}{2} \sin \frac{(a+b)}{2}$$

$$= 2 \sin \frac{(a+b)}{2} \left[\cos \frac{(a-b)}{2} - \cos \frac{(a+b+2c)}{2}\right]$$

$$= 2 \sin \frac{(a+b)}{2} \left[\left(-2 \sin \frac{(a-b+a+b+2c)}{2}\right) \sin \frac{(a-b-a-b-2c)}{4}\right]$$

$$= 2 \sin \left(\frac{a+b}{2}\right) \left[2 \sin \frac{(a+c)}{2} \sin \frac{(b+c)}{2}\right]$$

$$= 4 \sin \left(\frac{a+b}{2}\right) \sin \left(\frac{a+c}{2}\right) \sin \left(\frac{b+c}{2}\right) = \text{R.H.S.}$$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

42. **If $\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$, then show that $\sin \alpha + \cos \alpha = \sqrt{2} \cos \theta$.** **[NCERT Exemplar]**

Ans. Given, $\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$

Dividing by $\cos \alpha$

$$\tan \theta = \frac{\frac{\sin \alpha}{\cos \alpha} - \frac{\cos \alpha}{\cos \alpha}}{\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\cos \alpha}}$$

$$\Rightarrow \tan \theta = \frac{\tan \alpha - 1}{\tan \alpha + 1}$$

$$\Rightarrow \tan \theta = \frac{\tan \alpha - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{4} \tan \alpha} \quad [\because \tan \pi/4 = 1]$$

$$\Rightarrow \tan \theta = \tan \left(\alpha - \frac{\pi}{4}\right)$$

$$\Rightarrow \theta = \alpha - \frac{\pi}{4}$$

$\Rightarrow \quad \alpha = \theta + \frac{\pi}{4}$

$\therefore$ L.H.S. = $\sin\alpha + \cos\alpha$

$= \sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right)$

$= \left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right) + \left(\cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\right)$

$= \left(\sin\theta\cdot\frac{1}{\sqrt{2}} + \cos\theta\cdot\frac{1}{\sqrt{2}}\right) + \left(\cos\theta\frac{1}{\sqrt{2}} - \sin\theta\frac{1}{\sqrt{2}}\right)$

$\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}\right]$

$= \frac{1}{\sqrt{2}}[(\sin\theta + \cos\theta) + (\cos\theta - \sin\theta)]$

$= \frac{2\cos\theta}{\sqrt{2}}$

$= \sqrt{2}\cos\theta$ = R.H.S.

43. If $x = \sec\phi - \tan\phi$ and $y = \text{cosec}\,\phi + \cot\phi$, then show that $xy + x - y + 1 = 0$.

[NCERT Exemplar]

Ans. Given, $x = \sec\phi - \tan\phi$ and $y = \text{cosec}\,\phi + \cot\phi$

L.H.S. = $xy + x - y + 1$

$= (\sec\phi - \tan\phi)(\text{cosec}\,\phi + \cot\phi) + (\sec\phi - \tan\phi) - (\text{cosec}\,\phi + \cot\phi) + 1$

$= \left(\frac{1}{\cos\phi} - \frac{\sin\phi}{\cos\phi}\right)\left(\frac{1}{\sin\phi} + \frac{\cos\phi}{\sin\phi}\right) + \left(\frac{1}{\cos\phi} - \frac{\sin\phi}{\cos\phi}\right) - \left(\frac{1}{\sin\phi} + \frac{\cos\phi}{\sin\phi}\right) + 1$

$= \left(\frac{1-\sin\phi}{\cos\phi}\right)\left(\frac{1+\cos\phi}{\sin\phi}\right) + \left(\frac{1-\sin\phi}{\cos\phi}\right) - \left(\frac{1+\cos\phi}{\sin\phi}\right) + 1$

$= \frac{1 - \sin\phi + \cos\phi - \sin\phi\cos\phi}{\cos\phi\sin\phi} + \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi + \sin\phi\cos\phi}{\sin\phi\cos\phi}$

$= \frac{1 - \sin^2\phi - \cos^2\phi}{\cos\phi\sin\phi}$

$= \frac{1-1}{\cos\phi\sin\phi} = 0$

= RHS

44. If $\tan x = \frac{5}{12}$, $\pi < x < \frac{3\pi}{4}$ find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Ans. Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative

$\left[\pi - \frac{3\pi}{2}\right.$ shows third quadrant]

Also,

$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$ [Second quadrant]

Therefore, $\sin\frac{x}{2}$ is positive in second quadrants whereas $\cos\frac{x}{2}$ is negative.

Now,

$\sec^2 x = 1 + \tan^2 x$

$\sec^2 x = 1 + \left(\frac{5}{12}\right)^2$

$= 1 + \frac{25}{144}$

$= \frac{144+25}{144} = \frac{169}{144}$

$\therefore \quad \sec^2 x = \frac{169}{144} = \pm\frac{13}{12}$

And $\quad \cos x = \frac{-12}{13}$

[As in third and second quadrant cos is negative]

Now,

$2\sin^2\frac{x}{2} = 1 - \cos x$

$2\sin^2\frac{x}{2} = 1 - \left[\frac{-12}{13}\right]$

$2\sin^2\frac{x}{2} = \frac{25}{13}$

$\sin^2\frac{x}{2} = \frac{25}{26}$

$\sin\frac{x}{2} = \frac{5}{\sqrt{26}}$

[In second quadrant sin is positive]

$2\cos^2\frac{x}{2} = 1 + \cos x$

$= 1 - \frac{12}{13} = \frac{1}{13}$

$$\cos^2\frac{x}{2}=\frac{1}{26}$$

$$\cos\frac{x}{2}=\frac{-1}{\sqrt{26}}$$

Hence, $$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\frac{5}{\sqrt{26}}}{\frac{-1}{\sqrt{26}}}$$

$$\tan\frac{x}{2}=-5$$

45. Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, $\tan\frac{x}{2}$, when $\tan x=\frac{-4}{3}$, x lies in quadrant II. **[Diksha]**

Ans. Here, x is in quadrant II

i.e., $\frac{\pi}{2}<x<\pi$

$\Rightarrow \frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{2}$

Therefore, $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$

all lie in the first quadrant.

It is given that $\tan x=-\frac{4}{3}$.

$$\sec^2 x = 1+\tan^2 x$$

$$=1+\left(\frac{-4}{3}\right)^2=1+\frac{16}{9}=\frac{25}{9}$$

$\therefore$ $$\sec^2 x=\frac{25}{9}=\pm\frac{5}{3}$$

$\Rightarrow$ $$\cos x=\pm\frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

Thus, $$\cos x=\frac{-3}{5}$$

Now, $$\cos x=2\cos^2\frac{x}{2}-1$$

$\Rightarrow$ $$\frac{-3}{5}=2\cos^2\frac{x}{2}-1$$

$\Rightarrow$ $$2\cos^2\frac{x}{2}=1-\frac{3}{5}$$

$\Rightarrow$ $$2\cos^2\frac{x}{2}=\frac{2}{5}$$

$\Rightarrow$ $$\cos^2\frac{x}{2}=\frac{1}{5}$$

$\Rightarrow$ $$\cos\frac{x}{2}=\frac{1}{\sqrt{5}}\quad\left[\because \cos\frac{x}{2}\text{ is positive}\right]$$

$\therefore$ $$\cos\frac{x}{2}=\frac{\sqrt{5}}{5}\quad[\because \text{On rationalising}]$$

$$\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1$$

$\Rightarrow$ $$\sin^2\frac{x}{2}+\left(\frac{1}{\sqrt{5}}\right)^2=1$$

$\Rightarrow$ $$\sin^2\frac{x}{2}=1-\frac{1}{5}=\frac{4}{5}$$

$\Rightarrow$ $$\sin\frac{x}{2}=\frac{2}{\sqrt{5}}\quad\left[\because \sin\frac{x}{2}\text{ is positive}\right]$$

$\therefore$ $$\sin\frac{x}{2}=\frac{2\sqrt{5}}{5}$$

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)}=2$$

Thus, the respective value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{2\sqrt{5}}{5}$, $\frac{\sqrt{5}}{5}$, and 2.

46. $\cos 20°.\cos 40°.\cos 80° = \frac{1}{8}$. **[Delhi Gov. SQP 2022]**

Ans. L.H.S. $=\cos 20°\cos 40°\cos 80°$

$$=\frac{1}{2}[2\cos 20°\cos 40°]\cos 80°$$

$$=\frac{1}{2}[\cos(20°+40°)+\cos(20°-40°)]\cos 80°$$

$$=\frac{1}{2}[\cos 60°+\cos(-20°)]\cos 80°$$

$$=\frac{1}{2}\cos 80°\left[\frac{1}{2}+\cos 20°\right]$$

$$=\frac{1}{4}\cos 80°+\frac{1}{2}\cos 80°\cos 20°$$

$$=\frac{1}{4}\cos 80°+\frac{1}{4}[2\cos 80°\cos 20°]$$

$$=\frac{1}{4}\cos 80°+\frac{1}{4}[\cos(80°+20°)+\cos(80°-20°)]$$

$$= \frac{1}{4}\cos 80° + \frac{1}{4}[\cos 100° + \cos 60°]$$

$$= \frac{1}{4}\cos 80° + \frac{1}{4}\left[\cos(180° - 80°) + \frac{1}{2}\right]$$

$$= \frac{1}{4}\cos 80° - \frac{1}{4}\cos 80° + \frac{1}{8}$$

$[\because \cos(180° - 80°) = -\cos 80°]$

$$= \frac{1}{8} = \text{RHS}$$

47. Prove that $\cos\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2} = \sin 7\theta \sin 8\theta$. **[NCERT Exemplar]**

Ans. $\text{L.H.S.} = \cos\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2}$

$$= \frac{1}{2}\left[2\cos\theta\cos\frac{\theta}{2} - 2\cos 3\theta\cos\frac{9\theta}{2}\right]$$

$$= \frac{1}{2}\left[\cos\left(\theta + \frac{\theta}{2}\right) + \cos\left(\theta - \frac{\theta}{2}\right) - \left\{\cos\left(3\theta + \frac{9\theta}{2}\right) - \cos\left(3\theta - \frac{9\theta}{2}\right)\right\}\right]$$

$[\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$$= \frac{1}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2} - \cos\frac{15\theta}{2}\right)$$

$$= -\frac{1}{2}\left[2\sin\left(\frac{\theta + 15\theta}{2}\right)\sin\left(\frac{\theta - 15\theta}{2}\right)\right]$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= -\{\sin 8\theta \sin(-7\theta)\}$$

$= \sin 8\theta \sin 7\theta$ $[\because \sin(-\theta) = -\sin\theta]$

= R.H.S.

Hence, proved.

48. If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where α, β lie between 0 and $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$. **[NCERT Exemplar]**

Ans. Given, $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$

$$\Rightarrow \quad \sin(\alpha + \beta) = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$$\therefore \quad \sin(\alpha + \beta) = \frac{3}{5}$$

And $\cos(\alpha - \beta) = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \pm\frac{12}{13}$

$$\therefore \quad \cos(\alpha - \beta) = \frac{12}{13}$$

Now, $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$ $\left[\because 0 < \alpha < \frac{\pi}{4}\right]$

$$= \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

And $\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$

$$\therefore \quad \tan 2\alpha = \tan(\alpha + \beta + \alpha - \beta)$$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)}$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4}\cdot\frac{5}{12}} = \frac{\frac{(36+20)}{48}}{\frac{(48-15)}{48}} = \frac{56}{33}$$

49. Show that $\frac{1 + \sin\theta}{1 - \sin\theta} = \tan^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.

[Delhi Gov. SQP 2022]

Ans. $\text{L.H.S} = \frac{1 + \sin\theta}{1 - \sin\theta}$

$$\sin 2x = 2\sin x\cos x$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) = 1$$

$$\frac{1 + \sin\theta}{1 - \sin\theta}$$

$$= \frac{\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}$$

$$= \left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)^2$$

$$= \left[\frac{1+\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\right]^2$$

$$= \left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right]^2$$

$$\because \quad \tan\frac{\pi}{4} = 1$$

$$\frac{1+\sin\theta}{1-\sin\theta} = \left[\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right]^2$$

$$\left[\because \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$$\frac{1+\sin\theta}{1-\sin\theta} = \left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]^2$$

$$= \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Hence, proved.