30. PERMUTATIONS AND COMBINATIONS

IMPORTANT FACTS AND FORMULAE

Factorial Notation : Let n be a positive integer. Then, factorial n, denoted by $\underline{|n}$ or $n!$ is defined as :

$$n! = n(n-1)(n-2) \ldots\ldots 3.2.1.$$

Examples : (*i*) $5! = (5 \times 4 \times 3 \times 2 \times 1) = 120$; (*ii*) $4! = (4 \times 3 \times 2 \times 1) = 24$ etc.

We define, $0! = 1$.

Permutations : *The different arrangements of a given number of things by taking some or all at a time, are called permutations.*

Ex. 1. All permutations (or arrangements) made with the letters a, b, c by taking two at a time are (**ab, ba, ac, ca, bc, cb**).

Ex. 2. All permutations made with the letters a, b, c, taking all at a time are : (**abc, acb, bac, bca, cab, cba**).

Number of Permutations : Number of all permutations of n things, taken r at a time, is given by :

$$^nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Examples : (*i*) $^6P_2 = (6 \times 5) = 30$. (*ii*) $^7P_3 = (7 \times 6 \times 5) = 210$.

Cor. ***Number of all permutations of n things, taken all at a time = n !***

An Important Result : *If there are n objects of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of third kind and so on and p_r are alike of rth kind, such that $(p_1 + p_2 + \ldots\ldots + p_r) = n$.*

Then, ***number of permutations of these n objects is :***

$$\frac{n!}{(p_1!) \cdot (p_2!) \ldots\ldots (p_r!)}.$$

Combinations : *Each of the different groups or selections which can be formed by taking some or all of a number of objects, is called a combination.*

Ex. 1. Suppose we want to select two out of three boys A, B, C. Then, possible selections are AB, BC and CA.

Note that AB and BA represent the same selection.

Ex. 2. All the combinations formed by a, b, c, taking two at a time are **ab, bc, ca**.

Ex. 3. The only combination that can be formed of three letters a, b, c taken all at a time is **abc**.

Ex. 4. Various groups of 2 out of four persons A, B, C, D are :

AB, AC, AD, BC, BD, CD.

Ex. 5. Note that *ab* and *ba* are two different permutations but they represent the same combination.

Number of Combinations : The number of all combinations of n things, taken r at a time is :

$$^nC_r = \frac{n!}{(r!)(n-r)!} = \frac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r!}.$$

Note that : $^nC_n = 1$ *and* $^nC_0 = 1$.

An Important Result : $^nC_r = {^nC_{(n-r)}}$.

Example : (*i*) $^{11}C_4 = \frac{(11 \times 10 \times 9 \times 8)}{(4 \times 3 \times 2 \times 1)} = 330.$

(*ii*) $^{16}C_{13} = {^{16}C_{(16-13)}} = {^{16}C_3} = \frac{16 \times 15 \times 14}{3!} = \frac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560.$

SOLVED EXAMPLES

Ex. 1. ***Evaluate :*** $\frac{30!}{28!}$

Sol. We have, $\frac{30!}{28!} = \frac{30 \times 29 \times (28!)}{28!} = (30 \times 29) = 870.$

Ex. 2. ***Find the value of*** (*i*) $^{60}P_3$ (*ii*) $^4P_4$

Sol. (*i*) $^{60}P_3 = \frac{60!}{(60-3)!} = \frac{60!}{57!} = \frac{60 \times 59 \times 58 \times (57!)}{57!} = (60 \times 59 \times 58) = 205320.$

(*ii*) $^4P_4 = 4! = (4 \times 3 \times 2 \times 1) = 24.$

Ex. 3. ***Find the value of*** (*i*) $^{10}C_3$ (*ii*) $^{100}C_{98}$ (*iii*) $^{50}C_{50}$

Sol. (*i*) $^{10}C_3 = \frac{10 \times 9 \times 8}{3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$

(*ii*) $^{100}C_{98} = {^{100}C_{(100-98)}} = {^{100}C_2} = \left(\frac{100 \times 99}{2 \times 1}\right) = 4950.$

(*iii*) $^{50}C_{50} = 1.$ $[\because {^nC_n} = 1]$

Ex. 4. ***How many words can be formed by using all letters of the word 'BIHAR' ?***

Sol. The word BIHAR contains 5 different letters.

$\therefore$ Required number of words $= {^5P_5} = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

Ex. 5. ***How many words can be formed by using all the letters of the word 'DAUGHTER' so that the vowels always come together ?***

Sol. Given word contains 8 different letters. When the vowels AUE are always together, we may suppose them to form an entity, treated as one letter.

Then, the letters to be arranged are DGHTR (AUE).

These 6 letters can be arranged in $^6P_6 = 6! = 720$ ways.

The vowels in the group (AUE) may be arranged in $3! = 6$ ways.

$\therefore$ Required number of words $= (720 \times 6) = 4320.$

Ex. 6. ***How many words can be formed from the letters of the word 'EXTRA', so that the vowels are never together ?***

Sol. The given word contains 5 different letters.

Taking the vowels EA together, we treat them as one letter.

Then, the letters to be arranged are XTR (EA).

These letters can be arranged in $4! = 24$ ways.

The vowels EA may be arranged amongst themselves in $2! = 2$ ways.

Number of words, each having vowels together $= (24 \times 2) = 48$.

Total number of words formed by using all the letters of the given words

$$= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Number of words, each having vowels never together $= (120 - 48) = 72$.

Ex. 7. ***How many words can be formed from the letters of the word 'DIRECTOR' so that the vowels are always together?***

Sol. In the given word, we treat the vowels IEO as one letter.

Thus, we have DRCTR (IEO).

This group has 6 letters of which R occurs 2 times and others are different.

Number of ways of arranging these letters $= \frac{6!}{2!} = 360$.

Now 3 vowels can be arranged among themselves in $3! = 6$ ways.

$\therefore$ Required number of ways $= (360 \times 6) = 2160$.

Ex. 8. ***In how many ways can a cricket eleven be chosen out of a batch of 15 players?***

Sol. Required number of ways $= {}^{15}C_{11} = {}^{15}C_{(15-11)} = {}^{15}C_4$

$$= \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365.$$

Ex. 9. ***In how many ways, a committee of 5 members can be selected from 6 men and 5 ladies, consisting of 3 men and 2 ladies?***

Sol. (3 men out 6) and (2 ladies out of 5) are to be chosen.

$\therefore$ Required number of ways $= ({}^6C_3 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200$.

EXERCISE 30

(OBJECTIVE TYPE QUESTIONS)

Directions : ***Mark (✓) against the correct answer :***

1. The value of ${}^{75}P_2$ is :
   (*a*) 2775 (*b*) 150 (*c*) 5550 (*d*) None of these
2. How many 4-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed ?
   (*a*) 40 (*b*) 400 (*c*) 5040 (*d*) 2520
3. How many words with or without meaning, can be formed by using all the letters of the word, 'DELHI', using each letter exactly once ?
   (*a*) 10 (*b*) 25 (*c*) 60 (*d*) 120
4. In how many ways can the letters of the word 'APPLE' be arranged ?
   (*a*) 720 (*b*) 120 (*c*) 60 (*d*) 180 (*e*) None of these
5. In how many ways can the letters of the word 'LEADER' be arranged ?
   (*a*) 72 (*b*) 144 (*c*) 360 (*d*) 720 (*e*) None of these
   (Bank P.O. 2003)

6. In how many different ways can the letters of the word 'RUMOUR' be arranged ?
(*a*) 180 (*b*) 90 (*c*) 30 (*d*) 720 (*e*) None of these
(Bank P.O. 2003)

7. How many words can be formed by using all the letters of the word, 'ALLAHABAD' ?
(*a*) 3780 (*b*) 1890 (*c*) 7560 (*d*) 2520 (*e*) None of these

8. How many arrangements can be made out of the letters of the word 'ENGINEERING' ?
(*a*) 277200 (*b*) 92400 (*c*) 69300 (*d*) 23100 (*e*) None of these

9. How many words can be formed from the letters of the word 'SIGNATURE' so that the vowels always come together ? **(Bank P.O. 2003)**
(*a*) 720 (*b*) 1440 (*c*) 2880 (*d*) 3600 (*e*) 17280

10. In how many different ways can the letters of the word 'OPTICAL' be arranged so that the vowels always come together ? **(M.B.A. 2002)**
(*a*) 120 (*b*) 720 (*c*) 4320 (*d*) 2160 (*e*) None of these

11. In how many different ways can the letters of the word 'SOFTWARE' be arranged in such a way that the vowels always come together ? **(Bank P.O. 2003)**
(*a*) 120 (*b*) 360 (*c*) 1440 (*d*) 13440 (*e*) 720

12. In how many different ways can the letters of the word 'LEADING' be arranged in such a way that the vowels always come together ? **(Bank P.O. 2002)**
(*a*) 360 (*b*) 480 (*c*) 720 (*d*) 5040 (*e*) None of these

13. In how many different ways can the letters of the word 'JUDGE' be arranged in such a way that the vowels always come together ? **(S.B.I.P.O. 2001)**
(*a*) 48 (*b*) 120 (*c*) 124 (*d*) 160 (*e*) None of these

14. In how many different ways can the letters of the word 'AUCTION' be arranged in such a way that the vowels always come together ? **(S.B.I.P.O. 2000)**
(*a*) 30 (*b*) 48 (*c*) 144 (*d*) 576 (*e*) None of these

15. In how many different ways can the letters of the word 'BANKING' be arranged so that the vowels always come together ? **(Bank P.O. 2003)**
(*a*) 120 (*b*) 240 (*c*) 360 (*d*) 540 (*e*) 720

16. In how many different ways can the letters of the word 'CORPORATION' be arranged so that the vowels always come together ? **(S.B.I.P.O. 2003)**
(*a*) 810 (*b*) 1440 (*c*) 2880 (*d*) 50400 (*e*) 5760

17. In how many different ways can the letters of the word 'MATHEMATICS' be arranged so that the vowels always come together ?
(*a*) 10080 (*b*) 4989600 (*c*) 120960 (*d*) None of these

18. In how many different ways can the letters of the word 'DETAIL' be arranged in such a way that the vowels occupy only the odd positions ? **(Bank P.O. 2002)**
(*a*) 32 (*b*) 48 (*c*) 36 (*d*) 60 (*e*) 120

19. In how many different ways can the letters of the word 'MACHINE' be arranged so that the vowels may occupy only the odd positions ?
(*a*) 210 (*b*) 576 (*c*) 144 (*d*) 1728 (*e*) 3456

20. In how many ways can a group of 5 men and 2 women be made out of a total of 7 men and 3 women ? **(Bank P.O. 2003)**
(*a*) 63 (*b*) 90 (*c*) 126 (*d*) 45 (*e*) 135

21. In how many ways a committee, consisting of 5 men and 6 women can be formed from 8 men and 10 women ? **(Bank P.O. 2003)**
(*a*) 266 (*b*) 5040 (*c*) 11760 (*d*) 86400 (*e*) None of these

22. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done ? **(M.B.A. 2002)**
(*a*) 564 (*b*) 645 (*c*) 735 (*d*) 756 (*e*) None of these

23. In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there ?

(a) 159 (b) 194 (c) 205 (d) 209 (e) None of these

(S.B.I.P.O. 2000)

24. A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw ?

(a) 32 (b) 48 (c) 64 (d) 96 (e) None of these

(Bank P.O. 1998)

25. How many 3-digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9, which are divisible by 5 and none of the digits is repeated ? **(S.S.C. 2000)**

(a) 5 (b) 10 (c) 15 (d) 20

26. In how many ways can 21 books on English and 19 books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together ?

(a) 3990 (b) 1540 (c) 1995 (d) 3672 (e) None of these

27. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed ?

(a) 210 (b) 1050 (c) 25200 (d) 21400 (e) None of these

ANSWERS

1. (c)	2. (c)	3. (d)	4. (c)	5. (c)	6. (a)
7. (c)	8. (a)	9. (a)	10. (c)	11. (c)	12. (c)
13. (a)	14. (d)	15. (c)	16. (d)	17. (c)	18. (c)
19. (b)	20. (a)	21. (c)	22. (d)	23. (d)	24. (c)
25. (d)	26. (b)	27. (c)			

SOLUTIONS

1. $^{75}P_2 = \frac{75\,!}{(75-2)\,!} = \frac{75\,!}{73\,!} = \frac{75 \times 74 \times (73\,!)}{73\,!} = (75 \times 74) = 5550.$

2. 'LOGARITHM' contains 10 different letters.

Required number of words = Number of arrangements of 10 letters, taking 4 at a time

$$= {}^{10}P_4 = (10 \times 9 \times 8 \times 7) = 5040.$$

3. The word 'DELHI' contains 5 different letters.

Required number of words = Number of arrangements of 5 letters, taken all at a time

$$= {}^5P_5 = 5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

4. The word 'APPLE' contains 5 letters, 1A, 2P, 1L and 1E.

$\therefore$ Required number of ways $= \frac{5\,!}{(1\,!)(2\,!)(1\,!)(1\,!)} = 60.$

5. The word 'LEADER' contains 6 letters, namely 1L, 2E, 1A, 1D and 1R.

$\therefore$ Required number of ways $= \frac{6\,!}{(1\,!)(2\,!)(1\,!)(1\,!)(2\,!)} = 360.$

6. The word 'RUMOUR' contains 6 letters, namely 2R, 2U, 1M and 1U.

$\therefore$ Required number of ways $= \frac{6\,!}{(2\,!)(2\,!)(1\,!)(1\,!)} = 180.$

7. The word 'ALLAHABAD' contains 9 letters, namely 4A, 2L, 1H, 1B and 1D.

∴ Requisite number of words $= \dfrac{9\,!}{(4\,!)(2\,!)(1\,!)(1\,!)(1\,!)} = 7560.$

8. The word 'ENGINEERING' contains 11 letters, namely 3E, 3N, 2G, 2I and 1R.

∴ Required number of arrangements $= \dfrac{11\,!}{(3\,!)(3\,!)(2\,!)(2\,!)(1\,!)} = 277200.$

9. The word 'SIGNATURE' contains 9 different letters.

When the vowels IAUE are taken together, they can be supposed to form an entity, treated as one letter.

Then, the letters to be arranged are SGNTR (IAUE).

These 6 letters can be arranged in $^6P_6 = 6\,! = 720$ ways.

The vowels in the group (IAUE) can be arranged amongst themselves in

$^4P_4 = 4\,! = 24$ ways.

∴ Required number of words $= (720 \times 24) = 17280.$

10. The word 'OPTICAL' contains 7 different letters.

When the vowels OIA are always together, they can be supposed to form one letter.

Then, we have to arrange the letters PTCL (OIA).

Now, 5 letters can be arranged in $5\,! = 120$ ways.

The vowels (OIA) can be arranged among themselves in $3\,! = 6$ ways.

∴ Required number of ways $= (120 \times 6) = 720.$

11. The word 'SOFTWARE' contains 8 different letters.

When the vowels OAE are always together, they can be supposed to form one letter.

Thus, we have to arrange the letters SFTWR (OAE).

Now, 5 letters can be arranged in $6\,! = 720$ ways.

The vowels (OAE) can be arranged among themselves in $3\,! = 6$ ways.

∴ Required number of ways $= (720 \times 6) = 4320.$

12. The word 'LEADING' has 7 different letters.

When the vowels EAI are always together, they can be supposed to form one letter.

Then, we have to arrange the letters LDNG (EAI).

Now, 5 letters can be arranged in $5\,! = 120$ ways.

The vowels (EAI) can be arranged among themselves in $3\,! = 6$ ways.

∴ Required number of ways $= (120 \times 6) = 720.$

13. The word 'JUDGE' has 5 different letters.

When the vowels UE are always together, they can be supposed to form one letter.

Then, we have to arrange the letters JDG (UE).

Now, 4 letters can be arranged in $4\,! = 24$ ways.

The vowels (UE) can be arranged among themselves in $2\,! = 2$ ways.

∴ Required number of ways $= (24 \times 2) = 48.$

14. The word 'AUCTION' has 7 different letters.

When the vowels AUIO are always together, they can be supposed to form one letter.

Then, we have to arrange the letters CTN (AUIO).

Now, 4 letters can be arranged in $4\,! = 24$ ways.

The vowels (AUIO) can be arranged among themselves in $4\,! = 24$ ways.

∴ Required number of ways $= (24 \times 24) = 576.$

15. In the word 'BANKING', we treat the two vowels AI as one letter. Thus, we have BNKNG (AI).

This has 6 letters of which N occurs 2 times and the rest are different.

Number of ways of arranging these letters $= \dfrac{6\,!}{(2\,!)(1\,!)(1\,!)(1\,!)(1\,!)} = 360.$

Now, 2 vowels AI can be arranged in $2\,! = 2$ ways.

$\therefore$ Required number of ways $= (360 \times 2) = 720.$

16. In the word 'CORPORATION', we treat the vowels OOAIO as one letter. Thus, we have CRPRTN (OOAIO).

This has 7 letters of which R occurs 2 times and the rest are different.

Number of ways of arranging these letters $= \dfrac{7\,!}{2\,!} = 2520.$

Now, 5 vowels in which O occurs 3 times and the rest are different, can be arranged in $\dfrac{5\,!}{3\,!} = 20$ ways.

$\therefore$ Required number of ways $= (2520 \times 20) = 50400.$

17. In the word 'MATHEMATICS' we treat the vowels AEAI as one letter. Thus, we have MTHMTCS (AEAI).

Now, we have to arrange 8 letters, out of which M occurs twice, T occurs twice and the rest are different.

$\therefore$ Number of ways of arranging these letters $= \dfrac{8\,!}{(2\,!)(2\,!)} = 10080.$

Now, AEAI has 4 letters in which A occurs 2 times and the rest are different.

Number of ways of arranging these letters $= \dfrac{4\,!}{2\,!} = 12.$

$\therefore$ Required number of words $= (10080 \times 12) = 120960.$

18. There are 6 letters in the given word, out of which there are 3 vowels and 3 consonants. Let us mark these positions as under :

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)$$

Now, 3 vowels can be placed at any of the three places out of 4, marked 1, 3, 5.

Number of ways of arranging the vowels $= {}^3P_3 = 3\,! = 6.$

Also, the 3 consonants can be arranged at the remaining 3 positions.

Number of ways of these arrangements $= {}^3P_3 = 3\,! = 6.$

Total number of ways $= (6 \times 6) = 36.$

19. There are 7 letters in the given word, out of which there are 3 vowels and 4 consonants. Let us mark the positions to be filled up as follows :

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)$$

Now, 3 vowels can be placed at any of the three places, out of the four marked 1, 3, 5, 7.

$\therefore$ Number of ways of arranging the vowels $= {}^4P_3 = (4 \times 3 \times 2) = 24.$

Also, the 4 consonants at the remaining 4 positions may be arranged in

$= {}^4P_4 = 4\,! = 24$ ways.

$\therefore$ Required number of ways $= (24 \times 24) = 576.$

20. Required number of ways $= ({}^7C_5 \times {}^3C_2) = ({}^7C_2 \times {}^3C_1) = \left(\dfrac{7 \times 6}{2 \times 1} \times 3\right) = 63.$

21. Required number of ways $= ({}^{8}C_5 \times {}^{10}C_6)$

$$= ({}^{8}C_3 \times {}^{10}C_4) = \left(\frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}\right) = 11760.$$

22. We may have (3 men and 2 women) or (4 men and 1 woman) or (5 men only)

$\therefore$ Required number of ways $= ({}^{7}C_3 \times {}^{6}C_2) + ({}^{7}C_4 \times {}^{6}C_1) + ({}^{7}C_5)$

$$= \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{6 \times 5}{2 \times 1}\right) + ({}^{7}C_3 \times {}^{6}C_1) + ({}^{7}C_2)$$

$$= 525 + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 6\right) + \left(\frac{7 \times 6}{2 \times 1}\right)$$

$$= (525 + 210 + 21) = 756.$$

23. We may have (1 boy and 3 girls) or (2 boys and 2 girls) or (3 boys and 1 girl) or (4 boys).

$\therefore$ Required number of ways $= ({}^{6}C_1 \times {}^{4}C_3) + ({}^{6}C_2 \times {}^{4}C_2) + ({}^{6}C_3 \times {}^{4}C_1) + ({}^{6}C_4)$

$$= ({}^{6}C_1 \times {}^{4}C_1) + ({}^{6}C_2 \times {}^{4}C_2) + ({}^{6}C_3 \times {}^{4}C_1) + ({}^{6}C_2)$$

$$= (6 \times 4) + \left(\frac{6 \times 5}{2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) + \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times 4\right) + \left(\frac{6 \times 5}{2 \times 1}\right)$$

$$= (24 + 90 + 80 + 15) = 209.$$

24. We may have (1 black and 2 non-black) or (2 black and 1 non-black) or (3 black).

$\therefore$ Required number of ways $= ({}^{3}C_1 \times {}^{6}C_2) + ({}^{3}C_2 \times {}^{6}C_1) + ({}^{3}C_3)$

$$= \left(3 \times \frac{6 \times 5}{2 \times 1}\right) + \left(\frac{3 \times 2}{2 \times 1} \times 6\right) + 1 = (45 + 18 + 1) = 64.$$

25. Since each desired number is divisible by 5, so we must have 5 at the unit place. So, there is 1 way of doing it.

Tens place can be filled by any of the remaining 5 numbers.

So, there are 5 ways of filling the tens place.

The hundreds place can now be filled by any of the remaining 4 digits. So, there are 4 ways of filling it.

$\therefore$ Required number of numbers $= (1 \times 5 \times 4) = 20.$

26. In order that two books on Hindi are never together, we must place all these books as under :

X E X E X E X ..... X E X

where E denotes the position of an English book and X that of a Hindi book.

Since there are 21 books on English, the number of places marked X are therefore, 22.

Now, 19 places out of 22 can be chosen in ${}^{22}C_{19} = {}^{22}C_3 = \frac{22 \times 21 \times 20}{3 \times 2 \times 1} = 1540$ ways.

Hence, the required number of ways = 1540.

27. Number of ways of selecting (3 consonants out of 7) and (2 vowels out of 4)

$$= ({}^{7}C_3 \times {}^{4}C_2) = \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) = 210.$$

Number of groups, each having 3 consonants and 2 vowels = 210.

Each group contains 5 letters.

Number of ways of arranging 5 letters among themselves

$$= 5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

$\therefore$ Required number of words $= (210 \times 120) = 25200.$