4

Conics

INTRODUCTION

The curves known as *conics* were named after their historical discovery as the intersection of a plane with a *right circular cone.*

A **right circular cone** is the surface generated by a line moving about a circle and passing through a fixed point which lies on the normal to the circle through its centre. The fixed point V is called the **vertex** of the cone, and it separates the cone into two parts called **nappes.** Each moving line (through V) is called a **generator** of the cone. The circle about which the generators move is called the **base** of the cone and the normal to the base circle through V is called the **axis** of the cone. A portion of a right circular cone of two nappes is shown in figure 4.1

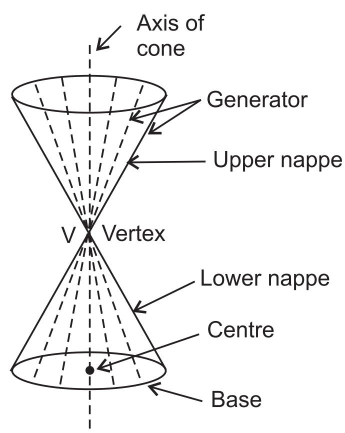

Fig. 4.1.

Applonius (before 200 B.C.) realised that a conic (or conic section) is a curve of intersection of a plane with a right circular cone of two nappes, and the three curves so obtained are *parabola, hyperbola* and *ellipse.*

The curve of intersection of a plane and a right circular cone is

(*i*) a **parabola** iff the cutting plane is parallel to one and only one generator of the cone (shown in fig. 4.2 (*i*)). Note that the plane intersects only one nappe of the cone.

(*ii*) A **hyperbola** iff the cutting plane is parallel to two generators of the cone (shown in fig. 4.2 (*ii*)). Note that the plane intersects both the nappes of the cone.

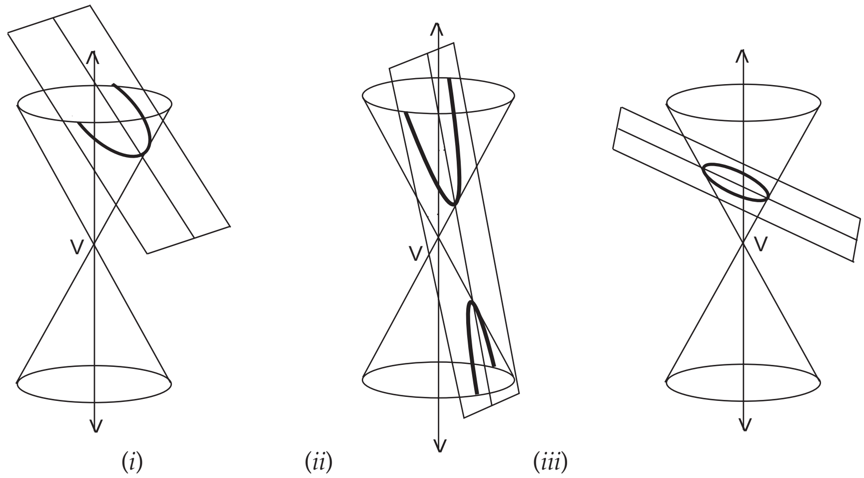

Fig. 4.2.

(*iii*) An *ellipse* iff the cutting plane is not parallel to any generator of the cone (shown in fig. 4.2 (*iii*)). Note that the plane intersects only one nappe of the cone, and it intersects every generator of the cone.

4.1 CONIC

We now define analytically a conic, and as special cases of this definition shall obtain three different types of curves — *parabola, ellipse and hyperbola.*

Let l be a fixed line and F be a fixed point not on l, and $e > 0$ be a fixed real number. Let $|MP|$ be the perpendicular distance from a point P (in the plane of the line l and point F) to the line l, then the locus of all points P such that

$$|FP| = e\,|MP|$$

is called a ***conic.***

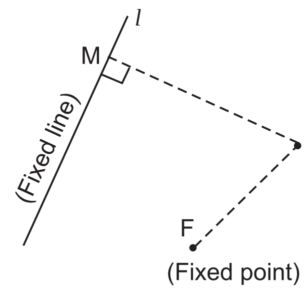

Fig. 4.3.

The fixed point F is called a **focus** of the conic and the fixed line l is called the **directrix** associated with F. The fixed real number e (> 0) is called **eccentricity** of the conic.

In particular, a conic with eccentricity e is called

(*i*) a **parabola** iff $e = 1$ (*ii*) an **ellipse** iff $e < 1$ (*iii*) a **hyperbola** iff $e > 1$.

4.2 SYMMETRY

Reflection of a point in a line

If a point P lies on the line l, then the **reflection** P in l is defined as the point P itself. If P does not lie on l, let M be the foot of perpendicular from P on l and produce it to a point P′ such that | MP | = | MP′ |, then P′ is called the **reflection** of P in l (shown in fig. 4.4).

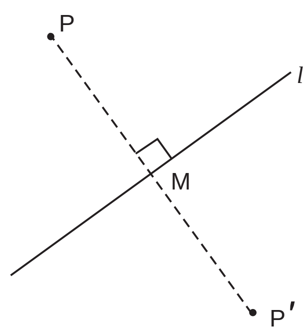

Fig. 4.4.

Note that if P′ is the reflection of P in l, then P is the reflection of P′ in l.

It follows that if P does not lie on l, then P′ is the reflection of P in l iff l is the perpendicular bisector of the segment PP′.

Reflection of a point in a point

The **reflection** of a point P in a fixed point M is the point P′ such that M is mid-point of the segment PP′ (shown in fig. 4.5).

In particular, if P = M then P′ = M = P.

Note that if P′ is the reflection of P in a point M then P is the reflection of P′ in M.

We leave it for the reader to see that the *reflection of a point* (α, β) *in the*

(i) origin is the point $(-\alpha, -\beta)$

(ii) x-axis is the point $(\alpha, -\beta)$.

(iii) y-axis is the point $(-\alpha, \beta)$.

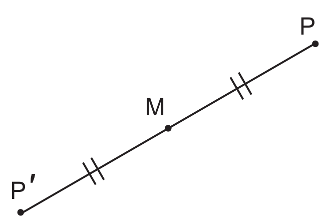

Fig. 4.5.

Symmetry of a curve about a line

A curve C is said to be ***symmetrical*** *about a line l iff for every point P on C, the reflection of P in l also lies on C. If a curve is symmetrical about a line l, then l is called a* ***line of symmetry*** *of C or an* ***axis*** *of C.*

Let $F(x, y) = 0$ be an equation of a curve C, then C is symmetrical about

(i) x-axis iff $F(x, y) = F(x, -y)$ ***(ii) y-axis iff*** $F(x, y) = F(-x, y)$.

Proof. (*i*) Let $P(\alpha, \beta)$ be a point on the curve C, then

$$F(\alpha, \beta) = 0 \qquad \ldots(1)$$

Now the curve C, is symmetrical about x-axis iff the reflection of P in the x-axis *i.e.* the point P′ $(\alpha, -\beta)$ lies on C *i.e.* iff $F(\alpha, -\beta) = 0$ ...(2)

From (1) and (2), it follows that C is symmetrical about x-axis iff

$$F(x, y) = F(x, -y).$$

We leave the proof of (*ii*) for the reader.

Symmetry of a curve about a point

A curve C is said to be ***symmetrical*** *about a point M iff for every point P on C, the reflection of P in M also lies on C. If a curve is symmetrical about a point M, then M is called* ***a centre of symmetry*** *of C or a* ***centre*** *of C.*

Let F (x, y) = 0 be an equation of a curve C, then C is symmetrical about the origin iff F (x, y) = F (– x, – y).

Proof. Let $P(\alpha, \beta)$ be a point on the curve C, then

$$F(\alpha, \beta) = 0 \qquad ...(1)$$

Now the curve C is symmetrical about the origin iff the reflection of P in the origin *i.e.* the point $P'(-\alpha, -\beta)$ lies on C

i.e. iff $F(-\alpha, -\beta) = 0$ ...(2)

From (1) and (2), it follows that C is symmetrical about origin iff

$$F(x, y) = F(-x, -y).$$

ILLUSTRATIVE EXAMPLE

Example. *Which of the following curves are symmetrical about the x-axis or y-axis or origin?*

(i) $7y^2 - 5x + 2 = 0$ *(ii)* $3x^2 - 4y^2 + 7 = 0$.

Solution. (*i*) The equation of the given curve is

$$F(x, y) = 7y^2 - 5x + 2 = 0.$$

Here,
$$F(x, -y) = 7(-y)^2 - 5x + 2 = 7y^2 - 5x + 2 = F(x, y);$$
$$F(-x, y) = 7y^2 - 5(-x) + 2 = 7y^2 + 5x + 2 \neq F(x, y);$$
$$F(-x, -y) = 7(-y)^2 - 5(-x) + 2 = 7y^2 + 5x + 2 \neq F(x, y).$$

Therefore, the given curve is symmetrical about x-axis but neither about y-axis nor about the origin.

(*ii*) The equation of the given curve is

$$F(x, y) = 3x^2 - 4y^2 + 7 = 0.$$

Here,
$$F(x, -y) = 3x^2 - 4(-y)^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y);$$
$$F(-x, y) = 3(-x)^2 - 4y^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y);$$
$$F(-x, -y) = 3(-x)^2 - 4(-y)^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y).$$

Therefore, the given curve is symmetrical about x-axis, y-axis and about the origin.

EXERCISE 4.1

1. Which of the following curves are symmetrical about the x-axis or y-axis or origin?

   (*i*) $5x^2 + 7y - 6 = 0$ (*ii*) $y^2 = x + 9$ (*iii*) $x^2 = 3y^2 + 7$ (*iv*) $x^2 + y^2 = 25$

   (*v*) $xy = 1$ (*vi*) $x^2 + y^2 - 4xy + 3 = 0$.

2. If a curve is symmetrical about each of the co-ordinate axes then prove that it is also symmetrical about the origin. What can you say about the converse?

   Hint. Let $P(x, y)$ lie on a curve C $\Rightarrow$ $(x, -y)$ lies on C $\Rightarrow$ $(-x, -y)$ lies on C.

4.3 PARABOLA

4.3.1 To find the equation of a parabola in the standard form $y^2 = 4ax$, $a > 0$.

Let F be the focus, l be the directrix and Z be the foot of perpendicular from F to the line l.

Take ZF as x-axis with positive direction from Z to F. Let A be the mid-point of ZF, take A as origin, then the line through A and perpendicular to ZF becomes y-axis (shown in figure 4.6).

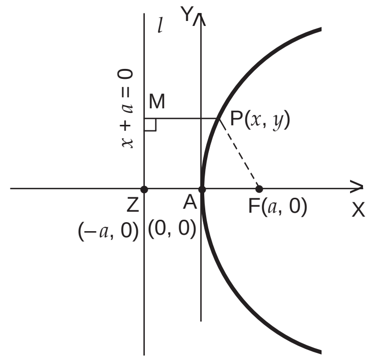

Fig. 4.6.

Let $|ZF| = 2a$

($a > 0$, because F does not lie on l),

then $|ZA| = |AF| = a$.

Since F lies to the right of A and Z lies to the left of A, co-ordinates of F, Z are $(a, 0)$, $(-a, 0)$ respectively. Therefore, the equation of the line l *i.e.* directrix is $x = -a$ *i.e.* $x + a = 0$.

Let P (x, y) be any point in the plane of the line l and the point F, and $|MP|$ be the perpendicular distance from P to the line l then P lies on ***parabola*** iff

$$|FP| = |MP| \qquad (e = 1 \text{ for parabola})$$

$$\Leftrightarrow \quad \sqrt{(x-a)^2 + y^2} = \frac{|x+a|}{1}$$

$$\Leftrightarrow \quad (x-a)^2 + y^2 = (x+a)^2 \qquad (\because |x|^2 = x^2)$$

$$\Leftrightarrow \quad x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$$

$$\Leftrightarrow \quad y^2 = 4ax.$$

Hence, the equation of a parabola in the standard form is $y^2 = 4ax$, $a > 0$, with focus F$(a, 0)$ and directrix $x + a = 0$.

Sometimes it is called a ***first standard form*** or a ***right hand parabola***.

4.3.2 To determine the shape of the parabola $y^2 = 4ax$, $a > 0$

The equation of the parabola is

$$F(x, y) = y^2 - 4ax = 0 \qquad \ldots(i)$$

We note the following facts about the given parabola :

1. $F(x, -y) = (-y)^2 - 4ax = y^2 - 4ax = F(x, y)$
   $\Rightarrow$ the given parabola is symmetrical about x-axis.
2. If $x < 0$, then $y^2 = 4ax$ has no real solutions in y and so there is no point on the curve with negative x-coordinate *i.e.* on the left of y-axis.

   As $x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$, $(0, 0)$ is the only point of the y-axis which lies on it, therefore, the entire curve, except the origin, lies to the right of y-axis.
3. If P (x, y) is a point of the parabola in the first quadrant, then the equation (i) gives $y = 2\sqrt{ax}$ so that as x increases, y also increases and the curve is unbounded.

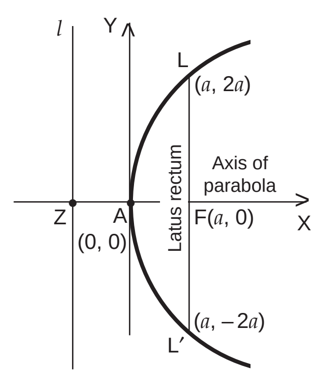

Fig. 4.7.

   Draw a rough diagram of a portion of the parabola in the first quadrant and then complete it with the help of symmetry.

   A portion of the given parabola is shown in fig. 4.7.

4.3.3 Some facts about the parabola $y^2 = 4ax,\ a > 0$

1. It is symmetrical about x-axis. This line is called ***axis of the parabola.***
2. Focus is F $(a, 0)$ and directrix is $x + a = 0$.
3. The point A $(0, 0)$ where the axis of parabola meets the parabola is called the **vertex of the parabola,** it is mid-point of ZF where F $(a, 0)$ is the focus and Z is the foot of perpendicular from focus to the directrix.
4. The curve lies entirely to the right of y-axis, except the origin.
5. y-axis meets the curve only at the origin.
6. The curve is unbounded.
7. Every line parallel to axis of parabola *i.e.* $y = c$ (c is any real number) meets the given parabola in a unique point $\left(\frac{c^2}{4a}, c\right)$.
8. A line segment joining any two points on the parabola is called a **chord** of the parabola.
9. Any chord passing through the focus F is called a **focal chord** of the parabola. The distance of any point on the parabola from the focus is called **focal distance** of the point.
10. A chord passing through the focus F and perpendicular to the axis of parabola is called **latus-rectum** and its length is called **length of latus-rectum.**
11. **Length of latus-rectum**

    Let chord L′L be the latus-rectum of the parabola, then L′L passes through focus F$(a, 0)$ and is perpendicular to x-axis.

    Let $|FL| = l\ (> 0)$, then points L, L′ are (a, l), $(a, -l)$ respectively.

    As L (a, l) lies on the parabola $y^2 = 4ax$, we get

    $$l^2 = 4a.a \Rightarrow l = 2a \qquad (\because\ l > 0)$$

    $\therefore$ L$(a, 2a)$, L′$(a, -2a)$

    and length of latus-rectum $= |L'L| = 2l = 4a$.

    Thus, the **length of latus rectum = $4a = 2|FZ| = 4|AF|$.**

    The end points of the latus-rectum are L $(a, 2a)$, L′ $(a, -2a)$ and the equation of the latus-rectum is $x - a = 0$.

4.3.4 To find the equation of a parabola in other standard forms

Find the equation of a parabola with

(i) focus F $(-a, 0)$, $a > 0$ and the line $x - a = 0$ as directrix.
(ii) focus F $(0, a)$, $a > 0$ and the line $y + a = 0$ as directrix.
(iii) focus F $(0, -a)$, $a > 0$ and the line $y - a = 0$ as directrix.

Solution. (*i*) Let P (x, y) be any point in the plane of directrix and focus, and $|MP|$ be the perpendicular distance from P to the directrix, then P lies on parabola

iff $|FP| = |MP|$ ($e = 1$ for parabola)

$$\Leftrightarrow \sqrt{(x+a)^2 + y^2} = \frac{|x-a|}{1}$$

$$\Leftrightarrow (x+a)^2 + y^2 = (x-a)^2$$

$$\Leftrightarrow x^2 + a^2 + 2ax + y^2 = x^2 + a^2 - 2ax$$

$$\Leftrightarrow \boldsymbol{y^2 = -4ax,\ a > 0.}$$

It is called ***2nd standard form*** or ***left hand parabola***.

(*ii*) Proceeding as above, it will be found that the equation of the parabola is

$$\boldsymbol{x^2 = 4ay,\ a > 0} \qquad \text{(Do it !)}$$

It is called ***3rd standard form*** or ***upward parabola.***

(*iii*) Proceeding as in part (*i*), it will be found that the equation of the parabola is

$$x^2 = -4ay,\ a > 0 \qquad \text{(Do it !)}$$

It is called ***4th standard form*** *or* ***downward parabola***.

4.3.5 Four standard forms of the parabola

The students are advised to trace the following four standard forms of parabolas and to fix in their memory the positions (figures) of these parabolas with respect to co-ordinate axes.

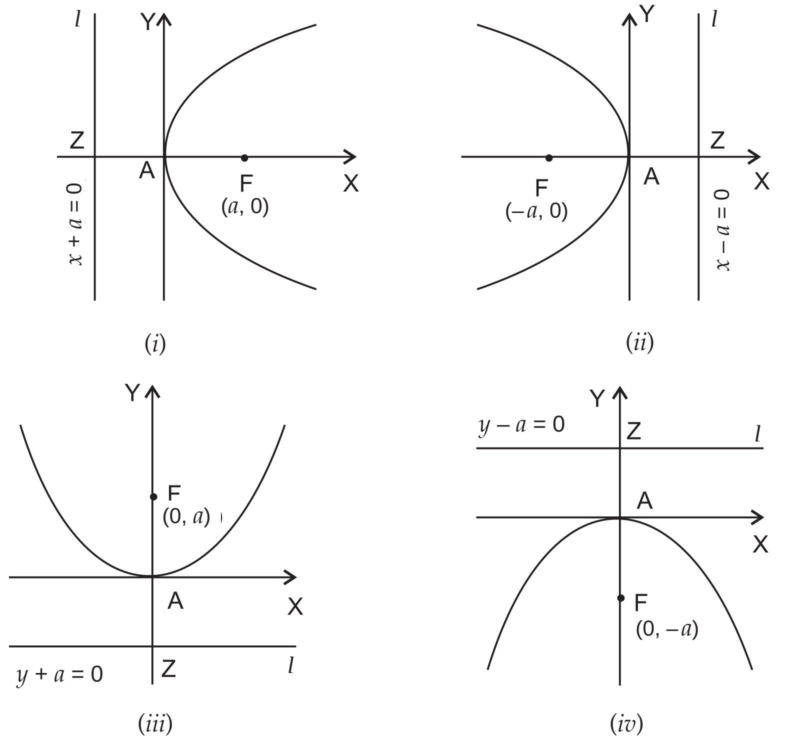

Fig. 4.8.

Main facts about the parabola

Equation	$y^2 = 4ax$ $(a > 0)$ *Right hand*	$y^2 = -4ax$ $a > 0$ *Left hand*	$x^2 = 4ay$ $a > 0$ *Upwards*	$x^2 = -4ay$ $a > 0$ *Downwards*
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Length of latus-rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus-rectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the equation of the parabola with focus at (–2, 0) and whose directrix is the line* $x + 2y - 3 = 0$.

Solution. The focus of the parabola is at F $(-2, 0)$ and directrix is the line $x + 2y - 3 = 0$.

Let P(x, y) be any point on the parabola and $|MP|$ be the perpendicular distance from P to the directrix, then by def. of parabola

$$|FP| = |MP| \qquad (e = 1 \text{ for parabola})$$

$$\Rightarrow \quad \sqrt{(x+2)^2 + y^2} = \frac{|x + 2y - 3|}{\sqrt{1^2 + 2^2}}, \text{ on squaring}$$

$$\Rightarrow \quad 5((x+2)^2 + y^2) = (x + 2y - 3)^2 \qquad (\because |x|^2 = x^2)$$

$$\Rightarrow \quad 5(x^2 + 4 + 4x + y^2) = x^2 + 4y^2 + 9 + 4xy - 6x - 12y$$

$\Rightarrow \quad 4x^2 - 4xy + y^2 + 26x + 12y + 11 = 0$, which is the required equation of the parabola.

Example 2. *Find the focus, directrix and eccentricity of the conic represented by the equation $3y^2 = 8x$.*

Solution. The given equation is $3y^2 = 8x$

i.e. $\quad y^2 = \frac{8}{3}x \qquad \ldots(i)$

which is comparable with $y^2 = 4ax$, so (*i*) represents a standard (right hand) parabola, and hence its eccentricity is 1. ($e = 1$ for parabola)

Also $\quad 4a = \frac{8}{3} \Rightarrow a = \frac{2}{3}$, therefore, focus is $(a, 0)$ *i.e.* $\left(\frac{2}{3}, 0\right)$ and the equation of directrix is $x + \frac{2}{3} = 0 \qquad | x + a = 0$

i.e. $\quad 3x + 2 = 0$.

Example 3. *Find the focus, directrix and eccentricity of the conic represented by the equation $5x^2 = -12y$.*

Solution. The given equation is $5x^2 = -12y$

i.e. $\quad x^2 = -\frac{12}{5}y \qquad \ldots(i)$

which is comparable with $x^2 = -4ay$, so (*i*) represents a parabola of fourth standard form and hence its eccentricity is 1. ($e = 1$ for parabola)

Also $\quad 4a = \frac{12}{5} \Rightarrow a = \frac{3}{5}$.

Therefore, focus is $(0, -a)$ *i.e.* $\left(0, -\frac{3}{5}\right)$

and the equation of the directrix is $y - \frac{3}{5} = 0 \qquad | y - a = 0$

i.e. $\quad 5y - 3 = 0$.

Example 4. *Find the equation of the parabola with vertex at origin and directrix the line $y + 3 = 0$. Also find its focus.*

Solution. Let A be the vertex and F be the focus of the parabola. Since vertex is at origin, so the point A is (0, 0).

The directrix of the parabola is the line $y + 3 = 0$ *i.e.* $y = -3$, which is a straight line parallel to x-axis at a distance 3 units below origin.

If the directrix meets y-axis at Z, then $|AZ| = 3$.

Therefore, the given parabola is of the third standard form and its equation is $x^2 = 4ay$, where

$$a = |AF| = |AZ| = 3.$$

$\therefore$ The equation of the required parabola is

$$x^2 = 4.3\,y \text{ i.e. } x^2 = 12y.$$

The focus of the parabola is F $(0, a)$ *i.e.* F (0, 3).

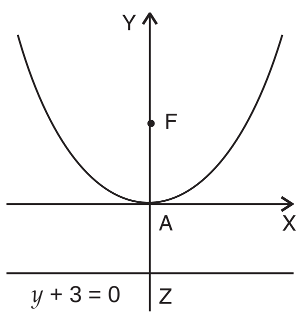

Fig. 4.9.

Example 5. *Find the equation of the parabola with vertex at origin, axis along x-axis and passing through the point (–1, 3).*

Solution. As the vertex of the parabola is at origin, its axis lies along x-axis and it passes through the point (–1, 3), which lies in the second quadrant, therefore, the parabola is of the second standard form.

Let its equation be $y^2 = -4ax$. But it passes through the point (–1, 3), we get

$$9 = -4a(-1) \Rightarrow 4a = 9.$$

∴ The equation of the parabola is $y^2 = -9x$.

Example 6. *Find the equation of the parabola with vertex at origin, symmetric with respect to y-axis and passing through (2, –3).*

Solution. The vertex of the parabola is at origin and it is symmetric with respect to y-axis *i.e.* axis of the parabola lies along y-axis.

Also it passes through (2, – 3), a point in the fourth quadrant. Therefore, the parabola is of the fourth standard form.

Let its equation be $x^2 = -4ay$ ...(i)

As it passes through (2, – 3), we get

$$2^2 = -4a \cdot (-3) \Rightarrow a = \frac{1}{3}.$$

Substituting this value of a in (i), the equation of the parabola is

$$x^2 = -4 \cdot \frac{1}{3} y \text{ i.e. } 3x^2 = -4y.$$

Example 7. *Find the equation of the parabola whose vertex is at the point (4, 1) and focus at the point (6, – 3)*

Solution. Given vertex is at A (4, 1) and focus at F (6, – 3), then the line joining A, F is the axis of the parabola.

Let the axis of parabola meet the directrix of the parabola at Z (shown in fig. 4.10).

Let Z be (α, β). Since A (4, 1) is mid-point of the segment ZF, we get

$$\frac{\alpha + 6}{2} = 4, \quad \frac{\beta - 3}{2} = 1$$

$$\Rightarrow \quad \alpha = 2, \beta = 5 \Rightarrow \text{Z is } (2, 5)$$

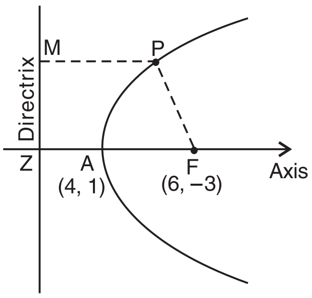

Fig. 4.10.

Slope of axis of parabola $= \frac{-3-1}{6-4} = -\frac{4}{2} = -2.$

As the directrix of the parabola is a straight line passing through Z (2, 5) and perpendicular to the axis of parabola, therefore, the equation of the directrix is

$$y - 5 = \frac{1}{2}(x - 2) \text{ or } x - 2y + 8 = 0.$$

Let P(x, y) be any point on the parabola and | MP | be the perpendicular distance from P to the directrix, then by def. of parabola

$$|\text{FP}| = |\text{MP}| \qquad (e = 1 \text{ for parabola})$$

$$\Rightarrow \quad \sqrt{(x-6)^2 + (y+3)^2} = \frac{|x - 2y + 8|}{\sqrt{1^2 + (-2)^2}} = \frac{|x - 2y + 8|}{\sqrt{5}}$$

$$\Rightarrow \quad 5[(x-6)^2 + (y+3)^2] = (x - 2y + 8)^2$$

$$\Rightarrow \quad 5(x^2 - 12x + 36 + y^2 + 6y + 9) = x^2 - 4xy + 4y^2 + 16x - 32y + 64$$

$\Rightarrow \quad 4x^2 + 4xy + y^2 - 76x + 62y + 161 = 0$, which is the required equation of the parabola.

Example 8. *The two lines $ty = x + t^2$ and $y + tx = 2t + t^3$ intersect at the point P. Show that P lies on the curve whose equation is $y^2 = 4x$.* (*I.S.C. 2001*)

Solution. Given straight lines are

$$ty - x = t^2 \quad \ldots(i)$$

and $$y + tx = 2t + t^3 \quad \ldots(ii)$$

To find the locus of the point of intersection P of the lines (*i*) and (*ii*), we have to eliminate the parameter t.

Multiplying (*i*) by t and adding it to (*ii*), we get

$$(t^2 + 1)\,y = 2t + 2t^3$$

$$\Rightarrow \quad y = 2t \quad \Rightarrow \quad t = \frac{y}{2} \quad \ldots(iii)$$

Substituting this value of t in (*i*), we get

$$\frac{y}{2}\cdot y - x = \frac{y^2}{4} \Rightarrow \frac{y^2}{2} - \frac{y^2}{4} = x \Rightarrow \frac{y^2}{4} = x \Rightarrow y^2 = 4x.$$

Hence, the point P lies on the curve $y^2 = 4x$.

EXERCISE 4.2

1. Write the equation of the parabola with the line $x + y = 0$ as directrix and the point (1, 0) as focus.
2. Find the equation of the parabola having focus at (3, – 4) and directrix as $x + y = 2$. (*I.S.C. 2007*)
3. Find the equation to the parabola whose focus is (–2, 1) and directrix is $6x - 3y = 8$.
4. The equation of the directrix of the parabola is $3x + 2y + 1 = 0$. The focus is (2, 1). Find the equation of the parabola. (*I.S.C. 2002*)
5. Find the equation to the parabola with the focus (a, b) and directrix $\frac{x}{a} + \frac{y}{b} = 1$.
6. Find the equation of the parabola whose focus is (–1, –2) and the equation of the directrix is given by $4x - 3x + 2 = 0$. Also find the equation of the axis. (*I.S.C. 2010*)
7. Find the equation of a parabola whose focus is the point (1, 1) and whose directrix is the line $3x + 4y - 2 = 0$. Also, find the equation of its axis and the co-ordinates of its vertex.
8. Find the focus, directrix and eccentricity of the conic represented by the equation $y^2 = 8\sqrt{3}x$.
9. In each of the parabolas

   (*i*) $y^2 = 2\sqrt{3}x$ (*ii*) $y^2 = -4x$ (*iii*) $3x^2 = 4y$ (*iv*) $x^2 = -12y$,

   find the length of latus-rectum, co-ordinates of focus and the equation of directrix.
10. Find the co-ordinates of the point on the parabola $y^2 = 18x$ whose ordinate is equal to three times the abscissa.
11. If the parabola $y^2 = 4px$ passes through the point (3, –2), find the length of latus-rectum and the co-ordinates of the focus. (*I.S.C. 2004*)
12. Find the equation of the parabolas with vertices at the origin and satisfying the following conditions :

    (*i*) Focus at (–4, 0) (*ii*) Directrix $y - 2 = 0$

    (*iii*) Passing through (2, 3) and axis along x-axis.
13. Find the equation of the parabola whose vertex is at (3, – 2) and focus is at (6, 2).

14. Find the equation to the parabola whose vertex is at (– 2, 2) and focus at (– 6, 6).
15. Given that the vertex is at (6, – 3) and directrix is $3x - 5y + 1 = 0$, find the focus and the equation of the parabola.

4.4 ELLIPSE

4.4.1 To find the equation of an ellipse in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2).$$

Let F be a focus, l be a directrix and Z be the foot of perpendicular from F to the line l.

Take FZ as x-axis with positive direction from F to Z. Divide the segment FZ internally and externally in the ratio $e : 1$ at the points A, A′ so that

$$\text{FA} = e\,.\,\text{AZ} \quad \ldots(i)$$

and $$\text{A'F} = e\,.\,\text{A'Z} \quad \ldots(ii)$$

As $e < 1$ and $\text{A'F} = e\,.\,\text{A'Z}$

⇒ A′ is closer to F than to Z

⇒ A′ lies towards the left of F.

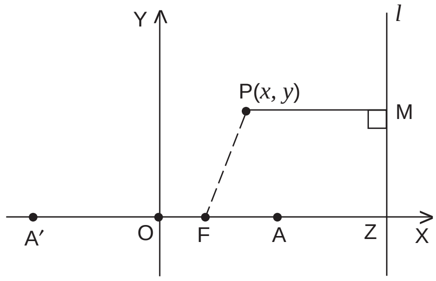

Fig. 4.11.

Let O be the mid-point of A′A, take O as origin, then the line through O and perpendicular to FZ becomes y-axis (shown in figure 4.11).

Let $\text{A'A} = 2a$ ($a > 0$), then $\text{A'O} = \text{OA} = a$ so the points A, A′ are $(a, 0)$, $(-a, 0)$ respectively.

The relations (i) and (ii) can be written as

$$\text{OA} - \text{OF} = e(\text{OZ} - \text{OA}) \quad \ldots(iii)$$

and $$\text{AO'} + \text{OF} = e(\text{A'O} + \text{OZ})$$

i.e. $$\text{OA} + \text{OF} = e(\text{OA} + \text{OZ}) \quad \ldots(iv) \quad (\because \text{A'O} = \text{OA})$$

Adding (iii) and (iv), we get

$$2\,.\,\text{OA} = 2e\,.\,\text{OZ} \Rightarrow a = e\,.\,\text{OZ} \Rightarrow \text{OZ} = \frac{a}{e}.$$

Subtracting (iii) from (iv), we get

$$2\,.\,\text{OF} = 2e\,.\,\text{OA} \Rightarrow \text{OF} = ae.$$

∴ The co-ordinate of focus F are $(ae, 0)$ and the equation of the directrix is

$$x = \frac{a}{e} \text{ i.e. } x - \frac{a}{e} = 0.$$

Let P (x, y) be any point in the plane of the line l and the point F, and $|\text{MP}|$ be the perpendicular distance from P to the line l, then P lies on ***ellipse*** iff

$$|\text{FP}| = e\,|\text{MP}| \quad (0 < e < 1)$$

$$\Leftrightarrow \sqrt{(x - ae)^2 + y^2} = e.\frac{\left|x - \frac{a}{e}\right|}{1}$$

$$\Leftrightarrow \sqrt{(x - ae)^2 + y^2} = |ex - a|$$

$$\Leftrightarrow x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 + a^2 - 2aex$$

$$\Leftrightarrow (1 - e^2)x^2 + y^2 = a^2(1 - e^2)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad (\because 0 < e < 1 \Rightarrow a^2(1 - e^2) \neq 0)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2).$$

4. Find the equation of the hyperbola satisfying the following conditions :

(*i*) Vertices (± 7, 0), $e = \frac{4}{3}$ (*ii*) Vertices (0, ± 5), foci (0, ± 8)

(*iii*) Foci (0, ± 4), length of transverse axis 6.

5. The focus of a parabola is (1, 5) and its directrix is $x + y + 2 = 0$. Find the equation of the parabola, its vertex and length of latus-rectum.

6. Find the equation of the ellipse with axes along x-axis and y-axis, which passes through the points (4, 3) and (6, 2).

7. Show that the equation $5x^2 + 30x + 2y + 59 = 0$ represents a parabola. Find its vertex, focus, length of latus-rectum and equations of directrix and axis of the parabola.

8. Find the equation of the parabola with its axis parallel to y-axis and passing through the points (4, 5), (–2, 11) and (–4, 21).

9. Identify the curves represented by the following equations :

(*i*) $2x^2 - 10xy + 12y^2 + 10x - 16y - 3 = 0$

(*ii*) $9x^2 - 24xy + 16y^2 - 6x + 8y - 5 = 0$

(*iii*) $6x^2 - 5xy - 6y^2 + 14x + 6y - 4 = 0$.

10. Show that the line $y = x + \sqrt{\frac{5}{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$. Also find the co-ordinates of the point of contact.

11. Determine k so that the line $2x + y + k = 0$ may touch the hyperbola $3x^2 - y^2 = 3$.

12. Find the equation of tangents to the ellipse $4x^2 + 5y^2 = 20$ which are perpendicular to the line $3x + 2y - 5 = 0$.

ANSWERS

EXERCISE 4.1

1. (*i*) Symmetric about y-axis only (*ii*) Symmetric about x-axis only

(*iii*) Symmetric about x-axis, y-axis and about origin

(*iv*) Symmetric about x-axis, y-axis and about origin

(*v*) Symmetric about origin only (*vi*) Symmetric about origin only.

2. May not be true; for example, $xy - 1 = 0$.

EXERCISE 4.2

1. $x^2 - 2xy + y^2 - 4x + 2 = 0$. 2. $x^2 - 2xy + y^2 - 8x + 20y + 46 = 0$.

3. $9x^2 + 36xy + 36y^2 + 276x - 138y - 161 = 0$.

4. $4x^2 - 12xy + 9y^2 - 58x - 30y + 64 = 0$.

5. $a^2x^2 - 2abxy + b^2y^2 - 2a^3x - 2b^3y + (a^4 - a^2b^2 + b^4) = 0$.

6. $9x^2 + 24xy + 16y^2 + 34x + 112x + 121 = 0$; $3x + 4y + 11 = 0$.

7. $16x^2 - 24xy + 9y^2 - 38x - 34y + 46 = 0$; $4x - 3y - 1 = 0$; $\left(\frac{7}{10}, \frac{3}{5}\right)$.

8. $(2\sqrt{3}, 0)$; $x + 2\sqrt{3} = 0$; 1.

9. (*i*) $2\sqrt{3}$; $\left(\frac{\sqrt{3}}{2}, 0\right)$; $2x + \sqrt{3} = 0$ (*ii*) 4 ; (– 1, 0) ; $x - 1 = 0$

(*iii*) $\frac{4}{3}$; $\left(0, \frac{1}{3}\right)$; $3y + 1 = 0$ (*iv*) 12 ; (0, – 3) ; $y - 3 = 0$.

10. (2, 6). 11. $\frac{4}{3}$; $\left(\frac{1}{3}, 0\right)$.

12. (*i*) $y^2 = -16x$ (*ii*) $x^2 = -8y$ (*iii*) $2y^2 = 9x$.

13. $16x^2 - 24xy + 9y^2 - 444x - 292y + 424 = 0$.

14. $x^2 + 2xy + y^2 + 32x - 32y + 128 = 0$.

15. (9, – 8) ; $25x^2 + 30xy + 9y^2 - 618x + 554y + 4929 = 0$.