Chapter **6**

APPLICATIONS OF DIFFERENTIATION

Learning Objectives

After studying this chapter, the students will be able to understand

- the concepts of demand and supply
- meaning and uses of cost, revenue and profit function
- average and marginal concepts
- elasticity of demand and supply
- relationship among average revenue, marginal revenue and elasticity of demand
- application of increasing and decreasing functions
- application of maxima and minima
- concept of Economic Order Quantity
- concepts of partial differentiation
- applications of partial derivatives in Economics
- partial elasticities of demand

Introduction


Leonhard Euler

Modern economic theory is based on both differential and integral calculus. In economics, differential calculus is used to compute marginal cost, marginal revenue, maxima and minima, elasticities, partial elasticities and also enabling economists to predict maximum profit (or) minimum loss in a specific condition. In this chapter, we will study about some important concepts and applications of differentiation in business and economics.

Euler's theorem in Economics

If factors of production function are paid as factors times of their marginal productivities, then the total factor payment is equal to the degree of homogeneity times the production function.

6.1 Applications of Differentiation in Business and Economics

In an economic situation, consider the variables are price and quantity. Let p be the unit price in rupees and x be the production (output / quantity) of a commodity demanded by the consumer (or) supplied by the producer.

6.1.1 Demand, supply, cost, revenue and profit functions

Demand function

In a market, the quantity of a commodity demanded by the consumer depends on its price. If the price of the commodity increases, then the demand decreases and if the price of the commodity decreases, then the demand increases.

The relationship between the quantity and the unit price of a commodity demanded by consumer is called as **demand function** and is defined as $x = f(p)$ or $p = f(x)$, where $x > 0$ and $p > 0$.

Graph of the demand function, $\boldsymbol{x = f(p)}$

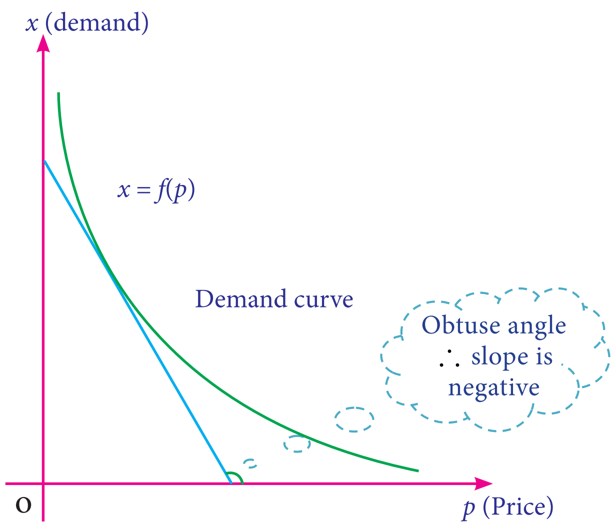


Fig : 6.1

DO YOU KNOW?

The "demand - price relationship" curve illustrates the negative relationship between price and quantity demanded.

Observations

(i) Price and quantity of the demand function are in inverse variation.

(ii) The graph of the demand function lies only in first quadrant.

(iii) Angle made by any tangent to the demand curve with respect to the positive direction of x – axis is always an obtuse angle.

(iv) Slope of the demand curve is negative(–ve).

Supply function

In a market, the quantity of a commodity supplied by producer depends on its price. If the price of the commodity increases, then quantity of supply increases and if the price of the commodity decreases, then quantity of supply decreases.

The relationship between the quantity and the unit price of a commodity supplied by producer is called as **supply function** and is defined as $x = g(p)$ or $p=g(x)$ where $x > 0$ and $p > 0$.

The graph of the supply function, $x= g(p)$

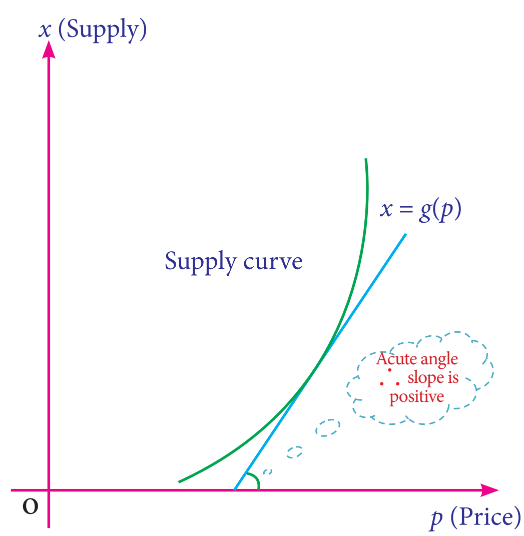


Fig : 6.2

DO YOU KNOW?

The "supply - price relationship" curve illustrates the positive relationship between price and quantity supplied.

Observations

(i) Price and quantity of the supply function are in direct variation.

(ii) The graph of supply function lies only in first quadrant.

(iii) Angle made by any tangent to the supply curve with respect to positive direction of x – axis is always an acute angle.

(iv) Slope of the supply curve is positive (+ve).

DO YOU KNOW?

The law of demand / supply tells us the direction of change, but not the rate at which the change takes place.

Equilibrium Price

The price at which the demand for a commodity is equal to its supply is called as Equilibrium Price and is denoted by p_E.

Equilibrium Quantity

The quantity at which the demand for a commodity is equal to its supply is called as Equilibrium Quantity and is denoted by x_E.

NOTE

Usually the demand and supply functions are expressed as x in terms of p, so the equilibrium quantity is obtained either from the demand function (or) from the supply function by substituting the equilibrium price.

Equilibrium Point

The point of intersection of the demand and supply function (p_E, x_E) is called as equilibrium point.

Diagrammatical explanation of equilibrium price, equilibrium quantity and equilibrium point

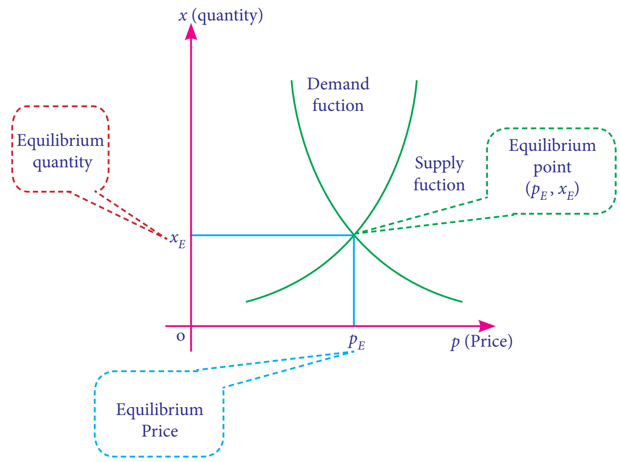


Fig: 6.3

Average and Marginal concepts

Usually, the variation in the dependent quantity 'y' with respect to the independent quantity 'x' can be described in terms of two concepts namely

(i) Average concept and

(ii) Marginal concept.

(i) Average concept

The average concept expressed as the variation of y over a whole range of x and is denoted by $\frac{y}{x}$.

(ii) Marginal concept

The marginal concept expressed as the instantaneous rate of change of y with respect to x and is denoted by $\frac{dy}{dx}$.

Remark:

If Δx be the small change in x and Δy be the corresponding change in y of the function $y=f(x)$, then

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$$

Instantaneous rate of change of y with respect to x is defined as the limiting case of ratio of the change in y to the change in x.

i.e. $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

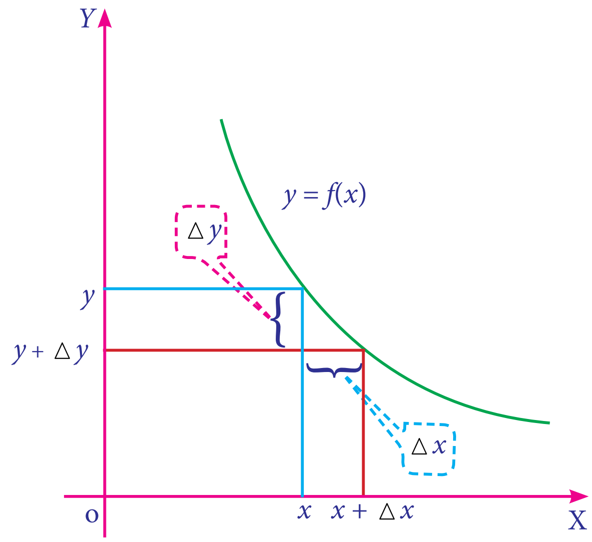


Fig : 6.4(a)

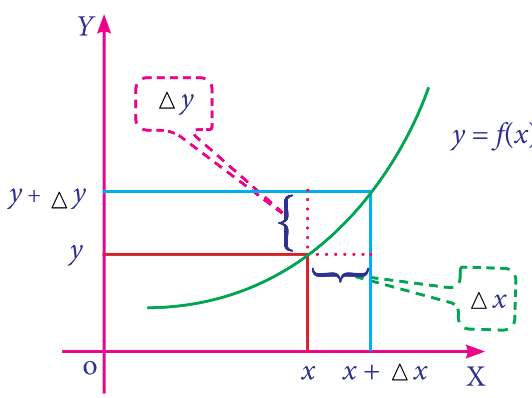


Fig : 6.4(b)

Cost function

The amount spent for the production of a commodity is called its cost function. Normally, total cost function [TC] consists of two parts.

(i) Variable cost; (ii) Fixed cost

Variable cost

Variable cost is the cost which varies almost in direct proportion to the volume of production.

Fixed cost

Fixed cost is the cost which does not vary directly with the volume of production.

If $f(x)$ be the variable cost and k be the fixed cost for production of x units, then total cost is $C(x) = f(x) + k$, $x > 0$.

NOTE

(i) Variable cost $f(x)$ is a single valued function.

(ii) Fixed cost k is independent of the level of output.

(iii) $f(x)$ does not contain constant term.

Some standard results

If $C(x) = f(x) + k$ be the total cost function, then

(i) Average cost:

$$AC = \frac{Total\ Cost}{Output} = \frac{C(x)}{x} = \frac{f(x)+k}{x}$$

(ii) Average variable cost:

$$AVC = \frac{Variable\ Cost}{Output} = \frac{f(x)}{x}$$

(iii) Average fixed cost:

$$AFC = \frac{Fixed\ Cost}{Output} = \frac{k}{x}$$

(iv) Marginal cost:

$$MC = \frac{dC}{dx} = \frac{d}{dx}[C(x)] = C'(x)$$

(v) Marginal average cost:

$$MAC = \frac{d}{dx}(AC)$$

(vi) Total cost:

TC = Average cost × output

(vi) Average cost [AC] is minimum, when $MC = AC$

Remark:

The marginal cost [MC] is approximately equal to the additional production cost of $(x+1)^{th}$ unit, when the production level is x units.

Diagrammatical explanation of marginal cost [MC]

Marginal cost is the change in aggregate cost when the volume of production is increased or decreased by one unit.

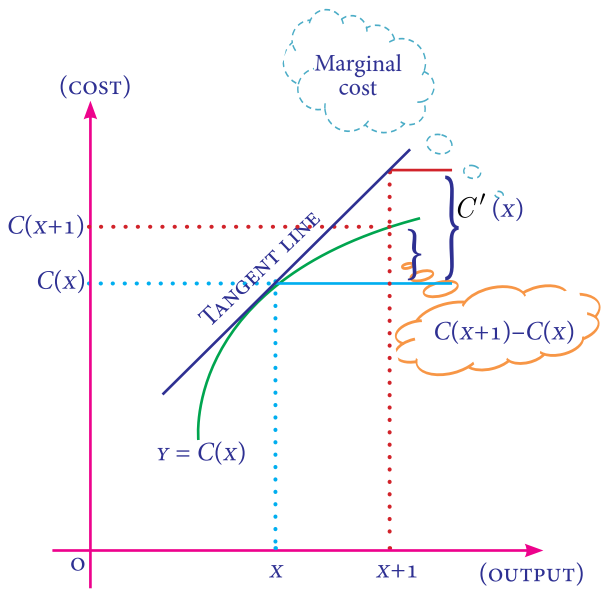


Fig: 6.5

Revenue function

Revenue is the amount realised on a commodity when it is produced and sold. If x is the number of units produced and sold and p is its unit price, then the total revenue function $R(x)$ is defined as $R(x) = px$, where x and p are positive.

DO YOU KNOW?

The revenue increases when the Producer, supplies more quantity at a higher price.

Some standard results

If $R(x) = px$ be the revenue function, then

(i) Average revenue:

$$AR = \frac{Total\ Revenue}{Output} = \frac{R(x)}{x} = p$$

(ii) Marginal revenue:

$$MR = \frac{dR}{dx} = \frac{d}{dx}(R(x)) = R'(x)$$

(iii) Marginal average revenue:

$$MAR = \frac{d}{dx}(AR) = AR'(x)$$

Remarks:

(i) Average revenue [AR] and price [p] are the same. [i.e. $AR=p$]

(ii) The marginal revenue [MR] is approximately equal to the additional revenue made on selling of $(x+1)^{th}$ unit, when the sales level is x units.

Diagrammatical explanation of Marginal Revenue [MR]

Marginal revenue is the change in aggregate revenue when the volume of selling unit is increased by one unit.

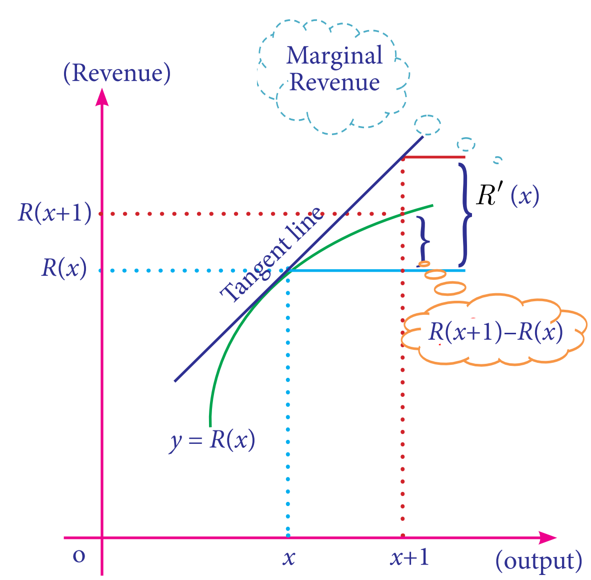


Fig : 6.6

Profit function

The excess of total revenue over the total cost of production is called the profit. If $R(x)$ is the total revenue and $C(x)$ is the total cost, then profit function $P(x)$ is defined as $P(x) = R(x) - C(x)$

Some standard results

If $P(x) = R(x) - C(x)$ be the profit function, then

(i) Average profit:

$$AP = \frac{Total\ Profit}{Output} = \frac{P(x)}{x}$$

(ii) Marginal profit:

$$MP = \frac{dP}{dx} = \frac{d}{dx}(P(x)) = P'(x)$$

(iii) Marginal average profit:

$$MAP = \frac{d}{dx}(AP) = AP'(x)$$

(iv) Profit $[P(x)]$ is maximum when $MR = MC$

6.1.2 Elasticity

Elasticity 'η' of the function $y = f(x)$ at a point x is defined as the limiting case of ratio of the relative change in y to the relative change in x.

$$\therefore \quad \eta = \frac{E_y}{E_x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\frac{dy}{y}}{\frac{dx}{x}}$$

$$\Rightarrow \quad \eta = \frac{x}{y} \cdot \frac{dy}{dx}$$

DO YOU KNOW?

$$\eta = \frac{\frac{dy}{dx}}{\frac{y}{x}} = \frac{\text{Marginal quantity of } y \text{ with respect to } x}{\text{Average quantity of } y \text{ with respect to } x}$$

(i) Price elasticity of demand

Price elasticity of demand is the degree of responsiveness of quantity demanded to a change in price.

If x is demand and p is unit price of the demand function $x = f(p)$, then the elasticity of demand with respect to the price is defined as $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$.

(ii) Price elasticity of supply :

Price elasticity of supply is the degree of responsiveness of quantity supplied to a change in price.

If x is supply and p is unit price of the supply function $x = g(p)$, then the elasticity of supply with respect to the price is defined as $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$.

NOTE

"price elasticity" is shortly called as "elasticity".

Some important results on price elasticity

(i) If $|\eta| > 1$, then the quantity demand or supply is said to be elastic.

(ii) If $|\eta| = 1$, then the quantity demand or supply is said to be unit elastic.

(iii) If $|\eta| < 1$, then the quantity demand or supply is said to be inelastic.

Remarks:

(i) Elastic : A quantity demand or supply is elastic when its quantity responds greatly to changes in its price. **Example: Cosumption of onion and its price.**

(ii) Inelastic: A quantity demand or supply is inelastic when its quantity responds very little to changes in its price. **Example: Consumption of rice and its price.**

(iii) Unit elastic : A quantity demand or supply is unit elastic when its quantity responds as the same ratio as changes in its price.

DO YOU KNOW?

Elasticity is predominantly used to assess the change in consumer demand as a result of a change in price.

Relationship among Marginal revenue [MR], Average revenue [AR] and Elasticity of demand[η_d].

We know that $R(x)=px$

i.e., $R = px$

and $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

Now, MR $= \frac{d}{dx}(R)$

$$= \frac{d}{dx}(px)$$

$$= p + x\frac{dp}{dx}$$

$$= p\left[1 + \frac{x}{p}\cdot\frac{dp}{dx}\right]$$

$$= p\left[1 + \frac{1}{\frac{p}{x}\cdot\frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{-\frac{p}{x}\cdot\frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{\eta_d}\right]$$

i.e. $MR = AR\left[1 - \frac{1}{\eta_d}\right]$ (or)

$$\eta_d = \frac{AR}{AR - MR}$$

Example 6.1

The total cost function for the production of x units of an item is given by $C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7$.

Find

(i) Average cost function

(ii) Average variable cost function

(iii) Average fixed cost function

(iv) Marginal cost function and

(v) Marginal Average cost function

Solution:

$$C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7$$

(i) Average cost:

$$AC = \frac{C}{x} = \frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}$$

(ii) Average variable cost:

$$AVC = \frac{f(x)}{x} = \frac{1}{3}x^2 + 4x - 25$$

(iii) Average fixed cost:

$$AFC = \frac{k}{x} = \frac{7}{x}$$

(iv) Marginal cost:

$$MC = \frac{dC}{dx} \text{ (or) } \frac{d}{dx}(C(x))$$

$$= \frac{d}{dx}\left[\frac{1}{3}x^3 + 4x^2 - 25x + 7\right]$$

$$= x^2 + 8x - 25$$

(v) Marginal Average cost:

$$MAC = \frac{d}{dx}[AC] = \frac{d}{dx}\left[\frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}\right]$$

$$= \frac{2}{3}x + 4 - \frac{7}{x^2}$$

Example 6.2

The total cost C in Rupees of making x units of a product is $C(x) = 50 + 4x + 3\sqrt{x}$. Find the marginal cost of the product at 9 units of output.

Solution:

$$C(x) = 50 + 4x + 3\sqrt{x}$$

Marginal cost $(MC) = \frac{dC}{dx} = \frac{d}{dx}[C(x)]$

$= \frac{d}{dx}\left[50 + 4x + 3\sqrt{x}\right] = 4 + \frac{3}{2\sqrt{x}}$

When $x = 9$, $\frac{dC}{dx} = 4 + \frac{3}{2\sqrt{9}} = 4\frac{1}{2}$ (or) ₹ 4.50

$\therefore$ MC is ₹ 4.50 , when the level of output is 9 units.

Example 6.3

Find the equilibrium price and equilibrium quantity for the following demand and supply functions.

Demand : $x = \frac{1}{2}(5 - p)$ and Supply : $x = 2p - 3$.

Solution:

At equilibrium, demand = supply

$\Rightarrow \quad \frac{1}{2}(5 - p) = 2p - 3$

$5 - p = 4p - 6$

$\Rightarrow p = \frac{11}{5}$

$\therefore$ Equilibrium price: $p_E = ₹\ \frac{11}{5}$

Now, put $p = \frac{11}{5}$ in $x = 2p - 3$

We get, $x = 2\left(\frac{11}{5}\right) - 3 = \frac{7}{5}$

$\therefore$ Equilibrium quantity: $x_E = \frac{7}{5}$ units.

Example 6.4

For the demand function $x = \frac{20}{p+1}$, p > 0, find the elasticity of demand with respect to price at a point $p = 3$. Examine whether the demand is elastic at $p = 3$.

Solution:

$x = \frac{20}{p+1}$

$\frac{dx}{dp} = \frac{-20}{(p+1)^2}$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$= -\frac{p}{\left(\frac{20}{(p+1)}\right)} \cdot \frac{-20}{(p+1)^2} = \frac{p}{p+1}$

When $p = 3$, $\eta_d = \frac{3}{4}$ (or) 0.75

Here $|\eta_d| < 1$

$\therefore$ demand is inelastic.

Example 6.5

Find the elasticity of supply for the supply function $x = 2p^2 - 5p + 1$, p> 3.

Solution:

$x = 2p^2 - 5p + 1$

$\frac{dx}{dp} = 4p - 5$

Elasticity of supply: $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$

$= \frac{p}{2p^2 - 5p + 1} \cdot (4p - 5) = \frac{4p^2 - 5p}{2p^2 - 5p + 1}$

Example 6.6

If $y = \frac{2x+1}{3x+2}$, then obtain the elasticity at $x = 1$.

Solution:

$y = \frac{2x+1}{3x+2}$

$\frac{dy}{dx} = \frac{(3x+2)(2) - (2x+1)(3)}{(3x+2)^2} = \frac{1}{(3x+2)^2}$

Elasticity: $\eta = \frac{x}{y} \cdot \frac{dy}{dx} = \frac{x}{\left(\frac{2x+1}{3x+2}\right)} \cdot \frac{1}{(3x+2)^2}$

$= \frac{x}{(2x+1)(3x+2)}$

When $x = 1$, $\eta = \frac{1}{15}$

Example 6.7

A demand function is given by $xp^n = k$ where n and k are constants. Prove that elasticity of demand is always constant.

Solution:

$xp^n = k \Rightarrow x = kp^{-n}$

$\frac{dx}{dp} = -nkp^{-n-1}$

Elasticity of demand : $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

$= -\frac{p}{kp^{-n}}(-nkp^{-n-1}) = n$, which is a constant.

Example 6.8

For the given demand function $p = 40-x$, find the output when $\eta_d = 1$

Solution:

$p = 40-x \Rightarrow x = 40-p$

$\frac{dx}{dp} = -1$

Elasticity of demand: $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp} = \frac{40-x}{x}$

Given that $\eta_d = 1$

$\therefore \frac{40-x}{x} = 1 \Rightarrow x = 20$

$\therefore$ output(x) = 20 units.

Example 6.9

Find the elasticity of demand in terms of x for the demand law $p = (a-bx)^{\frac{1}{2}}$. Also find the output(x) when elasticity of demand is unity.

Solution:

$p = (a-bx)^{\frac{1}{2}}$.

Differentiating with respect to the price 'p',

we get $1 = \frac{1}{2}(a-bx)^{\frac{-1}{2}}(-b)\cdot\frac{dx}{dp}$

$\therefore \frac{dx}{dp} = \frac{2(a-bx)^{\frac{1}{2}}}{-b}$

Elasticity of demand: $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

$= -\frac{(a-bx)^{\frac{1}{2}}}{x}\cdot\frac{2(a-bx)^{\frac{1}{2}}}{-b} = \frac{2(a-bx)}{bx}$

When $\eta_d = 1$, $\frac{2(a-bx)}{bx} = 1$

$2(a-bx) = bx \Rightarrow$ output$(x) = \frac{2a}{3b}$ units.

Example 6.10

Verify the relationship of elasticity of demand, average revenue and marginal revenue for the demand law $p = 50-3x$.

Solution:

$p = 50 - 3x$

$\frac{dp}{dx} = -3 \Rightarrow \frac{dx}{dp} = -\frac{1}{3}$

Elasticity of demand: $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

$= -\frac{50-3x}{x}\left(-\frac{1}{3}\right) = \frac{50-3x}{3x}$ ... (1)

Now, Revenue: $R = px$

$= (50-3x)x = 50x-3x^2$

Average revenue: $AR = p = 50-3x$

Marginal revenue: $MR = \frac{dR}{dx} = 50-6x$

$\therefore \frac{AR}{AR-MR} = \frac{50-3x}{(50-3x)-(50-6x)}$

$= \frac{50-3x}{3x}$ ... (2)

From (1) and (2), we get

$\eta_d = \frac{AR}{AR-MR}$, Hence verified.

Example 6.11

Find the elasticity of supply for the supply law $x = \frac{p}{p+5}$ when $p = 20$ and interpret your result.

Solution:

$x = \frac{p}{p+5}$

$$\frac{dx}{dp}=\frac{(p+5)-p}{(p+5)^2}=\frac{5}{(p+5)^2}$$

Elasticity of supply: $\eta_s=\frac{p}{x}\cdot\frac{dx}{dp}$

$$=(p+5)\frac{5}{(p+5)^2}=\frac{5}{p+5}$$

When $p=20$, $\eta_s=\frac{5}{20+5}=0.2$

Interpretation:

- If the price increases by 1% from $p=$ ₹ 20, then the quantity of supply increases by 0.2% approximately.
- If the price decreases by 1% from $p=$ ₹ 20, then the quantity of supply decreases by 0.2% approximately.

Example 6.12

For the cost function $C=2x\left(\frac{x+5}{x+2}\right)+7$, prove that marginal cost (MC) falls continuously as the output x increases.

Solution:

$$C=2x\left(\frac{x+5}{x+2}\right)+7=\frac{2x^2+10x}{x+2}+7$$

Marginal cost:

$$MC=\frac{dC}{dx}=\frac{d}{dx}\left[\frac{2x^2+10x}{x+2}+7\right]$$

$$=\frac{(x+2)(4x+10)-(2x^2+10x)}{(x+2)^2}$$

$$=\frac{2(x^2+4x+10)}{(x+2)^2}=\frac{2\left[(x+2)^2+6\right]}{(x+2)^2}$$

$$=2\left[1+\frac{6}{(x+2)^2}\right]$$

$\Rightarrow$ as x increases, MC decreases.

Hence proved.

Example 6.13

$\bar{C}=0.05x^2+16+\frac{100}{x}$ is the manufacturer's average cost function. What is the marginal cost when 50 units are produced and interpret your result.

Solution:

Total cost: $C=AC\times x=\bar{C}\times x$

$$=0.05x^3+16x+100$$

Marginal cost: $MC=\frac{dC}{dx}=0.15x^2+16$

$$\left(\frac{dC}{dx}\right)_{x=50}=0.15(50)^2+16$$

$$=375+16=₹\ 391$$

Interpretation:

If the production level is increased by one unit from $x=50$, then the cost of additional unit is approximately equal to ₹ 391.

Example 6.14

For the function $y=x^3+19$, find the values of x when its marginal value is equal to 27.

Solution:

$y=x^3+19$

$\frac{dy}{dx}=3x^2$ ... (1)

$\frac{dy}{dx}=27$ ... (2) [Given]

From (1) and (2), we get

$3x^2=27\Rightarrow x=\pm 3$

Example 6.15

The demand function for a commodity is $p=\frac{4}{x}$, where p is unit price. Find the instantaneous rate of change of demand with respect to price at $p=4$. Also interpret your result.

Solution:

$p = \frac{4}{x} \Rightarrow x = \frac{4}{p}$

$\therefore \frac{dx}{dp} = -\frac{4}{p^2}$

At $p = 4$, $\frac{dx}{dp} = -\frac{1}{4} = -0.25$

$\therefore$ Rate of change of demand with respect to the price at $p =$ ₹ 4 is -0.25.

Interpretation:

When the price increases by 1% from the level of $p =$ ₹ 4, the demand decreases (falls) by 0.25%.

Example 6.16

The demand and the cost function of a firm are $p = 497 - 0.2x$ and $C = 25x + 10000$ respectively. Find the output level and price at which the profit is maximum.

Solution:

We know that profit[P(x)] is maximum when marginal revenue [MR] = marginal cost [MC].

Revenue: $R = px$

$= (497 - 0.2x)x = 497x - 0.2x^2$

$MR = \frac{dR}{dx} = 497 - 0.4x$

Cost: $C = 25x + 10000$

$\therefore MC = 25$

$MR = MC \Rightarrow 497 - 0.4x = 25$

$\Rightarrow 472 - 0.4x = 0$

$\Rightarrow x = 1180$ units.

Now, $p = 497 - 0.2x$

at $x = 1180$, $p = 497 - 0.2(1180) =$ ₹ 261.

Example 6.17

The cost function of a firm is $C = \frac{1}{3}x^3 - 3x^2 + 9x$. Find the level of output $(x > 0)$ when average cost is minimum.

Solution:

We know that average cost [AC] is minimum when average cost [AC] = marginal cost [MC].

Cost: $C = \frac{1}{3}x^3 - 3x^2 + 9x$

$\therefore AC = \frac{1}{3}x^2 - 3x + 9$ and $MC = x^2 - 6x + 9$

Now, $AC = MC \Rightarrow \frac{1}{3}x^2 - 3x + 9 = x^2 - 6x + 9$

$\Rightarrow 2x^2 - 9x = 0 \Rightarrow x = \frac{9}{2}$ units. $[\because x > 0]$

Exercise 6.1

1. A firm produces x tonnes of output at a total cost of $C(x) = \frac{1}{10}x^3 - 4x^2 - 20x + 7$. Find the
   (i) average cost function
   (ii) average variable cost function
   (iii) average fixed cost function
   (iv) marginal cost function and
   (v) marginal average cost function.
2. The total cost of x units of output of a firm is given by $C = \frac{2}{3}x + \frac{35}{2}$. Find the
   (i) cost, when output is 4 units
   (ii) average cost, when output is 10 units
   (iii) marginal cost, when output is 3 units
3. Revenue function 'R' and cost function 'C' are $R = 14x - x^2$ and $C = x(x^2 - 2)$. Find the (i) average cost function (ii) marginal cost function, (iii) average revenue function and (iv) marginal revenue function.
4. If the demand law is given by $p = 10e^{-\frac{x}{2}}$, then find the elasticity of demand.

5. Find the elasticity of demand in terms of x for the following demand laws and also find the output (x), when the elasticity is equal to unity
(i) $p=(a-bx)^2$ (ii) $p=a-bx^2$

6. Find the elasticity of supply for the supply function $x=2p^2+5$ when p=3

7. The demand curve of a commodity is given by $p=\frac{50-x}{5}$, find the marginal revenue for any output x and also find marginal revenue at $x=0$ and $x=25$?

8. The supply function of certain goods is given by $x=a\sqrt{p-b}$ where p is unit price, a and b are constants with $p>b$. Find the elasticity of supply at $p=2b$.

9. Show that $MR=p\left[1-\frac{1}{\eta_d}\right]$ for the demand function $p=400-2x-3x^2$ where p is unit price and x is quantity demand.

10. For the demand function $p=550-3x-6x^2$ where x is quantity demand and p is unit price. Show that $MR=p\left[1-\frac{1}{\eta_d}\right]$

11. For the demand function $x=\frac{25}{p^4}, 1\le p\le 5$, determine the elasticity of demand.

12. The demand function of a commodity is $p=200-\frac{x}{100}$ and its cost is $C=40x+120$ where p is a unit price in rupees and x is the number of units produced and sold. Determine (i) profit function (ii) average profit at an output of 10 units (iii) marginal profit at an output of 10 units and (iv) marginal average profit at an output of 10 units.

13. Find the values of x, when the marginal function of $y=x^3+10x^2-48x+8$ is twice the x.

14. The total cost function y for x units is given by $y=3x\left(\frac{x+7}{x+5}\right)+5$. Show that the marginal cost [MC] decreases continuously as the output (x) increases.

15. Find the price elasticity of demand for the demand function $x=10-p$ where x is the demand and p is the price. Examine whether the demand is elastic, inelastic or unit elastic at $p=6$.

16. Find the equilibrium price and equilibrium quantity for the following functions.
Demand: $x=100-2p$ and supply: $x=3p-50$

17. The demand and cost functions of a firm are $x=6000-30p$ and $C=72000+60x$ respectively. Find the level of output and price at which the profit is maximum.

18. The cost function of a firm is $C=x^3-12x^2+48x$. Find the level of output $(x>0)$ at which average cost is minimum.

6.2 Maxima and Minima

We are using maxima and minima in our daily life as well as in every field such as chemistry, physics, engineering and in economics etc.,

In particular, we can use maxima and minima

(i) To maximize the beneficial values like profit, efficiency, output of a company etc.,

(ii) To minimize the negative values like, expenses, efforts etc.,

(iii) Used in the study of inventory control, economic order quantity etc.

6.2.1 Increasing and decreasing functions

Before learning the concept of maxima and minima, we will study the nature of the curve of a given function using derivative.

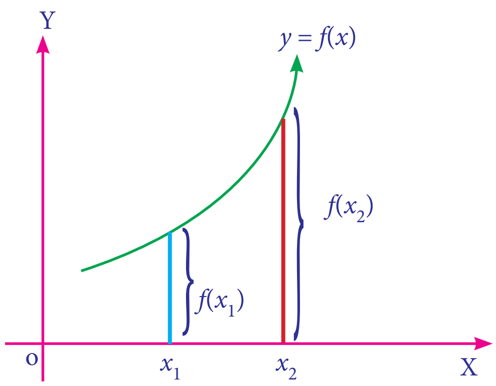
(i) Increasing function

A function $f(x)$ is said to be increasing function in the interval $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) \leq f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$

A function $f(x)$ is said to be strictly increasing in $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) < f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$



$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Strictly increasing function

Fig: 6.7

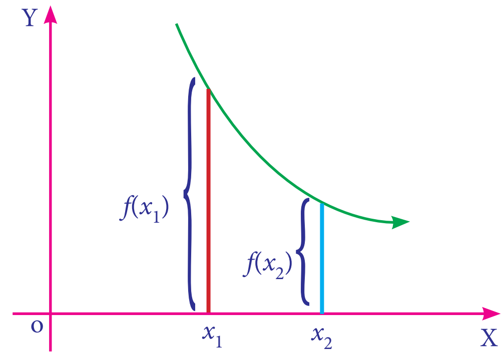
(ii) Decreasing function

A function $f(x)$ is said to be decreasing function in $[a, b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) \geq f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$

A function $f(x)$ is said to be strictly decreasing function in $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) > f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$



$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Strictly decreasing function

Fig: 6.8

NOTE

A function is said to be monotonic function if it is either an increasing function or a decreasing function.

Derivative test for increasing and decreasing function

Theorem: 6.1 (Without Proof)

Let $f(x)$ be a continuous function on $[a,b]$ and differentiable on the open interval (a,b), then

(i) $f(x)$ is increasing in $[a, b]$ if $f'\left(x\right) \geq 0$

(ii) $f(x)$ is decreasing in $[a, b]$ if $f'\left(x\right) \leq 0$

Remarks:

(i) $f(x)$ is strictly increasing in (a,b) if $f'\left(x\right) > 0$ for every $x \in \left(a,b\right)$

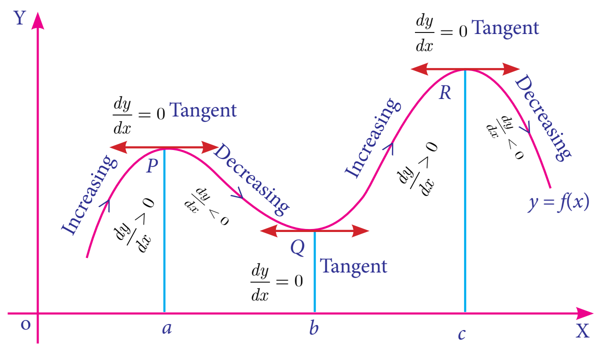
(ii) $f(x)$ is strictly decreasing in (a,b) if $f'\left(x\right) < 0$ for every $x \in \left(a,b\right)$

(iii) $f(x)$ is said to be a constant function if $f'\left(x\right) = 0$

6.2.2 Stationary Value of a function

Let $f(x)$ be a continuous function on $[a, b]$ and differentiable in (a, b). $f(x)$ is said to be stationary at $x = a$ if $f'(a)=0$.

The stationary value of $f(x)$ is $f(a)$. The point $(a, f(a))$ is called stationary point.



P, Q, R are called Stationary Points

Fig: 6.9

In figure 6.9 the function $y = f(x)$ has stationary at $x = a, x = b$ and $x = c$.

At these points, $\frac{dy}{dx} = 0$. The tangents at these points are parallel to x – axis.

NOTE

By drawing the graph of any function related to economics data, we can study the trend of the business related to the function and therefore, we can predict or forecast the business trend.

Example 6.18

Show that the function $f(x) = x^3 - 3x^2 + 4x, x \in R$ is strictly increasing function on R.

Solution :

$$f(x) = x^3 - 3x^2 + 4x, x \in R$$

$$f'(x) = 3x^2 - 6x + 4$$

$$= 3x^2 - 6x + 3 + 1$$

$$= 3(x-1)^2 + 1 > 0, \text{ for all } x \in R$$

Therefore, the function f is strictly increasing on $(-\infty, \infty)$.

Example 6.19

Find the interval in which the function $f(x) = x^2 - 4x + 6$ is strictly increasing and strictly decreasing.

Solution:

Given that $f(x) = x^2 - 4x + 6$

Differentiate with respect to x,

$f'(x) = 2x - 4$

When $f'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$.

Then the real line is divided into two intervals namely $(-\infty, 2)$ and $(2, \infty)$

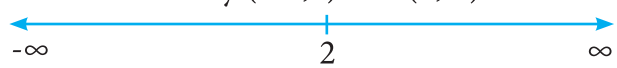


Fig :6.10

[To choose the sign of $f'(x)$ choose any values for x from the intervals ans substitute in $f'(x)$ and get the sign.]

Interval	Sign of $f'(x) = 2x - 4$	Nature of the function
$(-\infty, 2)$	< 0	$f(x)$ is strictly decreasing in $(-\infty, 2)$
$(2, \infty)$	> 0	$f(x)$ is strictly increasing in $(2, \infty)$

Table: 6.1

Example 6.20

Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is increasing or decreasing.

Solution :

$f(x) = 4x^3 - 6x^2 - 72x + 30$

$f'(x) = 12x^2 - 12x - 72$

$= 12(x^2 - x - 6)$

$= 12(x-3)(x+2)$

$f'(x) = 0 \Rightarrow 12(x-3)(x+2) = 0$

$x = 3$ (or) $x = -2$

$f(x)$ has stationary at $x = 3$ and at $x = -2$.

These points divides the whole interval into three intervals namely $(-\infty,-2),(-2,3)$ and $(3,\infty)$.

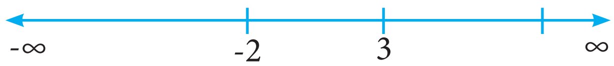


Fig : 6.11

Interval	Sign of $f'(x)$	Intervals of increasing/ decreasing
$(-\infty, -2)$	(–) (–) > 0	Increasing in $(-\infty, -2]$
$(-2, 3)$	(–) (+) < 0	Decreasing in $[-2, 3]$
$(3, \infty)$	(+) (+) > 0	Increasing in $[3, \infty)$

Table: 6.2

Example 6.21

Find the stationary value and the stationary points $f(x)=x^2+2x-5$.

Solution:

Given that $f(x)= x^2 + 2x - 5$ … (1)

$f'(x) = 2x + 2$

At stationary points, $f'(x) = 0$

$\Rightarrow 2x + 2 = 0 \Rightarrow x = -1$

$f(x)$ has stationary value at $x = -1$

When $x = -1$, from (1)

$f(-1) = (-1)^2+2(-1)-5 = -6$

Stationary value of $f(x)$ is -6

Hence stationary point is $(-1,-6)$

Example 6.22

Find the stationary values and stationary points for the function: $f(x)=2x^3+9x^2+12x+1$.

Solution :

Given that $f(x) = 2x^3+9x^2+12x+1$.

$f'(x) = 6x^2+18x+12$

$= 6(x^2+3x+2) = 6(x+2)(x+1)$

$f'(x) = 0 \Rightarrow 6\,(x+2)(x+1) = 0$

$\Rightarrow x + 2 = 0$ (or) $x + 1 = 0$.

$\Rightarrow x = -2$ (or) $x = -1$

$f(x)$ has stationary points at $x = -2$ and $x = -1$

Stationary values are obtained by putting $x = -2$ and $x = -1$.

When $x = -2, f(-2)=2(-8)+9(4)+12(-2)+1= -3$

When $x= -1, f(-1)= 2(-1)+9(1)+12(-1)+1=-4$

The stationary points are $(-2,-3)$ and $(-1,-4)$.

Example 6.23

The profit function of a firm in producing x units of a product is given by $P(x)=\frac{x^3}{3}+x^2+x$. Check whether the firm is running a profitable business or not.

Solution:

$P(x) = \frac{x^3}{3}+x^2+x.$

$P'(x) = x^2+2x+1=(x+1)^2$

It is clear that $P'(x)>0$ for all x.

$\therefore$ The firm is running a profitable business.

IMPORTANT NOTE

Let $R(x)$ and $C(x)$ are revenue function and cost function respectively when x units of commodity is produced. If $R(x)$ and $C(x)$ are differentiable for all $x > 0$, then $P(x) = R(x) - C(x)$ is maximized when Marginal Revenue = Marginal cost. i.e. when $R'(x)=C'(x)$ profit is maximum at its stationary point.

Example 6.24

Given $C(x)=\frac{x^2}{6}+5x+200$ and $p(x) = 40-x$ are the cost price and selling price when x units of commodity are produced. Find the level of the production that maximize the profit.

Solution:

Given $C(x)=\frac{x^2}{6}+5x+200$ … (1)

and $p(x) = 40-x$ …(2)

Profit is maximized when,

marginal revenue = marginal cost.

(i.e) $R'(x) = C'(x)$

$= \frac{x}{3}+5$

$R = p\cdot x$

$= 40x-x^2$

$R'(x) = 40-2x$

Hence $40-2x = \frac{x}{3}+5$

$x = 15$

At $x = 15$, the profit is maximum.

6.2.3 Local and Global (Absolute) Maxima and Minima

Definition 6.1

Local Maximum and local Minimum

A function f has a local maximum (or relative maximum) at c if there is an open interval (a,b) containing c such that $f(c) \geq f(x)$ for every $x \in (a,b)$.

Similarly, f has a local minimum at c if there is an open interval (a,b) containing c such that $f(c) \leq f(x)$ for every $x \in (a,b)$.

Definition 6.2

Absolute maximum and absolute minimum

A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in domain of f. The number $f(c)$ is called maximum value of f in the domain. Similarly f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in domain of f and the number $f(c)$ is called the minimum value of f on the domain.

The maximum and minimum value of f are called extreme values of f.

NOTE

Absolute maximum and absolute minimum values of a function f on an interval (a,b) are also called the global maximum and global minimum of f in (a,b).

Criteria for local maxima and local minima

Let f be a differentiable function on an open interval (a,b) containing c and suppose that $f''(c)$ exists.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$,then f has a local maximum at c.

NOTE

In Economics, if $y = f(x)$ represent cost function or revenue function, then the point at which $\frac{dy}{dx} = 0$, the cost or revenue is maximum or minimum.

Example 6.25

Find the extremum values of the function $f(x)=2x^3+3x^2-12x$.

Solution:

Given $f(x)= 2x^3+3x^2-12x$ … (1)

$f'(x) = 6x^2+6x-12$

$f''(x) = 12x + 6$

$f'(x)= 0 \Rightarrow 6x^2+6x-12 =0$

$\Rightarrow 6(x^2+x-2)= 0$

$\Rightarrow 6(x+2)(x-1)= 0$

$\Rightarrow x = -2 ; x = 1$

When $x = -2$, $f''(-2) = 12(-2) + 6 = -18 < 0$

$\therefore f(x)$ attains local maximum at $x = -2$ and local maximum value is obtained from (1) by substituting the value $x = -2$.

$f(-2) = 2(-2)^3+3(-2)^2-12(-2)$

$= -16+12+24 = 20.$

When $x = 1$, $f''(1) = 12(1) + 6 = 18$

$f(x)$ attains local minimum at $x = 1$ and the local minimum value is obtained by substituting $x = 1$ in (1).

$f(1) = 2(1) + 3(1) - 12(1) = -7$

Extremum values are -7 and 20.

Example 6.26

Find the absolute (global) maximum and absolute minimum of the function

$f(x)=3x^5-25x^3+60x+1$ in the interval $[-2,1]$

Solution :

$f(x) = 3x^5-25x^3+60x+1$ ... (1)

$f'(x) = 15x^4-75x^2+60 = 15(x^4-5x^2+4)$

$f'(x) = 0 \Rightarrow 15(x^4-5x^2+4) = 0$

$\Rightarrow (x^2-4)(x^2-1) = 0$

$\Rightarrow x = \pm 2$ (or) $x = \pm 1$

Of these four points $-2, \pm 1 \in [-2,1]$ and $2 \notin [-2,1]$

From (1)

$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1 = -15$

When $x = 1$

$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1 = 39$

When $x = -1$

$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1 = -37$

Absolute maximum is 39 and

Absolute minimum is -37.

6.3 Applications of Maxima and Minima

6.3.1 Problems on profit maximization and minimization of cost function:

Example 6.27

For a particular process, the cost function is given by $C = 56 - 8x + x^2$, where C is cost per unit and x, the number of unit's produced. Find the minimum value of the cost and the corresponding number of units to be produced.

Solution:

$C = 56 - 8x + x^2$

Differentiate with respect to x,

$\frac{dC}{dx} = -8 + 2x$ and $\frac{d^2C}{dx^2} = 2$

$\frac{dC}{dx} = 0 \Rightarrow -8 + 2x = 0 \Rightarrow x = 4$

When $x = 4$, $\frac{d^2C}{dx^2} = 2 > 0$

$\therefore C$ is minimum when $x = 4$

The minimum value of cost = 56–32+16 = 40

The corresponding number of units produced = 4.

Example 6.28

The total cost function of a firm is $C(x) = \frac{x^3}{3} - 5x^2 + 28x + 10$, where x is the output. A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where p is the price per unit of output, find the profit maximizing the output and price.

Solution:

Total revenue: $R = p\,x$

$= (2530 - 5x)x = 2530x - 5x^2$

Tax at the rate ₹ 2 per x unit = $2x$.

$\therefore\ C(x) + 2x = \frac{x^3}{3} - 5x^2 + 28x + 10 + 2x$

P = Total revenue – (Total cost + tax)

$= (2530 - 5x)x - (\frac{x^3}{3} - 5x^2 + 28x + 10 + 2x)$

$= -\frac{x^3}{3} + 2500x - 10$

$\frac{dP}{dx} = -x^2 + 2500$ and $\frac{d^2P}{dx^2} = -2x$

$\frac{dP}{dx} = 0 \Rightarrow 2500 - x^2 = 0 \Rightarrow x^2 = 2500$

$\therefore\ x = 50$ (–50 is not acceptable)

At $x = 50$ $\frac{d^2P}{dx^2} = -100 < 0$

P is maximum when $x = 50$.

$\therefore\ P = 2530 - 5(50) =$ ₹ 2280.

Example 6.29

The manufacturing cost of an item consists of ₹ 1,600 as over head material cost ₹ 30 per item and the labour cost ₹ $\left(\frac{x^2}{100}\right)$ for x items produced. Find how many items be produced to have the minimum average cost.

Solution:

As per given information for producing x units of certain item $C(x)$ = labour cost + material cost + overhead cost.

$= \frac{x^2}{100} + 30x + 1600$

$AC = \frac{C(x)}{x} = \frac{\frac{x^2}{100} + 30x + 1600}{x}$

$= \frac{x}{100} + 30 + \frac{1600}{x}$

$\frac{d(AC)}{dx} = \frac{1}{100} - \frac{1600}{x^2}$ & $\frac{d^2(AC)}{dx^2} = \frac{3200}{x^3}$

$\frac{d(AC)}{dx} = 0 \Rightarrow -\frac{1600}{x^2} + \frac{1}{100} = 0$

$\Rightarrow \frac{1}{100} = \frac{1600}{x^2} \Rightarrow x^2 = 160000$

$\therefore\ x = 400$ (–400 is not acceptable)

When $x = 400$, $\frac{d^2(AC)}{dx^2} = \frac{3200}{400^3} > 0$

AC is minimum at $x = 400$

Hence 400 items should be produced for minimum average cost.

Exercise 6.2

1. The average cost function associated with producing and marketing x units of an item is given by $AC = 2x - 11 + \frac{50}{x}$. Find the range of values of the output x, for which AC is increasing.

2. A television manufacturer finds that the total cost for the production and marketing of x number of television sets is $C(x) = 300x^2 + 4200x + 13{,}500$. If each product is sold for ₹ 8,400, show that the profit of the company is increasing.

3. A monopolist has a demand curve $x = 106 - 2p$ and average cost curve $AC = 5 + \frac{x}{50}$, where p is the price per unit output and x is the number of units of output. If the total revenue is $R = px$, determine the most profitable output and the maximum profit.

4. A tour operator charges ₹136 per passenger with a discount of 40 paisa for each passenger in excess of 100. The operator requires at least 100 passengers to operate the tour. Determine the number of passenger that will maximize the amount of money the tour operator receives.

5. Find the local minimum and local maximum of $y = 2x^3 - 3x^2 - 36x + 10$.

6. The total revenue function for a commodity is $R = 15x + \frac{x^2}{3} - \frac{1}{36}x^4$. Show that at the highest point average revenue is equal to the marginal revenue.

6.3.2 Inventory control

Inventory is any stored resource that is used to satisfy a current or a future need. Raw materials, finished goods are examples of inventory. The inventory problem involves placing and receiving orders of given sizes periodically so that the **total cost of inventory** is minimized.

An inventory decisions

1. How much to order ? 2. When to order ?

Costs involved in an inventory problems

(i) Holding cost (or) storage cost (or) inventory carrying cost (C_1) :

The cost associated with carrying or holding the goods in stock is known as holding cost per unit per unit time.

(ii) Shortage cost (C_2) :

The penalty costs that are incurred as a result of running out of stock are known as shortage cost.

(iii) Setup cost (or) ordering cost (or) procurement cost (C_3) :

This is the cost incurred with the placement of order or with the initial preparation of production facility such as resetting the equipment for production.

6.3.3 Economic Order Quantity (EOQ):

Economic order quantity is that size of order which minimizes total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty with the annual demands known. Economic order quantity (EOQ) is also called Economic lot size formula.

The derivation of this formula is given for better understanding and is exempted from examination.

The formula is to determine the optimum quantity ordered (or produced) and the optimum interval between successive orders, if the demand is known and uniform with no shortages.

Let us have the following assumptions.

(i) Let R be the uniform demand per unit time.

(ii) Supply or production of items to the inventory is instantaneous.

(iii) Holding cost is ₹ C_1 per unit time.

(iv) Let there be 'n' orders (cycles) per year, each time 'q' units are ordered (produced).

(v) Let ₹ C_3 be the ordering (set up) cost per order (cycle). Let 't' be the time taken between each order.

Diagrammatic representation of this model is given below:

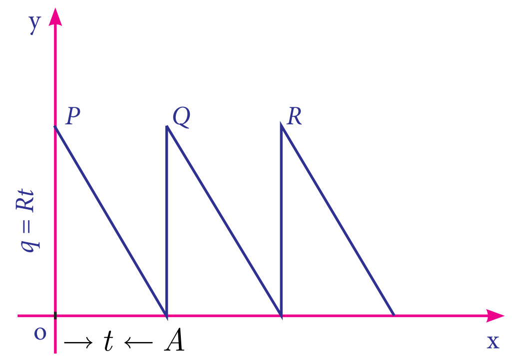


Fig. 6.12

If a production run is made at intervals t, a quantity $q = Rt$ must be produced in each run. Since the stock in small time dt is $Rt\,dt$, the stock in period t is

$$\int_0^t Rt\,dt = \frac{1}{2}Rt^2$$

$$= \frac{1}{2}qt \quad (Rt = q)$$

= Area of the inventory triangle OAP (Fig.6.12)

Cost of holding inventory per production run = $\frac{1}{2}C_1Rt^2$

Set up cost per production run = C_3.

Total cost per production run = $\frac{1}{2}C_1Rt^2 + C_3$

Average total cost per unit time:

$$C(t) = \frac{1}{2}C_1Rt + \frac{C_3}{t} \quad \text{... (1)}$$

$$\frac{d}{dt}C(t) = \frac{1}{2}C_1R - \frac{C_3}{t^2} \quad \text{... (2)}$$

$$\frac{d^2C(t)}{dt^2} = \frac{2C_3}{t^3} \quad \text{... (3)}$$

$C(t)$ is minimum if $\frac{d}{dt}C(t) = 0$ and $\frac{d^2}{dt^2}C(t) > 0$

$$\frac{d}{dt}C(t) = 0 \Rightarrow \frac{1}{2}C_1R - \frac{C_3}{t^2} = 0$$

$$\Rightarrow t = \sqrt{\frac{2C_3}{C_1R}}$$

When $\sqrt{\frac{2C_3}{C_1R}}$, $\frac{d^2C(t)}{dt^2} = \frac{2C_3}{\left(\frac{2C_3}{C_1R}\right)^{\frac{3}{2}}} > 0$

Thus $C(t)$ is minimum for optimum time interval $t_0 = \sqrt{\frac{2C_3}{C_1R}}$

Optimum quantity q_0 to be produced during each production run,

$$EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$$

This is known as the Optimal Lot-size formula due to Wilson.

(i) Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{\text{EOQ}} = R\sqrt{\frac{C_1}{2C_3R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

(ii) Minimum inventory cost per unit time, $C_0 = \sqrt{2C_1C_3R}$

(iii) Carrying cost $= \frac{q_0}{2} \times C_1$,

Ordering cost $= \frac{R}{q_0} \times C_3$

(iv) At EOQ, Ordering cost = Carrying cost

(v) Total optimum cost = Rp + $\sqrt{2C_1C_3R}$

Here, we will discuss EOQ problems only without shortage cost.

Example 6.30

A company uses 48000 units/year of a raw material costing ₹ 2.5 per unit. Placing each order costs ₹45 and the carrying cost is 10.8 % per year of the unit price. Find the EOQ, total number of orders per year and time between each order. Also verify that at EOQ carrying cost is equal to ordering cost.

Solution:

Here demand rate: $R = 48000$

Inventory cost: $C_1 = 10.8\%$ of $2.5 = \frac{10.8}{100} \times 2.5 = 0.27$

Ordering cost: $C_3 = 45$

Economic order quantity: $q_0 = \sqrt{\frac{2C_3R}{C_1}}$

$$= \sqrt{\frac{2 \times 45 \times 48000}{0.27}} = 4000 \text{ units}$$

Number of orders per year $= \frac{R}{q_0} = \frac{48000}{4000} = 12$

Time between orders: $t_0 = \frac{q_0}{R} = \frac{1}{12} = 0.083$ year

At EOQ, carrying cost:

$$= \frac{q_0}{2} \times C_1 = \frac{4000}{2} \times 0.27 = ₹\ 540$$

Ordering cost $= \frac{R}{q_0} \times C_3 = \frac{48000}{4000} \times 45 = ₹\ 540$

So at EOQ carrying cost is equal to ordering cost.

Example 6.31

A manufacturer has to supply 12,000 units of a product per year to his customer. The ordering cost (C_3) is ₹ 100 per order and carrying cost is ₹ 0.80 per item per month. Assuming there is no shortage cost and the replacement is instantaneous, determine the

(i) economic order quantity

(ii) time between orders

(iii) number of orders per year

Solution:

Demand per year: R = 12,000 units

Ordering cost: C_3 = ₹ 100/order

Carrying cost: C_1 = 0.80/item/month

= 0.80×12 per year

= ₹ 9.6 per year

(i) $\text{EOQ} = \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2\times100\times12000}{9.6}}$

= 500 units

(ii) Number of order per year = $\dfrac{Demand}{EOQ}$

$= \dfrac{12,000}{500} = 24$

(iii) Optimal time per order = $\dfrac{1}{t_0} = \dfrac{1}{24}$ year

$= \dfrac{12}{24}$ months $= \dfrac{1}{2}$ month = 15 days

Example 6.32

A company has to supply 1000 item per month at a uniform rate and for each time, a production run is started with the cost of ₹ 200. Cost of holding is ₹ 20 per item per month. The number of items to be produced per run has to be ascertained. Determine the total of setup cost and inventory carrying cost if the run size is 500, 600, 700, 800. Find the optimal production run size using EOQ formula.

Solution:

Demand : R = 1000 per month

Setup cost : C_3 = ₹ 200 per order

Carrying cost: C_1 = ₹ 20 per item per month.

Run size q	Set up cost $\frac{R}{q}\times C_3$	Carrying cost $\frac{q}{2}\times C_1$	Total Inventory cost (Set up cost + Carrying cost)
500	$\frac{1000}{500}\times200 = 400$	$\frac{500}{2}\times20 = 5000$	5400
600	$\frac{1000}{600}\times200 = 333.3$	$\frac{600}{2}\times20 = 6000$	6333.3
700	$\frac{1000}{700}\times200 = 285.7$	$\frac{700}{2}\times20 = 7000$	7285.7
800	$\frac{1000}{800}\times200 = 250$	$\frac{800}{2}\times20 = 8000$	8250

Table : 6.3

$$EOQ = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2\times1000\times200}{20}}$$

$= \sqrt{20000}$ = 141 units (app.)

Example 6.33

A manufacturing company has a contract to supply 4000 units of an item per year at uniform rate. The storage cost per unit per year amounts to ₹ 50 and the set up cost per production run is ₹ 160. If the production run can be started instantaneously and shortages are not permitted, determine the number of units which should be produced per run to minimize the total inventory cost.

Solution :

Annual demand: R = 4000

Storage cost: C_1 = ₹ 50

Setup cost per production: C_3 = ₹160

$$EOQ = \sqrt{\frac{2Rc_3}{c_1}} = \sqrt{\frac{2\times4000\times160}{50}} = 160.$$

∴ To minimize the production cost, number of units produced per run is 160 units.

Example 6.34

A company buys in lots of 500 boxes which is a 3 month supply. The cost per box is ₹125 and the ordering cost in ₹150. The annual inventory carrying cost is estimated at 20% of unit price.

(i) Determine the total inventory cost of existing inventory policy.

(ii) Determine EOQ in units

(iii) How much money could be saved by applying the economic order quantity?

Solution:

Given

Ordering cost per order: C_3= ₹150 per order.

Number of units per order: q = 500 units

Annual demand = 500 × 4 = 2000 units

∴ Demand rate: R = 2000 per year

Carrying cost: C_1= 20% of unit value

$$C_1 = \frac{20}{100} \times 125 = ₹25$$

(i) Total inventory cost of the existing inventory policy

$$= \frac{R}{q} \times C_3 + \frac{q}{2} C_1 = \frac{2000}{500} \times 150 + \frac{500}{2} \times 25$$

= ₹ 6850

(ii) $$EOQ = \sqrt{\frac{2RC_3}{C_1}}$$

$$= \sqrt{\frac{2 \times 2000 \times 150}{25}}$$

$$= \sqrt{12 \times 2000}$$

= 155 units (app.)

(iii) Minimum inventory cost = $\sqrt{2RC_3C_1}$

$$= \sqrt{2 \times 2000 \times 150 \times 25}$$

= ₹3873.

By applying the economic order quantity, money saved by a company = 6850−3873

= ₹2977.

Exercise 6.3

1. The following table gives the annual demand and unit price of 3 items

Items	Annual Demand (units)	Unit Price ₹
A	800	0.02
B	400	1.00
C	13,800	0.20

Ordering cost is Rs. 5 per order and annual holding cost is 10% of unit price.

Determine the following:

(i) EOQ in units

(ii) Minimum inventory cost

(iii) EOQ in rupees

(iv) EOQ in years of supply

(v) Number of orders per year.

2. A dealer has to supply his customer with 400 units of a product per every week. The dealer gets the product from the manufacturer at a cost of ₹ 50 per unit. The cost of ordering from the manufacturers in ₹ 75 per order. The cost of holding inventory is 7.5 % per year of the product cost. Find (i) EOQ and (ii) Total optimum cost.

6.4 Partial Derivatives

Partial derivative of a function of several variables is its derivative with respect to one of those variables, keeping other variables as constant. In this section, we will restrict our study to functions of two variables and their derivatives only.

Let $u = f(x, y)$ be a function of two independent variables x and y.

The derivative of u with respect to x when x varies and y remains constant is called the **partial derivative of u with respect to x,** denoted by $\frac{\partial u}{\partial x}$ (or) u_x and is defined as

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

provided the limit exists. Here Δx is a small change in x

The derivative of u with respect to y, when y varies and x remains constant is called the **partial derivative of u with respect to y,** denoted by $\frac{\partial u}{\partial y}$ (or) u_y and is defined as

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided the limit exists. Here Δy is a small change in y.

$\frac{\partial u}{\partial x}$ is also written as $\frac{\partial}{\partial x} f(x,y)$ (or) $\frac{\partial f}{\partial x}$.

Similarly $\frac{\partial u}{\partial y}$ is also written as $\frac{\partial}{\partial y} f(x,y)$ (or) $\frac{\partial f}{\partial y}$

The process of finding a partial derivative is called **partial differentiation.**

6.4.1 Successive partial derivatives

Consider the function $u = f(x, y)$. From this we can find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. If $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of x and y, then they may be differentiated partially again with respect to either of the independent variables, (x or y) denoted by $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y \partial x}$.

These derivatives are called second order partial derivatives. Similarly, we can find the third order partial derivatives, fourth order partial derivatives etc., if they exist. The process of finding such partial derivatives are called **successive partial derivatives.**

If we differentiate $u = f(x,y)$ partially with respect to x and again differentiating partially with respect to y,

we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ i.e., $\frac{\partial^2 u}{\partial y \partial x}$

Similarly, if we differentiate $u = f(x,y)$ partially with respect to y and again differentiating partially with respect to x,

we obtain $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ i.e., $\frac{\partial^2 u}{\partial x \partial y}$

NOTE

If $u(x,y)$ is a continuous function of x and y, then $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

Homogeneous functions

A function $f(x,y)$ of two independent variables x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for $t > 0$.

6.4.2 Euler's theorem and its applications

Euler's theorem for two variables:

If $u = f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Example 6.35

If $u = x^2(y-x) + y^2(x-y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2(x-y)^2$.

Solution:

$$u = x^2y - x^3 + xy^2 - y^3$$

$$\frac{\partial u}{\partial x} = 2xy - 3x^2 + y^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2xy - 3y^2$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2x^2 - 2y^2 + 4xy$$

$$= -2\left(x^2 + y^2 - 2xy\right)$$

$$= -2\,(x-y)^2$$

Example 6.36

If $u = \log(x^2+y^2)$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution:

$u = \log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right)\cdot 2 - 2x\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)} = \frac{2\,(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Example 6.37

If $u = xy + \sin(xy)$, then show that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$.

Solution:

$$u = xy + \sin(xy)$$

$$\frac{\partial u}{\partial x} = y + y\cos(xy)$$

$$\frac{\partial u}{\partial y} = x + x\cos(xy)$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= 1 + x\left(-\sin(xy)\cdot y\right) + \cos(xy)$$

$$= 1 - xy\sin(xy) + \cos(xy) \qquad \text{.... (1)}$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left(y + y\cos(xy)\right)$$

$$= 1 + \cos(xy) + y\left(-\sin(xy)\,x\right)$$

$$= 1 - xy\sin(xy) + \cos(xy)\ldots \qquad (2)$$

From (1) and (2), we get

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}.$$

Example 6.38

Verify Euler's theorem for the function

$$u = \frac{1}{\sqrt{x^2+y^2}}.$$

Solution:

$$u(x,y) = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$u(tx,ty) = \left(t^2x^2 + t^2y^2\right)^{-\frac{1}{2}} = t^{-1}\left(x^2+y^2\right)^{-\frac{1}{2}}$$

$\therefore$ u is a homogeneous function of degree -1

By Euler's theorem, $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = (-1)u = -u$

Verification:

$$u = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2x = \frac{-x}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$x\cdot\frac{\partial u}{\partial x} = \frac{-x^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2y = \frac{-y}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$y\cdot\frac{\partial u}{\partial y} = \frac{-y^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\therefore\ x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{-(x^2+y^2)}{(x^2+y^2)^{-\frac{3}{2}}}$$

$$= (-1)\frac{1}{\sqrt{x^2+y^2}} = (-1)u = -u$$

Hence Euler's theorem verified.

Example 6.39

Let $u = \log \dfrac{x^4+y^4}{x+y}$. By using Euler's theorem show that $x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = 3$.

Solution:

$$u = \log \frac{x^4+y^4}{x+y}$$

$$e^u = \frac{x^4+y^4}{x+y} = f(x,y) \qquad \text{... (1)}$$

Consider $f(x,y) = \dfrac{x^4+y^4}{x+y}$

$$f(tx,ty) = \frac{t^4x^4+t^4y^4}{tx+ty} = t^3\left(\frac{x^4+y^4}{x+y}\right) = t^3 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 3.

Using Euler's theorem we get,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3f$$

Consider $f(x,y) = e^u$

$$x \cdot \frac{\partial e^u}{\partial x} + y \cdot \frac{\partial e^u}{\partial y} = 3e^u$$

$$\therefore \quad e^u x \cdot \frac{\partial u}{\partial x} + e^u y \cdot \frac{\partial u}{\partial y} = 3e^u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3$$

Exercise 6.4

1. If $z = (ax+b)(cy+d)$, then find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
2. If $u = e^{xy}$, then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u(x^2+y^2)$.
3. Let $u = x\cos y + y\cos x$. Verify $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.
4. Verify Euler's theorem for the function $u = x^3 + y^3 + 3xy^2$.
5. Let $u = x^2y^3\cos\left(\dfrac{x}{y}\right)$. By using Euler's theorem show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 5u$.

6.5 Applications of Partial Derivatives

In this section we solve problems on partial derivatives which have direct impact on Industrial areas.

6.5.1 Production function and marginal productivities of two variables

(i) Production function:

Production P of a firm depends upon several economic factors like capital (K), labour (L), raw materials (R), machinery (M) etc... Thus $P = f(K,L,R,M,...)$ is known as production function. If P depends only on labour (L) and capital (K), then we write $P = f(L,K)$.

(ii) Marginal productivities:

Let $P = f(L,K)$ be a production function. Then $\dfrac{\partial P}{\partial L}$ is called the **Marginal productivity of labour** and $\dfrac{\partial P}{\partial K}$ is called the **Marginal productivity of capital**.

Euler's theorem for homogeneous production function $P(L,K)$ of degree 1 states that $L\dfrac{\partial P}{\partial L} + K\dfrac{\partial P}{\partial K} = P$

6.5.2 Partial elasticity of demand

Let $q = f(p_1, p_2)$ be the demand for commodity A, which depends upon the prices.

p_1 and p_2 of commodities A and B respectively.

The partial elasticity of demand q with respect to p_1 is defined to be

$$\eta_{qp_1} = \frac{Eq}{Ep_1} = \frac{-p_1}{q}\frac{\partial q}{\partial p_1}$$

The partial elasticity of demand q with respect to p_2 is defined to be

$$\eta_{qp_2} = \frac{Eq}{Ep_2} = \frac{-p_2}{q}\frac{\partial q}{\partial p_2}$$

Example 6.40

Find the marginal productivities of capital (K) and labour (L) if $P = 10L + 0.1L^2 + 5K - 0.3K^2 + 4KL$ when $K = L = 10$.

Solution:

We have $P = 10L + 0.1L^2 + 5K - 0.3K^2 + 4KL$

$$\frac{\partial P}{\partial L} = 10 + 0.2L + 4K$$

$$\frac{\partial P}{\partial K} = 5 - 0.6K + 4L$$

Marginal productivity of labour:

$$\left(\frac{\partial P}{\partial L}\right)_{(10,10)} = 10+2+40 = 52$$

Marginal productivity of capital:

$$\left(\frac{\partial P}{\partial K}\right)_{(10,10)} = 5-6+40 = 39$$

Example 6.41

The production function for a commodity is $P = 10L + 0.1L^2 + 15K - 0.2K^2 + 2KL$ where L is labour and K is Capital.

(i) Calculate the marginal products of two inputs when 10 units of each of labour and Capital are used

(ii) If 10 units of capital are used, what is the upper limit for use of labour which a rational producer will never exceed?

Solution:

(i) Given the production is

$$P = 10L - 0.1L^2 + 15K - 0.2K^2 + 2KL$$

$$\frac{\partial P}{\partial L} = 10 - 0.2L + 2K$$

$$\frac{\partial P}{\partial K} = 15 - 0.4K + 2L$$

When $L = K = 10$ units,

Marginal productivity of labour:

$$\left(\frac{\partial P}{\partial L}\right)_{(10,10)} = 10 - 2+20 = 28$$

Marginal productivity of capital :

$$\left(\frac{\partial P}{\partial K}\right)_{(10,10)} = 15-4+20 = 31$$

(ii) Upper limit for use of labour when $K = 10$ is given by $\left(\frac{\partial P}{\partial L}\right) \geq 0$

$10 - 0.2L + 20 \geq 0$

$30 \geq 0.2L$

i.e., $L \leq 150$

Hence the upper limit for the use of labour will be 150 units.

Example 6.42

For the production function, $P = 4L^{\frac{3}{4}}K^{\frac{1}{4}}$ verify Euler's theorem.

Solution:

$P = 4L^{\frac{3}{4}}K^{\frac{1}{4}}$ is a homogeneous function of degree 1.

Marginal productivity of labour is

$$\frac{\partial P}{\partial L} = 4 \times \frac{3}{4}L^{\frac{-1}{4}}K^{\frac{1}{4}} = 3\left(\frac{K}{L}\right)^{\frac{1}{4}}$$

Marginal productivity of capital is

$$\frac{\partial P}{\partial K} = 4L^{\frac{3}{4}} \times \frac{1}{4}K^{\frac{-3}{4}} = \left(\frac{L}{K}\right)^{\frac{3}{4}}$$

$$L\frac{\partial P}{\partial L}+K\frac{\partial P}{\partial K} = 3L\left(\frac{K}{L}\right)^{\frac{1}{4}}+K\left(\frac{L}{K}\right)^{\frac{3}{4}}$$

$$= 3L^{\frac{3}{4}}K^{\frac{1}{4}}+L^{\frac{3}{4}}K^{\frac{1}{4}}$$

$$= 4L^{\frac{3}{4}}K^{\frac{1}{4}} = P$$

Hence Euler's theorem is verified.

Example 6.43

The demand for a commodity x is $q = 5-2p_1+p_2-p_1^2p_2$. Find the partial elasticities $\frac{Eq}{EP_1}$ and $\frac{Eq}{EP_2}$ when $p_1 = 3$ and $p_2 = 7$.

Solution:

$$\frac{\partial q}{\partial p_1} = -2-2p_1p_2$$

$$\frac{\partial q}{\partial p_2} = 1-p_1^2$$

(i) $$\frac{Eq}{Ep_1} = -\frac{P_1}{q}\frac{\partial q}{\partial p_1}$$

$$= \frac{-p_1}{5-2p_1+p_2-p_1^2p_2}(-2-2p_1p_2)$$

$$= \frac{2p_1+2p_1^2p_2}{5-2p_1+p_2-p_1^2p_2}$$

When $p_1 = 3$ and $p_2 = 7$

$$\frac{Eq}{Ep_1} = \frac{2(3)+2(9)(7)}{5-6+7-(9)(7)} = \frac{132}{-57} = \frac{-132}{57}$$

(ii) $$\frac{Eq}{Ep_2} = -\frac{P_2}{q}\frac{\partial q}{\partial p_2} = \frac{-p_2(1-p_1^2)}{5-2p_1+p_2-p_1^2p_2}$$

$$= \frac{-p_2+p_2p_1^2}{5-2p_1+p_2-p_1^2p_2}$$

When $p_1 = 3$ and $p_2 = 7$

$$\frac{Eq}{Ep_2} = \frac{-7+7(9)}{5-6+7-(9)(7)} = \frac{56}{-57} = \frac{-56}{57}$$

Exercise 6.5

1. Find the marginal productivities of capital (K) and labour (L) if $P = 8L-2K+3K^2-2L^2+7KL$ when $K = 3$ and $L = 1$.

2. If the production of a firm is given by $P = 4LK-L^2+K^2$, $L > 0$, $K > 0$, Prove that $L\frac{\partial P}{\partial L}+K\frac{\partial P}{\partial K} = 2P$.

3. If the production function is $z = 3x^2-4xy+3y^2$ where x is the labour and y is the capital, find the marginal productivities of x and y when $x=1$, $y=2$.

4. For the production function $P = 3(L)^{0.4}(K)^{0.6}$, find the marginal productivities of labour (L) and capital (K) when $L = 10$ and $K = 6$. [use: $(0.6)^{0.6} = 0.736$, $(1.67)^{0.4} = 1.2267$].

5. The demand for a quantity A is $q = 13-2p_1-3p_2^2$. Find the partial elasticities $\frac{Eq}{Ep_1}$ and $\frac{Eq}{Ep_2}$ when $p_1 = p_2 = 2$.

6. The demand for a commodity A is $q = 80-p_1^2+5p_2-p_1p_2$. Find the partial elasticities $\frac{Eq}{Ep_1}$ and $\frac{Eq}{Ep_2}$ when $p_1=2$, $p_2=1$.

Exercise 6.6

Choose the Correct answer:

1. Average fixed cost of the cost function $C(x) = 2x^3+5x^2-14x+21$ is
   (a) $\frac{2}{3}$ (b) $\frac{5}{x}$ (c) $-\frac{14}{x}$ (d) $\frac{21}{x}$

2. Marginal revenue of the demand function $p = 20-3x$ is
   (a) $20-6x$ (b) $20-3x$
   (c) $20+6x$ (d) $20+3x$

3. If demand and the cost function of a firm are $p = 2 - x$ and $C = -2x^2 + 2x + 7$, then its profit function is
   (a) $x^2 + 7$ (b) $x^2 - 7$
   (c) $-x^2 + 7$ (d) $-x^2 - 7$

4. If the demand function is said to be elastic, then
   (a) $|\eta_d| > 1$ (b) $|\eta_d| = 1$
   (c) $|\eta_d| < 1$ (d) $|\eta_d| = 0$

5. The elasticity of demand for the demand function $x = \frac{1}{p}$ is
   (a) 0 (b) 1 (c) $-\frac{1}{p}$ (d) ∞

6. Relationship among MR, AR and η_d is
   (a) $\eta_d = \frac{AR}{AR - MR}$ (b) $\eta_d = AR - MR$
   (c) $MR = AR = \eta_d$ (d) $AR = \frac{MR}{\eta_d}$

7. For the cost function $C = \frac{1}{25}e^{5x}$, the marginal cost is
   (a) $\frac{1}{25}$
   (b) $\frac{1}{5}e^{5x}$
   (c) $\frac{1}{125}e^{5x}$
   (d) $25e^{5x}$

8. Instantaneous rate of change of $y = 2x^2 + 5x$ with respect to x at $x = 2$ is
   (a) 4 (b) 5 (c) 13 (d) 9

9. If the average revenue of a certain firm is ₹ 50 and its elasticity of demand is 2, then their marginal revenue is
   (a) ₹ 50 (b) ₹ 25 c) ₹ 100 (d) ₹ 75

10. Profit $P(x)$ is maximum when
   (a) $MR = MC$ (b) $MR = 0$
   (c) $MC = AC$ (d) $TR = AC$

11. The maximum value of $f(x) = \sin x$ is
   (a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

12. If $f(x,y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is equal to
   (a) $(n-1)f$ (b) $n(n-1)f$
   (c) nf (d) f

13. If $u = 4x^2 + 4xy + y^2 + 4x + 32y + 16$. then $\frac{\partial^2 u}{\partial y \partial x}$ is equal to
   (a) $8x + 4y + 4$ (b) 4
   (c) $2y + 32$ (d) 0

14. If $u = x^3 + 3xy^2 + y^3$, then $\frac{\partial^2 u}{\partial y \partial x}$ is
   (a) 3 (b) $6y$ (c) $6x$ (d) 2

15. If $u = e^{x^2}$, then $\frac{\partial u}{\partial x}$ is equal to
   (a) $2xe^{x^2}$ (b) e^{x^2} (c) $2e^{x^2}$ (d) 0

16. Average cost is minimum when
   (a) Marginal cost = Marginal revenue
   (b) Average cost = Marginal cost
   (c) Average cost = Marginal revenue
   (d) Average Revenue = Marginal cost

17. A company begins to earn profit at
   (a) Maximum point
   (b) Breakeven point
   (c) Stationary point
   (d) Even point

18. The demand function is always
   (a) Increasing function
   (b) Decreasing function
   (c) Non-decreasing function
   (d) Undefined function

19. If $q = 1000 + 8p_1 - p_2$, then $\frac{\partial q}{\partial p_1}$ is
   (a) -1 (b) 8
   (c) 1000 (d) $1000 - p_2$

20. If R = 5000 units / year, C_1 = 20 paise, C_3 = ₹ 20, then EOQ is

(a) 5000 (b) 100 (c) 1000 (d) 200

Miscellaneous Problems

1. The total cost function for the production of x units of an item is given by $C = 10 - 4x^3 + 3x^4$ find the (i) average cost function (ii) marginal cost function (iii) marginal average cost fuction.

2. Find out the indicated elasticity for the following functions

   (i) $p = xe^x, x > 0; \eta_s$

   (ii) $p = 10e^{-\frac{x}{3}}, x > 0; \eta_d$

3. Find the elasticity of supply when the supply function is given by $x = 2p^2 + 5$ at p=1.

4. For the demand function $p = 100 - 6x^2$, find the marginal revenue and also show that $MR = p\left[1 - \frac{1}{\eta_d}\right]$

5. The total cost function y for x units is given by $y = 4x\left(\frac{x+2}{x+1}\right) + 6$. Prove that marginal cost[MC] decreases as x increases.

6. For the cost function $C = 2000 + 1800x - 75x^2 + x^3$ find when the total cost (C) is increasing and when it is decreasing.

7. A certain manufacturing concern has total cost function $C = 15 + 9x - 6x^2 + x^3$. Find x, when the total cost is minimum.

8. Let $u = \log\frac{x^4 - y^4}{x - y}$. Using Euler's theorem show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

9. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = x^3 + 3x^2y^2 + y^3$

10. If $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 7$, then show that $f_{yy}(1,1) = 18$.

Summary

- Demand is the relationship between the quantity demanded and the price of a commodity.
- Supply is the relationship between the quantity supplied and the price of a commodity.
- Cost is the amount spent on the production of a commodity.
- Revenue is the amount realised by selling the output produced on commodity.
- Profit is the excess of total revenue over the cost of production.
- Elasticity of a function $y = f(x)$ at a point x is the limiting case of ratio of the relative change in y to the relative change in x
- Equilibrium price is the price at which the demand of a commodity is equal to its supply.
- Marginal cost is interpreted as the approximate change in production cost of $(x+1)^{th}$ unit, when the production level is x units.

- Marginal Revenue is interpreted as the approximate change in revenue made on by selling of $(x+1)^{th}$ unit, when the sale level is x units.
- A function $f(x)$ is said to be increasing function in the interval $[a, b]$ if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in [a, b]$
- A function $f(x)$ is said to be strictly increasing in $[a, b]$ if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in [a, b]$
- A function $f(x)$ is said to be decreasing function in $[a, b]$ if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in [a, b]$
- A function $f(x)$ is said to be strictly decreasing function in $[a, b]$if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in [a, b]$
- Let f be a differentiable function on an open interval (a, b) containing c and suppose that $f'(c)$ exists.

  (i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

  (ii) If $f'(c) = 0$ and $f''(c) < 0$,then f has a local maximum at c.
- A function $f(x, y)$ of two independent variables x and y is said to be homogeneous in x and y of degree n if for $t > 0$ $f(tx, ty) = t^n f(x, y)$
- If $u = f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$
- The partial elasticity of demand q with respect to p_1 is defined to be $\eta_{qp_1} = \frac{Eq}{Ep_1} = \frac{-p_1}{q}\frac{\partial q}{\partial p_1}$
- The partial elasticity of demand q with respect to p_2 is defined to be $\eta_{qp_2} = \frac{Eq}{Ep_2} = \frac{-p_2}{q}\frac{\partial q}{\partial p_2}$

Some standard results

1. Total cost: $C(x) = f(x) + k$
2. Average cost: $AC = \frac{f(x)+k}{x} = \frac{c(x)}{x}$
3. Average variable cost: AVC= $\frac{f(x)}{x}$
4. Average fixed cost: $AFC = \frac{k}{x}$
5. Marginal cost: $MC = \frac{dC}{dx}$
6. Marginal Average cost: $MAC = \frac{d}{dx}(AC)$
7. Total cost: $C(x) = AC \times x$
8. Revenue: $R = px$
9. Average Revenue: $AR = \frac{R}{x} = p$

10. Marginal Revenue: $MR = \frac{dR}{dx}$

11. Profit: $P(x) = R(x) - C(x)$

12. Elasticity: $\eta = \frac{x}{y} \cdot \frac{dy}{dx}$

13. Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

14. Elasticity of supply: $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$

15. Relationship between MR, AR and η_d :

$$MR = AR\left[1 - \frac{1}{\eta_d}\right] \text{ (or) } \eta_d = \frac{AR}{AR - MR}$$

16. Marginal function of y with respect to x (or) Instantaneous rate of change of y with respect to x is $\frac{dy}{dx}$

17. Average cost $[AC]$ is minimum when $MC = AC$

18. Total revenue $[TR]$ is maximum when $MR = 0$

19. Profit $[P(x)]$ is maximum when $MR = MC$

20. In price elasticity of a function,

    (a) If $|\eta| > 1$, then the function is elastic

    (b) If $|\eta| = 1$, then the function is unit elastic

    (c) If $|\eta| < 1$, then the function is inelastic.

21. $EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$

22. Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{\text{EOQ}} = R\sqrt{\frac{C_1}{2C_3R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

23. Minimum inventory cost per unit time, $C_0 = \sqrt{2C_1C_3R}$

24. Carrying cost $= \frac{q_0}{2} \times C_1$ and

    ordering cost $= \frac{R}{q_0} \times C_3$

25. At EOQ, Ordering cost = Carrying cost

26. If $u(x,y)$ is a continuous function of x and y then, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

GLOSSARY (கலைச்சொற்கள்)	
Approximately	தோரயமான
Average	சராசரி
Commodity	பொருள்
Consumer	நுகர்வோர்
Cost function	செலவுச் சார்பு
Demand	தேவை
Elasticity	நெகிழ்ச்சி
Equilibrium	சமநிலை
Excess	மிகுதியான
Fixed cost	ஒரே விலை / மாறா விலை
Marginal	இறுதிநிலை / விளிம்பு
Maximum	பெருமம்
Minimum	சிறுமம்
Production Output	உற்பத்தி வெளியீடு
Producer	உற்பத்தியாளர்
Profit	இலாபம்
Quantity	அளவு
Rate of change	மாறுவீதம்
Ratio	விகிதம்
Relative change	சார்ந்த மாற்றம்
Revenue function	வருவாய் சார்பு
Supply	அளிப்பு
Variable cost	மாறும் விலை