## 1.3 Laws of Conservation of Energy, Momentum and Angular Momentum.

**1.118** As $\vec{F}$ is constant so the sought work done

$$A = \vec{F} \cdot \Delta\vec{r} = \vec{F} \cdot (\vec{r_2} - \vec{r_1})$$

or, $$A = (3\vec{i} + 4\vec{j}) \cdot [(2\vec{i} - 3\vec{j}) - (\vec{i} + 2\vec{j})] = (3\vec{i} + 4\vec{j}) \cdot (\vec{i} - 5\vec{j}) = 17\text{ J}$$

**1.119** Differentating $v(s)$ with respect to time

$$\frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2} = w$$

(As locomotive is in unidrectional motion)

Hence force acting on the locomotive $F = m\,w = \dfrac{ma^2}{2}$

Let, *at* $v = 0$ at $t = 0$ then the distance covered during the first $t$ seconds

$$s = \frac{1}{2}wt^2 = \frac{1}{2}\frac{a^2}{2}t^2 = \frac{a^2}{4}t^2$$

Hence the sought work, $A = Fs = \dfrac{ma^2}{2}\dfrac{(a^2t^2)}{4} = \dfrac{m\,a^4t^2}{8}$

**1.120** We have

$$T = \frac{1}{2}mv^2 = as^2 \quad \text{or,} \quad v^2 = \frac{2\,as^2}{m} \tag{1}$$

Differentating Eq. (1) with respect to time

$$2\,vw_t = \frac{4\,as}{m}v \quad \text{or,} \quad w_t = \frac{2\,as}{m} \tag{2}$$

Hence net acceleration of the particle

$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{\left(\frac{2\,as}{m}\right)^2 + \left(\frac{2\,as^2}{mR}\right)^2} = \frac{2\,as}{m}\sqrt{1 + (s/R)^2}$$

Hence the sought force, $F = mw = 2as\sqrt{1 + (s/R)^2}$

**1.121** Let $\vec{F}$ makes an angle $\theta$ with the horizontal at any instant of time (Fig.). Newton's second law in projection form along the direction of the force, gives :

$F = kmg\cos\theta + mg\sin\theta$ (because there is no acceleration of the body.)

As $\vec{F} \uparrow\uparrow d\vec{r}$ the differential work done by the force $\vec{F}$,

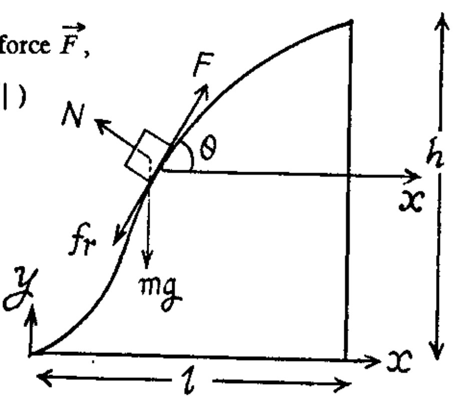
$$dA = \vec{F} \cdot d\vec{r} = F\,ds, \quad (\text{where } ds = |d\vec{r}|)$$

$$= kmg\,ds\,(\cos\theta) + mg\,ds\sin\theta$$

$$= kmg\,dx + mg\,dy.$$

Hence, $$A = kmg\int_0^l dx + mg\int_0^h dy$$

$$= kmg\,l + mgh = mg\,(k\,l + h).$$



1.122 Let $s$ be the distance covered by the disc along the incline, from the Eq. of increment of M.E. of the disc in the field of gravity : $\Delta T + \Delta U = A_{fr}$

$$0 + (-mgs\sin\alpha) = -kmg\cos\alpha\, s - kmg\, l$$

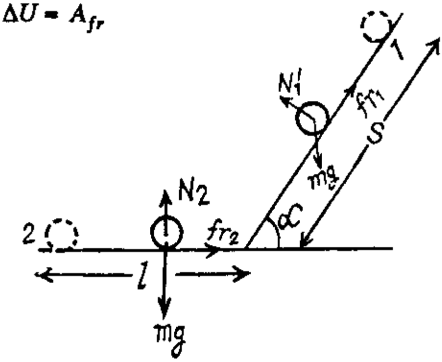
or, $$s = \frac{k\, l}{\sin\alpha - k\cos\alpha} \qquad (1)$$

Hence the sought work

$$A_{fr} = -kmg\,[s\cos\alpha + l\,]$$

$$A_{fr} = -\frac{k\, lmg}{1 - k\cot\alpha} \quad \text{[Using the Eqn. (1)]}$$

On puting the values $A_{fr} = -0.05$ J




1.123 Let $x$ be the compression in the spring when the bar $m_2$ is about to shift. Therefore at this moment spring force on $m_2$ is equal to the limiting friction between the bar $m_2$ and horizontal floor. Hence

$$\kappa x = k\, m_2\, g \quad \text{[where } \kappa \text{ is the spring constant (say)]} \qquad (1)$$

For the block $m_1$ from work-energy theorem : $A = \Delta T = 0$ for minimum force. ($A$ here indudes the work done in stretching the spring.)

so, $$Fx - \frac{1}{2}\kappa x^2 - kmg\, x = 0 \quad \text{or} \quad \kappa\frac{x}{2} = F - km_1 g \qquad (2)$$

From (1) and (2),

$$F = kg\left(m_1 + \frac{m_2}{2}\right).$$

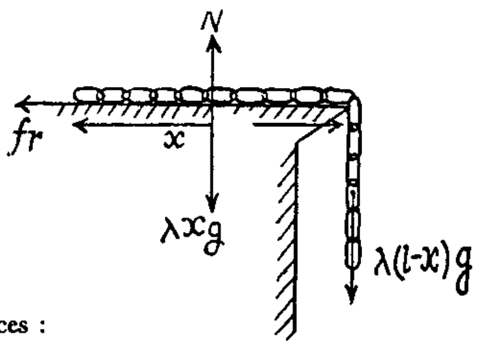
1.124 From the initial condition of the problem the limiting fricition between the chain lying on the horizontal table equals the weight of the over hanging part of the chain, i.e.

$\lambda\eta\, lg = k\lambda(1-\eta)\, lg$ (where $\lambda$ is the linear mass density of the chain)

So, $$k = \frac{\eta}{1-\eta} \qquad (1)$$

Let (at an arbitrary moment of time) the length of the chain on the table is $x$. So the net friction force between the chain and the table, at this moment :

$$f_r = kN = k\lambda x g \qquad (2)$$




The differential work done by the friction forces :

$$dA = \vec{f_r}\cdot d\vec{r} = -f_r\, ds = -k\lambda x g(-dx) = \lambda g\left(\frac{\eta}{1-\eta}\right)x\, dx \qquad (3)$$

(Note that here we have written $ds = -dx$., because $ds$ is essentially a positive term and as the length of the chain decreases with time, $dx$ is negative)

Hence, the sought work done

$$A = \int_{(1-\eta)l}^{0} \lambda g\frac{\eta}{1-\eta}x\, dx = -(1-\eta)\,\eta\frac{mgl}{2} = -1{\cdot}3 \text{ J}$$

1.125 The velocity of the body, $t$ seconds after the begining of the motion becomes $\vec{v} = \vec{v}_0 + \vec{g}t$. The power developed by the gravity ($m\vec{g}$) at that moment, is

$$P = m\vec{g}\cdot\vec{v} = m(\vec{g}\cdot\vec{v}_0 + g^2 t) = mg(gt - v_0 \sin\alpha) \quad (1)$$

As $m\vec{g}$ is a constant force, so the average power

$$<P> = \frac{A}{\tau} = \frac{m\vec{g}\cdot\Delta\vec{r}}{\tau}$$

where $\Delta\vec{r}$ is the net displacement of the body during time of flight.

As, $\quad m\vec{g} \perp \Delta\vec{r}$ so $<P> = 0$

1.126 We have $\quad w_n = \frac{v^2}{R} = at^2, \quad \text{or,} \quad v = \sqrt{aR}\,t,$

$t$ is defined to start from the begining of motion from rest.

So, $\quad w_t = \frac{dv}{dt} = \sqrt{aR}$

Instantaneous power, $P = \vec{F}\cdot\vec{v} = m(w_t\hat{u}_t + w_n\hat{u}_t)\cdot(\sqrt{aR}\,t\,\hat{u}_t),$

(where $\hat{u}_t$ and $\hat{u}_t$ are unit vectors along the direction of tangent (velocity) and normal respectively)

So, $\quad P = mw_t\sqrt{aR}\,t = maRt$

Hence the sought average power

$$<P> = \frac{\int_0^t P\,dt}{\int_0^t dt} = \frac{\int_0^t maRt\,dt}{t}$$

Hence $\quad <P> = \frac{maRt^2}{2t} = \frac{maRt}{2}$

1.127 Let the body $m$ acquire the horizontal velocity $v_0$ along positive $x$-axis at the point $O$.

(a) Velocity of the body $t$ seconds after the begining of the motion,

$$\vec{v} = \vec{v}_0 + \vec{w}t = (v_0 - kgt)\vec{i} \quad (1)$$

Instantaneous power $P = \vec{F}\cdot\vec{v} = (-kmg\vec{i})\cdot(v_0 - kgt)\vec{i} = -kmg(v_0 - kgt)$

From Eq. (1), the time of motion $\tau = v_0/kg$

Hence sought average power during the time of motion

$$<P> = \frac{\int_0^\tau -kmg(v_0 - kgt)\,dt}{\tau} = -\frac{kmg\,v_0}{2} = -2\text{ W (On substitution)}$$

From $F_x = mw_x$

$$-kmg = mw_x = mv_x\frac{dv_x}{dx}$$

or, $\quad v_x\,dv_x = -kg\,dx \doteq -\alpha g\,x\,dx$

To find $v(x)$, let us integrate the above equation

$$\int_{v_0}^{v} v_x\, dv_x = -\alpha g \int_0^x x\, dx \quad \text{or,} \quad v^2 = v_0^2 - \alpha g x^2 \tag{1}$$

Now, $$\vec{P} = \vec{F} \cdot \vec{v} = -m\alpha x g \sqrt{v_0^2 - \alpha g x^2} \tag{2}$$

For maximum power, $\frac{d}{dt}\left(\sqrt{v_0^2 x^2 - \lambda g x^4}\right) = 0$ which yields $x = \frac{v_0}{\sqrt{2\alpha g}}$

Putting this value of $x$, in Eq. (2) we get,

$$P_{max} = -\frac{1}{2} m v_0^2 \sqrt{\alpha g}$$

1.128 Centrifugal force of inertia is directed outward along radial line, thus the sought work

$$A = \int_{r_1}^{r_2} m\omega^2 r\, dr = \frac{1}{2} m\omega^2 \left(r_2^2 - r_1^2\right) = 0{\cdot}20\ \text{T} \quad \text{(On substitution)}$$

1.129 Since the springs are connected in series, the combination may be treated as a single spring of spring constant.

$$\kappa = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

From the equation of increment of M.E., $\Delta T + \Delta U = A_{ext}$

$$0 + \frac{1}{2}\kappa\, \Delta l^2 = A, \quad \text{or,} \quad A = \frac{1}{2}\left(\frac{\kappa\, \kappa_2}{\kappa_1 + \kappa_2}\right)\Delta l^2$$

1.130 First, let us find the total height of ascent. At the beginning and the end of the path of velocity of the body is equal to zero, and therefore the increment of the kinetic energy of the body is also equal to zero. On the other hand, in according with work-energy theorem $\Delta T$ is equal to the algebraic sum of the works $A$ performed by all the forces, i.e. by the force $F$ and gravity, over this path. However, since $\Delta T = 0$ then $A = 0$. Taking into account that the upward direction is assumed to coincide with the positive direction of the $y$ - axis, we can write

$$A = \int_0^h (\vec{F} + m\vec{g}) \cdot d\,\vec{r} = \int_0^h (F_y - mg)\, dy$$

$$= mg \int_0^h (1 - 2\,ay)\, dy = mgh\,(1 - ah) = 0.$$

whence $h = 1/a$.

The work performed by the force $F$ over the first half of the ascent is

$$A_F = \int_0^{h/2} F_y\, dy = 2mg \int_0^{h/2} (1 - ay)\, dy = 3\, mg/4a.$$

The corresponding increment of the potential energy is

$$\Delta U = mgh/2 = mg/2a.$$

1.131 From the equation $F_r = -\frac{dU}{dr}$ we get $F_r = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$

(a) we have at $r = r_0$, the particle is in equilibrium position. i.e. $F_r = 0$ so, $r_0 = \frac{2a}{b}$

To check, whether the position is steady (the position of stable equilibrium), we have to satisfy

$$\frac{d^2U}{dr^2} > 0$$

We have
$$\frac{d^2U}{dr^2} = \left[\frac{6a}{r^4} - \frac{2b}{r^3}\right]$$

Putting the value of $r = r_0 = \frac{2a}{b}$, we get

$\frac{d^2U}{dr^2} = \frac{b^4}{8a^3}$, (as $a$ and $b$ are positive constant)

So,
$$\frac{d^2U}{dr^2} = \frac{b^2}{8a^3} > 0,$$

which indicates that the potential energy of the system is minimum, hence this position is steady.

(b) We have
$$F_r = -\frac{dU}{dr} = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$$

For $F_r$ to be maximum, $\frac{dF_r}{dr} = 0$

So, $r = \frac{3a}{b}$ and then $F_{r\,(\max)} = \frac{-b^3}{27a^2}$,

As $F_r$ is negative, the force is attractive.

1.132 (a) We have

$$F_x = -\frac{\partial U}{dx} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{dy} = -2\beta y$$

So,
$$\vec{F} = 2\alpha x \vec{i} - 2\beta y \vec{i} \text{ and, } F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} \qquad (1)$$

For a central force, $\vec{r} \times \vec{F} = 0$

Here,
$$\vec{r} \times \vec{F} = (x\vec{i} + y\vec{j}) \times (-2\alpha x\vec{i} - 2\beta y\vec{j})$$
$$= -2\beta xy\vec{k} - 2\alpha xy(\vec{k}) \neq 0$$

Hence the force is not a central force.

(b) As $U = \alpha x^2 + \beta y^2$

So,
$$F_x = \frac{\partial U}{\partial x} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{\partial y} = -2\beta y.$$

So,
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4\alpha^2 x^2 + 4\beta^2 y^2}$$

According to the problem

$$F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} = C \text{ (constant)}$$

or, $$\alpha^2 x^2 + \beta^2 y^2 = \frac{C^2}{2}$$

or, $$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = \frac{C^2}{2\,\alpha^2\beta^2} = k\,(\text{say}) \tag{2}$$

Therefore the surfaces for which $F$ is constant is an ellipse.

For an equipotential surface $U$ is constant.

So, $$\alpha x^2 + \beta y^2 = C_0 \text{ (constant)}$$

or, $$\frac{x^2}{\sqrt{\beta^2}} + \frac{y^2}{\sqrt{\alpha^2}} = \frac{C_0}{\alpha\,\beta} = K_0 \text{ (constant)}$$

Hence the equipotential surface is also an ellipse.

1.133 Let us calculate the work performed by the forces of each field over the path from a certain point 1 $(x_1, y_1)$ to another certain point 2 $(x_2, y_2)$

(i) $dA = \vec{F}\cdot d\vec{r} = ay\,\vec{i}\cdot d\vec{r} = ay\,dx$ or, $A = a\int_{x_1}^{x_2} y\,dx$

(ii) $dA = \vec{F}\cdot d\vec{r} = (a\vec{xi} + by\,\vec{i})\cdot d\vec{r} = ax\,dx + by\,dy$

Hence $$A = \int_{x_1}^{x_2} a\,x\,dx + \int_{y_1}^{y_2} by\,dy$$

In the first case, the integral depends on the function of type $y(x)$, i.e. on the shape of the path. Consequently, the first field of force is not potential. In the second case, both the integrals do not depend on the shape of the path. They are defined only by the coordinate of the initial and final points of the path, therefore the second field of force is potential.

1.134 Let $s$ be the sought distance, then from the equation of increment of M.E. $\Delta T + \Delta U = A_{fr}$

$$\left(0 - \frac{1}{2}mv_0^2\right) + (+\,mg\,s\,\sin\alpha) = -\,kmg\cos\alpha\,s$$

or, $$s = \frac{v_0^2}{2g}\Big/(\sin\alpha + k\cos\alpha)$$

Hence $$A_{fr} = -\,kmg\cos\alpha\,s = \frac{-\,kmv_0^2}{2\,(k + \tan\alpha)}$$

1.135 Velocity of the body at height $h$, $v_h = \sqrt{2g\,(H-h)}$, horizontally (from the figure given in the problem). Time taken in falling through the distance $h$.

$t = \sqrt{\frac{2h}{g}}$ (as initial vertical component of the velocity is zero.)

Now $$s = v_h\,t = \sqrt{2g\,(H+h)} \times \sqrt{\frac{2h}{g}} = \sqrt{4\,(Hh - h^2)}$$

For $s_{max}$, $\frac{d}{ds}(Hh - h^2) = 0$, which yields $h = \frac{H}{2}$

Putting this value of $h$ in the expression obtained for $s$, we get,

$$s_{max} = H$$

1.136 To complete a smooth vertical track of radius $R$, the minimum height at which a particle starts, must be equal to $\frac{5}{2}R$ (one can proved it from energy conservation). Thus in our problem body could not reach the upper most point of the vertical track of radius $R/2$. Let the particle $A$ leave the track at some point $O$ with speed $v$ (Fig.). Now from energy conservation for the body $A$ in the field of gravity :

$$mg\left[h - \frac{h}{2}(1 + \sin\theta)\right] = \frac{1}{2}mv^2$$

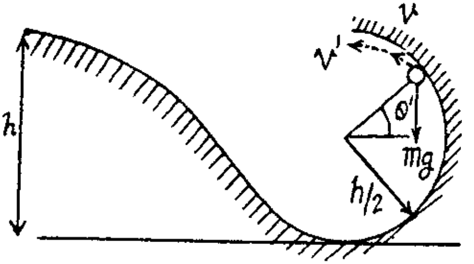
or, $\quad v^2 = gh(1 - \sin\theta) \qquad (1)$

From Newton's second law for the particle at the point $O$; $F_n = mw_n$,

$$N + mg\sin\theta = \frac{mv^2}{(h/2)}$$

But, at the point $O$ the normal reaction $N = 0$

So, $\quad v^2 = \frac{gh}{2}\sin\theta \qquad (2)$




From (3) and (4), $\sin\theta = \frac{2}{3}$ and $v = \sqrt{\frac{gh}{3}}$

After leaving the track at $O$, the particle $A$ comes in air and further goes up and at maximum height of it's trajectory in air, it's velocity (say $v'$) becomes horizontal (Fig.). Hence, the sought velocity of $A$ at this point.

$$v' = v\cos(90 - \theta) = v\sin\theta = \frac{2}{3}\sqrt{\frac{gh}{3}}$$

1.137 Let, the point of suspension be shifted with velocity $v_A$ in the horizontal direction towards left then in the rest frame of point of suspension the ball starts with same velocity horizontally towards right. Let us work in this, frame. From Newton's second law in projection form towards the point of suspension at the upper most point (say $B$) :

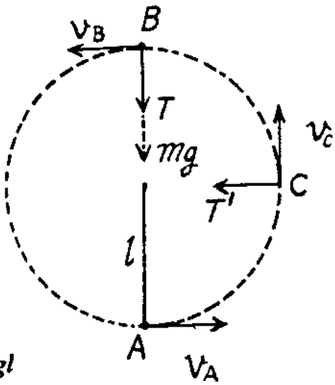
$$mg + T = \frac{mv_B^2}{l} \text{ or, } T = \frac{mv_B^2}{l} - mg \qquad (1)$$

Condition required, to complete the vertical circle is that $T \geq 0$. But (2)

$$\frac{1}{2}mv_A^2 = mg(2l) + \frac{1}{2}mv_B^2 \text{ So, } v_B^2 = v_A^2 - 4gl \qquad (3)$$

From (1), (2) and (3)

$$T = \frac{m\,(v_A^2 - 4\,gl)}{l} - mg \geq 0 \quad \text{or,} \quad v_A \geq \sqrt{5\,gl}$$




Thus $\qquad v_{A\,(\min)} = \sqrt{5\,gl}$

From the equation $F_n = mw_n$ at point $C$

$$T' = \frac{mv_c^2}{l} \qquad (4)$$

Again from energy conservation

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_c^2 + mgl \qquad (5)$$

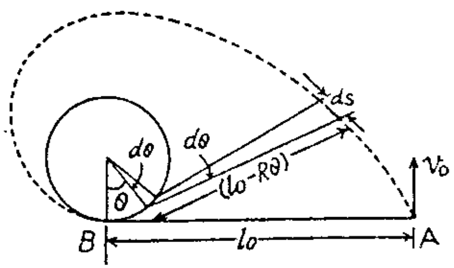
From (4) and (5)

$$T = 3\,mg$$

1.138 Since the tension is always perpendicular to the velocity vector, the work done by the tension force will be zero. Hence, according to the work energy theorem, the kinetic energy or velocity of the disc will remain constant during it's motion. Hence, the sought time $t = \frac{s}{v_0}$, where $s$ is the total distance traversed by the small disc during it's motion.

Now, at an arbitary position (Fig.)

$ds = (l_0 - R\,\theta)\,d\,\theta,$




so, $\quad s = \int_0^{l/R} (l_0 - R\,\theta)\,d\,\theta$

or, $\qquad s = \frac{l_0^2}{R} - \frac{R\,l_0^2}{2\,R^2} = \frac{l_0^2}{2R}$

Hence, the required time, $\quad t = \frac{l_0^2}{2\,R\,v_0}$

*It should be clearly understood that the only uncompensated force acting on the disc A in this case is the tension T, of the thread. It is easy to see that there is no point here, relative to which the moment of force T is invarible in the process of motion. Hence conservation of angular momentum is not applicable here.*

1.139 Suppose that $\Delta l$ is the elongation of the rubbler cord. Then from energy conservation,

$$\Delta U_{gr} + \Delta U_{el} = 0 \;(\text{as } \Delta T = 0)$$

or, $$-\,mg\,(l + \Delta l) + \frac{1}{2}\kappa\,\Delta l^2 = 0$$

or, $$\frac{1}{2}\kappa\,\Delta l^2 - mg\,\Delta l - mg\,l = 0$$

or, $$\Delta l = \frac{mg \pm \sqrt{(mg)^2 + 4 \times \frac{\kappa}{2} mgl}}{2 \times \frac{\kappa}{2}} \times \frac{\kappa}{2} = \frac{mg}{\kappa}\left[1 + \sqrt{1 \pm \frac{2\kappa l}{mg}}\right]$$

Since the value of $\sqrt{1 + \frac{2\kappa l}{mg}}$ is certainly greater than 1, hence negative sign is avoided.

So, $$\Delta l = \frac{mg}{\kappa}\left(1 + \sqrt{1 + \frac{2\kappa l}{mg}}\right)$$

1.140 When the thread $PA$ is burnt, obviously the speed of the bars will be equal at any instant of time until it breaks off. Let $v$ be the speed of each block and $\theta$ be the angle, which the elongated spring makes with the vertical at the moment, when the bar $A$ breaks off the plane. At this stage the elongation in the spring.

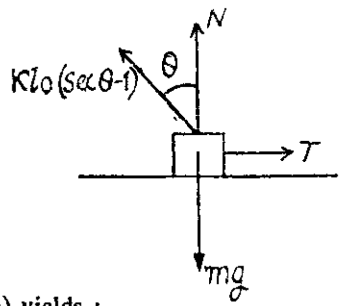
$$\Delta l = l_0 \sec\theta - l_0 = l_0(\sec\theta - 1) \quad (1)$$

Since the problem is concerned with position and there are no forces other than conservative forces, the mechanical energy of the system (both bars + spring) in the field of gravity is conserved, i.e. $\Delta T + \Delta U = 0$

So, $$2\left(\frac{1}{2} m v^2\right) + \frac{1}{2}\kappa l_0^2 (\sec\theta - 1)^2 - mgl_0 \tan\theta = 0 \quad (2)$$

From Newton's second law in projection form along vertical direction :

$mg = N + \kappa l_0 (\sec\theta - 1)\cos\theta$




But, at the moment of break off, $N = 0$.

Hence, $\kappa l_0 (\sec\theta - 1)\cos\theta = mg$

or, $$\cos\theta = \frac{\kappa l_0 - mg}{\kappa l_0} \quad (3)$$

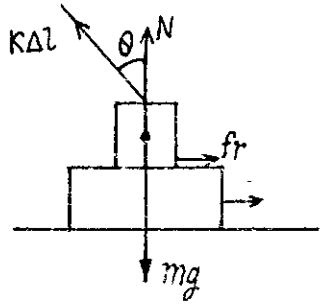
Taking $\kappa = \frac{5\,mg}{l_0}$, simultaneous solution of (2) and (3) yields :

$$v = \sqrt{\frac{19\,g\,l_0}{32}} = 1{\cdot}7 \text{ m/s}.$$

1.141 Obviously the elongation in the cord, $\Delta l = l_0(\sec\theta - 1)$, at the moment the sliding first starts and at the moment horizontal projection of spring force equals the limiting friction.

So, $$\kappa_1 \Delta l \sin\theta = kN \quad (1)$$

(where $\kappa_1$ is the elastic constant).




From Newton's law in projection form along vertical direction :

$$\kappa_1 \Delta l \cos\theta + N = mg.$$

or, $$N = mg - \kappa_1 \Delta l \cos\theta \quad (2)$$

From (1) and (2),

$$\kappa_1 \Delta l \sin\theta = k(mg - \kappa_1 \Delta l \cos\theta)$$

or, $$\kappa_1 = \frac{kmg}{\Delta l \sin\theta + k\,\Delta l \cos\theta}$$

From the equation of the increment of mechanical energy : $\Delta U + \Delta T = A_{fr}$

or, $$\left(\frac{1}{2}\kappa_1\,\Delta l^2\right) = A_{fr}$$

or, $$\frac{kmg\,\Delta l^2}{2\,\Delta l\,(\sin\theta + k\cos\theta)} = A_{fr}$$

Thus $$A_{fr} = \frac{kmg\,l_0\,(\sec\theta - 1)}{2\,(\sin\theta - k\cos\theta)} = 0{\cdot}09\text{ J} \quad \text{(on substitution)}$$

**1.142** Let the deformation in the spring be $\Delta l$, when the rod $AB$ has attained the angular velocity $\omega$. From the second law of motion in projection form $F_n = mw_n$ .

$$\kappa\,\Delta l = m\,\omega^2\,(l_0 + \Delta l) \quad \text{or,} \quad \Delta l = \frac{m\omega^2\,l_0}{\kappa - m\omega^2}$$

From the energy equation, $A_{ext} = \frac{1}{2}mv^2 + \frac{1}{2}\kappa\,\Delta l^2$

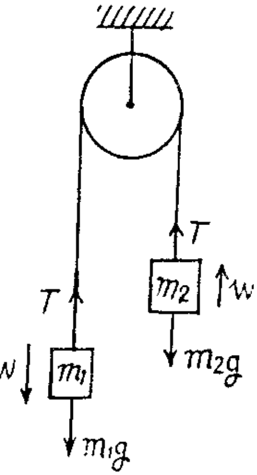
$$= \frac{1}{2}m\omega^2\,(l_0 + \Delta l)^2 + \frac{1}{2}\kappa\,\Delta l^2$$

$$= \frac{1}{2}m\omega^2\left(l_0 + \frac{m\omega^2\,l_0}{\kappa - m\omega^2}\right)^2 + \frac{1}{2}\kappa\left(\frac{m\omega^2\,l_0^2}{\kappa - m\omega^2}\right)^2$$

On solving $$A_{ext} = \frac{\kappa}{2}\,\frac{l_0^2\,\eta\,(1+\eta)}{(1-\eta)^2}, \quad \text{where } \eta = \frac{m\omega^2}{\kappa}$$

**1.143** We know that acceleration of centre of mass of the system is given by the expression.

$$\vec{w}_C = \frac{m_1\,\vec{w}_1 + m_2\,\vec{w}_2}{m_1 + m_2}$$




Since $\vec{w}_1 = -\vec{w}_2$

$$\vec{w}_C = \frac{(m_1 - m_2)\,\vec{w}_1}{m_1 + m_2} \qquad (1)$$

Now from Newton's second law $\vec{F} = m\vec{w}$, for the bodies $m_1$ and $m_2$ respectively.

$$\vec{T} + m_1\,\vec{g} = m_1\,\vec{w}_1 \qquad (2)$$

and $$\vec{T} + m_2\,\vec{g} = m_2\,\vec{w}_2 = -m_2\,\vec{w}_1 \qquad (3)$$

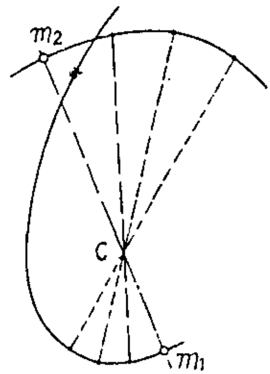
Solving (2) and (3)

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g}}{m_1 + m_2} \qquad (4)$$

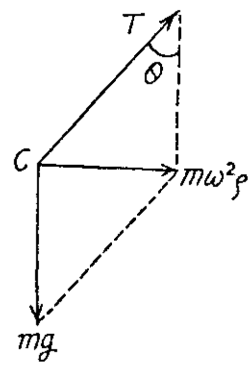
Thus from (1), (2) and (4),

$$\vec{w}_c = \frac{(m_1 - m_2)^2 \vec{g}}{(m_1 + m_2)^2}$$

1.144 As the closed system consisting two particles $m_1$ and of $m_2$ is initially at rest the C.M. of the system will remain at rest. Further as $m_2 = m_1/2$, the C.M. of the system divides the line joining $m_1$ and $m_2$ at all the moments of time in the ratio 1 : 2. In addition to it the total linear momentum of the system at all the times is zero. So, $\vec{p_1} = -\vec{p_2}$ and therefore the velocities of $m_1$ and $m_2$ are also directed in opposite sense. Bearing in mind all these thing, the sought trajectory is as shown in the figure.




1.145 First of all, it is clear that the chain does not move in the vertical direction during the uniform rotation. This means that the vertical component of the tension $T$ balances gravity. As for the horizontal component of the tension $T$, it is constant in magnitude and permanently directed toward the rotation axis. It follows from this that the C.M. of the chain, the point $C$, travels along horizontal circle of radius $\rho$ (say). Therefore we have,

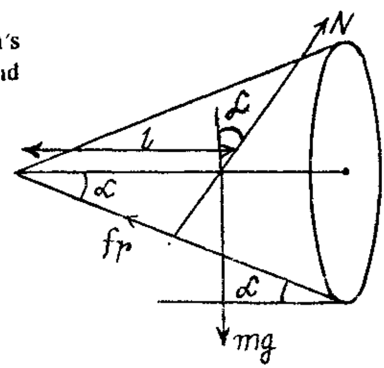



$$T \cos\theta = mg \quad \text{and} \quad T \sin\theta = m\omega^2 \rho$$

Thus $\quad \rho = \dfrac{g \tan\theta}{\omega^2} = 0{\cdot}8 \text{ cm}$

and $\quad T = \dfrac{mg}{\cos\theta} = 5 \text{ N}$

1.146 (a) Let us draw free body diagram and write Newton's second law in terms of projection along vertical and horizontal direction respectively.




$$N \cos\alpha - mg + fr \sin\alpha = 0 \qquad (1)$$

$$fr \cos\alpha - N \sin\alpha = m\omega^2 l \qquad (2)$$

From (1) and (2)

$$fr \cos\alpha - \frac{\sin\alpha}{\cos\alpha}(-fr \sin\alpha + mg) = m\omega^2 l$$

So, $$fr = mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) = 6\text{N} \tag{3}$$

(b) For rolling, without sliding,

$$fr \le k\,\text{N}$$

but, $N = mg\cos\alpha - m\,\omega^2 l\sin\alpha$

$$mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \le k\,(mg\cos\alpha - m\,\omega^2 l\sin\alpha)\ \text{[Using (3)]}$$

Rearranging, we get,

$$m\,\omega^2 l\,(\cos\alpha + k\sin\alpha) \le (k\,mg\cos\alpha - mg\sin\alpha)$$

Thus $$\omega \le \sqrt{g\,(k - \tan\alpha)/(1 + k\tan\alpha)\,l} = 2\ \text{rad/s}$$

1.147 (a) Total kinetic energy in frame $K'$ is

$$T = \frac{1}{2}m_1(\vec{v}_1 - \vec{V})^2 + \frac{1}{2}m_2(\vec{v}_2 - \vec{V})^2$$

This is minimum with respect to variation in $\vec{V}$, when

$$\frac{\delta T'}{\delta\vec{V}} = 0,\ \text{i.e.}\ m_1(\vec{v}_1 - \vec{V})^2 + m_2(\vec{v}_2 - \vec{V}) = 0$$

or $$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{V}_2}{m_1 + m_2} = \vec{v}_c$$

Hence, it is the frame of C.M. in which kinetic energy of a system is minimum.

(b) Linear momentum of the particle 1 in the $K'$ or $C$ frame

$$\tilde{\vec{p}}_1 = m_1(\vec{v}_1 - \vec{v}_c) = \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)$$

or, $$\tilde{\vec{p}}_1 = \mu(\vec{v}_1 - \vec{v}_2),\ \text{where},\ \mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Similarly, $$\tilde{\vec{p}}_2 = \mu(\vec{v}_2 - \vec{v}_1)$$

So, $$|\tilde{\vec{p}}_1| = |\tilde{\vec{p}}_2| = \tilde{p} = \mu\,v_{rel}\ \text{where},\ v_{rel} = |\vec{v}_1 - \vec{v}_2| \tag{3}$$

Now the total kinetic energy of the system in the $C$ frame is

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{p}^2}{2m_1} + \frac{\tilde{p}^2}{2m_2} = \frac{\tilde{p}^2}{2\mu}$$

Hence $$\tilde{T} = \frac{1}{2}\mu\,v_{rel}^2 = \frac{1}{2}\mu\,|\vec{v}_1 - \vec{v}_2|^2$$

**1.148** To find the relationship between the values of the mechanical energy of a system in the $K$ and $C$ reference frames, let us begin with the kinetic energy $T$ of the system. The velocity of the $i$-th particle in the $K$ frame may be represented as $\vec{v_i} = \vec{\tilde{v}_i} + \vec{v_C}$. Now we can write

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\vec{\tilde{v}_1} + \vec{v_C}) \cdot (\vec{\tilde{v}_i} + \vec{v_C})$$

$$= \sum \frac{1}{2} m_i \tilde{v}_i^2 + \vec{v_C} \sum m_i \vec{\tilde{v}_1} + \sum \frac{1}{2} m_i v_C^2$$

Since in the $C$ frame $\sum m_i \vec{\tilde{v}_i} = 0$, the previous expression takes the form

$$T = \tilde{T} + \frac{1}{2} m\, v_C^2 = \tilde{T} + \frac{1}{2} m\, V^2 \quad \text{(since according to the problem } v_C = V) \qquad (1)$$

Since the internal potential energy $U$ of a system depends only on its configuration, the magnitude $U$ is the same in all refrence frames. Adding $U$ to the left and right hand sides of Eq. (1), we obtain the sought relationship

$$E = \tilde{E} + \frac{1}{2} m\, V^2$$

**1.149** As initially $U = \tilde{U} = 0$, so, $\tilde{E} = \tilde{T}$

From the solution of 1.147 (b)

$$\tilde{T} = \frac{1}{2} \mu \, |\vec{v_1} - \vec{v_2}|,$$

As $$\vec{v_1} \perp \vec{v_2}$$

Thus $$\tilde{T} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$$

**1.150** Velocity of masses $m_1$ and $m_2$, after $t$ seconds are respectively.

$$\vec{v_1}' = \vec{v_1} + \vec{g} t \text{ and } \vec{v_2}' = \vec{v_2} + \vec{g} t$$

Hence the final momentum of the system,

$$\vec{p} = m_1 \vec{v_1}' + m_2 \vec{v_2}' = m_1 \vec{v_1} + m_2 \vec{v_2} + (m_1 + m_2)\, \vec{g} t$$

$$= \vec{p_0} + m\, \vec{g} t, \quad \text{(where, } \vec{p_0} = m_2 \vec{v_1} + m_2 \vec{v_2} \text{ and } m = m_1 + m_2)$$

And radius vector, $$\vec{r_C} = \vec{v_C} t + \frac{1}{2} \vec{w_C} t^2$$

$$\frac{(m_1 \vec{v_1} + m_2 \vec{v_2})\, t}{(m_1 + m_2)} + \frac{1}{2} \vec{g} t^2$$

$$= \vec{v_0} t + \frac{1}{2} \vec{g} t^2, \quad \text{where } \vec{v_0} = \frac{m_1 \vec{v_1} + m_2 \vec{v_2}}{m_1 + m_2}$$

1.151 After releasing the bar 2 acquires the velocity $v_2$, obtained by the energy, conservation :

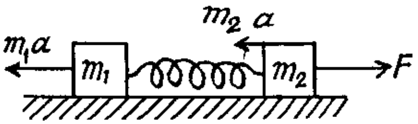
$$\frac{1}{2} m_2 v_2^2 = \frac{1}{2} \kappa x^2 \quad \text{or,} \quad v_2 = x\sqrt{\frac{\kappa}{m_2}} \tag{1}$$

Thus the sought velocity of C.M.

$$v_{cm} = \frac{0 + m_2 x \sqrt{\frac{\kappa}{m_2}}}{m_1 + m_2} = \frac{x\sqrt{m_2 \kappa}}{(m_1 + m_2)}$$

1.152 Let us consider both blocks and spring as the physical system. The centre of mass of the system moves with acceleration $a = \frac{F}{m_1 + m_2}$ towards right. Let us work in the frame of centre of mass. As this frame is a non-inertial frame (accelerated with respect to the ground) we have to apply a pseudo force $m_1 a$ towards left on the block $m_1$ and $m_2 a$ towards left on the block $m_2$

As the center of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The elongation of the spring will be maximum or minimum at this instant. Assume that the block $m_1$ is displaced by the distance $x_1$ and the block $m_2$ through a distance $x_2$ from the initial positions.




From the energy equation in the frame of C.M.

$$\Delta \tilde{T} + U = A_{ext},$$

(where $A_{ext}$ also includes the work done by the pseudo forces)

Here,

$$\Delta \tilde{T} = 0, \quad U = \frac{1}{2} k (x_1 + x_2)^2 \text{ and}$$

$$W_{ext} = \left(\frac{F - m_2 F}{m_1 + m_2}\right) x_2 + \frac{m_1 F}{m_1 + m_2} x_1 = \frac{m_1 F (x_1 + x_2)}{m_1 + m_2},$$

or, $$\frac{1}{2} k (x_1 + x_2)^2 = \frac{m_1 (x_1 + x_2) F}{m_1 + m_2}$$

So, $$x_1 + x_2 = 0 \text{ or, } x_1 + x_2 = \frac{2 m_1 F}{k (m_1 + m_2)}$$

Hence the maximum separation between the blocks equals : $l_0 + \frac{2 m_1 F}{k (m_1 + m_2)}$

Obviously the minimum sepation corresponds to zero elongation and is equal to $l_0$

1.153 (a) The initial compression in the spring $\Delta l$ must be such that after burning of the thread, the upper cube rises to a height that produces a tension in the spring that is atleast equal to the weight of the lower cube. Actually, the spring will first go from its compressed state to its natural length and then get elongated beyond this natural length. Let $l$ be the maximum elongation produced under these circumstances.

Then

$$\kappa l = mg \qquad (1)$$

Now, from energy conservation,

$$\frac{1}{2}\kappa \Delta l^2 = mg(\Delta l + l) + \frac{1}{2}\kappa l^2 \qquad (2)$$

(Because at maximum elongation of the spring, the speed of upper cube becomes zero)

From (1) and (2),

$$\Delta l^2 - \frac{2mg\,\Delta l}{\kappa} - \frac{3m^2g^2}{\kappa^2} = 0 \text{ or, } \Delta l = \frac{3mg}{\kappa}, \frac{-mg}{\kappa}$$

Therefore, acceptable solution of $\Delta l$ equals $\frac{3mg}{\kappa}$

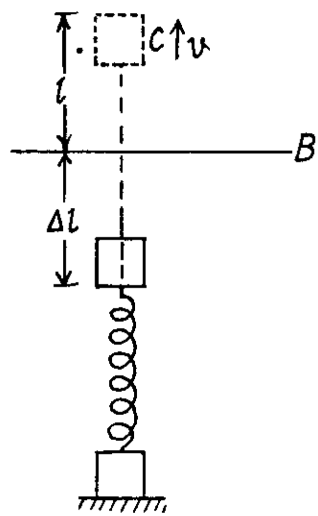
(b) Let $v$ the velocity of upper cube at the position (say, at $C$ ) when the lower block breaks off the floor, then from energy conservation.

$$\frac{1}{2}mv^2 = \frac{1}{2}\kappa(\Delta l^2 - l^2) - mg(l + \Delta l)$$

$$(\text{where } l = mg/\kappa \text{ and } \Delta l = 7\frac{mg}{\kappa})$$

or, $$v^2 = 32\frac{mg^2}{\kappa} \qquad (2)$$

At the position $C$, the velocity of C.M; $v_C = \frac{mv + 0}{2m} = \frac{v}{2}$ –Let, the C.M. of the system (spring+ two cubes) further rises up to $\Delta y_{C2}$.




Now, from energy conservation,

$$\frac{1}{2}(2m)v_C^2 = (2m)g\,\Delta r_{C2}$$

or, $$\Delta y_{C2} = \frac{v_C^2}{2g} = \frac{v^2}{8g} = \frac{4mg}{\kappa}$$

But, uptil position $C$, the C.M. of the system has already elevated by,

$$\Delta y_{C1} = \frac{(\Delta l + l)m + 0}{2m} = \frac{4mg}{\kappa}$$

Hence, the net displacement of the C.M. of the system, in upward direction

$$\Delta y_C = \Delta y_{C1} + \Delta y_{C2} = \frac{8mg}{\kappa}$$

1.154 Due to ejection of mass from a moving system (which moves due to inertia) in a direction perpendicular to it, the velocity of moving system does not change. The momentum change being adjusted by the forces on the rails. Hence in our problem velocities of buggies change only due to the entrance of the man coming from the other buggy. From the

Solving (1) and (2), we get

$$v_1 = \frac{mv}{M-m} \text{ and } v_2 = \frac{Mv}{M-m}$$

As $\vec{v_1} \uparrow\downarrow \vec{v}$ and $\vec{v_2} \uparrow\uparrow \vec{v}$

So, $$\vec{v_1} = \frac{-m\vec{v}}{(M-m)} \text{ and } \vec{v_2} = \frac{M\vec{v}}{(M-m)}$$

1.155 From momentum conservation, for the system "rear buggy with man"

$$(M+m)\vec{v_0} = m(\vec{u}+\vec{v_R}) + M\vec{v_R} \tag{1}$$

From momentum conservation, for the system (front buggy + man coming from rear buggy)

$$M\vec{v_0} + m(\vec{u}+\vec{v_R}) = (M+m)\vec{v_F}$$

So, $$\vec{v_F} = \frac{M\vec{v_0}}{M+m} + \frac{m}{M+m}(\vec{u}+\vec{v_R})$$

Putting the value of $\vec{v_R}$ from (1), we get

$$\vec{v_F} = \vec{v_0} + \frac{mM}{(M+m)^2}\vec{u}$$

1.156 (i) Let $\vec{v_1}$ be the velocity of the buggy after both man jump off simultaneously. For the closed system (two men + buggy), from the conservation of linear momentum,

$$M\vec{v_1} + 2m(\vec{u}+\vec{v_1}) = 0$$

or, $$\vec{v_1} = \frac{-2m\vec{u}}{M+2m} \tag{1}$$

(ii) Let $\vec{v}'$ be the velocity of buggy with man, when one man jump off the buggy. For the closed system (buggy with one man + other man) from the conservation of linear momentum :

$$0 = (M+m)\vec{v}' + m(\vec{u}+\vec{v}') \tag{2}$$

Let $\vec{v_2}$ be the sought velocity of the buggy when the second man jump off the buggy; then from conservation of linear momentum of the system (buggy + one man) :

$$(M+m)\vec{v}' = M\vec{v_2} + m(\vec{u}+\vec{v_2}) \tag{3}$$

Solving equations (2) and (3) we get

$$\vec{v_2} = \frac{m(2M+3m)\vec{u}}{(M+m)(M+2m)} \tag{4}$$

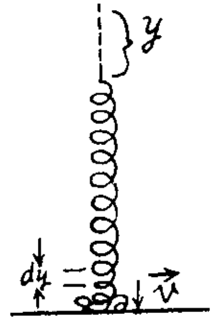
From (1) and (4)

$$\frac{v_2}{v_1} = 1 + \frac{m}{2(M+m)} > 1$$

Hence $v_2 > v_1$

1.157 The descending part of the chain is in free fall, it has speed $v = \sqrt{2gh}$ at the instant, all its points have descended a distance $y$. The length of the chain which lands on the floor during the differential time interval $dt$ following this instant is $vdt$.

For the incoming chain element on the floor :




From $dp_y = F_y\,dt$ (where $y$ – axis is directed down)

$$0 - (\lambda\, v dt)\, v = F_y\, dt$$

or $$F_y = -\lambda v^2 = -2\lambda g y$$

Hence, the force exerted on the falling chain equals $\lambda v^2$ and is directed upward. Therefore from third law the force exerted by the falling chain on the table at the same instant of time becomes $\lambda v^2$ and is directed downward.

Since a length of chain of weight $(\lambda y g)$ already lies on the table the total force on the floor is $(2\lambda y g) + (\lambda y g) = (3\lambda y g)$ or the weight of a length $3y$ of chain.

**1.158** Velocity of the ball, with which it hits the slab, $v = \sqrt{2gh}$

After first impact, $v' = ev$ (upward) but according to the problem $v' = \frac{v}{\eta}$, so $e = \frac{1}{\eta}$ (1)

and momentum, imparted to the slab,

$$= mv - (-mv') = mv(1+e)$$

Similarly, velocity of the ball after second impact,

$$v'' = ev' = e^2 v$$

And momentum imparted $= m(v' + v'') = m(1+e)\,ev$

Again, momentum imparted during third impact,

$$= m(1+e)\,e^2 v, \text{ and so on,}$$

Hence, net momentum, imparted $= mv(1+e) + mve(1+e) + mve^2(1+e) + \ldots$

$$= mv(1+e)(1+e+e^2+\ldots)$$

$$= mv\frac{(1+e)}{(1-e)}, \text{ (from summation of G.P.)}$$

$$= \sqrt{2gh}\,\frac{\left(1+\frac{1}{\eta}\right)}{\left(1-\frac{1}{\eta}\right)} = m\sqrt{2gh}\,/\,(\eta+1)\,/\,(\eta-1) \text{ (Using Eq. 1)}$$

$$= 0{\cdot}2 \text{ kg m/s. (On substitution)}$$

**1.159** (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e. $\vec{r}_C =$ constant or, $\Delta\vec{r}_C = 0$ i.e. $\sum m_i \Delta\vec{r}_i = 0$

or, $$m\left(\Delta\vec{r}_{mM} + \Delta\vec{r}_M\right) + M\,\Delta\vec{r}_M = 0$$

Thus, $$m(\vec{l}' + \vec{l}) + M\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m[\vec{v}'(t) + \vec{v}_2(t)] + M\vec{v}_2(t)$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

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Thus, $$m(\vec{l}' + \vec{l}) + M\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m[\vec{v}'(t) + \vec{v}_2(t)] + M\vec{v}_2(t)$$

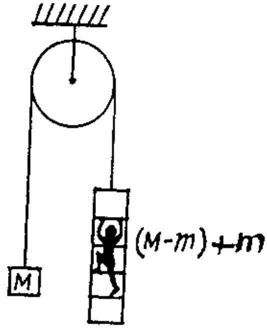
or, $$\vec{v}_2(t) = -\frac{m\vec{v}'(t)}{m+M} \qquad (1)$$

As $\vec{v}'(t)$ or $\vec{v}_2(t)$ is along horizontal direction, thus the sought force on the raft

$$= \frac{M\,d\vec{v}_2(t)}{dt} = -\frac{Mm}{m+M}\frac{d\vec{v}'(t)}{dt}$$

**Note :** *we may get the result of part* (a), *if we integrate Eq.* (1) *over the time of motion of man or raft.*

1.160 In the refrence frame fixed to the pulley axis the location of C.M. of the given system is described by the radius vector




$$\Delta \vec{r}_C = \frac{M\Delta \vec{r}_M + (M-m)\Delta \vec{r}_{(M-m)} + m\Delta \vec{r}_m}{2M}$$

But $\Delta \vec{r}_M = -\Delta \vec{r}_{(M-m)}$

and $\Delta \vec{r}_m = \Delta \vec{r}_{m(M-m)} + \Delta \vec{r}_{(M-m)}$

Thus $$\Delta \vec{r}_C = \frac{m\vec{l}'}{2M}$$

**Note :** *one may also solve this problem using momentum conservation.*

1.161 Velocity of cannon as well as that of shell equals $\sqrt{2gl\sin\alpha}$ down the inclined plane taken as the positive $x$ – axis. From the linear impulse momentum theorem in projection form along $x$ – axis for the system (connon + shell) i.e. $\Delta p_x = F_x \Delta t$ :

$p\cos\alpha - M\sqrt{2gl\sin\alpha} = Mg\sin\alpha\,\Delta t$ (as mass of the shell is neligible)

or, $$\Delta t = \frac{p\cos\alpha - M\sqrt{2gl\sin\alpha}}{Mg\sin\alpha}$$

1.162 From conservation of momentum, for the system (bullet + body) along the initial direction of bullet

$$mv_0 = (m+M)v, \text{ or, } v = \frac{mv_0}{m+M}$$

1.163 When the disc breaks off the body $M$, its velocity towards right (along $x$-axis) equals the velocity of the body $M$, and let the disc's velocity in upward direction (along $y$-axis) at that moment be $v'_y$

From conservation of momentum, along $x$-axis for the system (disc + body)

$$mv = (m+M)\, v'_x \quad \text{or} \quad v'_x = \frac{mv}{m+M} \tag{1}$$

And from energy conservation, for the same system in the field of gravity :

$$\frac{1}{2} mv^2 = \frac{1}{2}(m+M)\, v'^2_x + \frac{1}{2} m\, v'^2_y + mgh' ,$$

where $h'$ is the height of break off point from initial level. So,

$$\frac{1}{2} mv^2 = \frac{1}{2}(m+M)\frac{m^2 v^2}{(M+m)} + \frac{1}{2} m\, v'^2_y + mgh' , \qquad \text{using (1)}$$

or, $$v'^2_y = v^2 - \frac{mv^2}{(m+M)} - 2\, gh'$$

Also, if $h''$ is the height of the disc, from the break-off point,

then, $$v'^2_y = 2\, gh''$$

So, $$2g\,(h'' + h') = v^2 - \frac{m\, v^2}{(M+m)}$$

Hence, the total height, raised from the initial level

$$= h' + h'' = \frac{M\, v^2}{2g\,(M+m)}$$

1.164 (a) When the disc slides and comes to a plank, it has a velocity equal to $v = \sqrt{2\, gh}$. Due to friction between the disc and the plank the disc slows down and after some time the disc moves in one piece with the plank with velocity $v'$ (say).

From the momentum conservation for the system (disc + plank) along horizontal towards right :

$$mv = (m+M)\, v' \quad \text{or} \quad v' = \frac{mv}{m+M}$$

Now from the equation of the increment of total mechanical energy of a system :

$$\frac{1}{2}(M+m)\, v'^2 - \frac{1}{2} mv^2 = A_{fr}$$

or, $$\frac{1}{2}(M+m)\frac{m^2 v^2}{(m+M)^2} - \frac{1}{2} mv^2 = A_{fr}$$

so, $$\frac{1}{2} v^2 \left[\frac{m^2}{M+m} - m\right] = A_{fr}$$

Hence, $$A_{fr} = -\left(\frac{mM}{m+M}\right) gh = -\mu gh$$

$$\left(\text{where } \mu = \frac{mM}{m+M} = \text{reduced mass}\right)$$

(b) We look at the problem from a frame in which the hill is moving (together with the disc on it) to the right with speed $u$. Then in this frame the speed of the disc when it just gets onto the plank is, by the law of addition of velocities, $\bar{v} = u + \sqrt{2gh}$. Similarly the common speed of the plank and the disc when they move together is

$$\bar{v}' = u + \frac{m}{m+M}\sqrt{2gh}.$$

Then as above $\quad \bar{A}_{fr} = \frac{1}{2}(m+M)\bar{v}'^2 - \frac{1}{2}m\bar{v}^2 - \frac{1}{2}Mu^2$

$$= \frac{1}{2}(m+M)\left\{u^2 + \frac{2m}{m+M}u\sqrt{2gh} + \frac{m^2}{(m+M)^2}2gh\right\} - \frac{1}{2}(m+M)u^2 - \frac{1}{2}m\,2u\sqrt{2gh} - mgh$$

We see that $\bar{A}_{fr}$ is independent of $u$ and is in fact just $-\mu g h$ as in (a). Thus the result obtained does not depend on the choice of reference frame.

Do note however that it will be in correct to apply "conservation of enegy" formula in the frame in which the hill is moving. The energy carried by the hill is not negligible in this frame. See alsó the next problem.

**1.165** In a frame moving relative to the earth, one has to include the kinetic energy of the earth as well as earth's acceleration to be able to apply conservation of energy to the problem. In a reference frame falling to the earth with velocity $v_0$, the stone is initially going up with velocity $v_0$ and so is the earth. The final velocity of the stone is $0 = v_o - gt$ and that of the earth is $v_o + \frac{m}{M}gt$ ($M$ is the mass of the earth), from Newton's third law, where $t =$ time of fall. From conservation of energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 + mgh = \frac{1}{2}M\left(v_0 + \frac{m}{M}v_0\right)^2$$

Hence

$$\frac{1}{2}v_0^2\left(m + \frac{m^2}{M}\right) = mgh$$

Negecting $\frac{m}{M}$ in comparison with 1, we get

$$v_0^2 = 2gh \text{ or } v_o = \sqrt{2gh}$$

The point is this in earth's rest frame the effect of earth's accleration is of order $\frac{m}{M}$ and can be neglected but in a frame moving with respect to the earth the effect of earth's acceleration must be kept because it is of order one (i.e. large).

**1.166** From conservation of momentum, for the closed system "both colliding particles"

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}$$

or,

$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{1(3\vec{i} - 2\vec{j}) + 2(4\vec{j} - 6\vec{k})}{3} = \vec{i} + 2\vec{j} - 4\vec{k}$$

Hence

$$|\vec{v}| = \sqrt{1+4+16}\text{ m/s} = 4{\cdot}6\text{ m/s}$$

**1.167** For perfectly inelastic collision, in the C.M. frame, final kinetic energy of the colliding system (both spheres) becomes zero. Hence initial kinetic energy of the system in C.M. frame completely turns into the internal energy ($Q$) of the formed body. Hence

$$Q = \tilde{T}_i = \frac{1}{2}\mu \left|\vec{v}_1 - \vec{v}_2\right|^2$$

Now from energy conservation $\Delta T = -Q = -\frac{1}{2}\mu \left|\vec{v}_1 - \vec{v}_2\right|^2$,

In lab frame the same result is obtained as

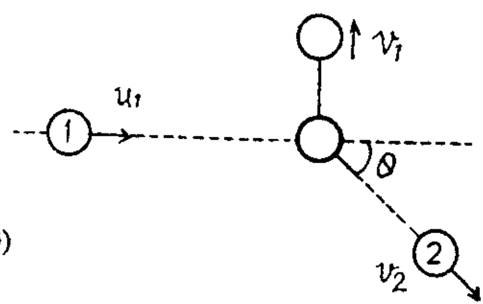
$$\Delta T = \frac{1}{2}\frac{(m_1\vec{v}_1 + m_2\vec{v}_2)^2}{m_1 + m_2} - \frac{1}{2} m_1 |\vec{v}_1|^2 + m_2 |\vec{v}_2|^2$$

$$= -\frac{1}{2}\mu |\vec{v}_1 - \vec{v}_2|^2$$

**1.168** (a) Let the initial and final velocities of $m_1$ and $m_2$ are $\vec{u}_1$, $\vec{u}_2$ and $\vec{v}$, $\vec{v}_2$ respectively. Then from conservation of momentum along horizontal and vertical directions, we get :

$$m_1 u_1 = m_2 v_2 \cos\theta \qquad (1)$$

and $$m_1 v_1 = m_2 v_2 \sin\theta \qquad (2)$$




Squaring (1) and (2) and then adding them,

$$m_2^2 v_2^2 = m_1^2 (u_1^2 + v_1^2)$$

Now, from kinetic energy conservation,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_1 v_1^2 \qquad (3)$$

or, $$m\,(u_1^2 - v_1^2) = m_2 v_2^2 = m_2 \frac{m_1^2}{m_2^2}(u_1^2 + v_1^2) \quad \text{[Using (3)]}$$

or, $$u_1^2\left(1 - \frac{m_1}{m_2}\right) = v_1^2\left(1 + \frac{m_1}{m_2}\right)$$

or, $$\left(\frac{v_1}{u_1}\right)^2 = \frac{m_2 - m_1}{m_1 + m_2} \qquad (4)$$

So, fraction of kinetic energy lost by the particle 1,

$$= \frac{\frac{1}{2} m_1 u_1^2 - \frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_1 u_1^2} = 1 - \frac{v_1^2}{u_1^2}$$

$$= 1 - \frac{m_2 - m_1}{m_1 + m_2} = \frac{2 m_1}{m_1 + m_2} \quad \text{[Using (4)]} \qquad (5)$$

(b) When the collision occurs head on,

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \qquad (1)$$

and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left[ \frac{m_1 (u_1 - v_1)^2}{m_2} \right]^2 \text{ [Using (5)]}$$

or, $$v_1 \left(1 + \frac{m_1}{m_2}\right) = u_1 \left(\frac{m_1}{m_2} - 1\right)$$

or, $$\frac{v_1}{u_1} = \frac{(m_1 / m_2 - 1)}{(1 + m_1 / m_2)} \qquad (6)$$

Fraction of kinetic energy, lost

$$= 1 - \frac{v_1^2}{u_1^2} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4 m_1 m_2}{(m_1 + m_2)^2} \text{ [Using (6)]}$$

**1.169** (a) When the particles fly apart in opposite direction with equal velocities (say $v$), then from conservatin of momentum,

$$m_1 u + 0 = (m_2 - m_1) v \qquad (1)$$

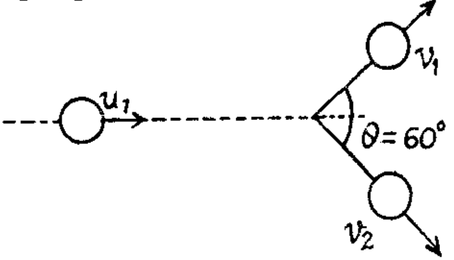
and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u^2 = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2$$

or, $$m_1 u^2 = (m_1 + m_2) v^2 \qquad (2)$$

From Eq. (1) and (2),

$$m_1 u^2 = (m_1 + m_2) \frac{m_1^2 u^2}{(m_2 - m_1)^2}$$




or, $$m_2^2 - 3 m_1 m_2 = 0$$

Hence $\frac{m_1}{m_2} = \frac{1}{3}$ as $m_2 \neq 0$

(b) When they fly apart symmetrically relative to the initial motion direction with the angle of divergence $\theta = 60°$,

From conservation of momentum, along horizontal and vertical direction,

$$m_1 u_1 = m_1 v_1 \cos(\theta/2) + m_2 v_2 \cos(\theta/2) \qquad (1)$$

and $$m_1 v_1 \sin(\theta/2) = m_2 v_2 \sin(\theta/2)$$

or, $$m_1 v_1 = m_2 v_2 \qquad (2)$$

Now, from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \qquad (3)$$

From (1) and (2),

$$m_1 u_1 = \cos(\theta/2) \left(m_1 v_1 + \frac{m_1 v_1}{m_2} m_2\right) = 2 m_1 v_1 \cos(\theta/2)$$

So, $$u_1 = 2\, v_1 \cos(\theta/2) \qquad (4)$$

From (2), (3), and (4)

$$4\, m_1 \cos^2(\theta/2)\, v_1^2 = m_1 v_1^2 + \frac{m_2 m_1^2 v_1^2}{m_2^2}$$

$$\text{or,} \quad 4\cos^2(\theta/2) = 1 + \frac{m_1}{m_2}$$

or, $$\frac{m_1}{m_2} = 4\cos^2\frac{\theta}{2} - 1$$

and putting the value of $\theta$, we get, $\dfrac{m_1}{m_2} = 2$

1.170 If $(v_{1x}, v_{1y})$ are the instantaneous velocity components of the incident ball and $(v_{2x}, v_{2y})$ are the velocity components of the struck ball at the same moment, then since there are no external impulsive forces (i.e. other than the mutual interaction of the balls)

We have $$u \sin\alpha = v_{1y} \;, \quad v_{2y} = 0$$

$$m\, u \cos\alpha = m\, v_{1x} + m\, v_{2x}$$

The impulsive force of mutual interaction satisfies

$$\frac{d}{dt}(v_{1x}) = \frac{F}{m} = -\frac{d}{dt}(v_{2x})$$

($F$ is along the $x$ axis as the balls are smooth. Thus $Y$ component of momentum is not transferred.) Since loss of K.E. is stored as deformation energy $D$, we have

$$D = \frac{1}{2} m u^2 - \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2$$

$$= \frac{1}{2} m u^2 \cos^2\alpha - \frac{1}{2} m v_{1x}^2 - \frac{1}{2} m v_{2x}^2$$

$$= \frac{1}{2m}\left[ m^2 u^2 \cos^2\alpha - m^2 v_{1x}^2 - (m u \cos\alpha - m v_{1x})^2 \right]$$

$$= \frac{1}{2m}\left[ 2m^2 u \cos\alpha\, v_{1x} - 2m^2 v_{1x}^2 \right] = m\,(v_{1x} u \cos\alpha - v_{1x}^2)$$

$$= m\left[ \frac{u^2\cos^2\alpha}{4} - \left( \frac{u\cos\alpha}{2} - v_{1x} \right)^2 \right]$$

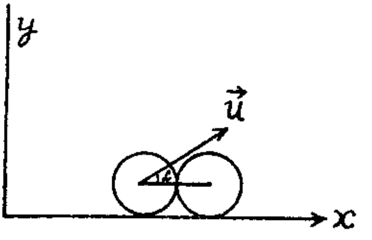
We see that $D$ is maximum when

$$\frac{u\cos\alpha}{2} = v_{1x}$$

and $$D_{max} = \frac{m u^2 \cos^2\alpha}{4}$$

Then $$\eta = \frac{D_{max}}{\frac{1}{2} m u^2} = \frac{1}{2}\cos^2\alpha = \frac{1}{4}$$

On substiuting $\alpha = 45°$



**1.171** From the conservation of linear momentum of the shell just before and after its fragmentation

$$3\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \quad (1)$$

where $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the velocities of its fragments.

From the energy conservation $\quad 3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \quad (2)$

Now
$$\vec{\tilde{v}}_i \text{ or } \vec{v}_{iC} = \vec{v}_i - \vec{v}_C = \vec{v}_i - \vec{v} \quad (3)$$

where $\vec{v}_C = \vec{v}$ = velocity of the C.M. of the fragments the velocity of the shell. Obviously in the C.M. frame the linear momentum of a system is equal to zero, so

$$\vec{\tilde{v}}_1 + \vec{\tilde{v}}_2 + \vec{\tilde{v}}_3 = 0 \quad (4)$$

Using (3) and (4) in (2), we get

$$3\eta v^2 = (\vec{v} + \vec{\tilde{v}}_1)^2 + (\vec{v} + \vec{\tilde{v}}_2)^2 + (\vec{v} - \vec{\tilde{v}}_1 - \vec{\tilde{v}}_2)^2 \quad = 3v^2 + 2\tilde{v}_1^2 + 2\tilde{v}_2^2 + 2\,\vec{\tilde{v}}_1 \cdot \vec{\tilde{v}}_2$$

or,
$$2\tilde{v}_1^2 + 2\tilde{v}_1\,\tilde{v}_2 \cos\theta + 2\tilde{v}_2^2 + 3\,(1-\eta)v^2 = 0 \quad (5)$$

If we have had used $\vec{\tilde{v}}_2 = -\vec{\tilde{v}}_1 - \vec{\tilde{v}}_3$, then Eq. 5 were contain $\tilde{v}_3$ instead of $\tilde{v}_2$ and so on.

The problem being symmetrical we can look for the maximum of any one. Obviously it will be the same for each.

For $\tilde{v}_1$to be real in Eq. (5)

$$4\,\tilde{v}_2^2 \cos^2\theta \geq 8(2\tilde{v}_2^2 + 3\,(1-\eta)\,v^2) \text{ or } 6(\eta - 1)v^2 \geq (4 - \cos^2\theta)\tilde{v}_2^2$$

So,
$$\tilde{v}_2 \leq v\sqrt{\frac{6\,(\eta-1)}{4-\cos^2\theta}} \quad \text{or} \quad \tilde{v}_{2(\max)} = \sqrt{2\,(\eta-1)}\;\; v$$

Hence $\;v_{2\,(\max)} = |\vec{v} + \vec{\tilde{v}}_2|_{\max} = v + \sqrt{2\,(\eta-1)}\;v = v\left(1 + \sqrt{2\,(\eta-1)}\right) = 1 \text{ km/s}$

Thus owing to the symmetry

$$v_{1(\max)} = v_{2\,(\max)} = v_{3(\max)} = v\left(1 + \sqrt{2(\eta-1)}\right) = 1 \text{ km/s}$$

**1.172** Since, the collision is head on, the particle 1 will continue moving along the same line as before the collision, but there will be a change in the magnitude of it's velocity vector. Let it starts moving with velocity $v_1$ and particle 2 with $v_2$ after collision, then from the conservation of momentum

$$mu = mv_1 + mv_2 \quad \text{or,} \quad u = v_1 + v_2 \quad (1)$$

And from the condition, given,

$$\eta = \frac{\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2\right)}{\frac{1}{2}mu^2} = 1 - \frac{v_1^2 + v_2^2}{u^2}$$

or,
$$v_1^2 + v_2^2 = (1-\eta)\,u^2 \quad (2)$$

From (1) and (2),

$$v_1^2 + (u - v_1)^2 = (1-\eta)\,u^2$$

or,
$$v_1^2 + u^2 - 2\,uv_1 + v_1^2 = (1-\eta)\,u^2$$

or, $$2v_1^2 - 2v_1 u + \eta u^2 = 0$$

So, $$v_1 = 2u \pm \frac{\sqrt{4u^2 - 8\eta u^2}}{4}$$

$$= \frac{1}{2}\left[u \pm \sqrt{u^2 - 2\eta u^2}\right] = \frac{1}{2}u(1 \pm \sqrt{1-2\eta})$$

Positive sign gives the velocity of the 2nd particle which lies ahead. The negative sign is correct for $v_1$.

So, $v_1 = \frac{1}{2}u(1 - \sqrt{1-2\eta}) = 5$ m/s will continue moving in the same direction.

Note that $v_1 = 0$ if $\eta = 0$ as it must.

1.173 Since, no external impulsive force is effective on the system "$M + m$", its total momentum along any direction will remain conserved.

So from $p_x =$ const.

$$mu = Mv_1 \cos\theta \quad \text{or,} \quad v_1 = \frac{m}{M}\frac{u}{\cos\theta} \qquad (1)$$

and from $p_y =$ const

$mv_2 = Mv_1 \sin\theta$ or, $v_2 = \frac{M}{m} v_1 \sin\theta = u\tan\theta$, [using (1)]

Final kinetic energy of the system

$$T_f = \frac{1}{2}mv_2^2 + \frac{1}{2}Mv_1^2$$

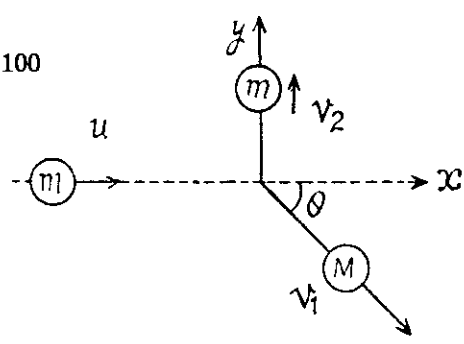
And initial kinetic energy of the system$= \frac{1}{2}mu^2$

So, $$\% \text{ change} = \frac{T_f - T_i}{T_i} \times 100$$

$$= \frac{\frac{1}{2}mu^2\tan^2\theta + \frac{1}{2}M\frac{m^2}{M^2}\frac{u^2}{\cos^2\theta} - \frac{1}{2}mu^2}{\frac{1}{12}mu^2} \times 100$$

$$= \frac{\frac{1}{2}u^2\tan^2\theta + \frac{1}{2}\frac{m}{M}u^2\sec^2\theta - \frac{1}{2}u^2}{\frac{1}{2}u^2} \times 100$$

$$= \left(\tan^2\theta + \frac{m}{M}\sec^2\theta - 1\right) \times 100$$




and putting the values of $\theta$ and $\frac{m}{M}$, we get % of change in kinetic energy$= -40\%$

1.174 (a) Let the particles $m_1$ and $m_2$ move with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively. On the basis of solution of problem 1.147 (b)

$$\tilde{p} = \mu v_{rel} = \mu \left|\vec{v_1} - \vec{v_2}\right|$$

As $$\vec{v_1} \perp \vec{v_2}$$

So, $$\tilde{p} = \mu\sqrt{v_1^2 + v_2^2} \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

(b) Again from 1.147 (b)

$$\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu\left|\vec{v_1} - \vec{v_2}\right|^2$$

So, $$\tilde{T} = \frac{1}{2}\mu(v_1^2 + v_2^2)$$

1.175 From conservation of momentum

$$\vec{p_1} = \vec{p_1}' + \vec{p_2}'$$

so $$\left(\vec{p_1} - \vec{p_1}'\right)^2 = p_1^2 - 2p_1 p_1' \cos\theta_1 + p_1'^2 = p_2'^2$$

From conservation of energy

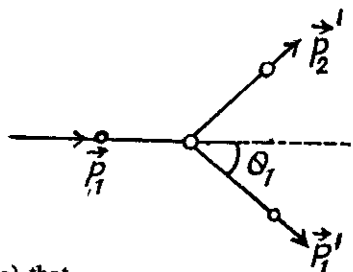
$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

Eliminating $p_2'$ we get

$$0 = p_1'^2\left(1 + \frac{m_2}{m_1}\right) - 2p_1' p_1 \cos\theta_1 + p_1^2\left(1 - \frac{m_2}{m_1}\right)$$

This quadratic equation for $p_1'$ has a real solution in terms of $p_1$ and $\cos\theta_1$ only if

$$4\cos^2\theta_1 \geq 4\left(1 - \frac{m_2^2}{m_1^2}\right)$$




or $$\sin^2\theta_1 \leq \frac{m_2^2}{m_1^2}$$

or $\sin\theta_1 \leq +\frac{m_2}{m_1}$ or $\sin\theta_1 \geq -\frac{m_2}{m_1}$

This clearly implies (since only + sign makes sense) that

$$\sin\theta_{1\,max} = \frac{m_2}{m_1}.$$

1.176 From the symmetry of the problem, the velocity of the disc $A$ will be directed either in the initial direction or opposite to it just after the impact. Let the velocity of the disc $A$ after the collision be $v'$ and be directed towards right after the collision. It is also clear from the symmetry of problem that the discs $B$ and $C$ have equal speed (say $v''$) in the directions, shown. From the condition of the problem,

$$\cos\theta = \frac{\eta\frac{d}{2}}{d} = \frac{\eta}{2} \text{ so, } \sin\theta = \sqrt{4 - \eta^2}\,/\,2 \qquad (1)$$

For the three discs, system, from the conservation of linear momentum in the symmetry direction (towards right)

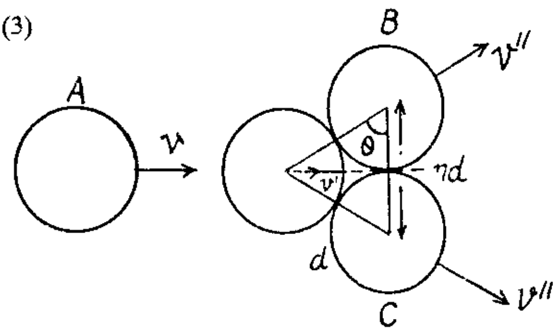
$$mv = 2m\,v''\sin\theta + m\,v' \text{ or, } v = 2\,v''\sin\theta + v' \qquad (2)$$

From the definition of the coefficeint of restitution, we have for the discs $A$ and $B$ (or $C$)

$$e = \frac{v'' - v' \sin\theta}{v \sin\theta - 0}$$

But $e = 1$, for perfectly elastic collision,

So, $v \sin\theta = v'' - v' \sin\theta$ (3)




From (2) and (3),

$$v' = \frac{v\,(1 - 2\sin^2\theta)}{(1 + 2\sin^2\theta)}$$

$$= \frac{v\,(\eta^2 - 2)}{6 - \eta^2} \quad \{\text{using (1)}\}$$

Hence we have,

$$v' = \frac{v\,(\eta^2 - 2)}{6 - \eta^2}$$

Therefore, the disc $A$ will recoil if $\eta < \sqrt{2}$ and stop if $\eta = \sqrt{2}$.

**Note :** *One can write the equations of momentum conservation along the direction perpendicular to the initial direction of disc A and the consevation of kinetic energy instead of the equation of restitution.*

1.177 (a) Let a molecule comes with velocity $\vec{v_1}$ to strike another stationary molecule and just after collision their velocities become $\vec{v}'_1$ and $\vec{v}'_2$ respectively. As the mass of the each molecule is same, conservation of linear momentum and conservation of kinetic energy for the system (both molecules) respectively gives :

$$\vec{v_1} = \vec{v}'_1 + \vec{v}'_2$$

and $$v_1^2 = v'^2_1 + v'^2_2$$

From the property of vector addition it is obvious from the obtained Eqs. that

$$\vec{v}'_1 \perp \vec{v}'_2 \quad \text{or} \quad \vec{v}'_1 \cdot \vec{v}'_2 = 0$$

(b) Due to the loss of kinetic energy in inelastic collision $v_1^2 > v'^2_1 + v'^2_2$

so, $\vec{v}'_1 \cdot \vec{v}'_2 > 0$ and therefore angle of divergence $< 90°$.

1.178 Suppose that at time $t$, the rocket has the mass $m$ and the velocity $\vec{v}$, relative to the reference frame, employed. Now consider the inertial frame moving with the velocity that the rocket has at the given moment. In this reference frame, the momentum increament that the rocket & ejected gas system acquires during time $dt$ is,

$$d\vec{p} = m\,d\vec{v} + \mu\,dt\,\vec{u} = \vec{F}\,dt$$

or, $$m\frac{d\vec{v}}{dt} = \vec{F} - \mu\vec{u}$$

or, $$m\vec{w} = \vec{F} - \mu\vec{u}$$

**1.179** According to the question, $\vec{F} = 0$ and $\mu = -dm/dt$ so the equation for this system becomes,

$$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $d\vec{v} \uparrow\downarrow \vec{u}$ so, $m\,dv = -u\,dm$.

Integrating within the limits :

$$\frac{1}{u}\int_0^v dv = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = \ln\frac{m_0}{m}$$

Thus, $v = u\ln\frac{m_0}{m}$

As $d\vec{v} \uparrow\downarrow \vec{u}$, so in vector form $\vec{v} = -\vec{u}\ln\frac{m_0}{m}$

**1.180** According to the question, $\vec{F}$ (external force) = 0

So, $$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $$d\vec{v} \uparrow\downarrow \vec{u},$$

so, in scalar form, $$m\,dv = -u\,dm$$

or, $$\frac{w\,dt}{u} = -\frac{dm}{m}$$

Integrating within the limits for $m(t)$

$$\frac{wt}{u} = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or,} \quad \frac{v}{u} = -\ln\frac{m}{m_0}$$

Hence, $$m = m_0\,e^{-(wt/u)}$$

**1.181** As $\vec{F} = 0$, from the equation of dynamics of a body with variable mass;

$$m\frac{d\vec{v}}{dt} = \vec{u}\frac{dm}{dt} \quad \text{or,} \quad d\vec{v} = \vec{u}\frac{dm}{m} \tag{1}$$

Now $d\vec{v} \uparrow\downarrow \vec{u}$ and since $\vec{u} \perp \vec{v}$, we must have $|d\vec{v}| = v_0\,d\alpha$ (because $v_0$ is constant) where $d\alpha$ is the angle by which the spaceship turns in time $dt$.

So, $$-u\frac{dm}{m} = v_0\,d\alpha \quad \text{or,} \quad d\alpha = -\frac{u}{v_0}\frac{dm}{m}$$

or, $$\alpha = -\frac{u}{v_0}\int_{m_0}^{m}\frac{dm}{m} = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right)$$

1.182 We have $\frac{dm}{dt} = -\mu$ or, $dm = -\mu\, dt$

Integrating $$\int_{m_0}^{m} dm = -\mu \int_0^t dt \quad \text{or,} \quad m = m_0 - \mu t$$

As $\vec{u} = 0$ so, from the equation of variable mass system :

$$(m_0 - \mu t)\frac{d\vec{v}}{dt} = \vec{F} \quad \text{or,} \quad \frac{d\vec{v}}{dt} = \vec{w} = \vec{F}/(m_0 - \mu t)$$

or, $$\int_0^{\vec{v}} d\vec{v} = \vec{F}\int_0^t \frac{dt}{(m_0 - \mu t)}$$

Hence $$\vec{v} = \frac{\vec{F}}{\mu} \ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

1.183 Let the car be moving in a reference frame to which the hopper is fixed and at any instant of time, let its mass be $m$ and velocity $\vec{v}$.

Then from the general equation, for variable mass system.

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}$$

We write the equation, for our system as,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \vec{v}\frac{dm}{dt} \quad \text{as,} \quad \vec{u} = -\vec{v} \tag{1}$$

So $$\frac{d}{dt}(m\vec{v}) = \vec{F}$$

and $$\vec{v} = \frac{\vec{F}t}{m} \text{ on integration.}$$

But $$m = m_0 + \mu t$$

so, $$\vec{v} = \frac{\vec{F}t}{m_0\left(1 + \frac{\mu t}{m_0}\right)}$$

Thus the sought acceleration, $$\vec{w} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m_0\left(1 + \frac{\mu t}{m_0}\right)^2}$$

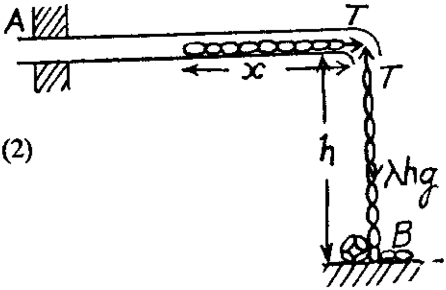
1.184 Let the length of the chain inside the smooth horizontal tube at an arbitrary instant is $x$. From the equation,

$$m\vec{w} = \vec{F} + \vec{u}\frac{dm}{dt}$$

as $\vec{u} = 0$, $\vec{F} \uparrow\uparrow \vec{w}$, for the chain inside the tube

$$\lambda x w = T \text{ where } \lambda = \frac{m}{l} \qquad (1)$$

Similarly for the overhanging part,




$$\vec{u} = 0$$

Thus $\qquad mw = F$

or $\qquad \lambda h w = \lambda h g - T \qquad (2)$

From (1) and (2),

$$\lambda (x + h)\, w = \lambda h g \quad \text{or,} \quad (x + h)\, v \frac{dv}{ds} = hg$$

or, $$(x + h)\, v \frac{dv}{(-dx)} = gh,$$

[As the length of the chain inside the tube decreases with time, $ds = -dx$.]

or, $$v\, dv = -g h \frac{dx}{x + h}$$

Integrating, $$\int_0^v v\, dv = -g h \int_{(l-h)}^{0} \frac{dx}{x + h}$$

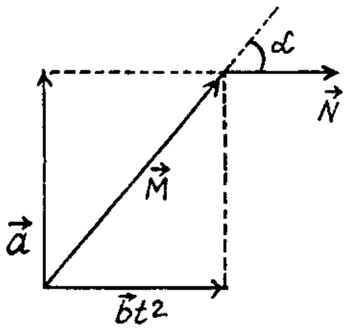
or, $$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right) \quad \text{or} \quad v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$

**1.185** Force moment relative to point $O$ ;

$$\vec{N} = \frac{d\vec{M}}{dt} = 2\,\vec{b}t$$

Let the angle between $\vec{M}$ and $\vec{N}$,

$\alpha = 45°$ at $t = t_0$ ,




$$\text{Then } \frac{1}{\sqrt{2}} = \frac{\vec{M} \cdot \vec{N}}{|\vec{M}||\vec{N}|} = \frac{(\vec{a} + \vec{b}\, t_0^2) \cdot (2\,\vec{b}t_0)}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0}$$

$$= \frac{2b^2 t_0^3}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0} = \frac{b\, t_0^2}{\sqrt{a^2 + b^2 t_0^4}}$$

So, $\quad 2\, b^2 t_0^4 = a^2 + b^2 t_0^4$ or, $t_0 = \sqrt{\frac{a}{b}}$ (as $t_0$ cannot be negative)

It is also obvious from the figure that the angle $\alpha$ is equal to 45° at the moment $t_0$, when $a = b\, t_0^2$, i.e. $t_0 = \sqrt{a/b}$ and $\vec{N} = 2\sqrt{\frac{a}{b}}\, \vec{b}$.

**1.186** $\vec{M}(t) = \vec{r} \times \vec{p} = \left(\vec{v_0}t + \frac{1}{2}\vec{g}t^2\right) \times m\left(\vec{v_0} + \vec{g}t\right)$

$$= mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(-\vec{k}) + \frac{1}{2} mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(\vec{k})$$

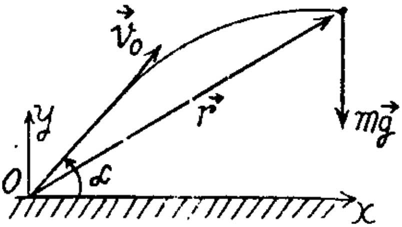
$$= \frac{1}{2} mv_0 gt^2 \cos\alpha\, (-\vec{k}):$$

Thus $M(t) = \dfrac{mv_0 g t^2 \cos\alpha}{2}$

Thus angular momentum at maximum height

i.e. at $t = \dfrac{\tau}{2} = \dfrac{v_0 \sin\alpha}{g}$,

$M\left(\dfrac{\tau}{2}\right) = \left(\dfrac{m v_0^3}{2g}\right) \sin^2\alpha \cos\alpha = 37 \text{ kg} - \text{m}^2/\text{s}$

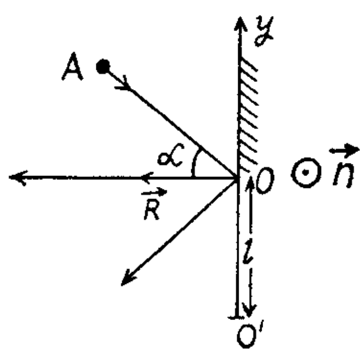



**Alternate :**

$$\vec{M}(0) = 0 \text{ so, } \vec{M}(t) = \int_0^t \vec{N}\,dt = \int_0^t (\vec{r} \times m\vec{g})$$

$$= \int_0^t \left[\left(\vec{v_0}t + \frac{1}{2}\vec{g}t^2\right) \times m\vec{g}\right] dt = \left(\vec{v_0} \times m\vec{g}\right)\frac{t^2}{2}$$

**1.187** (a) The disc experiences gravity, the force of reaction of the horizontal surface, and the force $\vec{R}$ of reaction of the wall at the moment of the impact against it. The first two forces counter-balance each other, leaving only the force $\vec{R}$. It's moment relative to any point of the line along which the vector $\vec{R}$ acts or along normal to the wall is equal to zero and therefore the angular momentum of the disc relative to any of these points does not change in the given process.




(b) During the course of collision with wall the position of disc is same and is equal to $\vec{r}_{oo'}$. Obviously the increment in linear momentum of the ball $\Delta\vec{p} = 2mv\cos\alpha\,\hat{n}$

Here, $\Delta\vec{M} = \vec{r}_{oo'} \times \Delta\vec{p} = 2\,mv\,l\cos\alpha\,\hat{n}$ and directed normally emerging from the plane of figure

Thus $|\Delta\vec{M}| = 2\,mv\,l\cos\alpha$

**1.188** (a) The ball is under the influence of forces $\vec{T}$ and $m\vec{g}$ at all the moments of time, while moving along a horizontal circle. Obviously the vertical component of $\vec{T}$ balance $m\vec{g}$ and so the net moment of these two about any point becoems zero. The horizontal component of $\vec{T}$, which provides the centripetal acceleration to ball is already directed toward the centre ($C$) of the horizontal circle, thus its moment about the point $C$ equals zero at all the moments of time. Hence the net moment of the force acting on the ball about point $C$ equals zero and that's why the angular mommetum of the ball is conserved about the horizontal circle.

(b) Let $\alpha$ be the angle which the thread forms with the vertical.

Now from equation of particle dynamics :

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 l\sin\alpha$

Hence on solving $\cos\alpha = \dfrac{g}{\omega^2 l}$ (1)

As $|\vec{M}|$ is constant in magnitude so from figure.

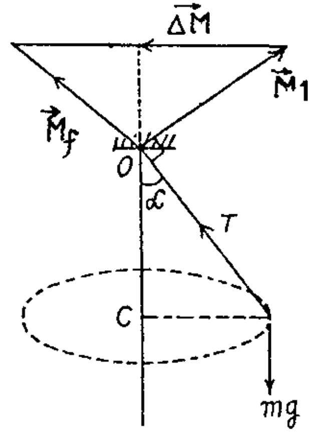
$|\Delta\vec{M}| = 2M\cos\alpha$ where

$M = |\vec{M}_i| = |\vec{M}_f|$

$= |\vec{r}_{bo}\times m\vec{v}| = mv\,l\left(\text{as } \vec{r}_{bo}\perp\vec{v}\right)$

Thus $|\Delta\vec{M}| = 2\,mv\,l\cos\alpha = 2\,m\omega\,l^2\sin\alpha\cos\alpha$

$$= \frac{2\,mgl}{\omega}\sqrt{1-\left(\frac{g}{\omega^2 t}\right)^2}\text{ (using 1).}$$



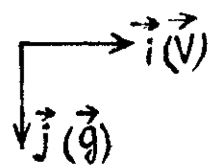

1.189 During the free fall time $t = \tau = \sqrt{\dfrac{2h}{g}}$, the reference point $O$ moves in hoizontal direction (say towards right) by the distance $V\tau$. In the translating frame as $\vec{M}(O) = 0$, so

$$\Delta\vec{M} = \vec{M}_f = \vec{r}$$

$$= (-V\tau\,\vec{i}+h\vec{j})\times m\{g\tau\vec{j}-V\vec{i}\}$$

$$= -m\,Vg\tau^2\,\vec{h}+mVh\,(+\vec{k})$$

$$= -m\,Vg\left(\frac{2h}{g}\right)\vec{k}+mV\,h\,(+\vec{k}) = -m\,Vh\,\vec{k}$$




Hence $|\Delta\vec{M}| = m\,Vh$

1.190 The Coriolis force is $(2m\,\vec{v}'\times\vec{\omega})$.

Here $\vec{\omega}$ is along the $z$-axis (vertical). The moving disc is moving with velocity $v_0$ which is constant. The motion is along the $x$-axis say. Then the Coriolis force is along $y$-axis and has the magnitude $2m\,v_0\,\omega$. At time $t$, the distance of the centre of moving disc from $O$ is $v_0 t$ (along $x$-axis). Thus the torque $N$ due to the coriolis force is

$N = 2m\,v_0\,\omega\cdot v_0 t$ along the $z$-axis.

Hence equating this to $\frac{dM}{dt}$

$$\frac{dM}{dt} = 2m\, v_0^2\, \omega t \quad \text{or} \quad M = m v_0^2 \omega t^2 + \text{constant.}$$

The constant is irrelevant and may be put equal to zero if the disc is originally set in motion from the point $O$.

This discussion is approximate. The Coriolis force will cause the disc to swerve from straight line motion and thus cause deviation from the above formula which will be substantial for large $t$.

**1.191** If $\dot{r} =$ radial velocity of the particle then the total energy of the particle at any instant is

$$\frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + kr^2 = E \qquad (1)$$

where the second term is the kinetic energy of angular motion about the centre $O$. Then the extreme values of $r$ are determined by $\dot{r} = 0$ and solving the resulting quadratic equation

$$k(r^2)^2 - Er^2 + \frac{M^2}{2m} = 0$$

we get

$$r^2 = \frac{E \pm \sqrt{E^2 - \frac{2M^2 k}{m}}}{2k}$$

From this we see that

$$E = k(r_1^2 + r_2^2) \qquad (2)$$

where $r_1$ is the minimum distance from O and $r_2$ is the maximum distance. Then

$$\frac{1}{2} m v_2^2 + 2k r_2^2 = k\,(\, r_1^2 + r_2^2\,)$$

Hence,
$$m = \frac{2kr^2}{v_2^2}$$

**Note** : Eq. (1) can be derived from the standard expression for kinetic energy and angular momentum in plane poler coordinates :

$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

$M$ = angular momentum = $mr^2 \dot{\theta}$

**1.192** The swinging sphere experiences two forces : The gravitational force and the tension of the thread. Now, it is clear from the condition, given in the problem, that the moment of these forces about the vertical axis, passing through the point of suspension $N_z = 0$. Consequently, the angular momentum $M_z$ of the sphere relative to the given axis ($z$) is constant.

Thus
$$m v_0\,(l \sin \theta) = m v\, l \qquad (1)$$

where $m$ is the mass of the sphere and $v$ is it s velocity in the position, when the thread forms an angle $\frac{\pi}{2}$ with the vertical. Mechanical energy is also conserved, as the sphere is under the influence if only one other force, i.e. tension, which does not perform any work, as it is always perpendicular to the velocity.

So, $$\frac{1}{2}mv_0^2 + mg\,l\cos\theta = \frac{1}{2}mv^2 \qquad (2)$$

From (1) and (2), we get,

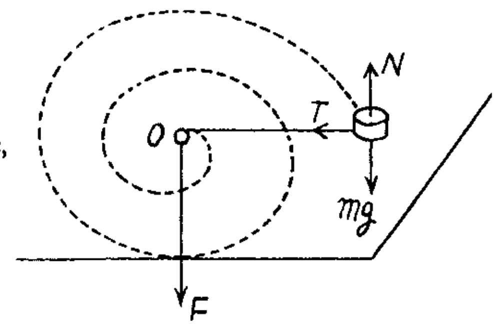
$$v_0 = \sqrt{2gl/\cos\theta}$$

**1.193** Forces, acting on the mass $m$ are shown in the figure. As $\vec{N} = m\vec{g}$, the net torque of these two forces about any fixed point must be equal to zero. Tension $T$, acting on the mass $m$ is a central force, which is always directed towards the centre $O$. Hence the moment of force $T$ is also zero about the point $O$ and therefore the angular momentum of the particle $m$ is conserved about $O$.

Let, the angular velocity of the particle be $\omega$, when the separation between hole and particle $m$ is $r$, then from the conservation of momentum about the point $O$, :

$$m(\omega_0 r_0) r_0 = m(\omega r) r,$$

or $$\omega = \frac{\omega_0 r_0^2}{r^2}$$




Now, from the second law of motion for $m$,

$$T = F = m\omega^2 r$$

Hence the sought tension;

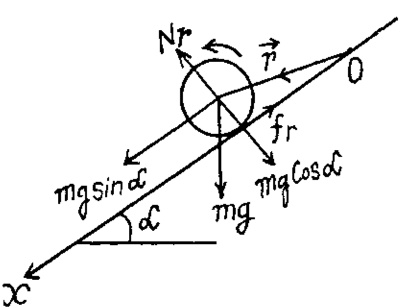
$$F = \frac{m\omega_0^2 r_0^4 r}{r^4} = \frac{m\omega_0^2 r_0^4}{r^3}$$

**1.194** On the given system the weight of the body $m$ is the only force whose moment is effective about the axis of pulley. Let us take the sense of $\vec{\omega}$ of the pulley at an arbitrary instant as the positive sense of axis of rotation ($z$-axis)

As $$M_z(0) = 0, \text{ so, } \Delta M_z = M_z(t) = \int N_z\,dt$$

So, $$M_z(t) = \int_0^t mgR\,dt = mgRt$$

**1.195** Let the point of contact of sphere at initial moment ($t = 0$) be at $O$. At an arbitrary moment, the forces acting on the sphere are shown in the figure. We have normal reaction $N_r = mg\sin\alpha$ and both pass through same line and the force of static friction passes through the point $O$, thus the moment about point $O$ becomes zero. Hence $mg\sin\alpha$ is the only force which has effective torque about point $O$, and is given by $|\vec{N}| = mgR\sin\alpha$ normally emerging from the plane of figure.




As $\vec{M}(t = 0) = 0$, so, $\Delta\vec{M} = \vec{M}(t) = \int \vec{N}\,dt$

Hence, $M(t) = Nt = mgR\sin\alpha t$

**1.196** Let position vectors of the particles of the system be $\vec{r_i}$ and $\vec{r_i}'$ with respect to the points $O$ and $O'$ respectively. Then we have,

$$\vec{r_i} = \vec{r_i}' + \vec{r_0} \qquad (1)$$

where $\vec{r_O}$ is the radius vector of $O'$ with respect to $O$.

Now, the angular momentum of the system relative to the point $O$ can be written as follows;

$$\vec{M} = \sum \left(\vec{r_i} \times \vec{p_i}\right) = \sum \left(\vec{r_i}' \times \vec{p_i}\right) + \sum \left(\vec{r_0} \times \vec{p_i}\right) \qquad \text{[using (1)]}$$

or, $$\vec{M} = \vec{M}' + \left(\vec{r_0} \times \vec{p}\right), \text{ where, } \vec{p} = \sum \vec{p_i} \qquad (2)$$

From (2), if the total linear momentum of the system, $\vec{p} = 0$, then its angular momentum does not depend on the choice of the point $O$.

*Note that in the C.M. frame, the system of particles, as a whole is at rest.*

**1.197** On the basis of solution of problem 1.196, we have concluded that; "in the C.M. frame, the angular momentum of system of particles is independent of the choice of the point, relative to which it is determined" and in accordance with the problem, this is denoted by $\vec{M}$.

We denote the angular momentum of the system of particles, relative to the point $O$, by $\vec{M}$. Since the internal and proper angular momentum $\tilde{\vec{M}}$, in the C.M. frame, does not depend on the choice of the point $O'$, this point may be taken coincident with the point $O$ of the $K$-frame, at a given moment of time. Then at that moment, the radius vectors of all the particles, in both reference frames, are equal ($\vec{r_i}' = \vec{r_i}$) and the velocities are related by the equation,

$$\vec{v_i} = \tilde{\vec{v}}_i + \vec{v_c}, \qquad (1)$$

where $\vec{v_c}$ is the velocity of C.M. frame, relative to the $K$-frame. Consequently, we may write,

$$\vec{M} = \sum m_i \left(\vec{r_i} \times \vec{v_i}\right) = \sum m_i \left(\vec{r_i}' \times \tilde{\vec{v}}_i\right) + \sum m_i \left(\vec{r_i} \times \vec{v_c}\right)$$

or, $$\vec{M} = \tilde{\vec{M}} + m\left(\vec{r_c} \times \vec{v_c}\right), \text{ as } \sum m_i \vec{r_i} = m\vec{r_c}, \text{ where } m = \sum m_i .$$

or, $$\vec{M} = \tilde{\vec{M}} + (\vec{r_c} \times m\vec{v_c}) = \tilde{\vec{M}} + (\vec{r_c} \times \vec{p})$$

**1.198** From conservation of linear momentum along the direction of incident ball for the system consists with colliding ball and phhere

$$mv_0 = mv' + \frac{m}{2} v_1 \qquad (1)$$

where $v'$ and $v_1$ are the velocities of ball and sphere 1 respectively after collision. (Remember that the collision is head on).

As the collision is perfectly elastic, from the definition of co-efficeint of restitution,

$$1 = \frac{v' - v_1}{0 - v_0} \text{ or, } v' - v_1 = -v_0 \qquad (2)$$

Solving (1) and (2), we get,

$v_1 = \frac{4\,v_0}{3}$, directed towards right.

In the C.M. frame of spheres 1 and 2 (Fig.)

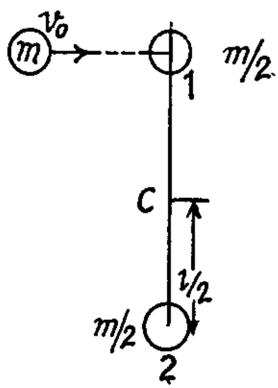
$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$ and $|\vec{\tilde{p}}_1| = |\vec{\tilde{p}}_2| = \mu\,|\vec{v}_1 - \vec{v}_2|$

Also, $\vec{r}_{1C} = -\vec{r}_{2C}$, thus $\vec{\tilde{M}} = 2\,[\vec{r}_{1C} \times \vec{\tilde{p}}_1]$

As $\vec{r}_{1C} \perp \vec{\tilde{p}}_1$, so, $\vec{\tilde{M}} = 2\left[\frac{l}{2}\,\frac{m/2}{2}\,\frac{4\,v_0}{3}\,\hat{n}\right]$

(where $\hat{n}$ is the unit vector in the sense of $\vec{r}_{1C} \times \vec{\tilde{p}}_1$)

Hence $\tilde{M} = \frac{m\,v_0\,l}{3}$




1.199 In the C.M. frame of the system (both the discs + spring), the linear momentum of the discs are related by the relation, $\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$, at all the moments of time.

where, $\tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \mu\,v_{rel}$

And the total kinetic energy of the system,

$$T = \frac{1}{2}\mu\,v_{rel}^2 \text{ [See solution of 1.147 (b)]}$$

Bearing in mind that at the moment of maximum deformation of the spring, the projection of $\vec{v}_{rel}$ along the length of the spring becomes zero, i.e. $v_{rel\,(x)} = 0$.

The conservation of mechanical energy of the considered system in the C.M. frame gives.

$$\frac{1}{2}\left(\frac{m}{2}\right)v_0^2 = \frac{1}{2}\kappa\,x^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_{rel\,(y)}^2 \qquad (1)$$

Now from the conservation of angular momentum of the system about the C.M.,

$$\frac{1}{2}\left(\frac{l_0}{2}\right)\left(\frac{m}{2}v_0\right) = 2\left(\frac{l_0+x}{2}\right)\frac{m}{2}\,v_{rel\,(y)}$$

or, $$v_{rel\,(y)} = \frac{v_0\,l_0}{(l_0+x)} = v_0\left(1+\frac{x}{l_0}\right)^{-1} \approx v_0\left(1-\frac{x}{l_0}\right), \text{ as } x << l_0 \qquad (2)$$

Using (2) in (1), $$\frac{1}{2}m\,v_0^2\left[1-\left(1-\frac{x}{l_0}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{1}{2}m\,v_0^2\left[1-\left(1-\frac{2x}{l_0}+\frac{x^2}{l_0^2}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{m\,v_0^2\,x}{l_0} \approx \kappa\,x^2, \text{ [neglecting } x^2/l_0^2]$$

As $$x \neq 0, \text{ thus } x = \frac{m\,v_0^2}{\kappa\,l_0}$$