HYPERBOLA [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Singel Correct Answer Type

1. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, $r > 1$, represents

   a. an ellipse b. a hyperbola
   c. a circle d. none of these

   (IIT-JEE 1981)

2. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and $(x_2; y_2)$ in the region, the point $((x_1 + x_2)/2, (y_1 + y_2)/2)$ is also in the region. The inequality defining this region is

   a. $x^2 + 2y^2 \le 1$ b. $\max\{|x|, |y|\} \le 1$
   c. $x^2 - y^2 \le 1$ d. $y^2 - x \le 0$

   (IIT-JEE 1981)

3. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents

   a. no locus if $k > 0$ b. an ellipse if $k < 0$
   c. a point if $k = 0$ d. a hyperbola if $k > 0$

   (IIT-JEE 1994)

4. Let $P(a \sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to

   a. $\frac{a^2 + b^2}{a}$ b. $-\left(\frac{a^2 + b^2}{a}\right)$
   c. $\frac{a^2 + b^2}{b}$ d. $-\left(\frac{a^2 + b^2}{b}\right)$

   (IIT-JEE 1999)

5. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

   a. $9x^2 - 8y^2 + 18x - 9 = 0$ b. $9x^2 - 8y^2 - 18x + 9 = 0$
   c. $9x^2 - 8y^2 - 18x - 9 = 0$ d. $9x^2 - 8y^2 + 18x + 9 = 0$

   (IIT-JEE 1999)

6. Which of the following is independent of α in the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, $(0 < \alpha < \pi/2)$?

   a. Eccentricity b. Abscissa of foci
   c. Directrix d. Vertex **(IIT-JEE 2003)**

7. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

   a. $(-2, \sqrt{6})$ b. $(-5, 2\sqrt{6})$
   c. $(1/2, 1/\sqrt{6})$ d. $(4, -\sqrt{6})$

   (IIT-JEE 2004)

8. A hyperbola having the transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

a. $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

b. $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$

c. $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$

d. $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$ **(IIT-JEE 2007)**

9. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the endpoints of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of triangle ABC is

a. $1-\sqrt{2/3}$ b. $\sqrt{3/2}-1$

c. $1+\sqrt{2/3}$ d. $\sqrt{3/2}+1$

(IIT-JEE 2008)

10. Let a and b be nonzero real numbers. Then the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

a. four straight lines, when $c = 0$ and a, b are of the same sign

b. two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a

c. two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

d. a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

(IIT-JEE 2008)

11. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at $(9, 0)$, then the eccentricity of the hyperbola is

a. $\sqrt{5/2}$ b. $\sqrt{3/2}$

c. $\sqrt{2}$ d. $\sqrt{3}$ **(IIT-JEE 2011)**

Multiple Correct Answers Type

1. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse. Also, the product of the eccentricities of the given ellipse and hyperbola is 1. Then,

a. the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

b. the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

c. the focus of the hyperbola is $(5, 0)$

d. the vertex of the hyperbola is $(5\sqrt{3}, 0)$

(IIT-JEE 2006)

2. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then

a. the equation of the ellipse is $x^2 + 2y^2 = 2$

b. the foci of the ellipse are $(\pm 1, 0)$

c. the equation of the ellipse is $x^2 + 2y^2 = 4$

d. the foci of the ellipse are $(\pm\sqrt{2}, 0)$ **(IIT-JEE 2009)**

3. Let the eccentricity of the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

a. the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

b. a focus of the hyperbola is $(2, 0)$

c. the eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$

d. the equation of the hyperbola is $x^2 - 3y^2 = 3$

(IIT-JEE 2011)

4. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are

a. $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ b. $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

c. $(3\sqrt{3}, -2\sqrt{2})$ d. $(3\sqrt{3}, 2\sqrt{2})$

(IIT-JEE 2012)

5. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle PMN, then the correct expression(s) is (are)

a. $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

b. $\frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$ for $x_1 > 1$

c. $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

d. $\frac{dm}{dy_1} = \frac{1}{3}$ for $x_1 > 0$ **(JEE Advanced 2015)**

Linked Comprehension Type

For Problems 1 and 2

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at points A and B. **(IIT-JEE 2010)**

1. The equation of a common tangent with positive slope to the circle as well as to the hyperbola is

a. $2x - \sqrt{5}y - 20 = 0$ b. $2x - \sqrt{5}y + 4 = 0$
c. $3x - 4y + 8 = 0$ d. $4x - 3y + 4 = 0$

2. The equation of the circle with AB as its diameter is

a. $x^2 + y^2 - 12x + 24 = 0$ b. $x^2 + y^2 + 12x + 24 = 0$
c. $x^2 + y^2 + 24x - 12 = 0$ d. $x^2 + y^2 - 24x - 12 = 0$

Matching Column Type

1. Match the conic in Column I with the statements/expressions in Column II. **(IIT-JEE 2009)**

Column I	Column II
(a) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(b) Parabola	(q) Point z in the complex plane satisfying $\|z + 2\| - \|z - 2\| = \pm 3$
(c) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(d) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \|z\|^2 + 1$

Integer Answer Type

1. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **(IIT-JEE 2010)**

Fill in the Blanks Type

1. An ellipse has eccentricity 1/2 and one focus at $S(1/2, 1)$. Its one directrix is the common tangent (nearer to S) to the circle $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. The equation of the ellipse in standard form is_________. **(IIT-JEE 1996)**

Subjective Type

1. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of the midpoint of the chord of contact. **(IIT-JEE 2005)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. d. 2. c. 3. c. 4. d. 5. b.
6. b. 7. d. 8. a. 9. b. 10. b.
11. b.

Multiple Correct Answers Type

1. a., c. 2. a., b. 3. b., d. 4. a., b.
5. a., b., d.

Linked Comprehension Type

1. b. 2. a.

Matching Column Type

1. (d) – (q), (s)

Integer Answer Type

1. 2

Fill in the Blanks Type

1. $9\left(x - \frac{1}{3}\right)^2 + 12(y - 1)^2 = 1$

Subjective Type

1. $\frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2$

Hints and Solutions

JEE Advaced

Single Correct Answer Type

1. d. Given that

$$\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$$

As $r > 1$, we have $1 - r < 0$ and $1 + r > 0$.

Let $1 - r = -a^2$, $1 + r = b^2$.

Then we get

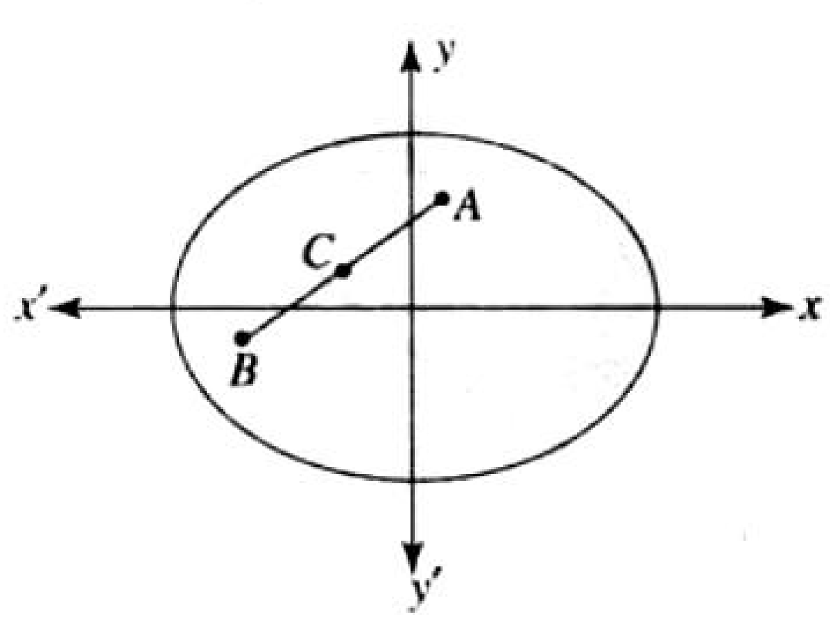
$$\frac{x^2}{-a^2} - \frac{y^2}{b^2} = 1$$

or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1$

which is not possible for any values of x and y.

2. c. We have given points $A(x_1, y_1)$ and $B(x_2, y_2)$ and its midpoint is $C\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

a. $x^2 + 2y^2 \le 1$ represents interior region of ellipse $x^2 + 2y^2 = 1$ as shown in the given figure.



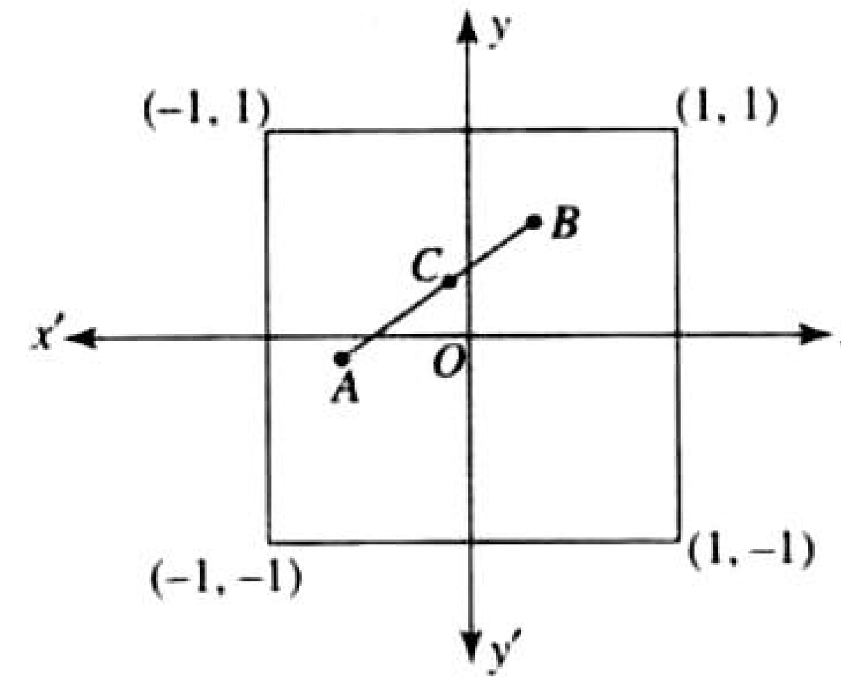
Clearly, for any two points A and B in the shaded region, mid point of AB also lies in the shaded region.

b. Max $\{|x|, |y|\} \le 1$

$\Rightarrow \quad |x| \le 1$ and $|y| \le 1$

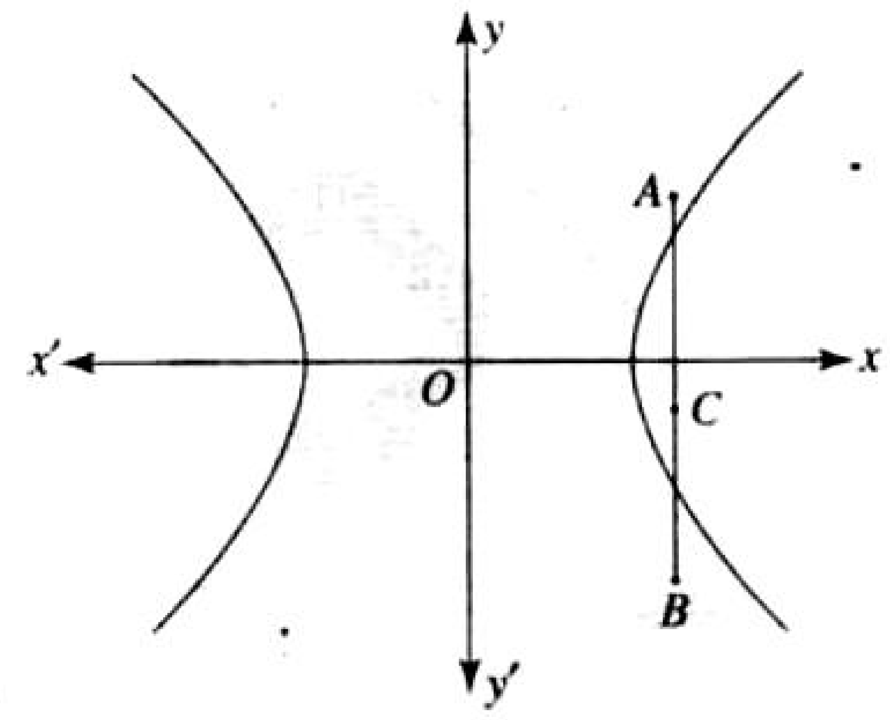
$\Rightarrow \quad -1 \le x \le 1$ and $-1 \le y \le 1$

This represents the interior region of a square with its sides $x = \pm 1$ and $y = \pm 1$ as shown in the given figure.



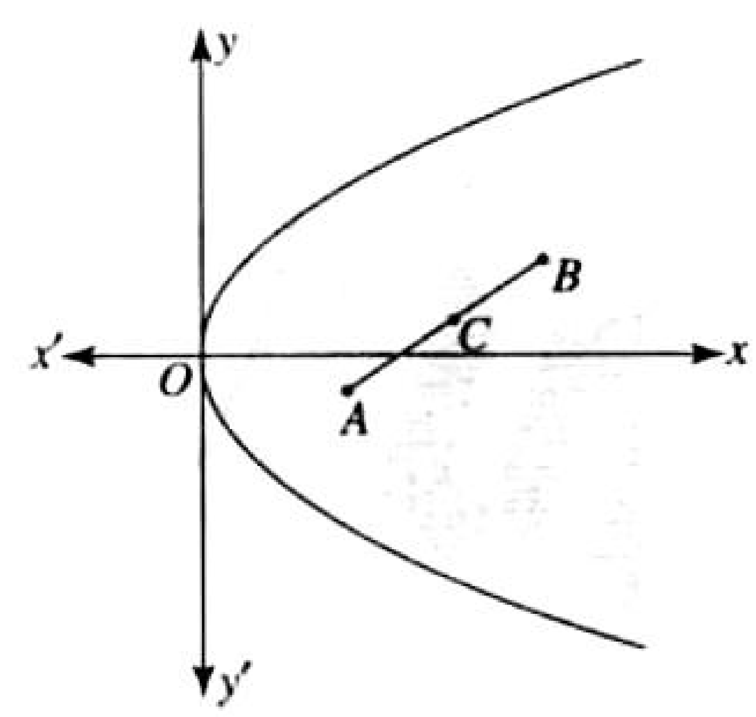
Clearly, for any two points A and B in the shaded region mid point of AB also lies in the shaded region

c. $x^2 - y^2 \le 1$ represents the exterior region of hyperbola as shown in the given figure.



As shown in the given figure for points A and B selected in the region, the mid point does not lie in the same region.

d. $y^2 \le x$ represents interior region of the parabola $y^2 = x$ as shown in the given figure.



Clearly, for any two points A and B in the shaded region, mid point of AB also lies in the shaded region.

3. c. We have

$2x^2 + 3y^2 - 8x - 18y + 35 = k$

or $2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$

or $2(x - 2)^2 + 3(y - 3)^2 = k$

For $k = 0$, we get

$2(x - 2)^2 + 3(y - 3)^2 = 0$

which represents the point (2, 3).

4. d. Normals at $P(\theta)$ and $Q(\phi)$ are

$ax \cos \theta + by \cot \theta = a^2 + b^2$

$ax \cos \phi + by \cot \phi = a^2 + b^2$

where $\phi = \dfrac{\pi}{2} - \theta$

These normals pass through (h, k). Therefore,

$ah \cos\theta + bk \cot\theta = a^2 + b^2$

and $ah \sin\theta + bk \tan\theta = a^2 + b^2$

Eliminating h, we have

$bk(\cot\theta \sin\theta - \tan\theta \cos\theta) = (a^2 + b^2)(\sin\theta - \cos\theta)$

or $k = -\left(\frac{a^2 + b^2}{b}\right)$

5. **b.** Let a pair of tangents be drawn from the point (x_1, y_1) to the hyperbola

$x^2 - y^2 = 9$

Then the chord of contact will be

$xx_1 - yy_1 = 9$ (i)

But the given chord of contact is

$x = 9$ (ii)

As (i) and (ii) represent the same line, these equations should be identical and, hence,

$\frac{x_1}{1} = -\frac{y_1}{0} = \frac{9}{9}$ or $x_1 = 1, y_1 = 0$

Therefore, the equation of pair of tangents drawn from $(1, 0)$ to $x^2 - y^2 = 9$ is

$(x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x \cdot 1 - y \cdot 0 - 9)^2$ (Using $SS_1 = T^2$)

or $(x^2 - y^2 - 9)(-8) = (x - 9)^2$

or $-8x^2 + 8y^2 + 72 = x^2 - 18x + 81$

or $9x^2 - 8y^2 - 18x + 9 = 0$

6. **b.** $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$

$a^2 = \cos^2\alpha$

$\therefore a^2e^2 = 1$

Hence, the foci are $(\pm ae, 0) \equiv (\pm 1, 0)$, which are independent of α.

7. **d.** The equation of tangent to the hyperbola $x^2 - 2y^2 = 4$ at any point (x_1, y_1) is $xx_1 - 2yy_1 = 4$.

Comparing with $2x + \sqrt{6}\,y = 2$ or $4x + 2\sqrt{6}\,y = 4$, we have

$x_1 = 4$ and $-2y_1 = 2\sqrt{6}$

Therefore, $(4, -\sqrt{6})$ is the required point of contact.

8. **a.** The length of transverse axis is

$2\sin\theta = 2a$

or $a = \sin\theta$

Also, for the ellipse

$3x^2 + 4y^2 = 12$

or $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$a^2 = 4, b^2 = 3$

$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

Hence, the focus of ellipse is $(2 \times 1/2, 0) \equiv (1, 0)$.

As the hyperbola is confocal with the ellipse, the focus of the hyperbola is $(1, 0)$. Now,

$ae' = 1$

or $\sin\theta \times e' = 1$

or $e' = \operatorname{cosec}\theta$

$\therefore b^2 = a^2(e'^2 - 1) = \sin^2\theta(\operatorname{cosec}^2\theta - 1) = \cos^2\theta$

Therefore, the equation of hyperbola is

$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

or $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

9. **b.** $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$

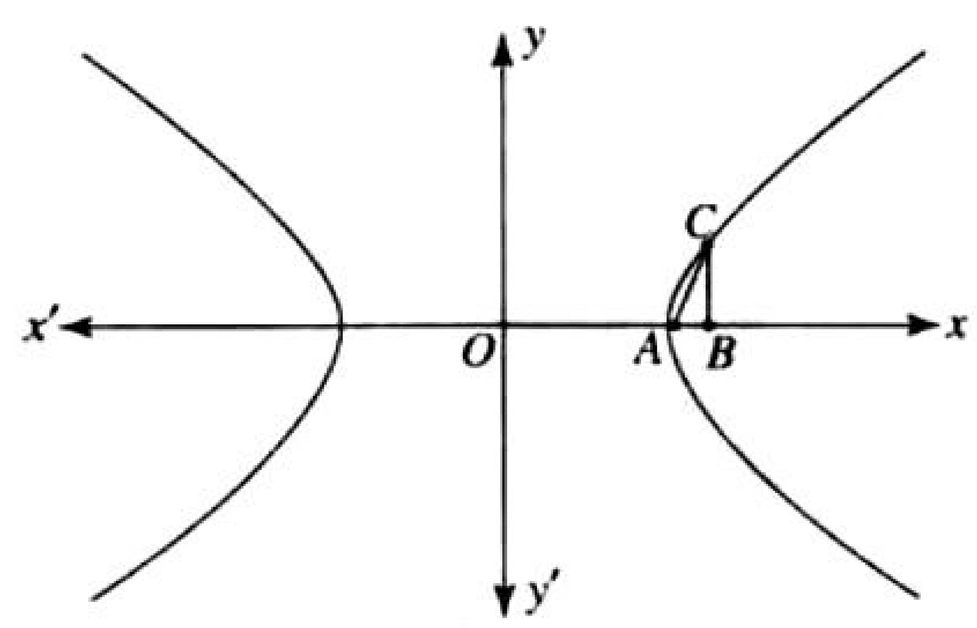
or

$(x^2 - 2\sqrt{2}\,x + 2) - 2(y^2 + 2\sqrt{2}\,y + 2) = 4$

or $\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$

Now B is one of the end points of its latus rectum and C is the focus of the hyperbola nearest to the vertex A.

Clearly, area of ΔABC does not change if we consider similar hyperbola with center at $(0, 0)$ or hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$



Here vertex is $A(2, 0)$.

$\therefore \quad a^2e^2 = a^2 + b^2 = 6$

So, one of the foci is $B(\sqrt{6}, 0)$, point C is $(ae, b^2/a)$ or $(\sqrt{6}, 1)$.

$\therefore \quad AB = \sqrt{6} - 2$ and $BC = 1$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} AB \times BC = \frac{1}{2}(\sqrt{6} - 2) \times 1$

$= \sqrt{\frac{3}{2}} - 1$ sq. units.

10. **b.**

We have $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

$\therefore ax^2 + by^2 + c = 0$ or $x^2 - 5xy + 6y^2 = 0$

From $x^2 - 5xy + 6y^2 = 0$ we have $(x - 2y)(x - 3y) = 0$, which represents a pair of straight lines.

If $c = 0$ and a and b have same sign then we have $ax^2 + by^2 = 0$, which is possible. Only if $(x, y) \equiv (0, 0)$.

If $a = b$ and c has sign opposite to a, we have $ax^2 + ay^2 = -c$ or $x^2 + y^2 = -c/a$, which represents the circle.

If a and b have same sign opposite to that of c, then $ax^2 + by^2 = -c$ is equation of ellipse.

11. **b.** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating w.r.t. x.

$\frac{dy}{dx} = \frac{xb^2}{ya^2}$

Therefore, the slope of normal at (6, 3) is $-a^2/2b^2$.

The equation of normal is

$$(y-3) = \frac{-a^2}{2b^2}(x-6)$$

It passes through the point (9, 0). Therefore,

$$\frac{a^2}{2b^2} = 1 \text{ or } \frac{b^2}{a^2} = \frac{1}{2} \therefore \; e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2}$$

$$\therefore e = \sqrt{\frac{3}{2}}$$

Multiple Correct Answers Type

1. **a., c.** For the given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

we have

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

Hence, the eccentricity of the hyperbola is 5/3.

Let the hyperbola be

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

Then $B^2 = A^2(e^2 - 1) = A^2\left(\frac{25}{9} - 1\right) = \frac{16}{9}A^2$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{A^2} - \frac{9y^2}{16A^2} = 1$$

As it passes through (3, 0), we get $A^2 = 9$ and $B^2 = 16$.

The equation is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

The foci of the hyperbola are $(\pm ae, 0) \equiv (\pm 5, 0)$.

The vertex of the hyperbola is (3, 0).

2. **a., b.** Since ellipse intersects hyperbola orthogonally. The ellipse and hyperbola will be confocal. So, comparing coordinates of foci

$$\left(\pm a \times \frac{1}{\sqrt{2}}, 0\right) \equiv (\pm 1, 0)$$

or $a = \sqrt{2}$ and $e = \frac{1}{\sqrt{2}}$

or $b^2 = a^2(1 - e^2)$

or $b^2 = 1$

Therefore, the equation of the ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

3. **b., d.** For the ellipse

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

we have $1^2 = 2^2(1 - e^2)$

or $e = \frac{\sqrt{3}}{2}$

Therefore, the eccentricity of the hyperbola is $2/\sqrt{3}$. So, for hyperbola

$$b^2 = a^2\left(\frac{4}{3} - 1\right)$$

or $3b^2 = a^2$

One of the foci of the ellipse is $(\sqrt{3}, 0)$.

The hyperbola passes through $(\sqrt{3}, 0)$. Therefore,

$$\frac{3}{a^2} = 1$$

or $a^2 = 3$ and $b^2 = 1$

Therefore, the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$ or $x^2 - 3y^2 = 3$.

The focus of the hyperbola is

$$(ae, 0) \equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) \equiv (2, 0)$$

4. **a., b.** Slope of tangent = 2

The tangents are

$y = 2x \pm \sqrt{9 \times 4 - 4}$ (using $y = mx \pm \sqrt{a^2m^2 - b^2}$)

i.e., $2x - y = \pm 4\sqrt{2}$

or $\frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$ and $\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$ (eqn. of tangent to hyperbola at point (x_1, y_1) on it) we get the points of contact as $(9/2\sqrt{2}, 1/\sqrt{2})$ and $(-9/2\sqrt{2}, -1/\sqrt{2})$.

Alternate Method:

The equation of tangent at $P(\theta)$ is

$$\left(\frac{\sec\theta}{3}\right)x - \left(\frac{\tan\theta}{2}\right)y = 1$$

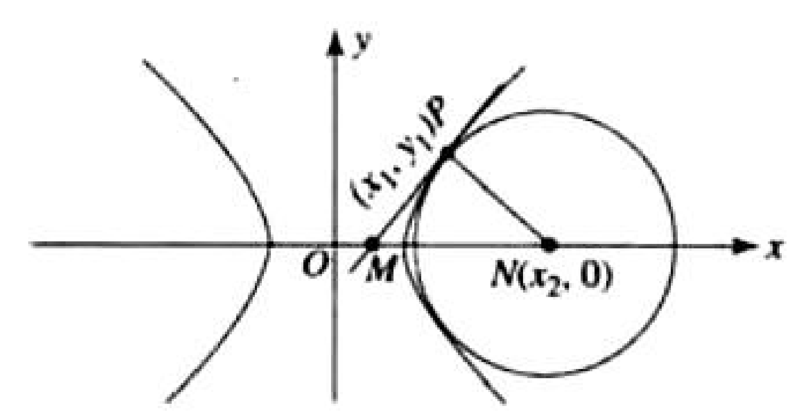
$$\therefore \text{Slope} = \frac{2\sec\theta}{3\tan\theta} = 2$$

$$\text{or } \sin\theta = \frac{1}{3}$$

$\therefore \sec\theta = \pm\frac{3}{2\sqrt{2}}$ and corresponding by $\tan\theta = \pm\frac{1}{2\sqrt{2}}$

Therefore, the points are $(9/2\sqrt{2}, 1/\sqrt{2})$ and $(-9/2\sqrt{2}, -1/\sqrt{2})$.

5. **a., b., d.**



As shown in figure, hyperbola and circle touch at $P(x_1, y_1)$.

Equation of tangent to H at P is $xx_1 - yy_1 = 1$.

It meets the x-axis at $M(1/x_1, 0)$.

Now, centroid of ΔPMN is (l, m).

So, $l = \dfrac{x_1 + x_2 + \dfrac{1}{x_1}}{3}$ and $m = \dfrac{y_1}{3} = \dfrac{\sqrt{x_1^2 - 1}}{3}$

Now, $\left.\dfrac{dy}{dx}\right|_{H \text{ at } P} = \left.\dfrac{dy}{dx}\right|_{S \text{ at } P}$

$\Rightarrow \quad \dfrac{x_1}{y_1} = \dfrac{x_2 - x_1}{y_1}$

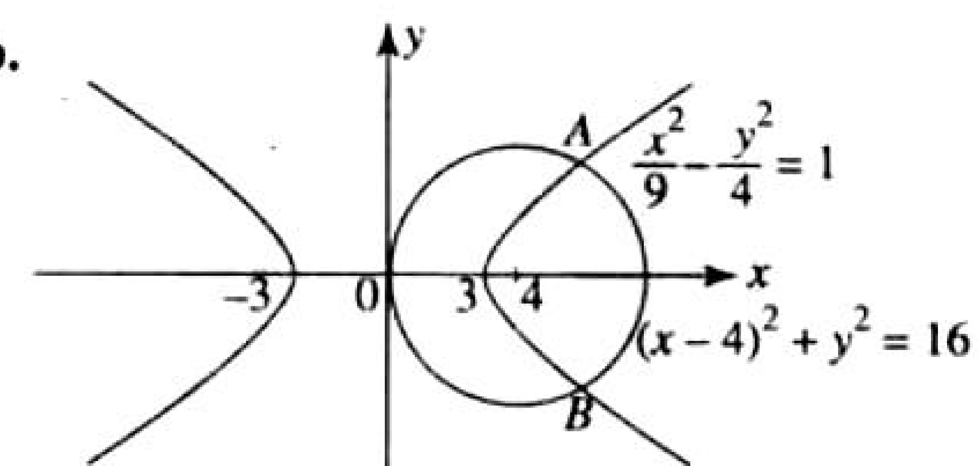
$\Rightarrow \quad x_2 = 2x_1$

So, $l = x_1 + \dfrac{1}{3x_1}$

$$\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \frac{dm}{dy_1} = \frac{1}{3}, \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 3}}$$

Linked Comprehension Type

1. b.



A tangent to $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, having slope m, is

$$y = mx + \sqrt{9m^2 - 4}, m > 0$$

It is tangent to $x^2 + y^2 - 8x = 0$. Therefore, its distance from the center of the circle is equal to radius of circle

$$\therefore \quad \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1 + m^2}} = 4$$

or $495m^4 + 104m^2 - 400 = 0$

or $m^2 = \dfrac{4}{5}$ or $m = \dfrac{2}{\sqrt{5}}$

Therefore, the tangent is

$$y = \frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}}$$

or $2x - \sqrt{5}y + 4 = 0$

2. a. A point on the hyperbola is $(3 \sec \theta, 2 \tan \theta)$.

It lies on the circle. So $9 \sec^2\theta + 4 \tan^2\theta - 24 \sec \theta = 0$, i.e.,

$13 \sec^2\theta - 24 \sec \theta - 4 = 0$

or $\sec \theta = 2, -\dfrac{2}{13}$

Clearly circle cuts the hyperbola in first and fourth quadrants.

$\therefore \quad \sec \theta = 2, \tan \theta = \sqrt{3}$

The points of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$.

Therefore, the circle with AB as diameter is

$(x - 6)^2 + y^2 = (2\sqrt{3})^2$

or $x^2 + y^2 - 12x + 24 = 0$

Matching Column Type

1. (d) – (q), (s)

(q) If $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$, then locus of variable point 'z' is on branch of the hyperbola with fixed points z_1 and z_2 foci.

Given $|z + 2| - |z - 2| = 3$

Clearly distance between complex numbers '2' and '−2' is 4 which less than 3.

So, locus of z is a branch of the hyperbola

(s) If eccentricity is $[1, \infty)$, then the conic can be a parabola (if $e = 1$) and a hyperbola if $e \in (1, \infty)$.

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (2) Substituting $(a/e, 0)$ in $y = -2x + 1$, we get

$0 = -\dfrac{2a}{e} + 1$

or $\dfrac{2a}{e} = 1$

or $a = \dfrac{e}{2}$

Also, $1 = \sqrt{a^2m^2 - b^2}$

or $1 = a^2m^2 - b^2$

or $1 = 4a^2 - b^2$

or $1 = \dfrac{4e^2}{4} - b^2$

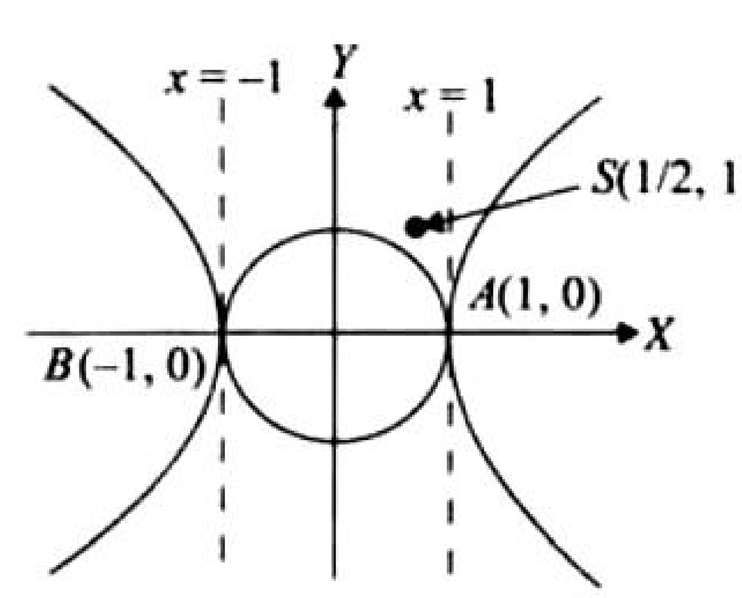
or $b^2 = e^2 - 1$

Also, $b^2 = a^2(e^2 - 1)$

$\therefore a = 1, e = 2$

Fill in the Blanks Type

1.



For the circle $x^2 + y^2 = 1$ and the rectangular hyperbola $x^2 - y^2 = 1$, one common tangent is evidently $x = 1$, the other being $x = -1$. The required standard form of the ellipse with focus at $S(1/2, 1)$ and directrix $x = 1$ is

$$\sqrt{\left(x-\frac{1}{2}\right)^2+(y-1)^2}=\frac{1}{2}\left|\frac{x-1}{\sqrt{1}}\right| \quad \text{(Using } SP = ePM\text{)}$$

$$\text{or } \left(x-\frac{1}{2}\right)^2+(y-1)^2=\left(\frac{1}{2}\right)^2(1-x)^2$$

$$\text{or } \frac{3x^2}{4}-\frac{x}{2}+(y-1)^2=0$$

$$\text{or } \frac{3}{4}\left(x-\frac{1}{3}\right)^2+(y-1)^2=\frac{1}{12}$$

$$\text{or } 9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$$

Subjective Type

1. Consider any point $P(3 \sec \theta, 2 \tan \theta)$ on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$

   Then the equation of chord of contact to the circle $x^2+y^2=9$ w.r.t. point P is

   $(3 \sec \theta)\, x+(2 \tan \theta)\, y=9$ (i)

   If (h, k) is the midpoint of chord of contact, then the equation of chord of contact will be

   $hx+ky-9=h^2+k^2-9$ (From $T=S_1$)

   or $hx+ky=h^2+k^2$ (ii)

   But (i) and (ii) represent the same straight line. Hence,

$$\frac{3 \sec \theta}{h}=\frac{2 \tan \theta}{k}=\frac{9}{h^2+k^2}$$

$$\text{or } \sec \theta=\frac{3h}{h^2+k^2},\ \tan \theta=\frac{9k}{2(h^2+k^2)}$$

Eliminating θ, we have

$$\frac{9h^2}{(h^2+k^2)^2}-\frac{81k^2}{4(h^2+k^2)^2}=1$$

$$\text{or } 4h^2-9k^2=\frac{4}{9}(h^2+k^2)^2$$

$$\text{or } \frac{h^2}{9}-\frac{k^2}{4}=\left(\frac{h^2+k^2}{9}\right)^2$$

Therefore, the locus of (h, k) is

$$\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$$