NOTES

ALGEBRA

The part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations.

Introduction to Algebra

- **Variable:** A letter symbol which can take various numerical values is called variable or literal.
  Example: x, y, z etc.

- **Constant:** A symbol which can take a fixed numerical value is called a constant.
  Example: $-1, \frac{1}{2}, 2, 4, 3, 5$ etc.

- **Term:** Numericals or literals or their combinations by operation of multiplication are called terms.
  Example: $5x^2, 7x, \frac{x}{7}, \frac{y^2}{9}, \frac{5}{2}x$ etc.

- **Constant Term:** A term of an expression having no literal is called a constant term.
  Example: $4, \frac{-1}{2}, \frac{7}{4}$ etc.

- Algebraic expression: A combination of constant and variable connected by mathematical operations $(+, -, \times, \div)$ is called an algebraic expression.
  Example: $2a+3,\ 2a+3b,\ 7n+4, -p+-r+2$ etc.

Types of Algebraic Expressions;-

- **Monomial:** An expression containing only one term is called a monomial.
  Example: $7x, -11a^2b^2, \frac{-7}{5}$ etc.

- **Binomial:** An expression containing two terms is called a binomial.
  Example: $2x+3,\ 6x-5y$ etc.

- **Trinomial:** An expression containing only three terms is called trinomial.
  Example: $2x+3y-\frac{5}{2}, \frac{a}{2}-\frac{b}{3}+4$ etc.

- **Multinomial:** An expression containing only two or more terms is called a multinomial.

  Example: $2x+3y-2,\ 7+6x+3y,\ 3\sqrt{x}+4$ etc.

- **Degree of a monomial:** The degree of a monomial is the sum of the indices (power) in each of its variables.

  Example: The degree of $6x^2y\ z^3$ is $2+1+3=6$

- **Degree of polynomial:** The highest power of terms in a polynomial is called the degree of a polynomial.

  Example: Degree of $6x^2-5x^2 2x-3$ is '3'. The degree of $6x^6+5x^5\ y^7+9$ is 12.

- **Zero polynomial:** If all the coefficients in a polynomial are zeros, then it is called a zero polynomial.

- **Zero of the polynomial:** The number for which the value of a polynomial is zero, is called zero of the polynomial.

  Example: $x=1$, is called a zero of $x-1$.

- **Polynomial:** An expression containing one or more terms with positive integral powers is called a polynomial.

  Example: $8x^2+2x+3,\ \ 2p-3q+\frac{5}{2}r$ etc.

- **Factors:** In a product, each of the literals or numerical values is called a factor of the product.

  Example: $15=3\times5$, where 3, 5 are called the factors of 15.

  $10xy=2\times5\times x\times y$, where $2,\ 5,\ x,\ y$ are called the factors of $10\ xy$.

- **Coefficient:** In a product containing two or more than two factors, each factor is called coefficient of the product of other factors.

  Example: In 6xy, 6 is called numerical coefficient of $'xy'$ and $'x'$ is the literal coefficient of $'7y'$ and $'y'$ is the literal coefficient of $7x$.

  Note: Degree of zero polynomial is not defined but some of the famous mathematicians claim the degree of zero polynomial is defined as -1 or $-\infty$.

- **Substitutions:** The method of replacing numerical values in the place of literal numbers is called substitution.

  Example: The value of $9x$ at $x=4$ is $9\times4=36$

Operation of Algebraic Expression

- **Addition:** Addition of algebraic expressions means adding the like terms of the expressions.

  Note: Unlike terms cannot be combined or added.

  Example: $2x+5x=7x,\ 8xy+9xy=17xy$ etc.

- Combining the coefficients of like terms of an expression through addition or subtraction is called simplication of an algebraic expression.

- There are two methods of adding algebraic expression. They are
  - Horizontal method
  - Vertical method

Horizontal method

In this method, like terms should be added and unlike terms should be written separately by using associative law of addition.

Example: $7x+4y$ and $3x-5y$

Solution: $7x+4y+3x-5y$

$=7x+3x+4y-5y$

$=10x-y$

Vertical method: following the steps.

- In this method the expressions to be added, are written one below the other.
- The like terms of each type are placed in separate columns.
- The sum will be written below that column.

Example:

$$\begin{array}{r} 5x^2-3y^2+4 \\ 7x^2-8y^2-10 \\ \hline 12x^2-11y^2-6 \\ \hline \end{array}$$

Subtraction of Algebraic Expressions:

The additive inverse of a number

- The additive inverse of any number is obtained by a simple change of its sign, so additive inverse of a number is also called the negative of that number.

  Example: Additive inverse of 8 is -8.

Additive Inverse of Expression

- The additive inverse or the negative of an expression is obtained by replacing each term of the expression by its additive inverse.

Example: 1 Additive inverse of $-9x$ is $9x$.

Example: 2 Subtract $16x-5y$ from $7x+4y$

Solution: $(7x+4y)-(16x-5y)$

$=7x+4y-16x+5y$

$=-9x+9y$

Example: 3 Subtract $3\times 2-5x-4$ from $5x^2+6x+8$

Solution: $(5x^2+6x+8)-(3x^2-5x-4)$ additive inverse of $\left(3x^2-5x-4\right)$ is $-3x^2+5x-4$

then, $5x^2+6x+8-3x^2+5x-4$

$=5x^2-3x^2+6x+5x+8-4$

$=2x^2+11x+4$

Subtraction can also be done in two ways;

- **Horizontal method:**

  Example: $(x+y)-(2x+3y)$

  $=x+y-2x-3y$

  $=-x-2y$

- **Vertical method:**

  Example:

$$\begin{array}{r} x^2+2xy+y^2 \\ -2x^2+5xy+2y^2 \\ +\quad -\quad - \\ \hline 3x^2-3xy-y^2 \\ \hline \end{array}$$

Multiplication of Algebraic Expression

- First remember the Rule of Exponent.

  $a^m \times a^n = a^{m+n}$ (a is any variable and m, n are positive integers.)

  e.g. $x^6 \times x^7 = x^{6+7} = x^{13}$

Multiplication of monomials

- Product of two monomials = (Product of their numerical coefficients) × (product of their variables).

  Example: $6xy$ and $-3x^2y$

  Solution: $(6xy)\times(-3x^2y)$

  $=\{6\times -3\}\times(x^2y\times xy)$

  $=-18x^3y^2$

Multiplication of a Binomial and a monomial;

- Use distributive properly $x\,(y+z)=xy+xz$.

  Example: $2x(3y+z)=2x\times 3y+2xz$

  $=6xy+2xz$

Multiplication can also done in two ways

- **Horizontal method:**

  Example: $2x(3x+5z)$

  $=2x\times 3x+2x\times 5z$

  $=6x^2+10xz$

- **Column method:**

$$\begin{array}{r} 3x+5z \\ \times\quad 2x \\ \hline 6x^2+10xz \\ \hline \end{array}$$

Multiplication of two binomials

- Suppose two binomials $(x+y)$ and $(a+b)$ and using the distributive law of multiplication over addition twice.

  $(x+y)\times(a+b)=x(a+b)+y(a+b)$

  $=(x\times a+x\times b)+(y\times a+y\times b)$

  $=ax+bx+ay+by$

  Example: $(3x-5y)\times(7x+4y)$

  $=3x(7x+4y)-5y(7x+4y)$

  $=(2x\times 7x+3x\times 4y)+(-5y\times 7x+4y\times -5y)$

  $=21x^2+12xy-35xy-20y^2$

  $=21x^2-23xy-20y^2$

Division of Algebraic Expression

- First remember the rule of exponent $a^m\div a^n=a^{m-n}$ (Where $m>n$ and m and n are positive integers and a is variable)

  Example: Divide x^{10} by x^5

  Solution: $x^{10}\div x^5=x^{10-5}=x^5$

Division of monomials

- **Rule:** Quotient of two monomials

  = (quotient of their numerical coefficients) $\times$ (quotient of their variables)

Example: $38x^2y^2$ by $19xy$

Solution: $38x^2y^2 \div 19xy$

$$= \frac{38}{19} \times \frac{x^2y^2}{xy}$$

$$= 2 \times x^{2-1}.y^{2-1}$$

$$= 2xy$$

- Division of a polynomial by a monomial

Example: $12x^4 - 6x^2 + 3x$ by $3x$

Solution: $\left(12x^4 - 6x^2 + 3x\right) \div 3x$

$$= \frac{12x^4}{3x} - \frac{6x^2}{3x} + \frac{3x}{3x}$$

$$4x^3 - 2x + 1$$

Division of a polynomial by a polynomial

- Arrange the terms of the dividend and divisor in decreasing order of powers keeping zero for missing term.
- Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.
- Multiply the entire divisor by this first term of the quotient and put the product under the dividend, keeping like terms under each other.
- Subtract the product from the dividend and bring down the rest of the dividend.
- Step 4 gives use the new dividend, repeat steps 1 to 4.
- Continue the process till the degree of the remainder becomes not defined or less than that of the divisor.

Example: 1 $\left(x^2 + 2x + 1\right)$ by $\left(x + 1\right)$

$$\begin{array}{r|l} x+1 & x^2 + 2x + 1 \;(x + 1 \\ & \underline{\underset{-}{x^2} \underset{-}{+} x} \\ & \quad x + 1 \\ & \quad \underline{\underset{-}{x} \underset{-}{+} 1} \\ & \quad\quad \underline{0} \end{array}$$

Example: 2 Divide $x^3 + 8$ by $x + 2$

$$\begin{array}{r|l} x+2 & x^3 + 8 \;(x^2 - 2x + 4 \\ & \underline{\underset{-}{x^3} \underset{-}{+} 2x^2} \\ & -2x^2 + 8 \\ & \underline{\underset{+}{-2x^2} \underset{+}{-} 4x} \\ & \quad 4x + 8 \\ & \quad \underline{\underset{-}{4x} \underset{-}{+} 8} \\ & \quad\quad \underline{0} \end{array}$$

Special Products

- Special products and its proof:

(i) $(a+b)^2 + a^2 + 2ab + b^2$

LHS $= (a+b)^2$

$= (a+b)(a+b)$

$= a(a+b) + b(a+b)$

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

LHS = RHS

(ii) $(a+b)^2 = a^2 - 2ab + b^2$

LHS $= (a-b)(a-b)$

$= a(a-b) - b(a-b)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

LHS = RHS

(iii) $(a+b)(a-b) = a^2 - b^2$

LHS $= (a+b)(a-b)$

$= a(a-b) + b(a-b)$

$= a^2 - ab + ab - b^2$

$= a^2 - b^2$

LHS = RHS

Example: 1 If $x + \frac{1}{x} = 5$, find $x^2 + \frac{1}{x^2}$

Solution: $x + \frac{1}{x} = 5$

Squaring both sides

$$\left(x + \frac{1}{x}\right)^2 = (5)^2$$

$$= x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x} = 25$$

$$= x^2 + \frac{1}{x^2} + 2 = 25$$

$$= x^2 + \frac{1}{x} = 25 - 2$$

$\therefore\ x^2+\frac{1}{x^2}=23$

Example: 2 Expand $(x-3y)^2$

Solution: $(x-3y)(x-3y)$

$=x^2-2\times 3y\times x+(3y)^2$

$=x^2-6xy+9y^2$

Linear equation in one variable

Equation:

- If two numerical expressions are joined or connected by the symbol "is equal to (=)", then the combination is called as an equation.

LHS and RHS Notations

- The sign of equality '=' in an equation divides it into two sides namely, the left hand side and the right hand side, written as LHS and RHS respectively.

Example: $7x+2=5x+3$

Here, $7x+2=\text{LHS}$ and $5x+3=\text{RHS}$

Linear Equation in one variable

- Linear equation which involves one variable is called linear equation in one variable or simple equation.

  Example: 1 $4x-3=x+5$

  Example: 2 $y+8=11$

- General form of linear equation is $ax+b=0$, where $(a\neq 0)$ and a, b are real numbers.

Solution of an Equation

- The values of variables which satisfies (LHS = RHS) the equation is called the solution or root of an equation.

Example: $2x+10=4$

Here, LHS $=2x+10,\ \text{RHS}=14$

Now, above equation is true only when $x=2$ i.e. $x=2$

$=LHS=2\times 2+10=14$

$RHS=14$

$\therefore$ LHS = RHS

$\therefore$ Root of $2x+10=14$ is 2.

Type of finding the solutions of linear equations

- Transposition: Any term of an equation may be taken to the other side with a change in its sign. This process is called transposition.

  Example: (i) $x + 5 = 7$

  Solution: $x = 7 - 5$

  $\therefore\ x = 2$

  Example: (ii) $\frac{x}{8} = 5$

  Solution: $x = 5 \times 8$

  $x = 40$

  Example: (iii) $2x = 40$

  Solution: $2x40$

  $x = \frac{\cancel{40}}{\cancel{2}}$

  $\therefore\ x = 20$

  Example: (iv) $3x + 2 = x - 2$

  Solution: $2x = -4$

  $x = \frac{\cancel{4}}{\cancel{2}} = -2$

  $\therefore\ x = -2$

  Example: The sum of two numbers is 100 and their difference is 10. Find the numbers.

  Solution: Let the numbers be x , $100 - x$

  $x - (100 - x) = 10$

  $x - 100 + x = 10$

  $2x = 110$

  $x = 55$

  Then the numbers are 55 and 45.