Ch-3 Fourier Series

$x(s) \longrightarrow \boxed{H(s)} \longrightarrow y(s)$

$x(t)$ $\quad$ $h(t)$ $\quad$ $y(t)$

depends on M_s, ϕ_s → phase

M_s ↓ magnitude

$x(t) = \sin(\omega t + \phi)$

m_i = magnitude of i/p signal

ϕ_i = phase of i/p signal

ω_i = frequency of i/p signal

$M_s|_{\omega = \omega_i}$

$\phi_s|_{\omega = \omega_i}$

Eg:- $H(s) = \frac{1}{s+2}$

$x(t) = 2\sin(2t + 45^\circ)$

$y(t) = ?$

$m_i = 2$

$\omega_i = 2$ rad/sec, $\phi_i = 45^\circ$

$H(j\omega) = \frac{1}{2+j\omega}$

$M = \frac{1}{\sqrt{\omega^2+4}}$

$\phi = -\tan^{-1}(\omega/2)$

$z = x + iy$

$\theta = \tan^{-1}(y/x)$

$M = \sqrt{x^2+y^2}$

$M_o = M_i \times M_s|_{\omega=\omega_i}$

$\phi_o = \phi_i + \phi_s|_{\omega=\omega_i}$

$M|_{\omega_i=2} = 1/\sqrt{8}$

$\phi_i|_{\omega_i=2} = -45^\circ$

$M_o = M_i \times M_s|_{\omega=\omega_i}$

$M_o = 2 \times \frac{1}{\sqrt{8}} = 1/\sqrt{2}$

$\phi_o = \phi_i + \phi_s|_{\omega=\omega_i}$

$\phi_o = 45^\circ - 45^\circ = 0$

$$\boxed{y(t) = \frac{1}{\sqrt{2}}\sin\omega t}$$

Periodic Power Signal $\Rightarrow$ Fourier Series

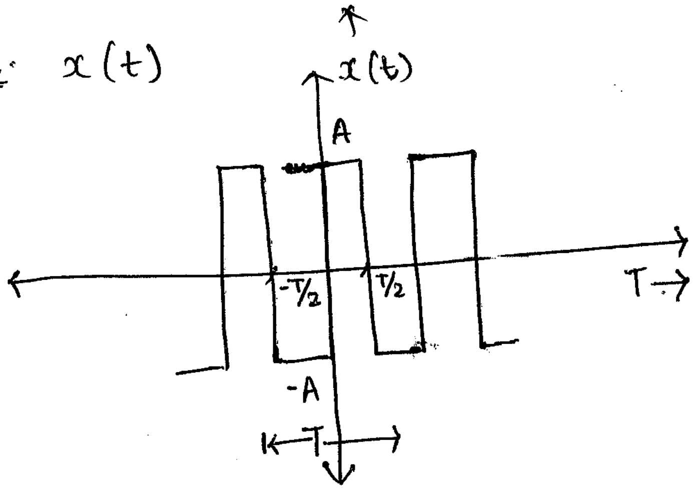
Energy Signal $\Rightarrow$ Fourier transform

$\Rightarrow$ There are 2 reasons for evaluating the fourier series:-

① To obtain an expression for $f(t)$ that applies everywhere, rather than only over a single period.

② To obtain phasor, which indirectly tells how much power is available at each harmonic of the waveform.

Eg: $x(t)$



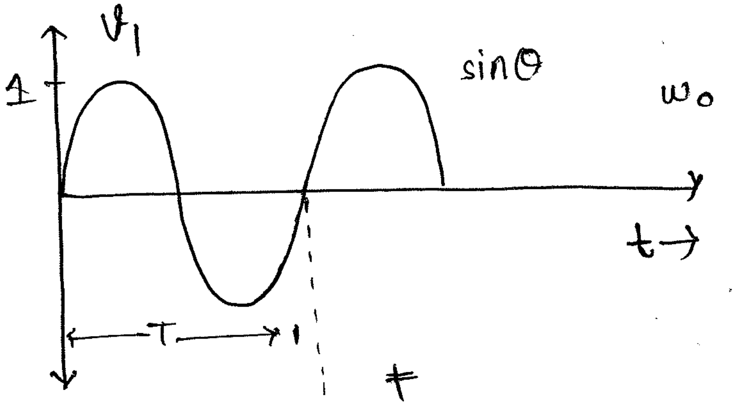
$$x(t) = A \quad , 0 < t < T/2$$
$$= -A \quad , -T/2 < t < 0$$

$$x(t) = \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \frac{1}{4}\sin 4\theta + \frac{1}{5}\sin 5\theta$$

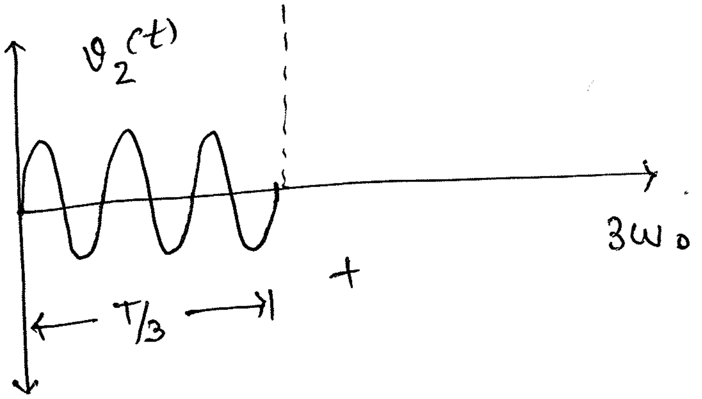
$$v_1(t) = \sin\theta$$

$$v_2(t) = \sin\theta + \frac{1}{3}\sin 3\theta$$

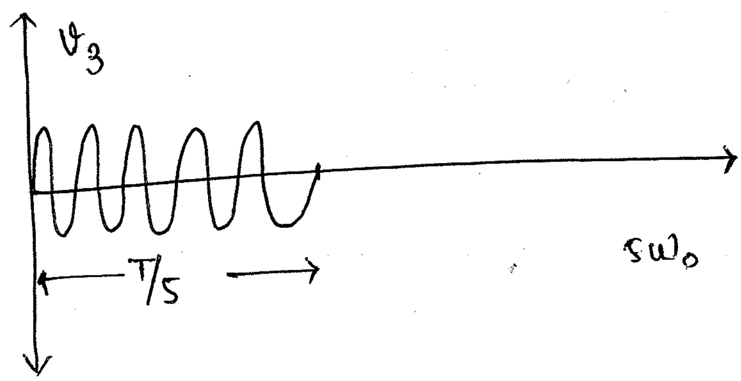
$$v_3(t) = \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta$$



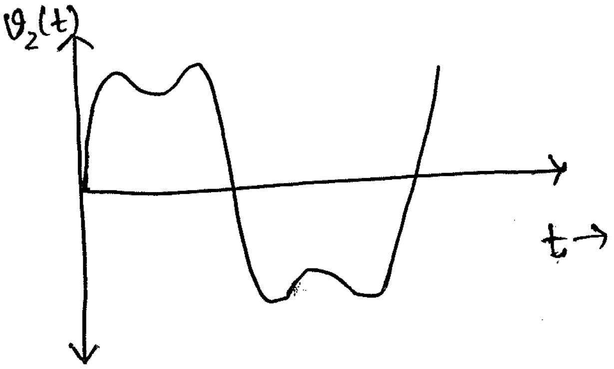
v_1
1
$\sin\theta$
ω_0
$t \rightarrow$
T
+



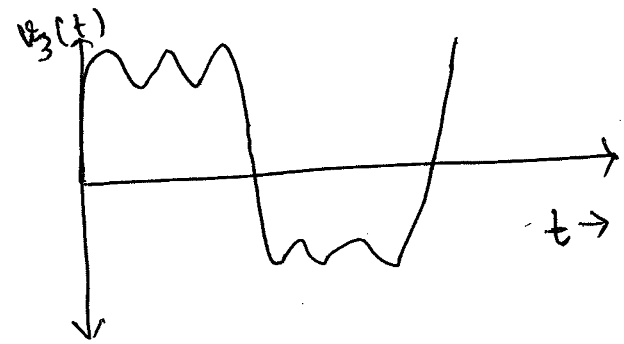
$v_2(t)$
$3\omega_0$
$T/3$
+



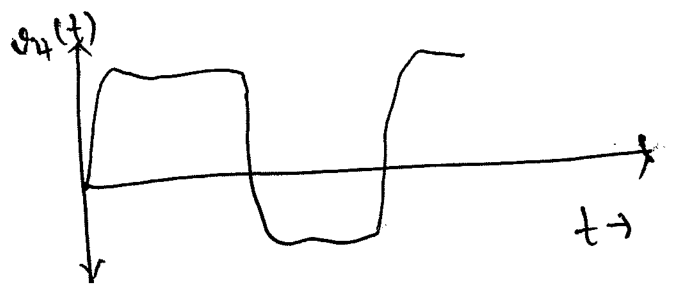
v_3
$5\omega_0$
$T/5$



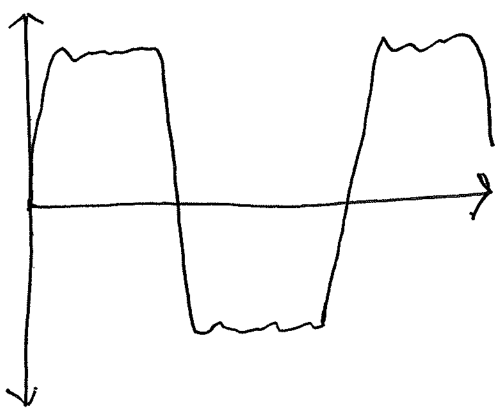
$v_2(t)$
$t \rightarrow$



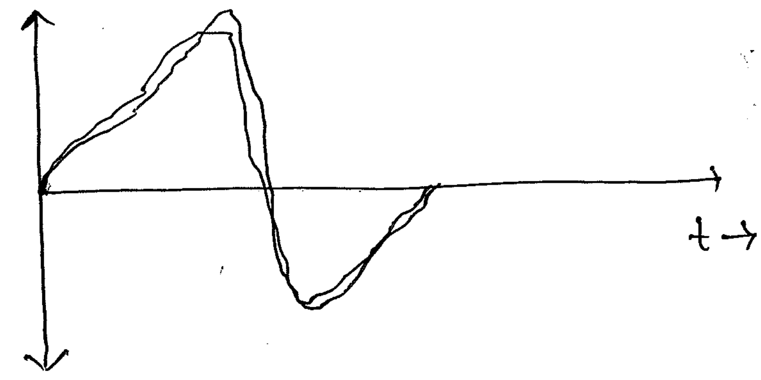
$v_3(t)$
$t \rightarrow$

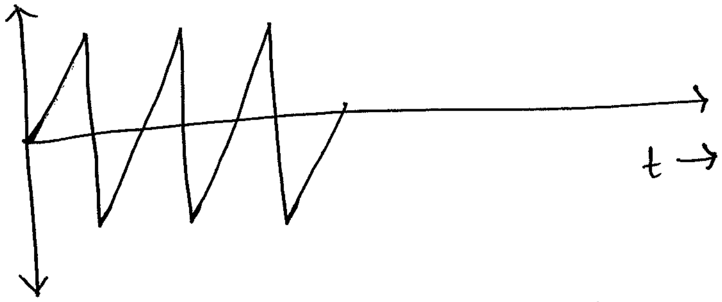


$v_4(t)$
$t \rightarrow$



$x(t) = \sin\theta - \frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta - \frac{1}{6}\sin 6\theta - \cdots\cdots$





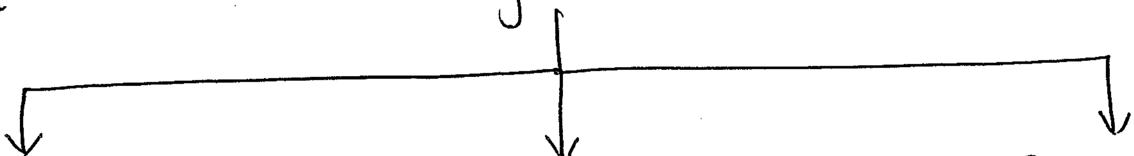
There are 2 reasons for obtaining Fourier series:-

(i) To obtain expression of $x(t)$ which is applicable everywhere rather than overall single period.

(ii) To obtain phasors which tells how the total power of $x(t)$ is disturbed over different frequency component.

What is Fourier series?

- It is an approximation process in which ~~for~~ periodic non-sinusoidal signals are converted into harmonically ~~converted~~ related sinusoidal signals.



Trigonometric F.S. | Polar form of T.F.S. | Exponential F.S. (E.F.S.)

Periodic ⟶ Power signal ⟶ Fourier Series

Non-periodic ⟶ Energy signal ⟶ Fourier Transform

* Trigonometric Fourier Series

- We can find the F.S. representation of any signal if that signal satisfies principle of orthogonality

$$\phi(t) = \{ \phi_1(t), \phi_2(t), \phi_3(t), \ldots \} \Leftarrow \text{Principle of orthogonality}$$

$$\int_{-\infty}^{\infty} \phi_1(t) \cdot \phi_2(t)\, dt = 0$$

$$\Rightarrow \left(\begin{array}{l} a_0, a_1 \cos \omega_0 t, a_2 \cos 2\omega_0 t, \ldots\ldots, a_n \cos n\omega_0 t, \ldots \\ b_1 \sin \omega_0 t, b_2 \sin 2\omega_0 t, \ldots\ldots, b_n \sin n\omega_0 t, \ldots\ldots \end{array} \right)$$

① $\int_T \sin n\omega_0 t\, dt = 0$

② $\int_T \cos n\omega_0 t\, dt = 0$

③ $\int_T \sin^2 n\omega_0 t\, dt = T/2$

④ $\int_T \cos^2 n\omega_0 t\, dt = T/2$

⑤ $\int_T \sin n\omega_0 t \cdot \sin m\omega_0 t = 0, \; m \neq n$
$= T/2, \; m = n$

⑥ $\int_T \cos n\omega_0 t \cdot \cos m\omega_0 t\, dt = 0, \; m \neq n$
$= T/2, \; m = n$

⑦ $\int_T \sin n\omega_0 t \cdot \cos n\omega_0 t\, dt = 0$

*Fourier Series Representation

$$x(t) = a_0 + a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + a_3\cos 3\omega_0 t + \dots + a_n\cos n\omega_0 t$$
$$+ \dots + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \dots b_n\sin n\omega_0 t + \dots$$

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n \sin n\omega_0 t)} \quad \text{—— (1)}$$

a_0, a_n and $b_n \rightarrow$ Fourier series co-efficient

(1) $a_0 \rightarrow$ integrate eq^n (1) on both sides

$$\int_T x(t)\,dt = \int_T a_0\,dt + \sum_{n=1}^{\infty}\Big(\int_T a_n \underbrace{\cos n\omega_0 t\,dt}_{0} + \int_T b_n \underbrace{\sin n\omega_0 t\,dt}_{0}\Big)$$

$$\int_T x(t)\,dt = a_0 T + 0 + 0$$

$$\boxed{a_0 = \frac{1}{T}\int_T x(t)\,dt} \Rightarrow \text{DC value}$$

(2) $a_n \rightarrow$ multiply by $\cos n\omega_0 t$ and then integrate of eq^n (1)

$$\int_T x(t)\cos n\omega_0 t\,dt = \int_T a_0 \underbrace{\cos n\omega_0 t\,dt}_{0} + \int_T \sum_{n=1}^{\infty} a_n \cos^2 n\omega_0 t\,dt$$
$$+ \int_T b_n \underbrace{\sin n\omega_0 t \cos n\omega_0 t\,dt}_{0}$$

$$\int_T x(t)\cos n\omega_0 t\,dt = \frac{a_n T}{2}$$

$$\boxed{a_n = \frac{2}{T}\int_T x(t)\cos n\omega_0 t\,dt}$$

(5) $b_n \longrightarrow$ multiply by $\sin n\omega_o t$ and then integrate eqⁿ (1

$$\boxed{b_n = \frac{2}{T}\int_T x(t)\sin n\omega_o t \, dt}$$

* Trigonometric Fourier Series

Polar form of TFS:-

$$x(t) = d_o + \sum_{n=1}^{\infty} d_n \cos(n\omega_o t + \theta_n)$$

$$x(t) = d_o + \sum_{n=1}^{\infty} (d_n \cos n\omega_o t \cos\theta_n - d_n \sin n\omega_o t \sin\theta_n)$$

$$\boxed{a_o = d_o} \qquad \boxed{a_n = d_n \cos\theta_n} \qquad \boxed{b_n = -d_n \sin\theta_n}$$

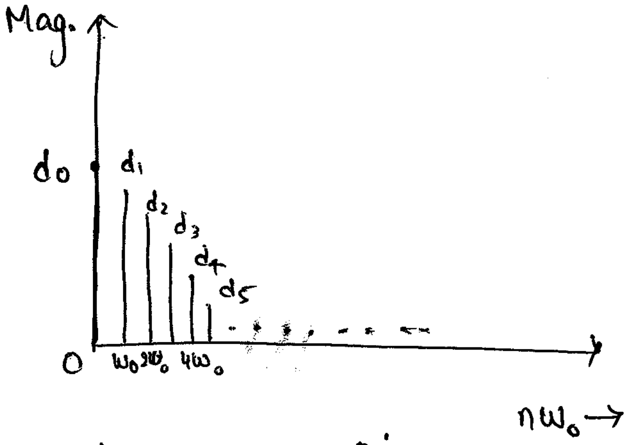
$$a_n^2 + b_n^2 = d_n^2\cos^2\theta_n + d_n^2\sin^2\theta_n$$

$$\boxed{d_n = \sqrt{a_n^2 + b_n^2}} \qquad \tan\theta_n = \frac{-b_n}{a_n}$$

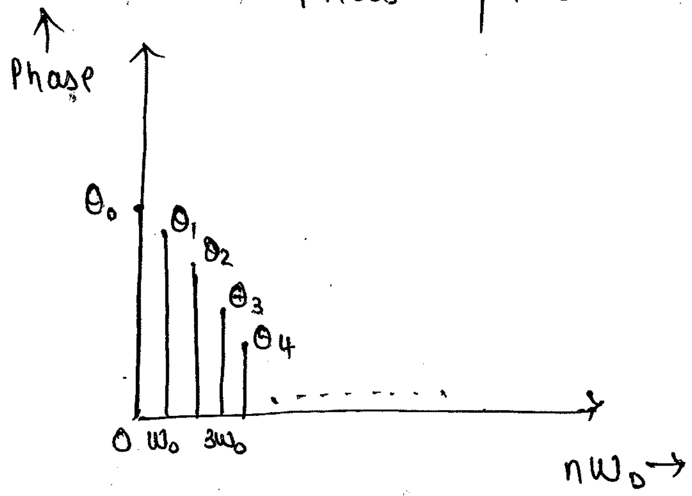
$$\boxed{\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)} \qquad d_n = a_n - j b_n$$

* Spectrums of polar form:

Magnitude plot



Phase plot



$d_0 \rightarrow +ve \rightarrow 0^\circ$

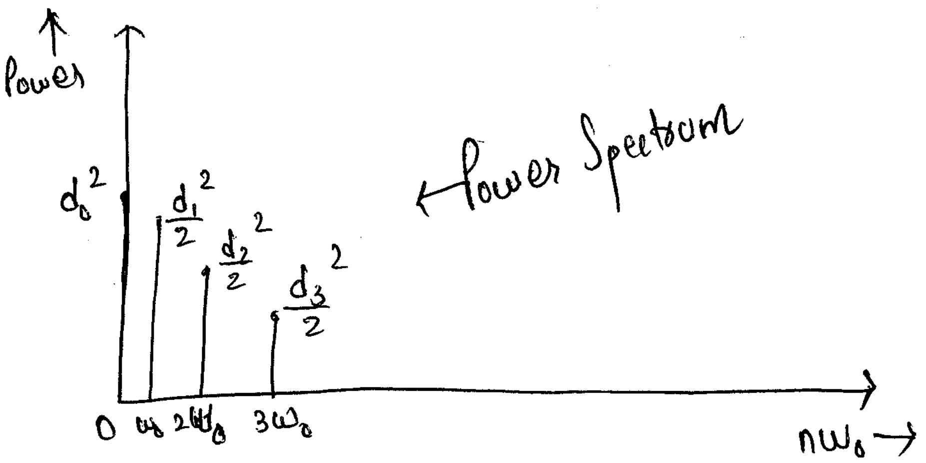
$d_0 \rightarrow -ve \rightarrow 180^\circ$

$$x(t) = a_0 + a_1 \cos\omega_0 t + b_1 \sin\omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + \ldots$$

$$P_{Total} = a_0^2 + \frac{a_1^2}{2} + \frac{b_1^2}{2} + \frac{a_2^2}{2} + \frac{b_2^2}{2} + \ldots$$

$$\Rightarrow \quad x(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)$$

$$P_{Total} = d_0^2 + \frac{d_1^2}{2} + \frac{d_2^2}{2} + \ldots$$



*Exponential Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2}\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right) + \sum_{n=1}^{\infty} e^{-jn\omega_0 t}\left(\frac{a_n}{2} - \frac{b_n}{2j}\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t}\underbrace{\left(\frac{a_n - jb_n}{2}\right)}_{C_n} + \sum_{n=1}^{\infty} e^{-jn\omega_0 t}\underbrace{\left(\frac{a_n + jb_n}{2}\right)}_{C_{-n}}$$

Let $C_n = \frac{a_n - jb_n}{2}$ $\qquad C_{-n} = \frac{a_n + jb_n}{2}$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t}\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t}$$

Let $n = -m$

$$x(t) = a_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{m=-1}^{-\infty} C_m e^{jm\omega_0 t}$$

Let $m = n$

$$x(t) = a_0 + \sum_{n=-\infty}^{n=-1} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$ ← exponential fourier series

$$\boxed{C_n = \frac{1}{T}\int_T x(t)\cdot e^{-jn\omega_0 t}\,dt}$$

T.F.S.		E.F.S.
a_0	$\longrightarrow$	C_0
a_n	$\longrightarrow$	C_n
b_n	$\longrightarrow$	C_{-n}

Let, $C_n = \dfrac{a_n - jb_n}{2}$ $\qquad C_{-n} = \dfrac{a_n + jb_n}{2}$

$C_0 = a_0$, $\quad C_n = \dfrac{a_n - jb_n}{2}$, $\quad C_{-n} = \dfrac{a_n + jb_n}{2}$

$C_n + C_{-n} = a_n$ $\qquad C_n - C_{-n} = -jb_n$

$a_0 = C_0$ $\qquad b_n = \dfrac{C_n - C_{-n}}{-j}$

$a_n = C_n + C_{-n}$

$b_n = j(C_n - C_{-n})$ $\qquad b_n = \dfrac{jC_n - jC_{-n}}{1}$

$$|C_n| = |C_{-n}| = \frac{\sqrt{a_n^2 + b_n^2}}{2} = \frac{d_n}{2}$$

$$d_n = 2|C_n| = 2|C_{-n}|$$

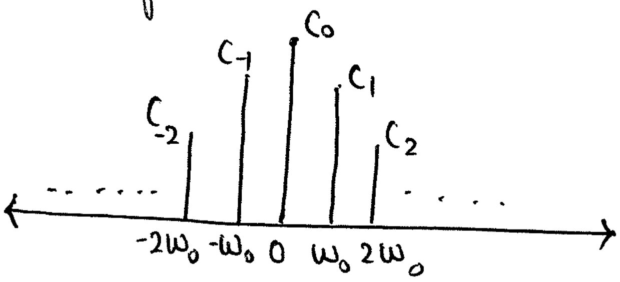
$$\boxed{\angle C_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \theta_n} \qquad \boxed{\angle C_{-n} = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \theta_{-n}}$$

$$\int u\cdot v\,dt = u\int v\,dt - u'\iint v\,dt + u''\iiint v\,dt - \ldots\ldots\ldots$$

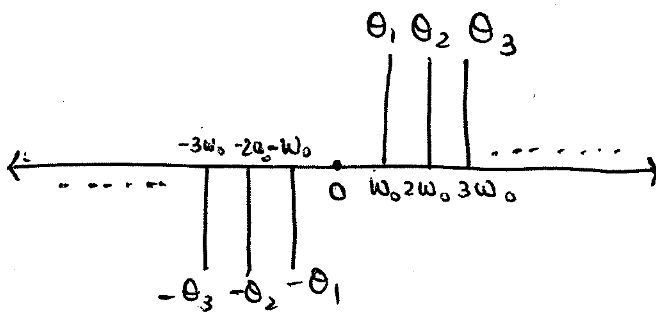
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

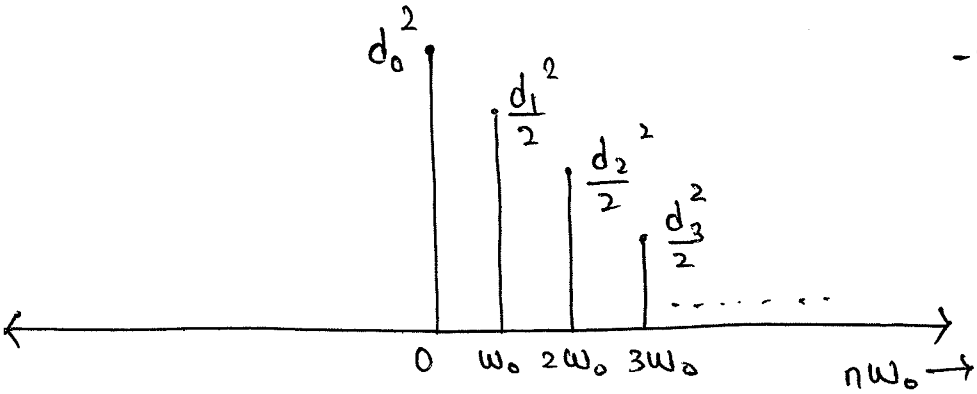
$$\left| C_n e^{jn\omega_0 t} \right| = |C_n| = |C_{-n}|$$

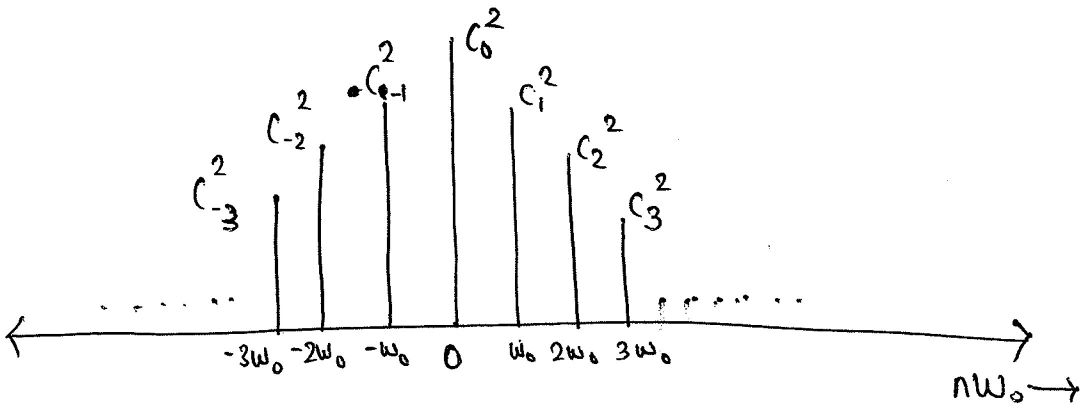
(1) Magnitude



(2) Phase







Q:- Find trigonometric fourier series exponential series?

① $x(t) = 10\sin^2\left(\frac{3\pi}{7}t\right)$

$x(t) = 5 - 5\cos\left(\frac{6\pi}{7}t\right)$

$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$

$\therefore \underline{\omega_0 = \frac{6\pi}{7}}$, $a_0 = 5$, $a_1 = -5$, $b_n = 0$

② $x(t) = 4\cos\frac{\pi}{10}t - 5\sin(10t - 30^\circ)$

$\omega_1 = \frac{\pi}{10}$ $\qquad$ $\omega_2 = 10$

$\frac{2\cancel{\pi}}{T_1} = \frac{\cancel{\pi}}{10}$ $\qquad$ $\frac{\cancel{2}\pi}{T_2} = \cancel{10}\,5$

$T_1 = 20$ $\qquad$ $T_2 = \frac{\pi}{5}$

$\frac{T_1}{T_2} = \frac{20 \cdot 5}{\pi} = \frac{100}{\pi}$ ← irrational → non-periodic

T.F.S. not exists.

③ $x(t) = 15\cos\left(\frac{3\pi}{4}t - 30^\circ\right) - 10\cos(4\pi t) + 15\sin(0.7\pi t) - 25\sin(15\pi t)$

$\omega_1 = \frac{3\pi}{4}$ $\qquad$ $\omega_2 = 4\pi$ $\qquad$ $\omega_3 = 0.7\pi$ $\qquad$ $\omega_4 = 15\pi$

$\frac{2\cancel{\pi}}{T_1} = \frac{3\cancel{\pi}}{4}$ $\qquad$ $\frac{2\cancel{\pi}}{T_2} = 4\cancel{\pi}$ $\qquad$ $\frac{2\cancel{\pi}}{T_3} = \frac{7\cancel{\pi}}{10}$ $\qquad$ $\frac{2\cancel{\pi}}{T_4} = 15\cancel{\pi}$

$T_1 = \frac{8}{3}$ $\qquad$ $T_2 = \frac{1}{2}$ $\qquad$ $T_3 = \frac{20}{7}$ $\qquad$ $T_4 = \frac{2}{15}$

$T_1 = 8/3$, $T_2 = 1/2$, $T_3 = 20/7$, $T_4 = 2/15$

$$T = \frac{LCM(8,1,20,2)}{HCF(3,2,7,15)}$$

$$T = \frac{40}{1} = 40s$$

$\frac{T_1}{T_2}, \frac{T_1}{T_3}, \frac{T_1}{T_4} \Rightarrow$ rational

T.F.S. exists.

$$\omega_0 = \frac{HCF(3\pi, 4\pi, 7\pi, 15\pi)}{LCM(7,1,10,1)}$$

$$\boxed{\omega_0 = \frac{\pi}{70}}$$

Comparing it with $15\cos\left(\frac{3\pi}{7}t - 30^\circ\right)$

$n\omega_0 = \frac{3\pi}{7}$

$n \cdot \frac{\cancel{\pi}}{\cancel{70}_{10}} = \frac{3\cancel{\pi}}{\cancel{7}}$

$n = 30$

$n\omega_0 = 4\pi$

$n \cdot \frac{\cancel{\pi}}{70} = 4\cancel{\pi}$

$n = 280$

$n\omega_0 = 0.7\pi$

$\frac{n\cancel{\pi}}{7\cancel{0}} = \frac{7\cancel{\pi}}{1\cancel{0}}$

$n = 49$

$n\omega_0 = 15\pi$

$\frac{n\cancel{\pi}}{70} = 15\cancel{\pi}$

$n = 1050$

$a_{30} = 15$ $\quad b_{49} = 15$

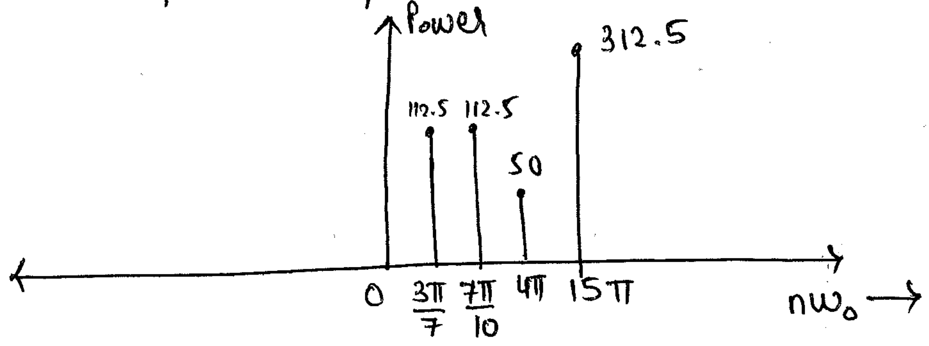
$a_{280} = -30$ $\quad b_{1050} = -25$

$$C_{30} = \frac{a_{30} - j b_{30}}{2} = \frac{15}{2} = 7.5$$

What is the power at 30^{th} harmonic? $P = \frac{A^2}{2} = \frac{(15)^2}{2} = 112.5W$

At 20^{th} harmonic power? $= 0W$ (zero)

Draw power spectrum?



Q:- Find E.F.S. and also find C_n.

1) $x(t) = 10\sin(5t - 30^\circ)$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Here, $a_0 = 0$, $\omega_0 = 5$

$$x(t) = \frac{10}{2j} e^{j(5t-30^\circ)} - \frac{10}{2j} e^{-j(5t-30^\circ)}$$

$$\left[\sin \omega t = \frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$x(t) = \underbrace{-5j e^{-j30^\circ}}_{C_1} \cdot \underbrace{e^{j5t}}_{n=1} + \underbrace{5j e^{j30^\circ}}_{C_{-1}} \cdot \underbrace{e^{-j5t}}_{n=-1}$$

$\theta_1 = -90^\circ - 30^\circ$

$\theta_1 = -120^\circ$

$\theta_{-1} = 90^\circ + 30^\circ$

$\theta_{-1} = 120^\circ$

$$C_1 = -5j e^{-j30^\circ} = -5j(\cos 30^\circ - j\sin 30^\circ)$$

$$C_{-1} = 5j e^{j30^\circ} = 5j(\cos 30^\circ + j\sin 30^\circ)$$

$$a_n = C_n + C_{-n}$$

$$a_1 = c_1 + c_{-1}$$

$$a_1 = -5j\,e^{-j30^\circ} - 5j\,e^{j30^\circ}$$

$$a_1 = -5j^2\left[\frac{e^{j30^\circ} - e^{-j30^\circ}}{2j}\right]$$

$$a_1 = -10\sin 30^\circ = -10\cdot \tfrac{1}{2} = -5$$

$$b_1 = j\,(c_1 - c_{-1})$$

$$b_1 = j\left[-5j\,e^{-j30^\circ} - 5j\,e^{j30^\circ}\right]$$

$$b_1 = -5j^2\cdot 2\left[\frac{e^{j30^\circ} - e^{-j30^\circ}}{2}\right]$$

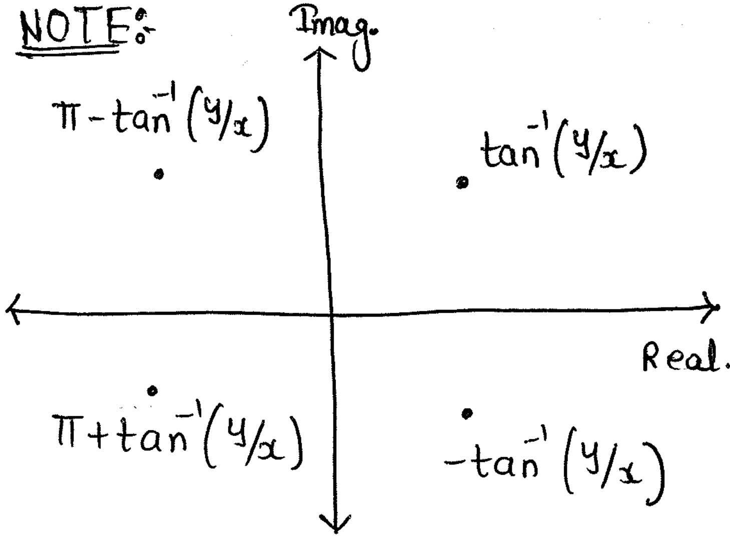
$$b_1 = 10\cos 30^\circ = 10\cdot\frac{\sqrt{3}}{2} = 5\sqrt{3}$$

For verifying the answer:-

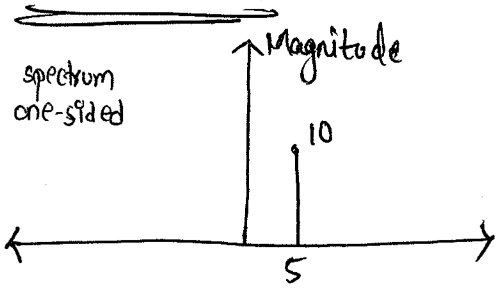
$$x(t) = 10\sin(5t - 30^\circ)$$

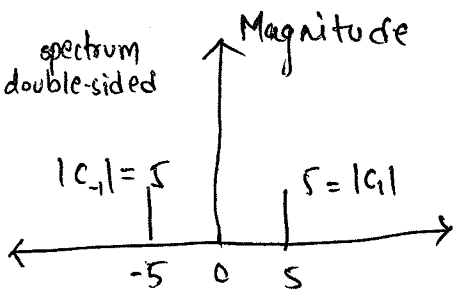
$$= \underbrace{10\sin 5t\cos 30^\circ}_{b_1} - \underbrace{10\cos 5t\sin 30^\circ}_{a_1}$$

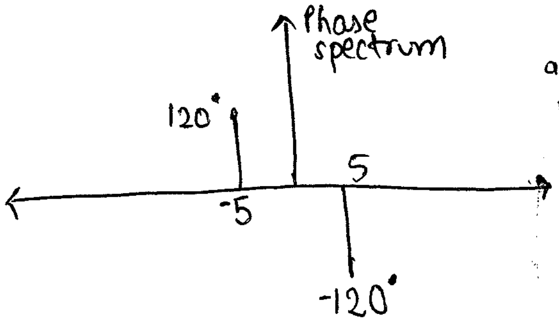
NOTE:



Observation







angle finded by c_1 and c_{-1}.

(2) $x(t) = 15\cos\left(\frac{\pi}{10}t\right) - 10\sin 3t$

$$\omega_1 = \frac{\pi}{10}, \quad \omega_2 = 3$$

$$\frac{2\pi}{T_1} = \frac{\pi}{10}, \quad \frac{2\pi}{T_2} = 3$$

$$T_1 = 20, \quad T_2 = \frac{2\pi}{3}$$

$$\frac{T_1}{T_2} = \frac{\overset{10}{\cancel{20}} \cdot 3}{\cancel{2}\pi} = \frac{30}{\pi} \rightarrow \text{irrational}$$

$\therefore$ Non-periodic   We can't find fourier series

(3) $x(t) = -4\sin 14t + 10\cos\left(\frac{5t}{7} - 30^\circ\right) - 15\cos(0.3t - 45^\circ)$

$$\omega_1 = 14 \qquad \omega_2 = 5/7 \qquad \omega_3 = \frac{0.3}{10}$$

$$\omega = \frac{HCF(14, 5, 3)}{LCM(1, 7, 10)} = \frac{1}{70}$$

$x(t) = -4\sin 14t + 10\cos\left(\frac{5t}{7} - 30^\circ\right) - 15\cos(0.3t - 45^\circ)$

This series is in exponential, so we have to convest it into exponential domain.

$$x(t) = -\frac{4}{2j}\left[e^{j14t} - e^{-j14t}\right] + 5\left[e^{j\left(\frac{5}{7}t - 30^\circ\right)} + e^{-j\left(5/7t - 30^\circ\right)}\right]$$
$$-7.5\left[e^{j(0.3t - 45^\circ)} - e^{-j(0.3t - 45^\circ)}\right]$$

$n\omega_0 = 14$

$n \cdot \frac{1}{70} = 14$

$n = 980$

$n\omega_0 = \frac{5}{7}$

$n \cdot \frac{1}{\cancel{70}_{10}} = \frac{5}{\cancel{7}}$

$n = 50$

$n\omega_0 = \frac{0.3}{10}$

$n \cdot \frac{1}{7\cancel{0}} = \frac{3}{1\cancel{0}}$

$n = 21$

$C_{980} = 2j$, $C_{-980} = -2j$ $\qquad [C_n = C_{-n}^*]$

Phase $= 90°$ $\qquad$ Phase $= -90°$

$C_{21} = -7.5e^{-j45°}$, $C_{-21} = 7.5\,e^{j45°} \longrightarrow$ Phase $= -135°$

Phase $= -7.5\cos 45° + j7.5 \sin 45°$

$\pi - \tan^{-1}(\tan 45°)$

$\pi - \tan^{-1}(1)$

$= 135°$

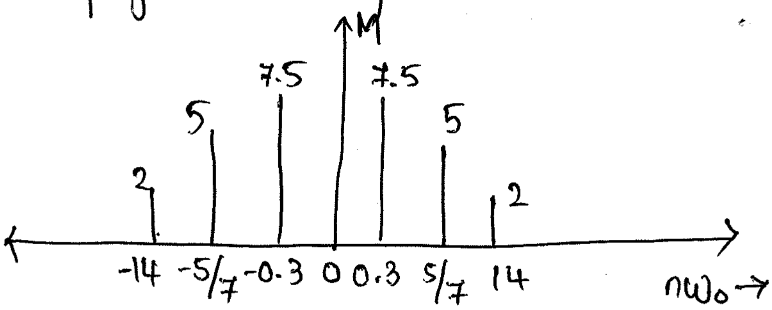
$C_{50} = 5\,e^{-j30°}$ $\qquad C_{-50} = +5e^{j30°}$

$= 5(\cos 30° - j\sin 30°)$ $\qquad$ Phase $= +30°$

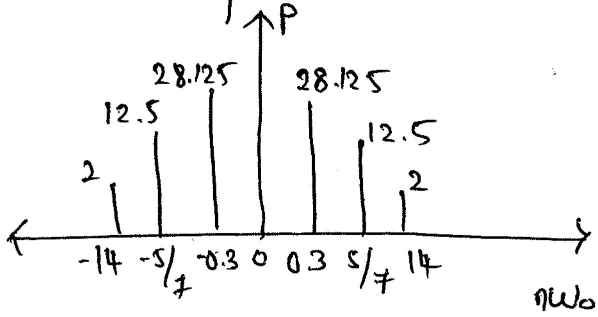
$= 5\left(\sqrt{3}/2 - j\,1/2\right)$

Phase $= -30°$

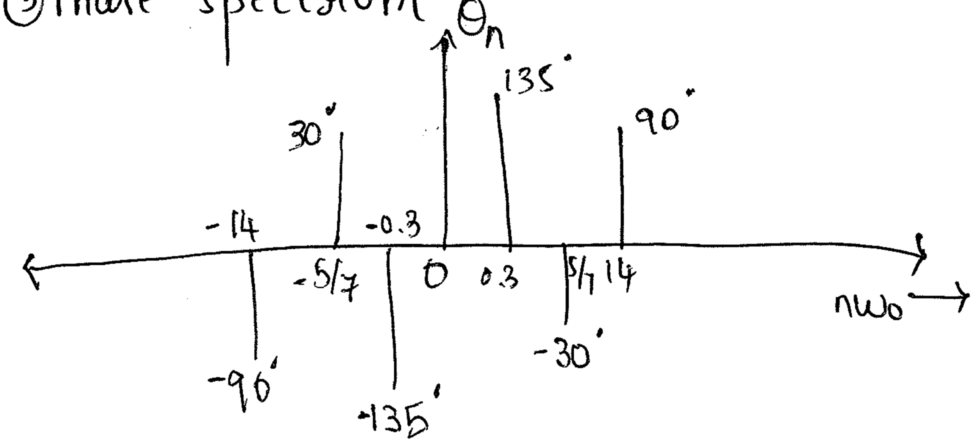
① Magnitude Spectrum



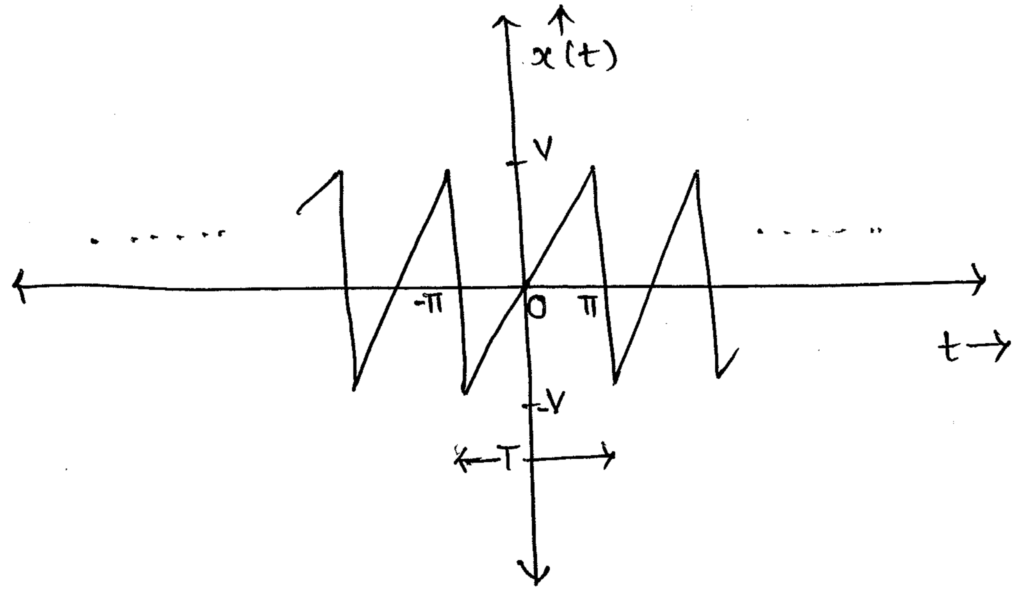
② Power Spectrum



③ Phase spectrum



Q:- For the given periodic signal. Find trigonometric fourier series co-efficient.



Sol^n:- Slope $= \frac{V}{\pi}$, Area for 1 period $= 0$

$T = 2\pi$

$\omega_0 = \frac{2\pi}{T} = \frac{\cancel{2\pi}}{\cancel{2\pi}} = 1$ rad./sec.

$x(t) = \frac{V}{\pi} t \quad ; -\pi \leq t \leq \pi$

$$a_0 = \frac{\int_T x(t)\,dt}{T}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{V}{\pi} \cdot t\right) dt$$

$$= \frac{1}{2\pi} \cdot \frac{V}{\pi} \left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{V}{2\pi^2} \cdot \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right)$$

$$a_0 = 0$$

$$a_n = \frac{2}{T} \int_{-\pi}^{\pi} \underbrace{\frac{V}{\pi} \cdot t}_{\text{odd}} \underbrace{\cos nt}_{\text{even}}\, dt$$

$$a_n = \frac{V}{\pi^2} \left[t \frac{\sin nt}{n} - 1\left(\frac{-\cos nt}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{V}{\pi^2} \left[0 + \frac{\cos \pi n}{n^2} - \frac{\cos(-\pi n)}{n^2}\right]$$

$$a_n = 0$$

$$b_n = \frac{2}{T}\int_{-\pi}^{\pi} x(t)\sin nt\,dt$$

$$= \frac{2}{2\pi}\cdot\frac{V}{\pi}\int_{-\pi}^{\pi} \underset{o}{\underset{\uparrow}{t}}\cdot\underset{o}{\underset{\uparrow}{\sin nt}}\,dt \quad = \text{even}$$

$$= \frac{V}{\pi^2}\; 2\cdot\int_{0}^{\pi} t\sin nt\,dt$$

$$= \frac{2V}{\pi^2}\left[t\left(\frac{-\cos nt}{n}\right) - 1\left(\frac{-\sin nt}{n^2}\right)\right]_0^{\pi}$$

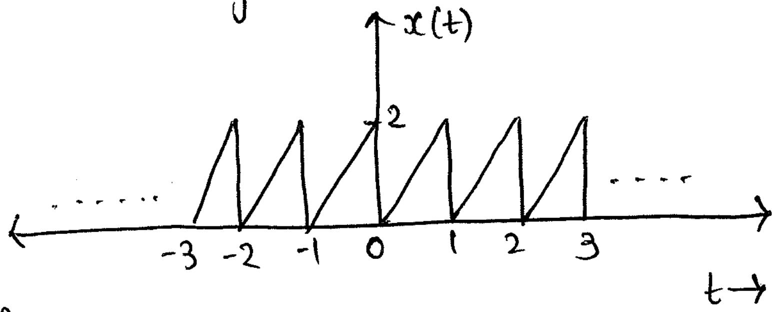
$$= \frac{2V}{\pi^2}\left[\frac{\pi}{n}(-\cos n\pi)\right]$$

$$b_n = \frac{2V}{n\pi}(-1)\cos n\pi$$

$$\therefore b_n = \frac{2V}{n\pi}, \quad n = \text{odd}$$

$$\therefore b_n = \frac{-2V}{n\pi}, \quad n = \text{even}$$

Q:- Find Trigonometric Fourier series co-efficient?



Sol^n:- $T = 1s$

$W_0 = \frac{2\pi}{T} = 2\pi$ rad/sec.

Area $= \frac{1}{2} \times 2 \times 1 = 1$

$$\therefore a_0 = \frac{\int_T x(t)\,dt}{T} = \frac{1}{1} = 1$$

$$a_n = \frac{2}{T}\int_T x(t)\cos n\omega_0 t\, dt$$

$$= \frac{2}{1}\int_0^1 2t\cos n2\pi t \cdot dt$$

$$= 2\left[2t\left(\frac{\sin n2\pi t}{2\pi n}\right) + 2\left(\frac{\cos n2\pi t}{(2\pi n)^2}\right)\right]_0^1$$

$$= 2\left[0 + \frac{2\cdot 1}{(2\pi n)^2} - \frac{2}{(2\pi n)^2}\right]$$

$$a_n = 0$$

$$b_n = \frac{2}{T}\int_T x(t)\sin n\omega_0 t\, dt$$

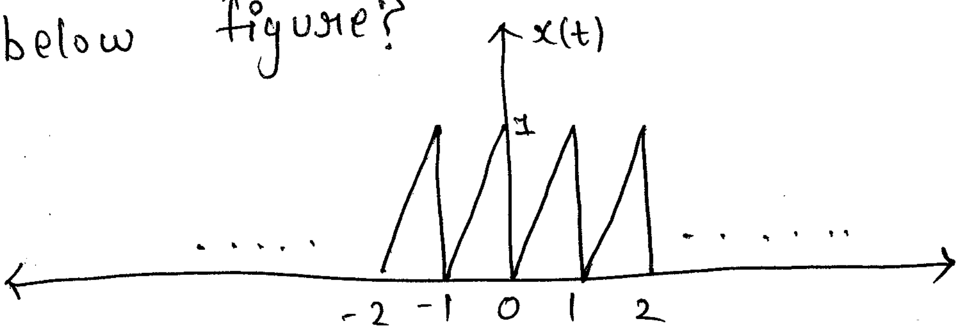
$$= \frac{2}{1}\int_0^1 2t\sin n2\pi t\, dt$$

$$= 2\left[2t\left(\frac{-\cos n2\pi t}{n2\pi}\right) + 2\left(\frac{\sin n2\pi t}{(n2\pi)^2}\right)\right]_0^1$$

$$= 2\left[\frac{2(-1)}{2\pi n}\right]$$

$$b_n = \frac{-2}{\pi n}$$

Q:- Find exponential fourier series co-efficient of below figure?



Sol^n:- $T = 1$ sec.

$$w_0 = \frac{2\pi}{T} = 2\pi \text{ rad/sec.}$$

$$C_n = \frac{1}{T}\int_T x(t)\, e^{-jn w_0 t}\, dt$$

$$= \frac{1}{1}\int_0^1 t \cdot e^{-jn2\pi t} \cdot dt$$

$$= \left[t \cdot \frac{e^{-jn2\pi t}}{-jn2\pi} - 1 \cdot \frac{e^{-jn2\pi t}}{(-jn2\pi t)^2} \right]_0^1$$

$$= \frac{e^{-jn2\pi}}{-jn2\pi} - \frac{e^{-jn2\pi}}{-n^2 \cdot 4\pi^2} - \left[0 - \frac{1 \cdot e^0}{-1 n^2 4\pi^2} \right]$$

$$= \frac{e^{-jn2\pi}}{-jn2\pi} + \frac{e^{-jn2\pi}}{n^2 4\pi^2} - \frac{e^0}{n^2 4\pi^2}$$

$$C_n = \frac{1}{n^2 4\pi^2}\left(e^{-jn2\pi} - 1\right) + \frac{j e^{-jn2\pi}}{n2\pi}$$

Now, $e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1$

$$C_n = \frac{1}{n^2 4\pi^2}(1-1) + \frac{j \cdot 1}{n2\pi}$$

$$\boxed{C_n = \frac{j}{2n\pi}} \qquad \boxed{C_{-n} = \frac{-j}{2n\pi}}$$

$$\theta_n = \tan^{-1}\left(\frac{\text{Real}}{\text{Ima.}}\right) = \tan^{-1}\left(\frac{1}{2\pi n \cdot 0}\right)$$

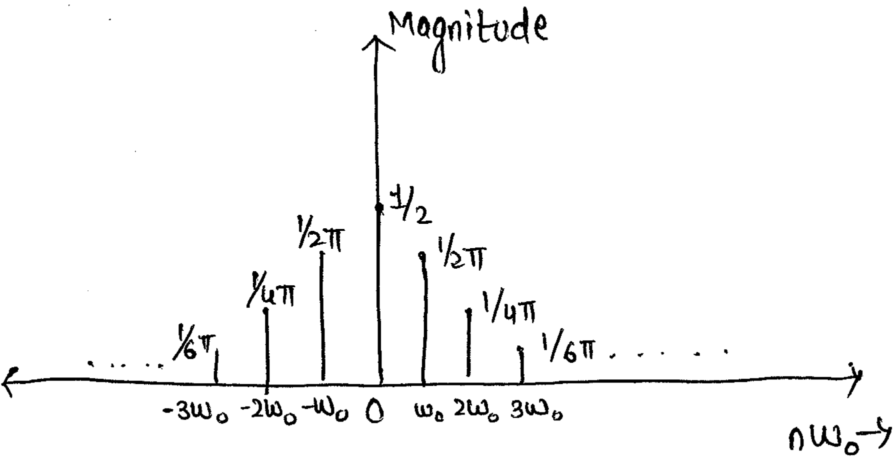
$$\theta_n = \pi/2$$

$$\theta_{-n} = \tan^{-1}\left(\frac{-1}{2\pi n \cdot 0}\right) = -\pi/2$$

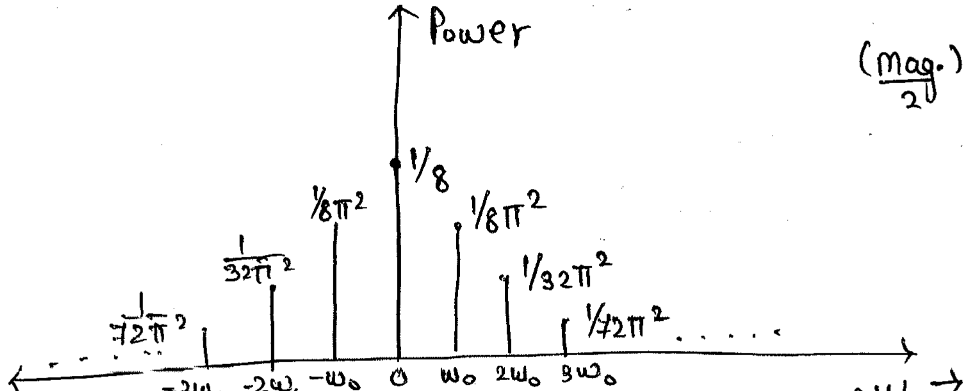
$$\theta_n = \pi/2, \quad \theta_{-n} = -\pi/2$$

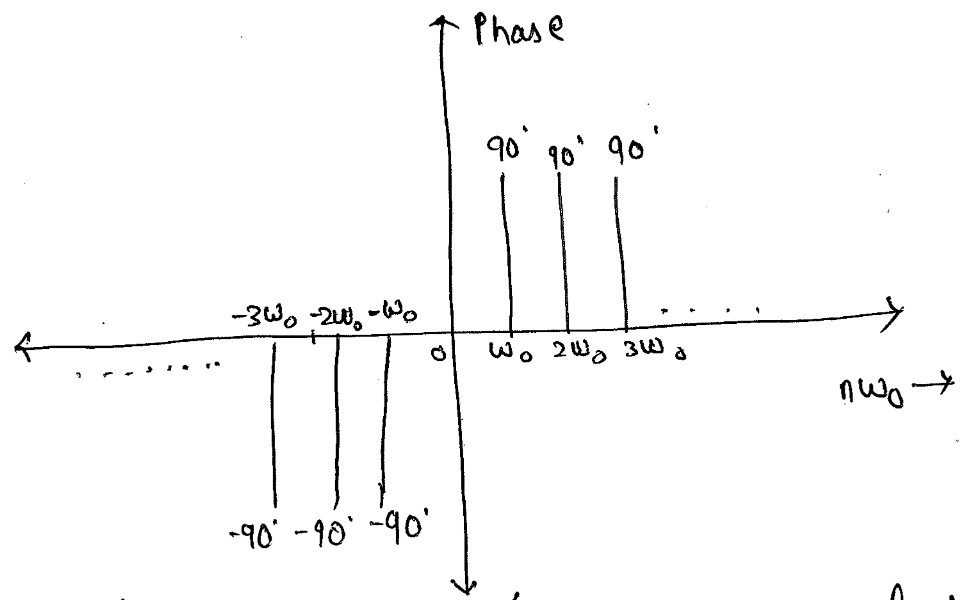
$$a_0 = d_0 = \frac{1}{T}\int x(t)\,dt = \frac{1}{1}\left(\frac{1}{2}\cdot 1\cdot 1\right) = \frac{1}{2}$$

$$|c_n| = |c_{-n}| = \frac{1}{2\pi n}$$

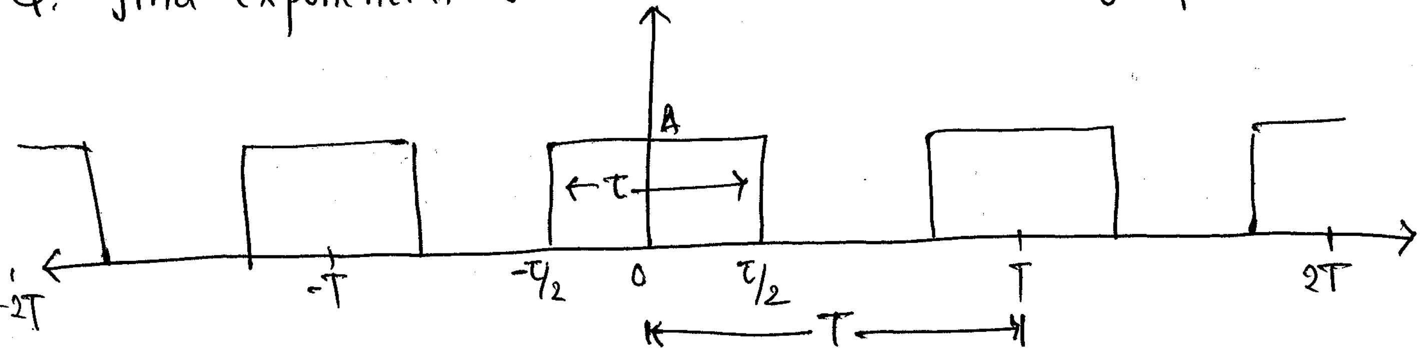


$$\frac{(\text{mag.})^2}{2} = \text{Power}$$





Q:- Find exponential Fourier series of below graph:-



$\underline{Sol^n}$:- $x(t) = A \quad , -\tau/2 < t < \tau/2$

$$T = T \quad , \quad \omega_0 = \frac{2\pi}{T}$$

$$a_0 = C_0 = \frac{Area}{T} = \frac{A\cdot\tau}{T}$$

$$C_n = \frac{1}{T}\int_0^{\tau/2} x(t)\, e^{-jn\omega_0 t}\, dt + \int_{3T/2}^{T} x(t)\cdot e^{-jn\omega_0 t}\cdot dt$$

$$\cong \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\cdot e^{-jn\omega_0 t}\cdot dt$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 \tau/2} - e^{+jn\omega_0\tau/2}}{-jn\omega_0}\right]$$

$$= \frac{2A}{T\cdot n\omega_0}\left[\frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{2j}\right]$$

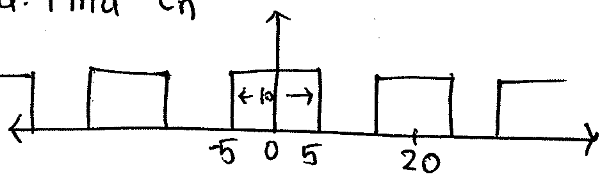
$$= \frac{2A}{T\cdot n\omega_0}\left(\sin n\omega_0 \tau/2\right)$$

$$= \frac{\cancel{2}A}{\cancel{T}\cdot n\cdot\frac{2\pi}{\cancel{T}}}\sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{\tau}{\cancel{2}}\right)$$

$$= \frac{A}{n\pi}\,\frac{\sin\left(\frac{n\pi\cdot\tau}{T}\right)}{\frac{n\pi\cdot\tau}{T}}\cdot\frac{n\pi\cdot\tau}{T}$$

$$\boxed{C_n = \frac{A\tau}{T}\,\mathrm{sinc}\left(\frac{n\tau}{T}\right)}$$

Q:- Find C_n



$$C_n = \frac{A}{2}\,\mathrm{sinc}(n/2)$$

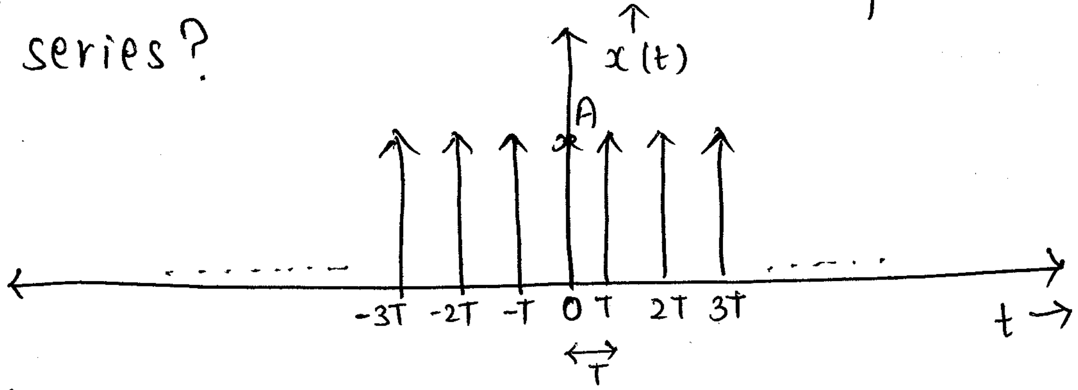
$$= \frac{A}{2}\,\frac{\sin\frac{\pi n}{2}}{\frac{\pi n}{2}}$$

$$C_n = \frac{A}{\pi n}\sin\left(\frac{\pi n}{2}\right)$$

$$C_2 = 0$$

$$C_5 = \frac{A}{\pi\cdot 5}\cdot 1$$

Q:- Find the co-efficient of exponential fourier series?



Sol^n:- $T = T$

$w = \frac{2\pi}{T}$

$x(t) = A\cdot\delta(t), t = 0, \pi$

$= 0, \ 0 < t < \pi$

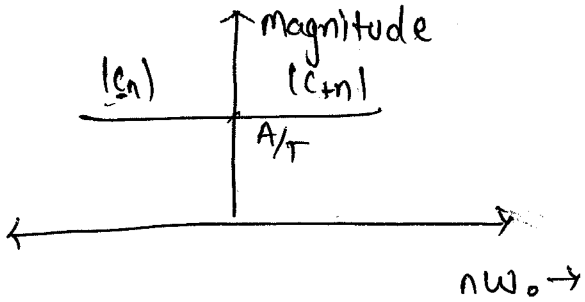
$$C_n = \frac{1}{T}\int_T x(t)\cdot e^{-jnw_0 t}\,dt$$

$$C_n = \frac{1}{T}\int_0^T \delta(t)\cdot A\cdot e^{-jnw_0 t}\,dt$$

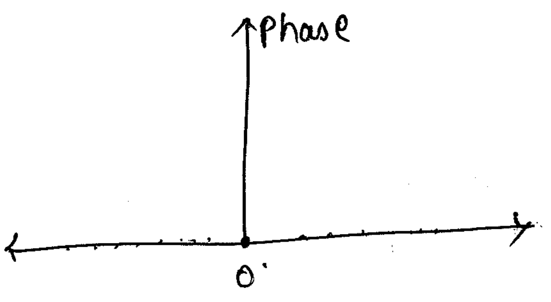
$$\left[\int_{-\infty}^{\infty} x(t)\cdot\delta(t-t_0)\,dt = x(t_0)\right]$$

$$C_n = \frac{1}{T}\cdot(A)$$

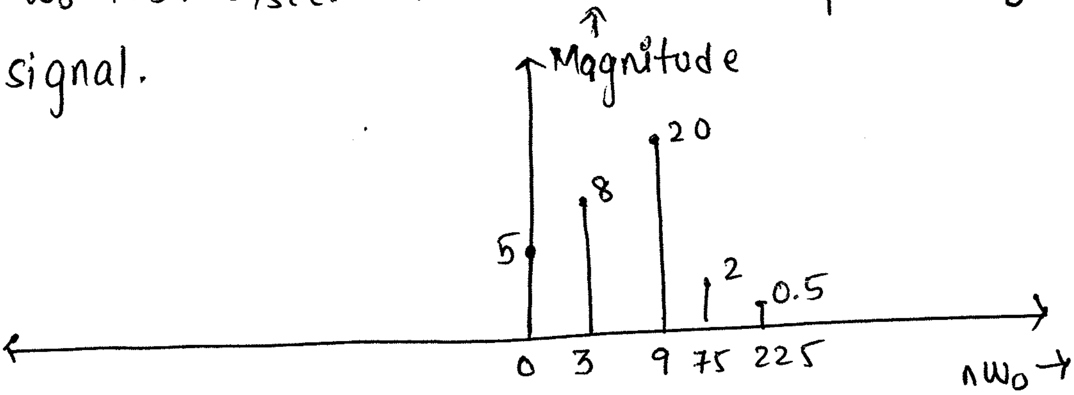
$$C_{-n} = \frac{A}{T}$$

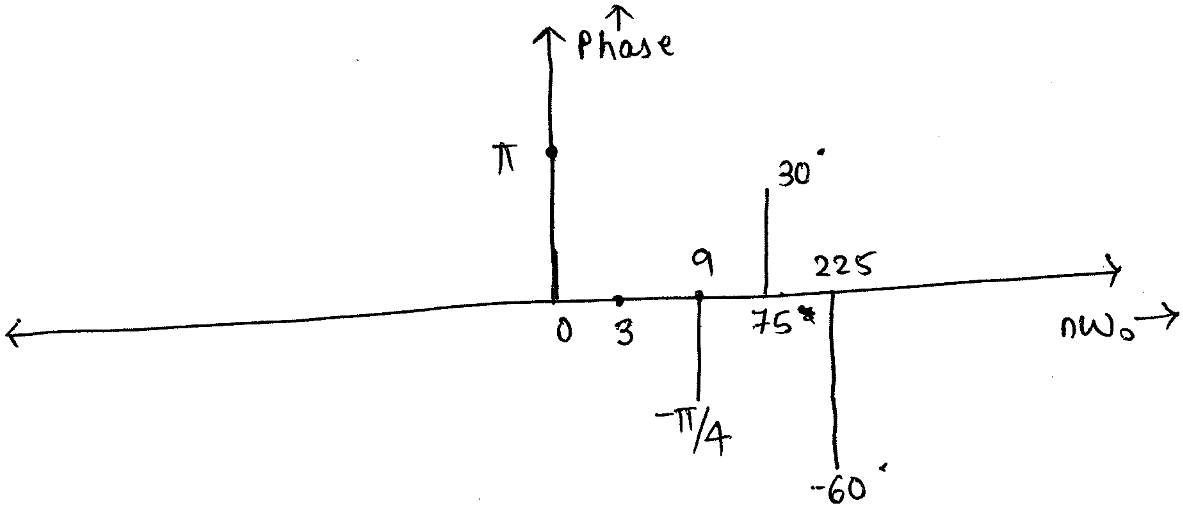


$$\theta_n = 0^\circ = \theta_{-n}$$



Q:- The spectra of periodic signal $x(t)$ with $w_0 = 1.5$ rad/sec. Write the corresponding time domain signal.





Sol^n:- At π, $d_0 = -ve = -5$

$a_1 = d_1 \cos\theta_1$ $\qquad$ $b_1 = -d_1 \sin\theta_1$

$a_1 = 8\cos 0° = 8$ $\qquad$ $b_1 = -3\sin 0° = 0$

~~x(t) = 5 cos~~ $x(t) = d_0 + \sum_{n=1}^{\infty} d_1 \cos(nw_0 t + \theta_n)$

$x(t) = 5\cos(0+\pi) + 8\cos(3t+0°) + 20\cos(9t - \pi/4)$
$\quad + 2\cos(75t + 30°) + 0.5\cos(225t - 60°)$

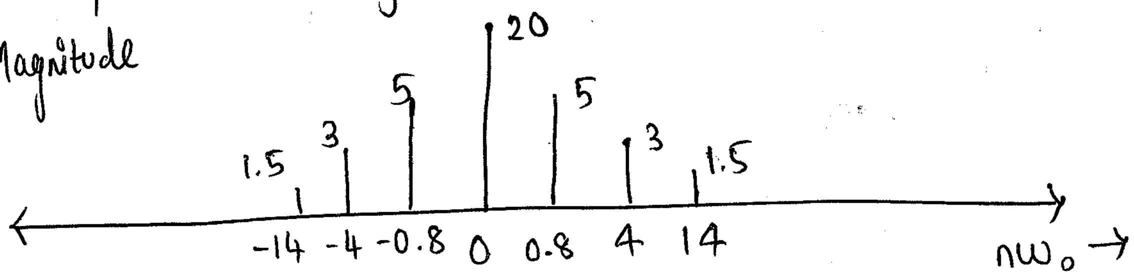
$w_0 = HCF(3, 9, 75, 225)$

$\boxed{w_0 = 3 \text{ rad/sec.}}$

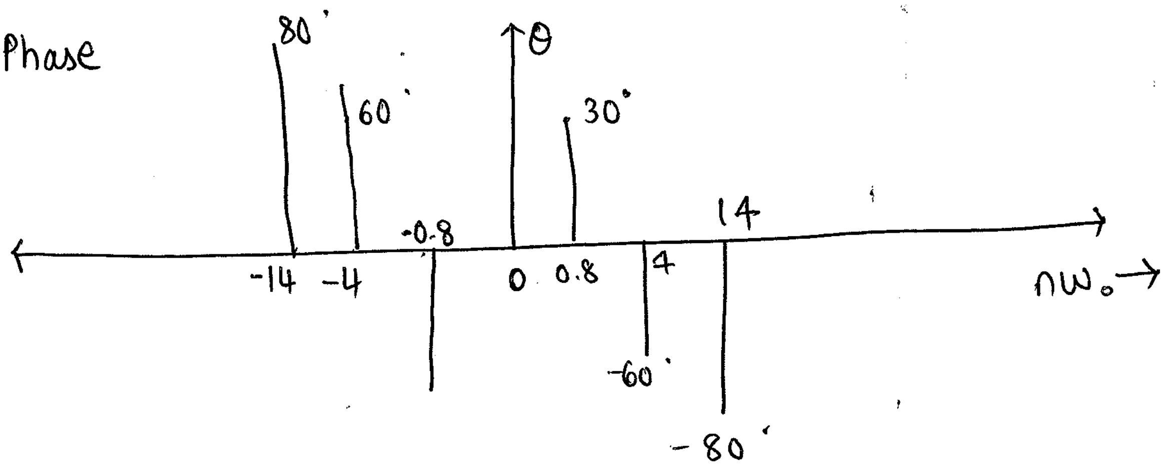
We will not include 0 value bcz. it indicates DC value.

Q:- Spectra is given find series?

Magnitude



Phase



Solⁿ:- $M = |C_n| + |C_{-n}|$

$$x(t) = 20\cos(0°) = 10\cos(0.8t + 30°) + 6\cos(4t - 60°) + 3\cos(14t - 80°)$$

$\omega_1 = 0.8$, $\omega_2 = 4$, $\omega_3 = 14$

$\omega = HCF\,(0.8, 4, 14)$

$= HCF\left(\frac{8}{10}, 4, 14\right)$

$$\boxed{\omega = \frac{HCF\,(8,4,14)}{LCM\,(10,1,1)} = \frac{2}{10} = 0.2\,rad/sec.}$$

Now, Observation:-

$n\omega_0 = 0.8$

$n \cdot 0.2 = 0.8$

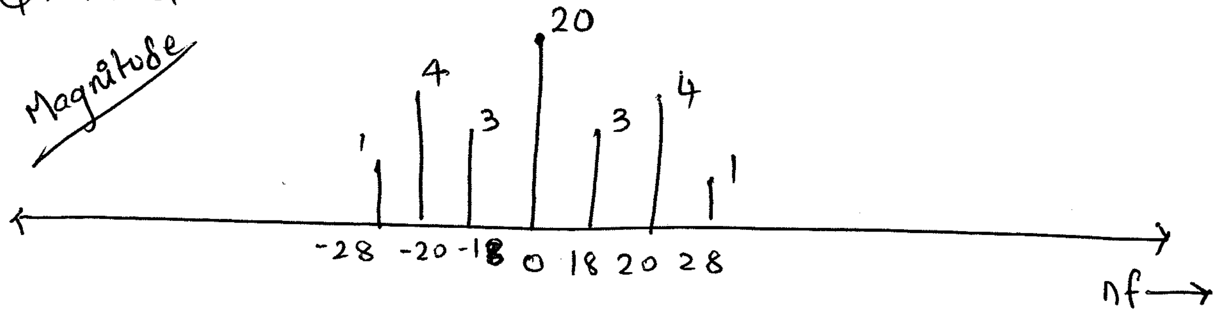
$n = 4^{th}$ harmonic

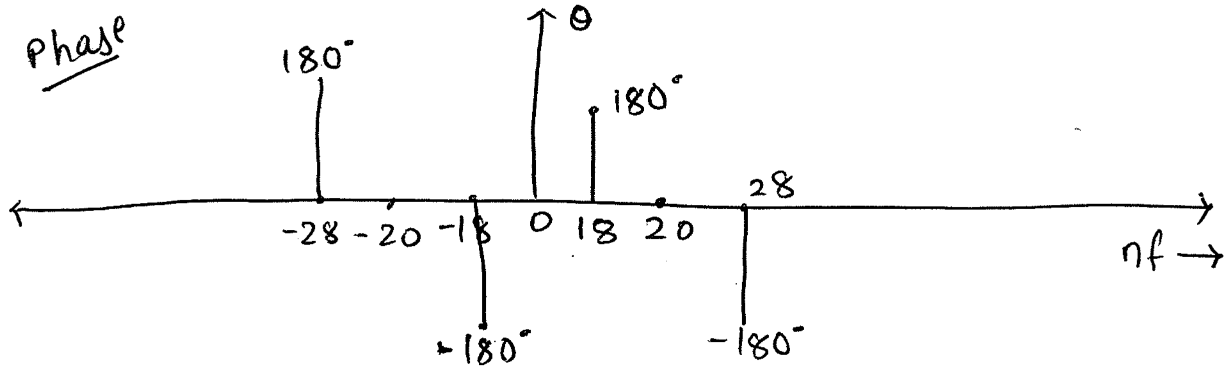
So, At 0.8 the signal is 4^{th} harmonic.

Q:- What is power of 20^{th} harmonic?

$$P = \frac{A^2}{2} = \frac{(6)^2}{2} = 18W.$$

Q:- Find series





Sol:- $x(t) = 20\cos 0° + 6\cos(2\pi(18)t + 180°) + 8\cos(2\pi(20)t + 0°)$
$+ 2\cos(2\pi(28)t - 180°)$

$x(t) = 20 + 6\cos(32\pi t + 180°) + 8\cos(40\pi t) + 2\cos(56\pi t - 180°)$

$\omega_1 = 32\pi$, $\omega_2 = 40\pi$, $\omega_3 = 56\pi$

$\omega = HCF(32\pi, 40\pi, 56\pi)$

$\omega = 4\pi$ rad/sec.

NOTE:-

Knowledge of symmetry can reduce complexity of solving series.

*Effect of symmetry on Fourier coefficient

① Even symmetry

$x(t) = x(-t) \Rightarrow$ Even signal

$$\boxed{a_0 = \frac{1}{T}\int_T x(t)\,dt}$$

$$a_n = \frac{2}{T}\int_T \underset{E}{x(t)}\,\underset{E}{\cos n\omega_0 t}\,dt \qquad E \cdot E = E$$

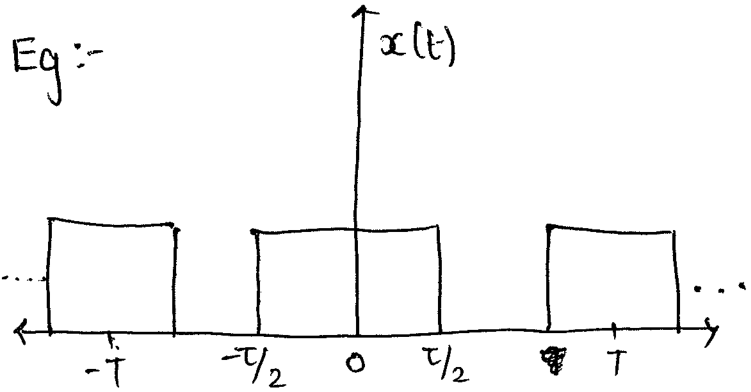
$$a_n = \frac{2 \cdot 2}{T}\int_{(0,T/2)} x(t)\cdot\cos n\omega_0 t\,dt$$

$$\boxed{a_n = \frac{4}{T}\int_{T/2} x(t)\cdot\cos n\omega_0 t\,dt}$$

$$b_n = \frac{2}{T}\int_T \underset{E}{x(t)}\cdot\underset{O}{\sin n\omega_0 t}\,dt \qquad E \cdot O = O$$

$$\boxed{b_n = 0}$$

Eg:-



② Odd symmetry

$x(t) = -x(-t) \Rightarrow$ odd signal

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$\boxed{a_0 = 0}$$

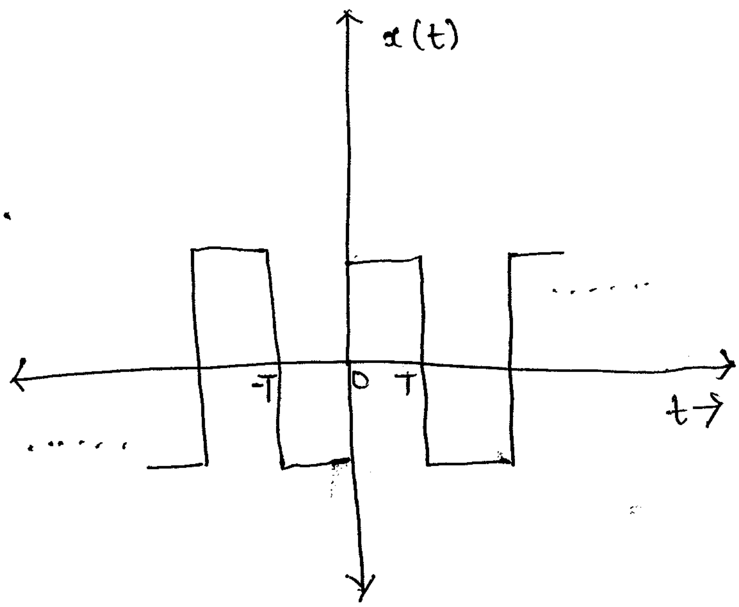
$$a_n = \frac{2}{T}\int_T \underset{O}{x(t)}\cdot\underset{E}{\cos n\omega_0 t}\,dt \qquad O \cdot E = 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_T \underset{O}{x(t)}\cdot\underset{O}{\sin n\omega_0 t}\,dt \qquad O \cdot O = E$$

$$= \frac{2 \cdot 2}{T}\int_{T/2} x(t)\cdot\sin n\omega_0 t\,dt$$

$$\boxed{b_n = \frac{4}{T}\int_{T/2} x(t)\cdot\sin n\omega_0 t\,dt}$$



③ Half-wave symmetry $\boxed{a_0 = 0}$ In this case $a_0 = 0$ always

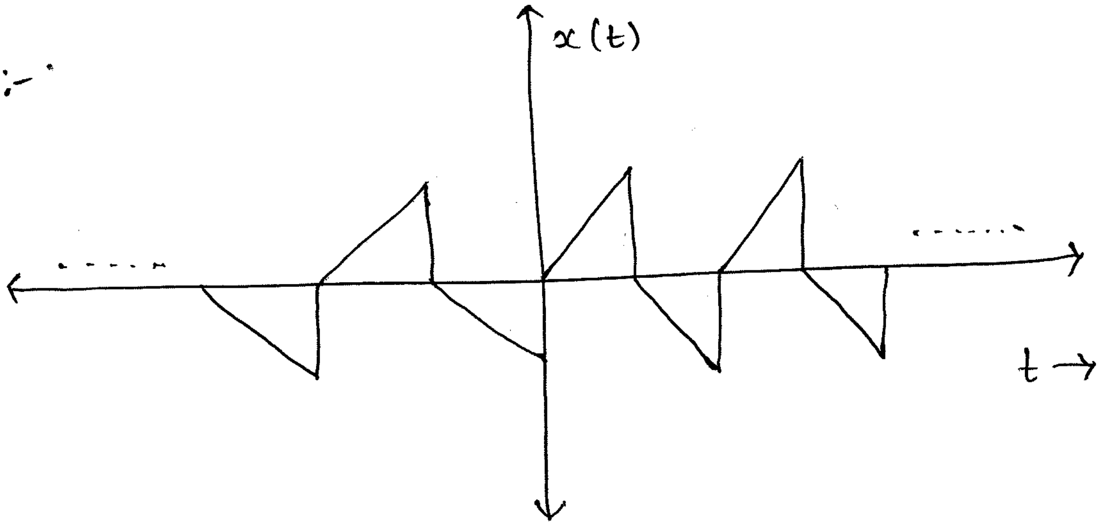
$$x(t) = \pm x(t + T/2)$$

only odd components

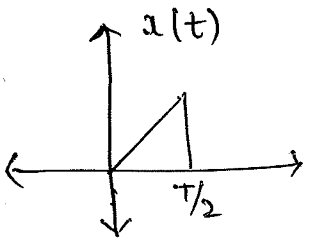
HWE → odd components of cosine → $x(t) = +x(t \pm T/2)$

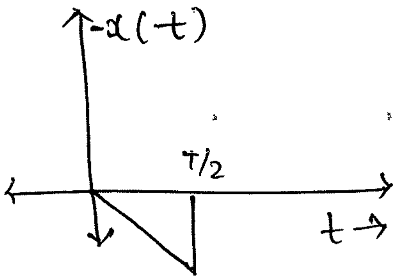
HWO → odd components of sine → $x(t) = -x(t \pm T/2)$

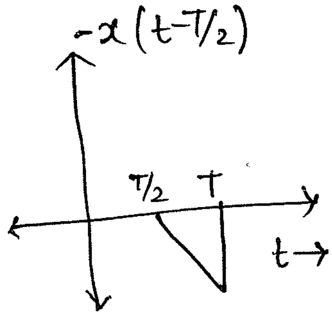
Eg:-



HWO







④ Hidden sy

a_0 ✓
a_n (odd, even
b_n ✗

Q:- Find T.F.S
figure.

Sol:- $a_0 = \int_T$

DC val

∴ subtra

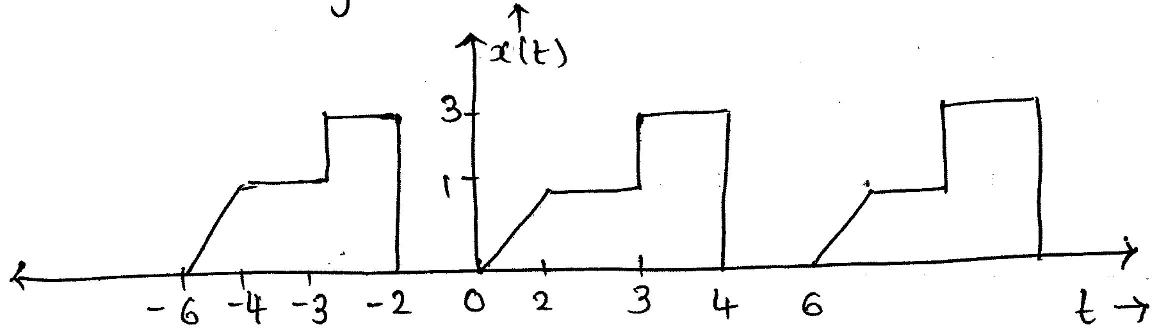
$x(t)$

- ODD (sine) → b_n → H.W. (odd) → odd harmonics of sine
- EVEN (dc, cosine)
  - $a_0 = 0$ → H.W. → odd harmonics of cosine
  - $a_0 \neq 0$ → Hidden → H.W. (odd) → dc + odd harmonics of cosine
- NENO
  - $a_0 = 0$ → H.W. → odd harmonics of sine & cosine
  - $a_0 \neq 0$ → Hidden
    - Even → HW → dc + odd harmonic of cosine
    - Odd → HW → dc + odd harmonic of sine

H.W. :- $x(t) = -x(t \pm T/2)$ → only odd harmonics

H.W. :- $x(t) = x(t \pm T/2)$ → only even harmonics

Q:- The average value of periodic signal $x(t)$ is shown in figure is ______.



Sol^n:-

$$\text{Average value} = \frac{\text{Area under the curve over period}}{\text{Fundamental time period}}$$

$$= \frac{\frac{1}{2} \times 2 \times 1 + 1 \times 1 + 3 \times 1}{6}$$

$$= \frac{1+1+3}{6} = \underline{\frac{5}{6}}$$

Q:- The Fourier series representation of periodic signal is $x(t)$ equals to $x(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t}$.

If one of the component is $a \cos 3\pi t$ the value of a is ______.

Sol^n: $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{+jn\omega_0 t}$

$\omega_0 = \pi \leftarrow$ fundamental period

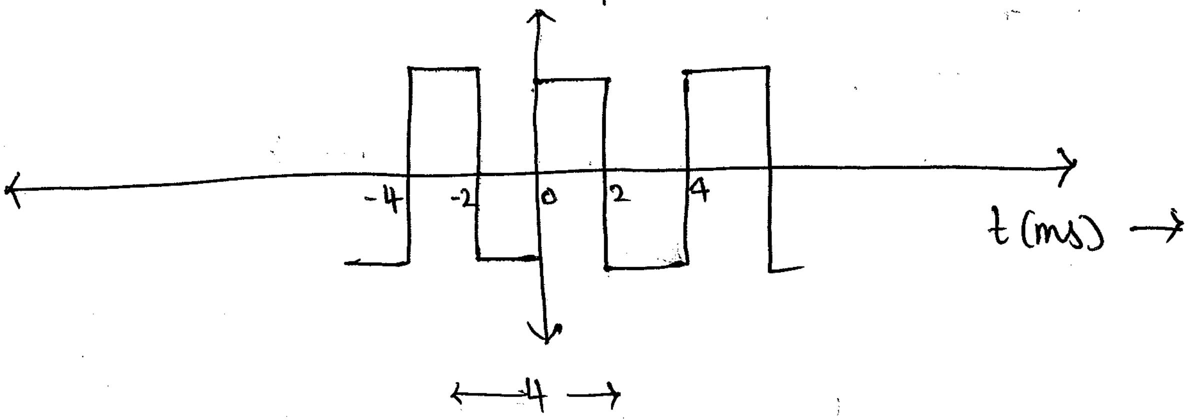
$$C_n = \frac{3}{4+(n\pi)^2} \qquad C_{-n} = \frac{3}{4+(n\pi)^2}$$

$$C_n = \frac{a_n - jn}{2} \qquad C_{-n} = \frac{a_n + jn}{2}$$

$$a_n = C_n + C_{-n}$$

3rd harmonic $\quad a_3 = C_3 + C_{-3} = 2 \cdot \frac{3}{4+(3\pi)^2} = \boxed{\frac{6}{4+9\pi^2} = a}$

Q:- A periodic rectangular signal $x(t)$ has the waveform shown in figure. Frequency of 5^{th} harmonic of its spectrum is ___ Hz.



Sol:- $T = 4 \times 10^{-3}$ s

$\frac{1}{f} = 4 \times 10^{-3}$

$f = \frac{1000}{4} = 250\text{Hz}$

for 5^{th} harmonic $= 5f = 5 \cdot 250 = 1250\text{Hz}$

Q:- The fundamental frequency of composite signal $x(t)$

$x(t) = 2 + 3\cos(0.2t) + \cos(0.25t + \pi/2) + 4\cos(0.3t - \pi)$

Sol:- $\omega = 0.2$, $\omega = 0.25$ $\omega = 0.3$

$\omega = HCF(0.2, 0.25, 0.3)$

$= HCF\left(\frac{2}{10}, \frac{25}{100}, \frac{3}{10}\right)$

$= \frac{HCF(1,1,3)}{LCM(5,4,10)} = \frac{1}{20} = 0.05\,\text{rad/sec.}$

Q:- Find fundamental frequency of $x(t) = 30\sin 100t + 10\cos 300t + 6\sin(500t + \pi/4)$

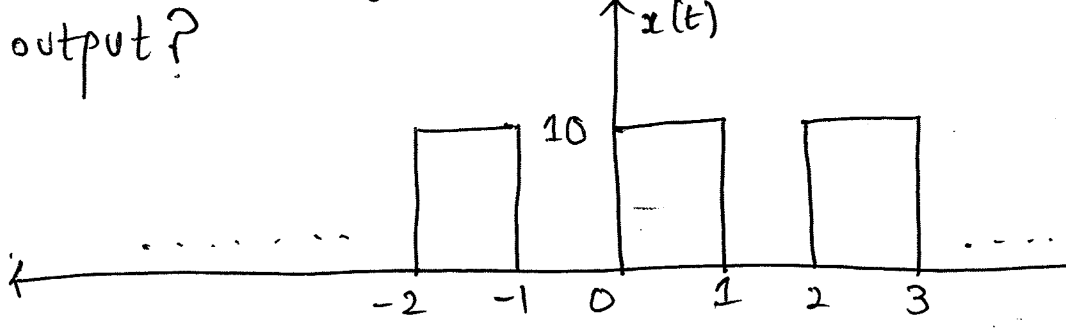
$\omega = 100$, $\omega = 300$, $\omega = 500$

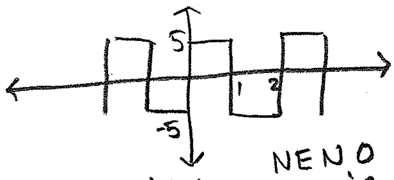
$HCF(100, 300, 500)$

$\omega_0 = 100\,\text{rad/sec.}$

Q:- $H(\omega) = \begin{cases} 1 & , |\omega| < 4\pi \\ 0 & , |\omega| > 4\pi \end{cases}$

A periodic signal $x(t)$ is shown in figure is applied to an LTI system with frequency response. Find output?





NENO
odd harmonic
↓
odd harmonic of sine

Sol^n:- $a_0 = \frac{10 \times 1}{2} = 5$, $T = 2$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t \, dt$$

$$= \frac{2}{2}\int_0^1 10\cos n\omega_0 t \, dt + \int_1^2 0.$$

$$= 1\left[10\,\frac{\sin \omega_0 n t}{n\omega_0}\right]_0^1$$

$$a_n = \left[\frac{10 \sin \frac{2\pi}{T}\cdot n t}{n\cdot\frac{2\pi}{T}}\right]_0^1 = 0$$

$$b_n = \frac{2}{T}\int_0^T x(t)\cdot \sin n\omega_0 t \, dt$$

$$= \frac{2}{2}\int_0^1 10 \sin n\omega_0 t \, dt + \frac{2}{2}\int_1^2 0$$

$$b_n = \left[\frac{-10\cos n\cdot\frac{2\pi}{2}\cdot t}{n\cdot\frac{2\pi}{2}}\right]_0^1 = \frac{10}{n\pi}\left[1-(-1)^n\right]$$

$\therefore b_n = 0$, $n =$ even

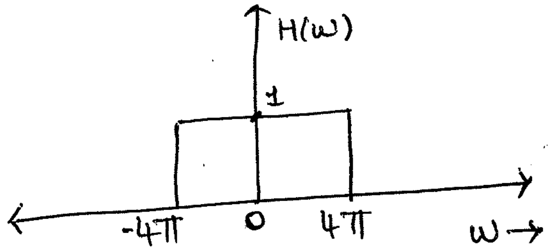
$b_n = \frac{20}{n\pi}$, $n =$ odd

$H(\omega) = 1$, $|\omega| < 4\pi$

$= 0$, $|\omega| > 4\pi$

$H(\omega) = 1$, $-4\pi < \omega < 4\pi$

$= 0$, otherwise



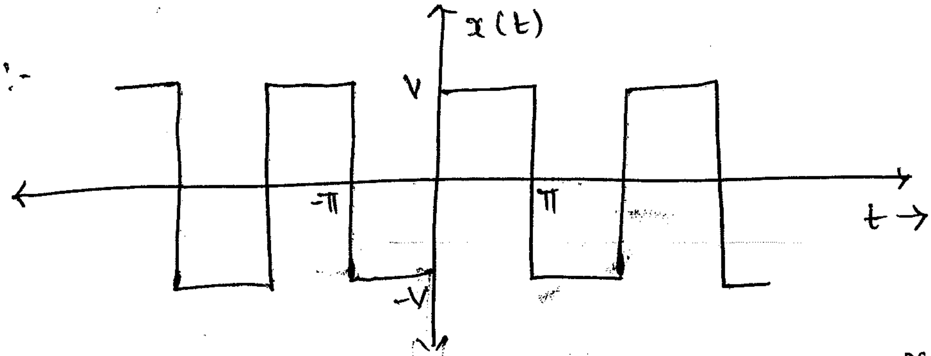
$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

$$x(t) = 5 + \frac{20}{\pi}\sin \pi t + \frac{20}{3\pi}\sin 3\pi t + \frac{20}{5\pi}\sin 5\pi t + \frac{20}{7\pi}\sin 7\pi t$$

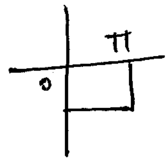
But due to filter, terms $\omega_0 > 4\pi$ will be rejected.

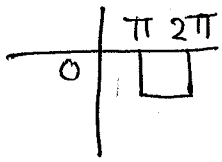
$$\therefore \boxed{x(t) = 5 + \frac{20}{\pi}\sin \pi t + \frac{20}{3\pi}\sin 3\pi t}$$

Q:-



Sol^n:- only b_n will be there



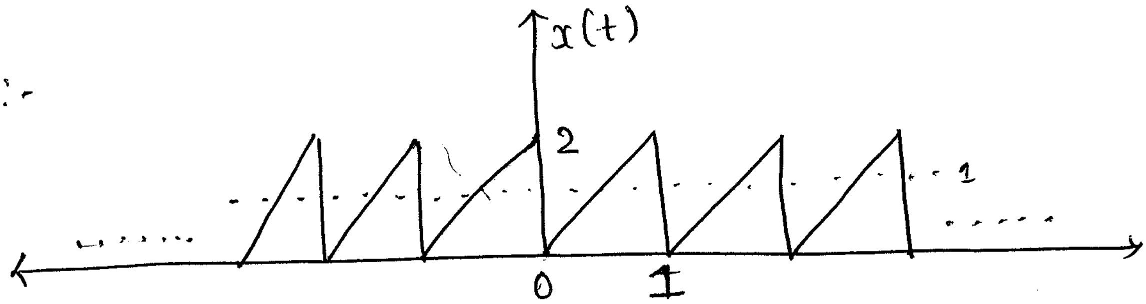


Check H.W.

It is odd H.W. symmetry

$\therefore b_n \in$ only odd components / harmonics of sine

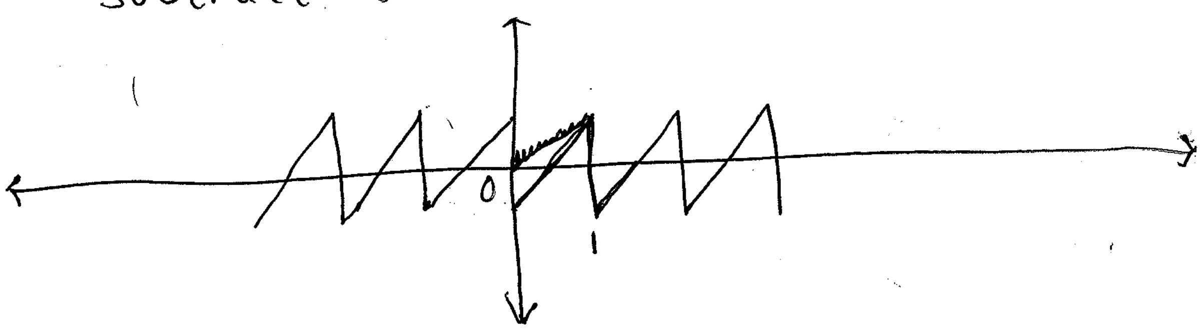
Q:-



Sol^n:- NENO

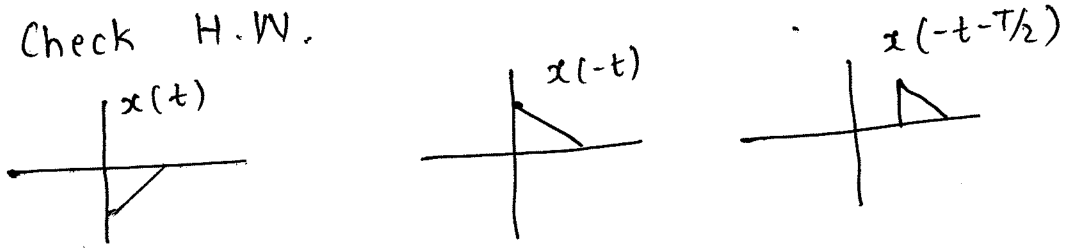
$$a_0 = \frac{\text{Area}}{T} = \frac{\frac{1}{2}\times 2 \times 1}{1} = 1 \text{ (DC value)}$$

Subtract DC value from $x(t)$



It is odd (sine harmonics will be there)

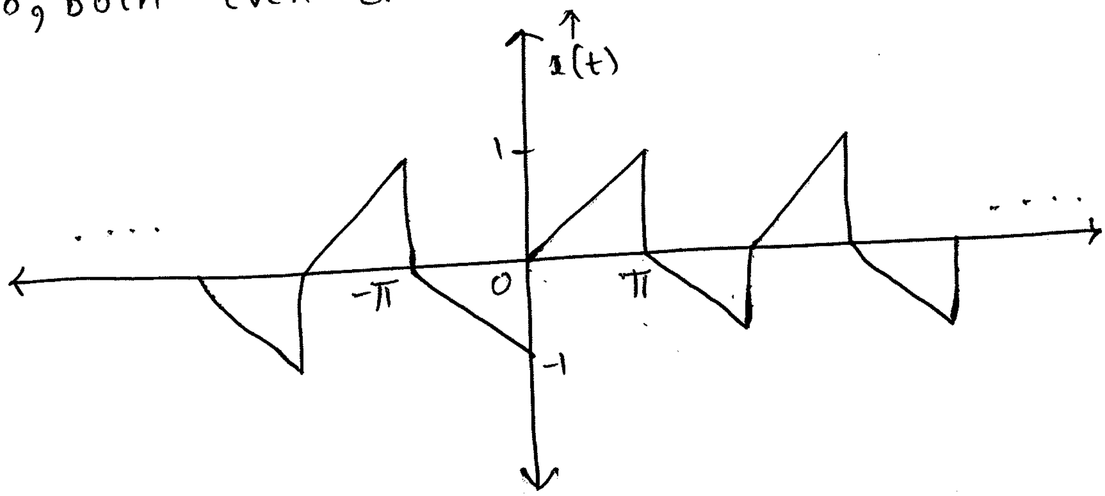
Check H.W.



Not H.W.

So, both even & odd harmonics of sine are there.

Q:-



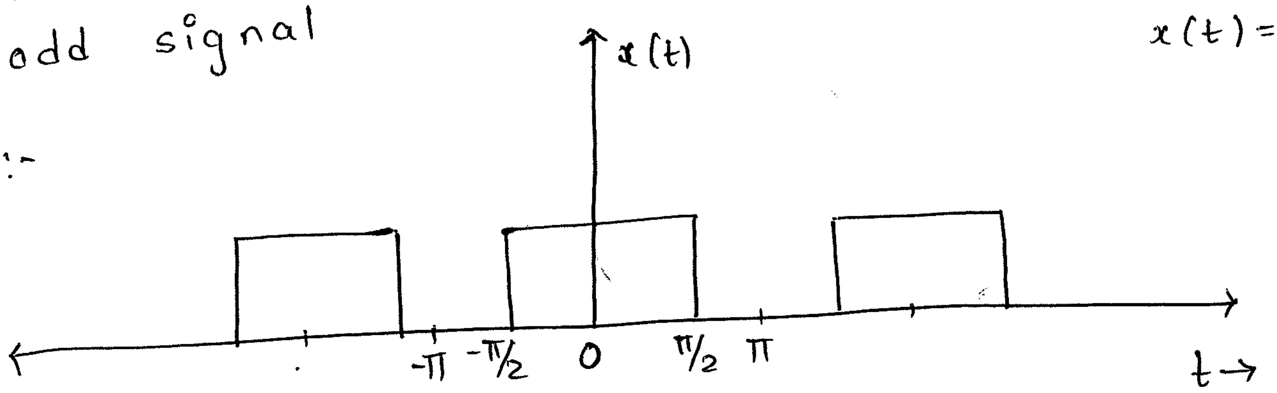
Sol^n:-

$a_0 = 0$

odd signal

$x(t) = x(-t)$

Q:-

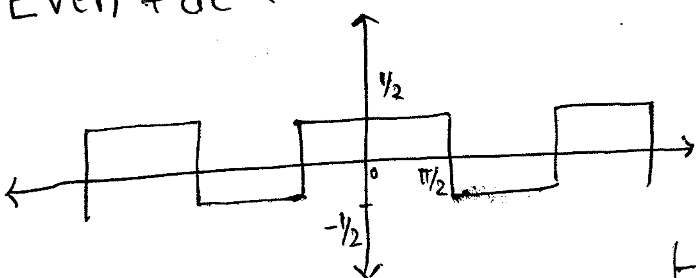


Sol^n:-

← T = 2π →

$a_0 = \frac{1}{T}$ Area $= \frac{1}{2\pi} \cdot \pi = \frac{1}{2}$ (dc. value) $a \neq 0$ Hidden

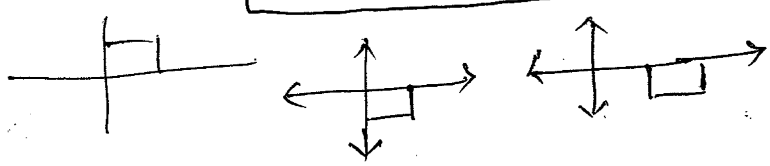
Even + dc (cosine)



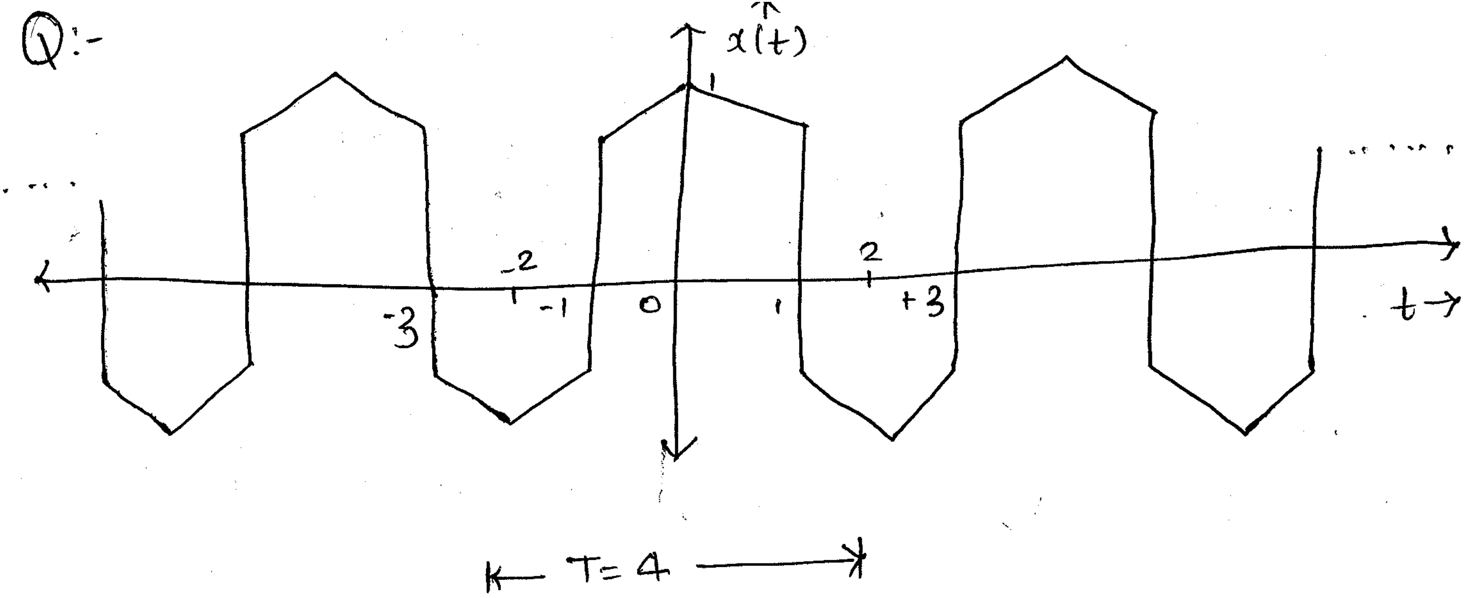
H.W. (odd)

$\boxed{\text{odd harmonics of cosine}}$

H.W.



Q:-



Sol:- $a_0 = \frac{Area}{T} = 0$

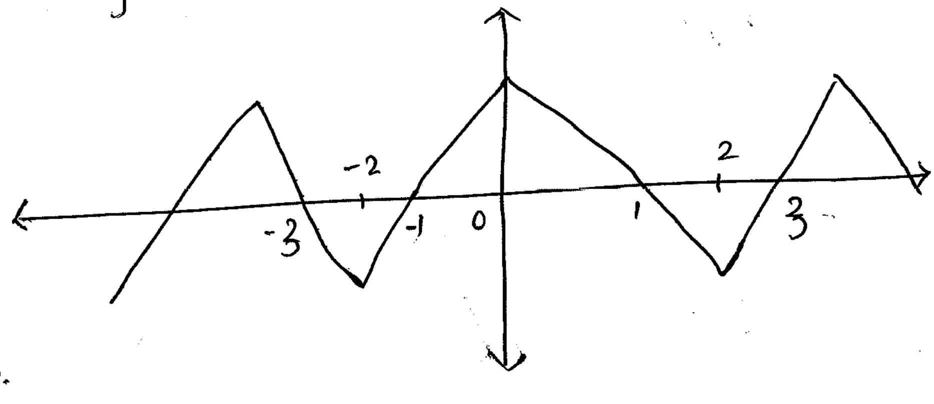
It is even signal but $a_0 = 0$

H.W. symmetry

odd H.W. symmetry

∴ only odd harmonics of cosine

Q:-



Sol:-

Even function ← T= 4 →

Area = 0

$a_0 = 0$

odd harmonics of cosine (H.W.) ⇒ odd

Q:-

List - I	List - II
(A) $f(t) + f(-t) = 0$	1. Even harmonics can exist
(B) $f(t) - f(-t) = 0$	2. Odd harmonics can exist
(c) $f(t) + f(t - t/2) = 0$	3. The d c & cosine terms can exist
(d) $f(t) - f(t - t/2) = 0$	4. sine terms can exist
	5. cosine terms of even harmonics can exist.

Properties of Fourier Series

① Linearity

$$\alpha x(t) + \beta y(t) \xleftrightarrow{F.S.} \alpha C_n + \beta B_n$$

② Time shifting

$$x(t-t_0) \xleftrightarrow{F.S.} e^{-j\omega_0 n t_0} \cdot C_n$$

③ Freq. shifting

$$e^{jm\omega_0 t} x(t) \xleftrightarrow{F.S.} C_{n-m}$$

④ Time scaling

if $x(t) \xleftrightarrow{F.S.} C_n |_T$

$$x(at) \xleftrightarrow{F.S.} C_n |_{T/a}, \quad \omega_0' = a\omega_0$$

Proof:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}$$

$$x(at) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 at}$$

$$x(at) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega' t}, \quad \omega' = a\omega_0$$

⑤ Time reversal

if $x(t) \xleftrightarrow{F.S.} C_n$

then $x(-t) \xleftrightarrow{F.S.} C_{-n}$

⑥ Conjugation

if $x(t) \xleftrightarrow{F.S.} C_n$

$$x^*(t) \xleftrightarrow{F.S.} C_{-n}^*$$

Proof:- $C_n = \frac{1}{T}\int_0^T x(t) \cdot e^{-jn\omega_0 t} dt$

$$C_{-n} = \frac{1}{T}\int_0^T x(t) \cdot e^{jn\omega_0 t} dt$$

$$C_{-n}^* = \frac{1}{T}\int_0^T x^*(t) e^{-jn\omega_0 t} dt$$

$$\therefore x^*(t) \xleftrightarrow{F.S.} C_{-n}^*$$

⑦ Periodic Convolution

if $x(t) \xleftrightarrow{F.S.} C_n$ &

$h(t) \xleftrightarrow{F.S.} B_n$

then

$y(t) = x(t) * h(t)$.

$$y(t) \xleftrightarrow{F.S.} T[C_n \cdot B_n]$$

$$x(t) * h(t) \xleftrightarrow{F.S.} T[C_n \cdot B_n]$$

⑧ Multiplication

if $x(t) \xleftrightarrow{F.S.} C_n$

$h(t) \xleftrightarrow{F.S.} B_n$

then

$$x(t) \cdot h(t) \xleftrightarrow{F.S.} \sum_{k=-\infty}^{\infty} C_k B_{n-k}$$

⑨ Differentiation

if $x(t) \xleftrightarrow{F.S.} C_n$

then $\dfrac{d\,x(t)}{dt} \xleftrightarrow{F.S.} jn\omega_o C_n$

Proof: $x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_o t}$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_o t} \cdot (jn\omega_o)$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} (jn\omega_o C_n)\, e^{jn\omega_o t}$$

$$\boxed{\frac{d^k x(t)}{dt^k} \xleftrightarrow{F.S.} (jn\omega_o)^k \cdot C_n}$$

(10) Integration

$$\boxed{\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.S.} \frac{C_n}{jn\omega_0}}$$

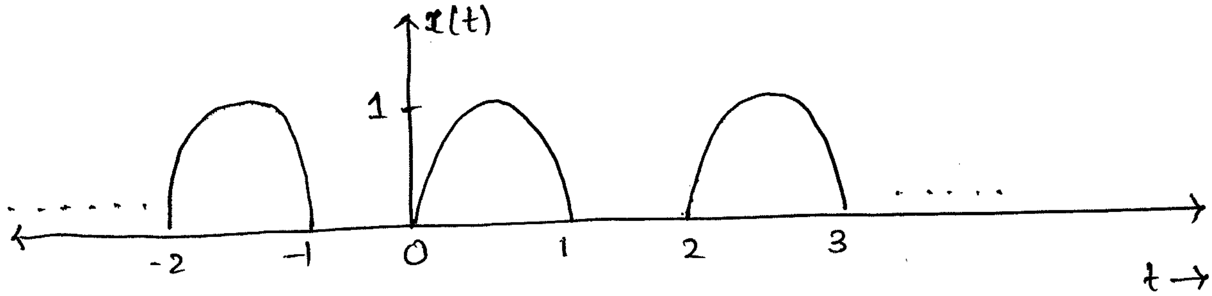
(11) Parseval's Power Theorem

$$\boxed{\frac{1}{T}\int_{T} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

Time domain

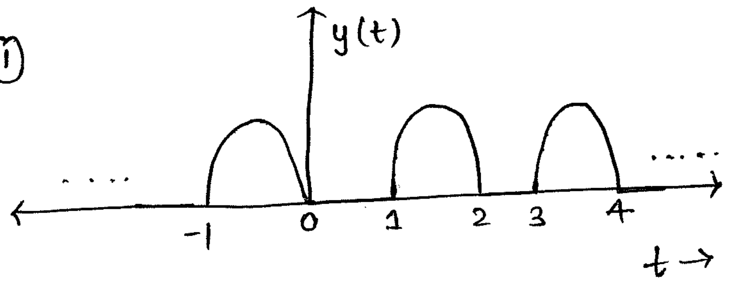
a_0^2	C_0^2
$\frac{a_n^2}{2}$	$\frac{a_n^2+b_n^2}{4} = \lvert C_n\rvert^2$
$\frac{b_n^2}{2}$	$\frac{a_n^2+b_n^2}{4} = \lvert C_{-n}\rvert^2$

Q:- The fourier series co-efficient of signal $x(t)$ shown in figure are $C_0 = \frac{1}{\pi}$, $C_1 = -j0.25$, $C_n = \frac{1}{\pi(1-n^2)}$ (n=even)



Find Fourier series co-efficient of $y(t)$, $f(t)$ and $g(t)$.

(1)



$y(t) = x(t-1)$
$y(t) = x(t+1)$
$y(t) = x(-t)$

(1) $y(t) = x(t-1)$ OR (2) $x(t+1)$

$d_n = e^{-j\omega_0 n t_0} C_n$

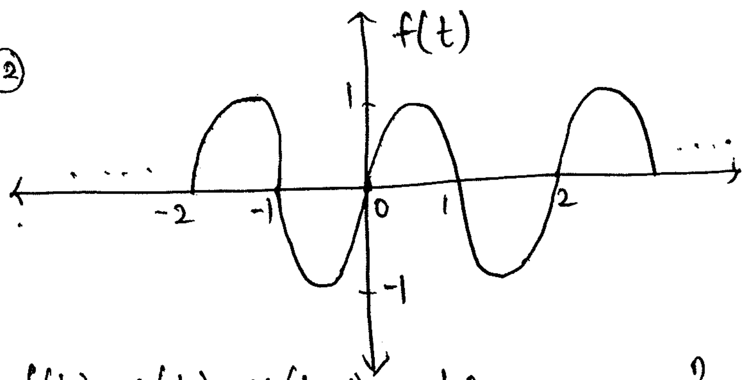
$d_n = e^{-j\pi \cdot n \cdot 1} \cdot C_n$

$d_n = (\cos\pi n + j\sin\pi n) C_n = (-1)^n \cdot C_n$

$d_0 = 1/\pi = C_0$

$d_1 = -C_1 = +j0.25$

(2)



$f(t) = x(t) - x(t-1)$

↓

$f_n = C_n - (-1)^n \cdot C_n$

$f_0 = C_0 - C_0 = 0$

$f_1 = C_1 + C_1 = -0.5j$

$f_n = \frac{1}{\pi(1-n^2)} - (-1)^n \cdot \frac{1}{\pi(1-n^2)}$

$f_n = 0$; n=even

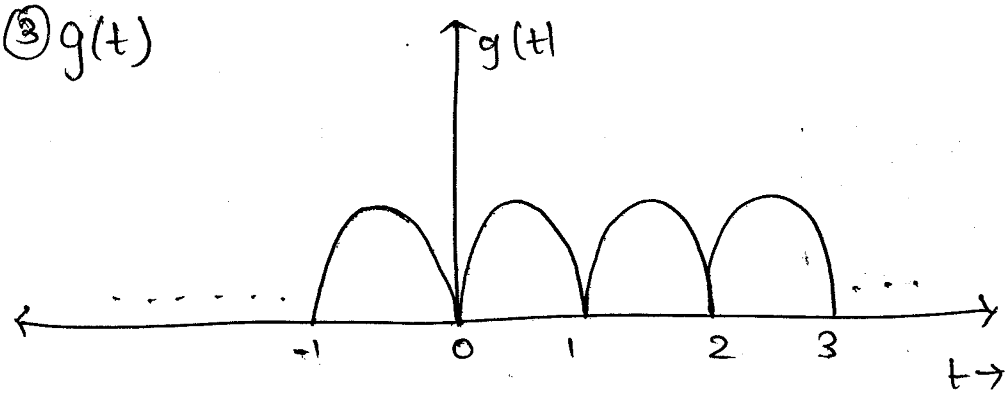
(3) $y(t) = x(-t)$

$d_n = C_{-n}$

$d_0 = C_0 = 1/\pi$

$d_1 = C_{-1} = C_1^* = +j0.25$

③ $g(t)$



$g(t) = x(t) + x(t-1)$

$\downarrow$

$g_n = C_n + (-1)^n C_n$

$g_0 = C_0 + (-1)^0 C_0$

$g_0 = \frac{2}{\pi}$

$g_1 = C_1 - C_1 = 0$

$g_n = \frac{1}{\pi(1-n^2)} + \frac{(-1)^n \cdot 1}{\pi(1-n^2)}$ $\left[\because g_n = C_n + C_{-n}\right]$

$g_n = \frac{2}{\pi(1-n^2)}$

Q:- Let $x(t)$ be a periodic signal with period T and fourier series co-efficient C_n. Let $y(t)=$ $y(t) = x(t-t_0) + x(t+t_0)$. The Fourier series co-efficient of $y(t)$ is D_n. If $D_n = 0$ for all odd n then t_0 can be (a) $T/8$ (b) $T/4$ (c) $T/2$ (d) $2T$

Sol:- $y(t) = x(t-t_0) + x(t+t_0)$

$C_n = D_n e^{-jn\omega_0 t_0} + D_n e^{+jn\omega_0 t_0}$

$C_n = \frac{D_n}{2} \cdot 2 \left[e^{-jn\omega_0 t_0} + e^{jn\omega_0 t_0}\right]$

$C_n = 2d_n \cos n\omega_0 t_0$

$\cos n\omega_0 t_0 = 0$

$\cancel{n}\omega_0 t_0 = \frac{\cancel{n}\pi}{2}$

$\Rightarrow d_n = 0$

$\frac{2\pi}{T} t_0 = \frac{\pi}{2}$

$\boxed{t_0 = T/4}$

Q:- A periodic signal has fourier series representation $x(t) \xleftrightarrow{F.S.} C_n = -n \cdot 2^{|n|}$. Without finding $x(t)$ find Fourier series representation of $y(t)$

(a) $y(t) = x(3t)$

(b) $y(t) = \frac{d}{dt} x(t)$

(c) $y(t) = x(t-1)$

(d) $y(t) = Re\{x(t)\}$

(e) $y(t) = \cos 4\pi t \cdot x(t)$

Sol^n:- Let $y(t) \longrightarrow [d_n, \omega_0]$

(a) $y(t) = x(t) \Big|_{C_n, T}, \omega_0$

$y(t) = x(3t) \Big|_{C_n, T/3}, \omega_0' = 3\omega_0 = 3\pi$

(b) $y(t) = \frac{d}{dt} x(t)$

$d_n \cancel{y(t)} = (jn\omega_0) C_n$

Hint:

$x(t) = X_R(t) + jX_I(t)$

$x^*(t) = X_R(t) - jX_I(t)$

$X_R(t) = \frac{x(t) + x^*(t)}{2}$

$X_I(t) = \frac{x(t) - x^*(t)}{2j}$

(c) $y(t) = x(t-1)$

$$y(t) = C_n e^{-jn\omega_0 t}$$

$$y(t-1) = C_n e^{-jn\omega_0}$$

(d) $y(t) = \mathrm{Re}\{x(t)\}$

$$x_R(t) = \frac{x(t) + x^*(t)}{2}$$

$$\Big\updownarrow \text{F.S.} \qquad \Big\updownarrow \text{F.S.}$$

$$d_n = \frac{C_n + C^*_{-n}}{2}$$

~~(e) $y(t) = \cos 4\pi t \cdot x(t)$ $= e^{4\pi}$~~

$$d_n = \frac{-n\cdot 2^{|n|} + +n\cdot 2^{|n|}}{2}$$

$$\boxed{d_n = 0}$$

$$x_I(t) = \frac{C_n - C^*_{-n}}{2j}$$

$$d_{im} = j(n2^{|n|})$$

(e) $y(t) = \cos 4\pi t \cdot x(t)$

$$= \left(\frac{e^{j4\pi t} + e^{-j4\pi t}}{2}\right) x(t)$$

$$y(t) = \frac{e^{j4\pi t}\cdot x(t)}{2} + \frac{e^{-j4\pi t}\cdot x(t)}{2}$$

$$\Big\updownarrow \text{F.S.} \qquad \Big\updownarrow \text{F.S.}$$

$$d_n = \frac{1}{2} C_{n-4} + \frac{1}{2} C_{n+4}$$

$$= \frac{1}{2}\left[-(n-4)2^{|n-4|} + -(n+4)2^{|n+4|}\right]$$

$$= \frac{1}{2}\left[-n\cdot 2^{|n-4|} + 4\cdot 2^{|n-4|} - n\cdot 2^{|n+4|} - 4\cdot 2^{|n+4|}\right]$$

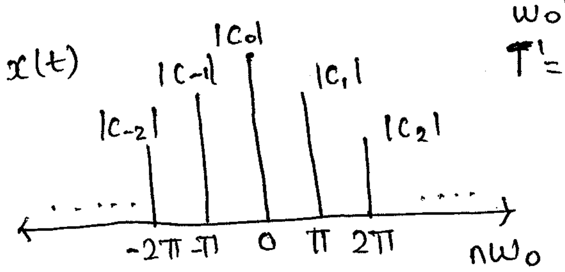
(a) $y(t) = x(3t)$

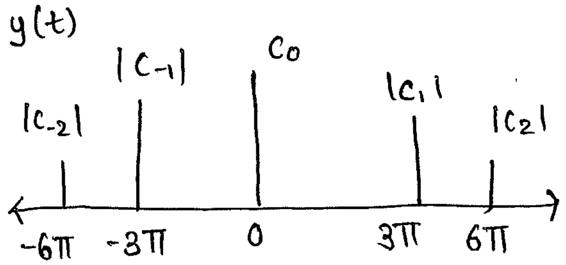
$$y(t) \xleftrightarrow{\text{F.S.}} C_n = -n\cdot 2^{|n|}$$

$|\omega_0'| = 3\omega_0$

$\omega_0' = 3\pi$

$T' = T/3$





(b) $y(t) = \frac{d}{dt} x(t)$

$$\Big\updownarrow \text{F.S.}$$

$$d_n = (jn\omega_0) C_n$$

$$\boxed{(d_n) = (jn\pi)\cdot(-n)\, 2^{|n|}}$$

(c) $y(t) = x(t-1)$

$$\Big\updownarrow \text{F.S.}$$

$$d_n = e^{-jn\omega_0 t_0} C_n$$

$$d_n = e^{-jn\pi} C_n$$

$$d_n = (\cos n\pi - j\sin n\pi) C_n$$

$$d_n = (-1)^n \cdot (-1)\, n\, 2^{|n|}$$

$$\boxed{d_n = (-1)^{n+1}\cdot n\cdot 2^{|n|}}$$

Q:- Find value of d_6 in (e).

(e) $d_n = \frac{1}{2} C_{n-4} + \frac{1}{2} C_{n+4}$

$$d_6 = \frac{1}{2} C_2 + \frac{1}{2} C_{10}$$

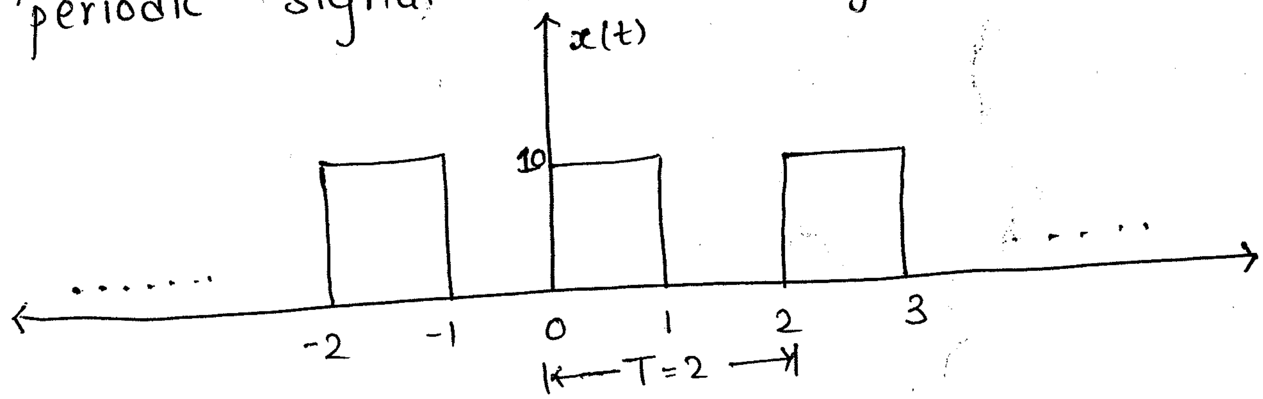
$$= \frac{1}{2}\left[-2\cdot|2|^{2} + (-10)\, 2^{10}\right]$$

$$= \frac{1}{2}\left[-2(4) - 10(1024)\right]$$

$$= \frac{1}{2}(-8 - 10240)$$

$$d_6 = \frac{-10248}{2} = \underline{-5124}$$

** Q:- Find the power upto 2^{nd} harmonic for the *periodic signal shown in figure?



Sol^n:- Approach: Find C_n, a_n, b_n

Then $P = \sum_{n=-\infty}^{\infty} |C_n|^2$ ← Parseval's theorem

NENO

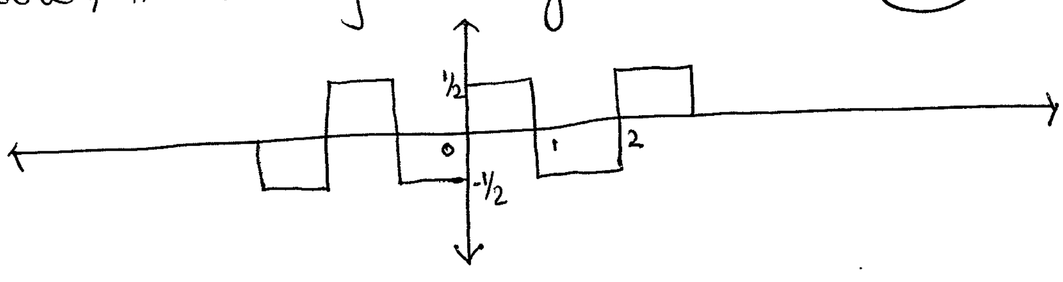
- $DC = a_0 = \frac{\text{Area}}{\text{Timeperiod}} = \frac{10\times 1}{2} = \frac{10}{2} = 5$

Now, hidden symmetry

odd

$a_n = 0$

$b_n = ?$



$$b_n = \frac{2}{T}\int_T x(t)\sin n\omega_0 t \, dt = \frac{20}{n\pi} \quad , n = \text{odd}$$

$$= 0 \quad , n = \text{even}$$

$$C_n = \frac{a_n - j b_n}{2}$$

$$C_n = \frac{0 - j\frac{20}{n\pi}}{2} = -j\frac{10}{n\pi}$$

Now,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$= \sum_{n=-2}^{2} |C_n|^2$$

$$= \left|\left(\frac{5j}{\pi}\right)^2\right| + \left|\left(\frac{10j}{\pi}\right)^2\right| + \left|\left(\frac{-10j}{\pi}\right)^2\right| + \left|\left(\frac{-5j}{\pi}\right)^2\right|$$

$$= \left|\frac{-25}{\pi^2}\right| + \left|\frac{-100}{\pi^2}\right| + \left|\frac{-100}{\pi^2}\right| + \left|\frac{-25}{\pi^2}\right|$$

$$P = \left|\frac{-250}{\pi^2}\right| = \frac{250}{\pi^2} = 25.33\,W$$

NOTE:

$$x(t) = a_0 + \sum_{n=-\infty}^{\infty} b_n \sin n\omega_0 t$$

$$= a_0 + \underbrace{b_1 \sin \pi t}_{1^{st}\text{ harm.}} + \underbrace{b_3 \sin 3\pi t}_{3^{rd}\text{ harm.}} + b_5 \sin 5\pi t + \dots$$

(actually 2^{nd} harmonic is zero)

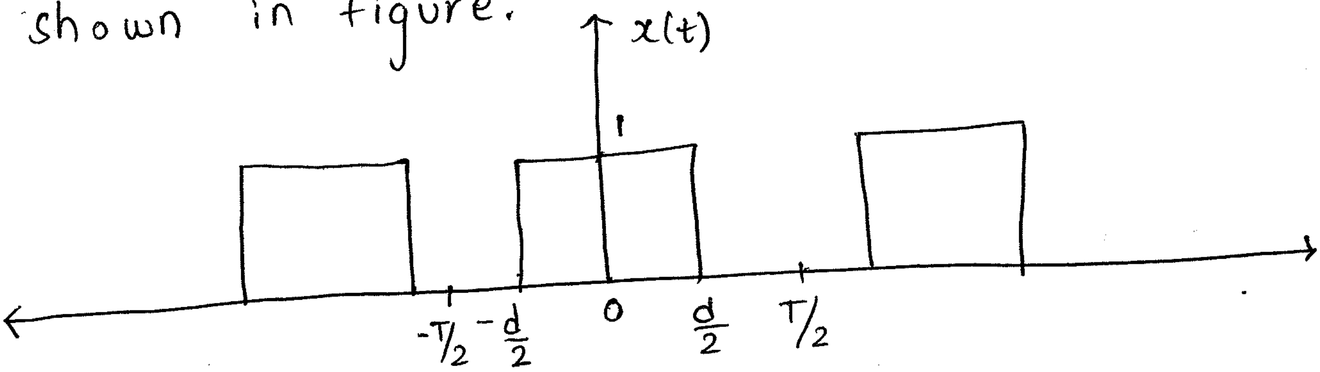
$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 = \sum_{n=-2}^{2} |C_n|^2 = \cancel{|C_{-2}|^2} + |C_{-1}|^2 + |C_0|^2 + |C_1|^2 + \cancel{|C_2|^2}$$

$$P = a_0^2 + \frac{b_1^2}{2} = 25 + \frac{400}{2\times\pi^2} = 45.26\,W$$

$P_{\text{at } 2^{nd}\text{ harmonic}} = 0$

$P_{\text{at (DC)}} = 25\,W$

Q:- By using derivative method find the exponential fourier signal co-efficient of the signal shown in figure.



Sol^n:- Let fourier co-efficient of $x(t)$ is C_n

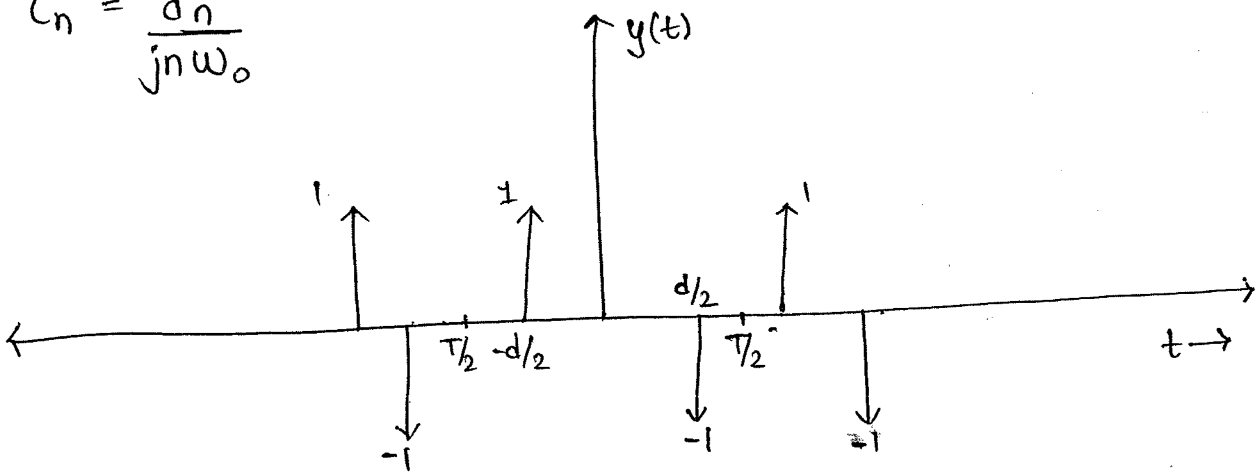
$|\leftarrow T \rightarrow|$

$$y(t) = \frac{d\,x(t)}{dt}$$

$\downarrow$

$$d_n = (jn\omega_0)C_n$$

$$C_n = \frac{d_n}{jn\omega_0}$$



$$x(t) = u(t+d/2) - u(t-d/2)$$

$$y(t) = \frac{d\,x(t)}{dt} = \delta(t+d/2) - \delta(t-d/2)$$

$$d_n = \frac{1}{T}\int_0^T y(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\left[\int_0^T \delta(t+d/2)e^{-jn\omega_0 t} - \int_0^T \delta(t-d/2)\cdot e^{-jn\omega_0 t}\right]$$

$$d_n = \frac{1}{T}\left[e^{jn\omega_0 \frac{d}{2}} - e^{-jn\omega_0 d/2}\right]$$

$$= \frac{1}{T}\cdot 2j\left[\frac{e^{jn\omega_0 d/2} - e^{-jn\omega_0 d/2}}{2j}\right]$$

$$\boxed{d_n = \frac{2j}{T}\sin(n\omega_0 d/2)}$$

$$C_n = \frac{d_n}{\cancel{2}jn\omega_0} = \frac{2j\sin\left(\frac{n\omega_0 d}{2}\right)}{T\cdot jn\omega_0} = \frac{\cancel{2}\sin\left(\frac{n\omega_0 d}{2}\right)}{\cancel{T}\cdot n\cdot\frac{\cancel{2}\pi}{\cancel{T}}}$$

$$C_n = \frac{\sin\left(\frac{n\omega_0 d}{2}\right)\cdot\frac{\omega_0 d}{2}}{\pi\ \frac{n\omega_0 d}{2}}$$

$$= \frac{\omega_0 d}{2\pi}\, sa(n\omega_0 d/2) \qquad \left[\because sa(x) = \frac{\sin x}{x}\right]$$

$$= \frac{\cancel{2\pi}\, d}{T\cdot\cancel{2\pi}}\, sa\left(\frac{n\omega_0 d}{2}\right)$$

$$\boxed{C_n = \frac{d}{T}\, sa\left(\frac{n\omega_0 d}{2}\right)}$$

OR

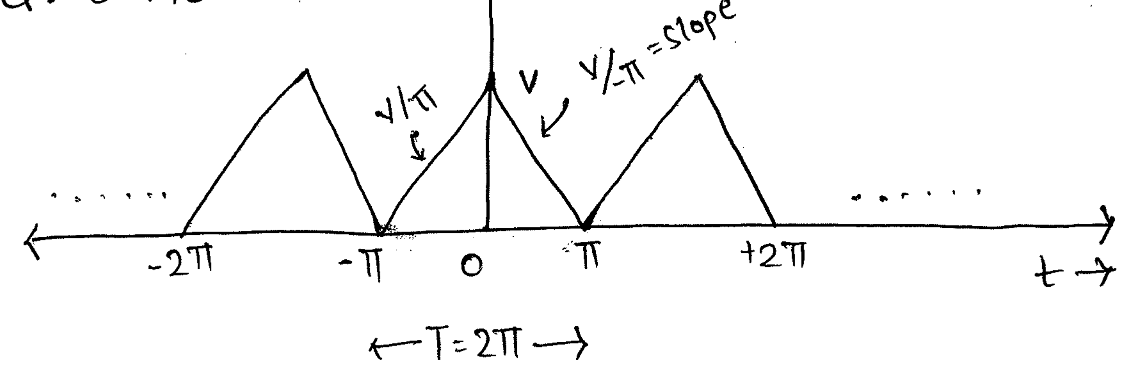
$$C_n = \frac{d}{T}\ \frac{\sin n\omega_0 d/2}{n\omega_0 d/2}$$

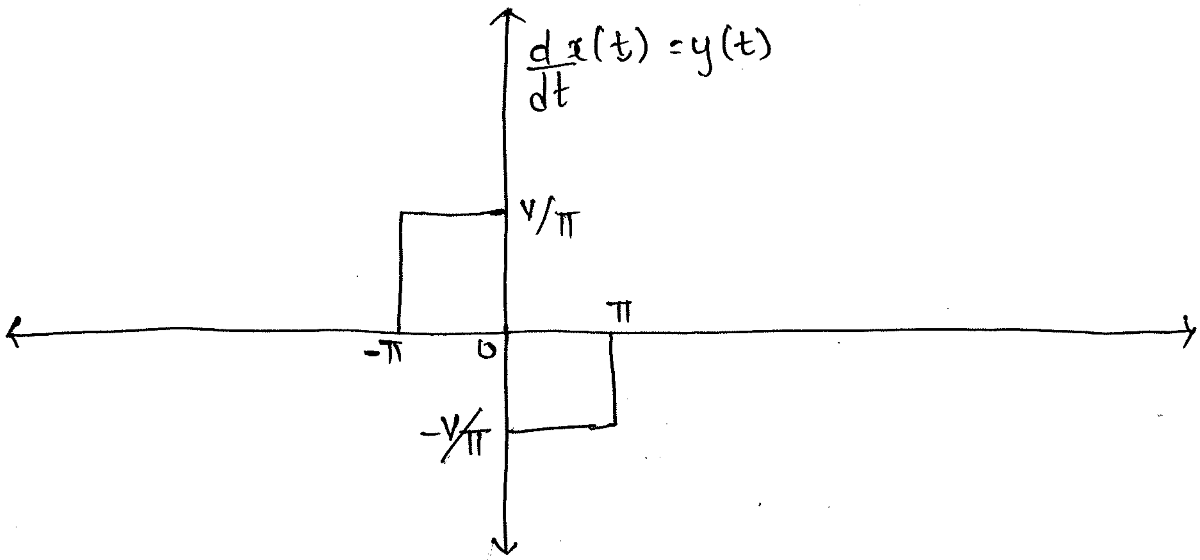
$$= \frac{d}{T}\ \frac{\sin n\frac{2\pi}{T}\cdot d/2}{n\cdot\frac{2\pi}{T} d/2} \qquad \left[\because \operatorname{sinc} x = \frac{\sin\pi x}{\pi x}\right]$$

$$\boxed{C_n = \frac{d}{T}\operatorname{sinc}\left(\frac{nd}{T}\right)}$$

Q:- Same as before

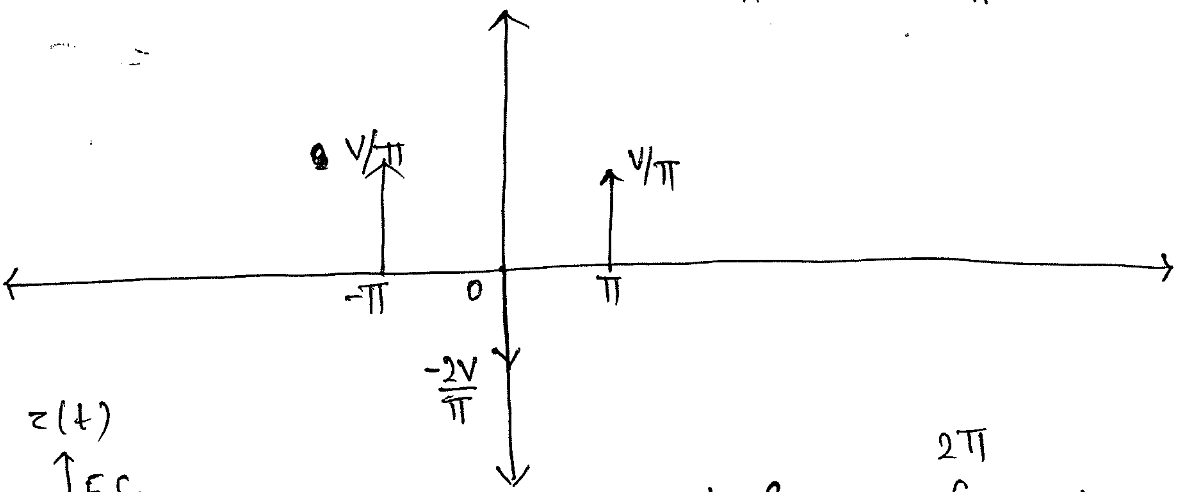
$$\text{Slope} = \frac{\text{Change in Amp}}{\text{Change in Tim}} = \frac{\text{Next} - \text{Previous}}{\text{next} - (-\pi)}$$





$$y(t) = \frac{V}{\pi} u(t+\pi) - \frac{2V}{\pi} u(t) + \frac{V}{\pi} u(t-\pi)$$

$$z(t) = \frac{dy(t)}{dt} = \frac{V}{\pi} \delta(t+\pi) - \frac{2V}{\pi} \delta(t) + \frac{V}{\pi} \cdot \delta(t-\pi)$$



$z(t) \xleftrightarrow{\text{F.S.}} f_n = (jn\omega_0)^2 C_n$

$$C_n = \frac{f_n}{(jn\omega_0)^2}$$

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{V}{\pi} \delta(t+\pi) - \frac{2V}{\pi} \delta(t) + \frac{V}{\pi} \delta(t-\pi) \right) e^{-jn\omega}$$

$$f_n = \frac{1}{2\pi} \left[\frac{V}{\pi} e^{+jn\pi\omega_0} - \frac{2V}{\pi} e^{0} + \frac{V}{\pi} e^{-jn\omega_0\pi} \right]$$

$$f_n = \frac{1}{2\pi}\left[\frac{V}{\pi}\cos n\pi \cdot \frac{2\pi}{T}\cdot t - \frac{2V}{\pi} + \frac{V}{\pi}^2\left(\frac{-2V}{\pi} + \frac{V}{\pi}\left(\frac{e^{+jn\frac{2\pi}{T}\cdot\pi} - e^{-jn\frac{2\pi}{T}}}{2}\right)\right)\right]$$

$$= \frac{-V}{\pi^2} + \frac{V}{\pi^2}\cos n\pi$$

$$f_n = \frac{-V}{\pi^2} + \frac{V}{\pi^2}(-1)^n = \frac{V}{\pi^2}\left(-1 + (-1)^n\right)$$

$$\boxed{C_n = \frac{f_n}{(jn\omega_0)^2}}$$

$$\cancel{C_n = \frac{V}{\pi^2}\left(\frac{1-(-1)^n}{n^2}\right)}$$

$$C_n = \frac{-V/\pi^2 + V/\pi^2(-1)^n}{-(n^2 \times 1)}$$

$$C_n = \frac{V/\pi^2 - V/\pi^2(-1)^n}{n^2}$$

$$C_n = \frac{V}{\pi^2}\frac{(1-(-1)^n)}{n^2}$$

$$\boxed{C_n = \frac{2V}{\pi^2 n^2}, \quad n = \text{odd}}$$

$\cancel{C_0 = \frac{V}{2}}$

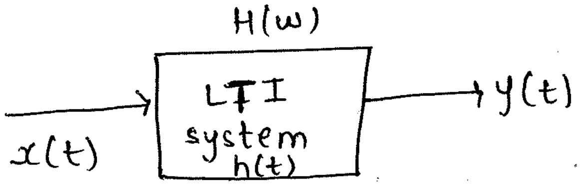
Limitation of Fourier Series

- It can be used only for periodic inputs and thus not applicable for aperiodic one.
- It cannot be used for unstable or even marginally stable systems, applicable only for stable

Sol^n:- Laplace transform by using e^{st}

Sol^n:- Fourier Transform (applicable to aperiodic also)

* SYSTEM RESPONSE



$$x(t) = \sum_{n=-\infty}^{\infty} x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\cdot h(\tau)\,d\tau$$

if $x(t) = e^{j\omega t}$

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\cdot h(\tau)\,d\tau$$

$$y(t) = e^{j\omega t}\underbrace{\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega\tau}\cdot d\tau}_{\text{F.T.}}$$

$$y(t) = e^{j\omega_0 t}H(\omega)$$

if i/p $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$$\downarrow$$

$$y(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\cdot H(n\omega_0)$$

$$y(t) = \sum_{n=-\infty}^{\infty} (C_n H(n\omega_0))e^{jn\omega_0 t}$$

$$x(t) \xleftrightarrow{\text{F.S.}} C_n$$

$$\boxed{y(t) \xleftrightarrow{\text{F.S.}} C_n\cdot H(n\omega_0)}$$