Example 10 : If A + B + C = π, then prove that

$sin2A + sin2B + sin2C = 4sinA\ sinB\ sinC.$

Solution : L.H.S. $= sin2A + sin2B + sin2C$

$= 2sin(A + B)\ cos(A - B) + 2sinC \cdot cosC$

$= 2sin(\pi - C)\ cos(A - B) + 2sinC \cdot cosC$ **(A + B + C = π)**

$= 2sinC \cdot cos(A - B) + 2sinC \cdot cosC$

$= 2sinC\ [cos(A - B) + cosC]$

$= 2sinC\ [cos(A - B) - cos(A + B)]$ **(A + B + C = π)**

$= 2sinC\ [-2sinA \cdot sin(-B)]$

$= 4sinA\ sinB\ sinC$ **(sin(−B) = −sinB)**

= R.H.S.

Example 11 : If A + B + C = $\frac{\pi}{2}$, then prove that

$cos^2A + cos^2B + cos^2C = 2[1 + sinA\ sinB\ sinC].$

Solution : L.H.S. $= cos^2A + cos^2B + cos^2C$

$= \frac{1+cos2A}{2} + \frac{1+cos2B}{2} + \frac{1+cos2C}{2}$

$= \frac{1}{2}[3 + cos2A + cos2B + cos2C]$

$= \frac{1}{2}[3 + 2cos(A + B)\ cos(A - B) + 1 - 2sin^2C]$

$= \frac{1}{2}[4 + 2cos(A + B) \cdot cos(A - B) - 2sin^2C]$

$= 2 + cos\left(\frac{\pi}{2} - C\right) \cdot cos(A - B) - sin^2C$ **$\left(A + B = \frac{\pi}{2} - C\right)$**

$= 2 + sinC\ [cos(A - B) - sinC]$

$= 2 + sinC\ [cos(A - B) - cos(A + B)]$ **$\left(A + B = \frac{\pi}{2} - C\right)$**

$= 2 + sinC\ [-2sinA \cdot sin(-B)]$

$= 2 + 2sinA\ sinB\ sinC$

$= 2\ [1 + sinA\ sinB\ sinC]$

or second method :

L.H.S. $= cos^2A + cos^2B + cos^2C$

$= cos^2A + 1 - sin^2B + 1 - sin^2C$

$= 2 + (cos^2A - sin^2B) - sin^2C$

$= 2 + cos(A + B)\ cos(A - B) - sin^2C$

$= 2 + cos\left(\frac{\pi}{2} - C\right) \cdot cos(A - B) - sin^2C$

$= 2 + sinC \cdot cos(A - B) - sin^2C$

$= 2 + sinC\ [cos(A - B) - sinC]$

$= 2 + sinC\ [cos(A - B) - cos(A + B)]$

$= 2 + sinC\ [-2sinA \cdot sin(-B)]$

$= 2\ [1 + sinA\ sinB\ sinC]$ = R.H.S.

Exercise 5.3

1. If A + B + C = π, prove that

(1) $cos2A + cos2B + cos2C = -1 - 4cosA\ cosB\ cosC$

(2) $sinA + sinB + sinC = 4cos\frac{A}{2}\ cos\frac{B}{2}\ cos\frac{C}{2}$

(3) $cosA + cosB + cosC = 1 + 4sin\frac{A}{2}\ sin\frac{B}{2}\ sin\frac{C}{2}$

(4) $sin^2A + sin^2B + sin^2C = 2(1 + cosA\ cosB\ cosC)$

(5) $cos^2A + cos^2B + cos^2C = 1 - 2cosA\ cosB\ cosC$

(6) $sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2} = 1 - 2sin\frac{A}{2}\ sin\frac{B}{2}\ sin\frac{C}{2}$

(7) $cos^2\frac{A}{2} + cos^2\frac{B}{2} + cos^2\frac{C}{2} = 2\left(1 + sin\frac{A}{2}\ sin\frac{B}{2}\ sin\frac{C}{2}\right)$

(8) $sin^2A + sin^2B - sin^2C = 2sinA\ sinB\ cosC$

2. If A + B + C = $\frac{\pi}{2}$, prove that

(1) $sin^2A + sin^2B + sin^2C = 1 - 2sinA\ sinB\ sinC$

(2) $sin2A + sin2B + sin2C = 4cosA\ cosB\ cosC$

(3) $sin^2A - sin^2B + sin^2C = 1 - 2cosA\ sinB\ cosC$

*

Miscellaneous Problems :

Example 12 : Prove that $tan142\frac{1}{2}^o = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

Solution : $tan142\frac{1}{2}^o = tan\left(90^o + 52\frac{1}{2}^o\right)$

$$= -cot52\frac{1}{2}^o$$

$$= -cot\left(45^o + 7\frac{1}{2}^o\right)$$

$$= -\frac{cot\,7\frac{1}{2}^o - 1}{cot\,7\frac{1}{2}^o + 1}$$

$$= -\frac{cos\,7\frac{1}{2}^o - sin\,7\frac{1}{2}^o}{cos\,7\frac{1}{2}^o + sin\,7\frac{1}{2}^o}$$

$$= -\frac{cos\,7\frac{1}{2}^o - sin\,7\frac{1}{2}^o}{cos\,7\frac{1}{2}^o + sin\,7\frac{1}{2}^o} \times \frac{cos\,7\frac{1}{2}^o - sin\,7\frac{1}{2}^o}{cos\,7\frac{1}{2}^o - sin\,7\frac{1}{2}^o}$$

$$= -\frac{\left(cos\,7\frac{1}{2}^o - sin\,7\frac{1}{2}^o\right)^2}{cos^2\,7\frac{1}{2}^o - sin^2\,7\frac{1}{2}^o}$$

$$= -\frac{1 - 2sin\,7\frac{1}{2}^o \times cos\,7\frac{1}{2}^o}{cos^2\,7\frac{1}{2}^o - sin^2\,7\frac{1}{2}^o}$$

$$= -\frac{1 - \sin 15^\circ}{\cos 15^\circ}$$

$$= -\frac{1 - \sin(45^\circ - 30^\circ)}{\cos(45^\circ - 30^\circ)}$$

$$= -\frac{1 - \frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$= -\frac{2\sqrt{2} - \sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= -\frac{(2\sqrt{2} - \sqrt{3} + 1)}{(\sqrt{3} + 1)}\frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)}$$

$$= -\frac{(2\sqrt{6} - 2\sqrt{2} - 3 + \sqrt{3} + \sqrt{3} - 1)}{2}$$

$$= -\sqrt{6} + \sqrt{2} + 2 - \sqrt{3}$$

$$= 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$$

Example 13 : If A + B + C = π, then prove that

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right).$$

Solution : R.H.S. $= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$

$$= 1 + 4\sin\left(\frac{B + C}{4}\right)\sin\left(\frac{A + C}{4}\right)\sin\left(\frac{A + B}{4}\right) \qquad \textbf{(A + B + C = }\boldsymbol{\pi}\textbf{)}$$

$$= 1 + 2\left(2\sin\left(\frac{B + C}{4}\right)\sin\left(\frac{A + C}{4}\right)\right)\sin\left(\frac{A + B}{4}\right)$$

$$= 1 + 2\sin\left(\frac{A + B}{4}\right)\left[\cos\left(\frac{B - A}{4}\right) - \cos\left(\frac{A + B + 2C}{4}\right)\right]$$

$$= 1 + 2\sin\left(\frac{A + B}{4}\right)\cos\left(\frac{B - A}{4}\right) - 2\sin\left(\frac{\pi - C}{4}\right)\cos\left(\frac{\pi + C}{4}\right)$$

$$= 1 + \left(\sin\frac{B}{2} + \sin\frac{A}{2}\right) - \left(\sin\frac{\pi}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + \sin\frac{B}{2} + \sin\frac{A}{2} - \sin\frac{\pi}{2} + \sin\frac{C}{2}$$

$$= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = \text{L.H.S.}$$

Example 14 : If α and β be the roots of the equation $a\cos\theta + b\sin\theta = c$, prove that $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{2b}{a + c}$. Hence, deduce that $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{b}{a}$.

Solution : $a\cos\theta + b\sin\theta = c$

$$\therefore \quad a\left(\frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}\right) + b\left(\frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}\right) = c$$

$$\therefore \quad a - a\tan^2\frac{\theta}{2} + 2b\tan\frac{\theta}{2} = c + c\tan^2\frac{\theta}{2}$$

$$\therefore \quad (a + c)\tan^2\frac{\theta}{2} - 2b\tan\frac{\theta}{2} + (c - a) = 0$$

This is a quadratic equation in $tan\frac{\theta}{2}$ and its roots are $tan\frac{\alpha}{2}$ and $tan\frac{\beta}{2}$.

$$\therefore \quad tan\frac{\alpha}{2} + tan\frac{\beta}{2} = -\left(\frac{-2b}{a+c}\right) = \frac{2b}{a+c} \text{ and } tan\frac{\alpha}{2} \cdot tan\frac{\beta}{2} = \frac{c-a}{c+a}$$

$$\text{Now, } tan\left(\frac{\alpha+\beta}{2}\right) = \frac{tan\frac{\alpha}{2} + tan\frac{\beta}{2}}{1 - tan\frac{\alpha}{2}\, tan\frac{\beta}{2}}$$

$$= \frac{\frac{2b}{a+c}}{1 - \frac{c-a}{c+a}} = \frac{2b}{a+c-c+a} = \frac{2b}{2a} = \frac{b}{a}$$

Example 15 : Prove using principle of mathematical induction,

$$cos\theta + cos2\theta + cos3\theta + ... + cosn\theta = sin\frac{(n+1)\theta}{2} \cdot cos\frac{n\theta}{2} \cdot cosec\frac{\theta}{2} - 1.$$

Solution :

Let, $P(n) : cos\theta + cos2\theta + cos3\theta + ... + cosn\theta = sin\frac{(n+1)\theta}{2} \cdot cos\frac{n\theta}{2} \cdot cosec\frac{\theta}{2} - 1$

For $n = 1$, L.H.S. $= cos\theta$, R.H.S. $= sin\theta \cdot cos\frac{\theta}{2} \cdot cosec\frac{\theta}{2} - 1$

$$= \frac{sin\theta \cdot cos\frac{\theta}{2}}{sin\frac{\theta}{2}} - 1$$

$$= \frac{2sin\frac{\theta}{2} \cdot cos\frac{\theta}{2} \cdot cos\frac{\theta}{2}}{sin\frac{\theta}{2}} - 1$$

$$= 2cos^2\frac{\theta}{2} - 1$$

$$= cos\theta = \text{R.H.S.} \qquad (cos2\theta = 2cos^2\theta - 1)$$

$\therefore$ P(1) is true.

Let P(k) is true.

$$\therefore \quad cos\theta + cos2\theta + cos3\theta + ... + cosk\theta = sin(k+1)\frac{\theta}{2} \cdot cos\frac{k\theta}{2} \cdot cosec\frac{\theta}{2} - 1$$

Let, $n = k + 1$

L.H.S. $= cos\theta + cos2\theta + cos3\theta + ... + cosk\theta + cos(k+1)\theta$

$$= \frac{sin\left(\frac{k+1}{2}\right)\theta \cdot cos\frac{k\theta}{2}}{sin\frac{\theta}{2}} - 1 + cos(k+1)\theta$$

$$= \frac{1}{2sin\frac{\theta}{2}}\left(2sin\left(\frac{k+1}{2}\right)\theta\ cos\frac{k\theta}{2} + 2sin\frac{\theta}{2} \cdot cos(k+1)\theta\right) - 1$$

$$= \frac{1}{2sin\frac{\theta}{2}}\left[sin\frac{(2k+1)\theta}{2} + sin\frac{\theta}{2} + sin\frac{(2k+3)\theta}{2} - sin\frac{(2k+1)\theta}{2}\right] - 1$$

$$= \frac{1}{sin\frac{\theta}{2}}\left[\frac{1}{2}\left(sin\frac{(2k+3)\theta}{2} + sin\frac{\theta}{2}\right)\right] - 1$$

$$= \frac{1}{sin\frac{\theta}{2}}\left[\frac{1}{2} \cdot 2sin\frac{(k+2)\theta}{2} \cdot cos\frac{(k+1)\theta}{2}\right] - 1$$

$$= \sin\frac{(k+2)\theta}{2} \cdot \cos\frac{(k+1)\theta}{2} \cdot \operatorname{cosec}\frac{\theta}{2} - 1$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true. $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true for $\forall n \in$ N by P.M.I.

Exercise 5

Prove : **(1 to 15)**

1. $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \sec\theta + \tan\theta$

2. $\frac{\cot\theta + \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1} = \cot\frac{\theta}{2}$

3. $\tan\alpha = \sqrt{5}\tan\beta \Rightarrow \cos2\alpha = \frac{3\cos2\beta - 2}{3 - 2\cos2\beta}$

4. $\tan\frac{\alpha}{2} = \cos\theta \Rightarrow \sin\alpha = \frac{1-\tan^4\frac{\theta}{2}}{1+\tan^4\frac{\theta}{2}}$

5. If $\sin\theta = a$, then the roots of $a(1 + x^2) = 2x$ are $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$.

6. If $\cos\theta = a$, then the roots of $4x^2 - 4x + 1 = a^2$ are $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$.

7. If α and β are the roots of the equation $a\cos\theta + b\sin\theta = c$, then

(1) $\cos\alpha + \cos\beta = \frac{2ac}{a^2+b^2}$ and $\cos\alpha \cdot \cos\beta = \frac{c^2-b^2}{a^2+b^2}$

(2) $\tan\alpha + \tan\beta = \frac{-2ab}{b^2-c^2}$ and $\tan\alpha \cdot \tan\beta = \frac{a^2-c^2}{b^2-c^2}$

(3) $\sin(\alpha + \beta) = \frac{2ab}{a^2+b^2}$.

8. $\cos^5\theta = \frac{1}{16}[10\cos\theta + 5\cos3\theta + \cos5\theta]$

9. $(2\cos\theta + 1)(2\cos\theta - 1)(2\cos2\theta - 1)(2\cos4\theta - 1) = 2\cos8\theta + 1$

10. $\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta + \cot4\theta = \cot\frac{\theta}{2}$

11. $(\cos^2 48^\circ - \sin^2 12^\circ) - (\cos^2 66^\circ - \sin^2 6^\circ) = \frac{1}{4}$

12. $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$

13. $\cot\frac{\pi}{24} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$

14. $\tan\frac{\pi}{16} = \sqrt{4+2\sqrt{2}} - (\sqrt{2} + 1)$

15. $4\sin27^\circ = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$

16. If $x = \sin\theta + \cos\theta \cdot \sin2\theta$ and $y = \cos\theta + \sin\theta \cdot \sin2\theta$,

then prove that $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2$.

17. If A + B + C = π, prove that

(1) $sin(B + 2C) + sin(C + 2A) + sin(A + 2B) = 4sin\left(\frac{B-C}{2}\right) \cdot sin\left(\frac{C-A}{2}\right) \cdot sin\left(\frac{A-B}{2}\right)$

(2) $cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2} = 4cos\left(\frac{\pi-A}{4}\right) \cdot cos\left(\frac{\pi-B}{4}\right) \cdot cos\left(\frac{\pi-C}{4}\right)$

18. Prove : ΔABC is right angled triangle $\Leftrightarrow$

$cos^2A + cos^2B + cos^2C = 1 \Leftrightarrow sin^2A + sin^2B + sin^2C = 2$

Prove by principle of mathematical induction : **(19 to 22)**

19. $sinx + sin3x + sin5x + ... + sin(2n - 1)x = \frac{sin^2nx}{sinx}$

20. $\frac{1}{2}tan\frac{x}{2} + \frac{1}{4}tan\frac{x}{4} + ... + \frac{1}{2^n}\, tan\frac{x}{2^n} = \frac{1}{2^n}\, cot\, \frac{x}{2^n} - cotx$

21. $sin\theta + sin2\theta + ... + sinn\theta = sin\frac{(n+1)\theta}{2} \cdot sin\frac{n\theta}{2} \cdot cosec\frac{\theta}{2}$

22. $cos\alpha \cdot cos2\alpha \cdot cos4\alpha \cdot ... \cdot cos2^{n-1}\alpha = \frac{sin^n 2\alpha}{2^n \cdot sin\alpha}$

23. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) One root of $4x^3 - 3x = \frac{1}{2}$ is ... ☐

(a) $sin70^o$ (b) $sin10^o$ (c) $sin20^o$ (d) $cos70^o$

(2) The range of the function $cos^4\theta - sin^4\theta$ is ... ☐

(a) [0, 1] (b) [−1, 1] (c) (0, 1) (d) (−1, 1)

(3) The range of $sec^4\theta + cosec^4\theta$ is ... ☐

(a) $[1, \alpha]$ (b) R^+ (c) $[8, \infty)$ (d) R − (−1, 1)

(4) The value of $cos67\frac{1}{2}^o$ is ... ☐

(a) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (b) $\frac{\sqrt{2-\sqrt{2}}}{2}$ (c) $\sqrt{2} - 1$ (d) $\sqrt{2} + 1$

(5) The value of $3sin\frac{\pi}{9} - 4sin^3\frac{\pi}{9}$ is ... ☐

(a) $\frac{1}{2}$ (b) -1 (c) $\frac{\sqrt{3}}{2}$ (d) $-\frac{1}{2}$

(6) If $sin\theta = \frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$, then P(2θ) is in the ...... quadrant. ☐

(a) 1st (b) 2nd (c) 3rd (d) 4th

(7) One root of the equation $6x - 8x^3 = \sqrt{3}$ is ... ☐

(a) $sin20^o$ (b) $sin30^o$ (c) $sin10^o$ (d) $cos10^o$

(8) If α is the root of $25cos^2\theta + 5cos\theta - 12 = 0$, $\frac{\pi}{2} < \theta < \pi$ then $sin2\alpha$ is ... ☐

(a) $\frac{-24}{25}$ (b) $\frac{-13}{18}$ (c) $\frac{13}{18}$ (d) $\frac{24}{25}$

(9) $\dfrac{\sin 3\theta}{1+2\cos 2\theta}$ is equal to ...

(a) $-\sin\theta$ (b) $-\cos\theta$ (c) $\cos\theta$ (d) $\sin\theta$

(10) The value of $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2$ is ...

(a) $\tan^2\frac{\theta}{2}$ (b) $2\cot\frac{\theta}{2}$ (c) $\cot^2\frac{\theta}{2}$ (d) $2\mathrm{cosec}\frac{\theta}{2}$

(11) The value of $12\sin 40^\circ - 16\sin^3 40^\circ$ is ...

(a) $-3\sqrt{2}$ (b) $2\sqrt{3}$ (c) $-2\sqrt{3}$ (d) $3\sqrt{2}$

(12) If $\sin\alpha = \frac{-3}{5}$, $\pi < \alpha < \frac{3\pi}{2}$, then the value of $\cos\frac{\alpha}{2}$ is ...

(a) $\frac{-3}{\sqrt{10}}$ (b) $\frac{-1}{\sqrt{10}}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$

(13) If $\dfrac{1+\cos A}{1-\cos A} = \dfrac{m^2}{n^2}$, then $\tan A$ is equal to ...

(a) $\pm\dfrac{2mn}{m^2-n^2}$ (b) $\pm\dfrac{2mn}{m^2+n^2}$ (c) $\dfrac{m^2+n^2}{m^2-n^2}$ (d) $\dfrac{m^2-n^2}{m^2+n^2}$

(14) $\cos^4\left(\frac{\pi}{24}\right) - \sin^4\left(\frac{\pi}{24}\right)$ is equal to ...

(a) $\dfrac{\sqrt{5}-1}{2\sqrt{2}}$ (b) $\dfrac{\sqrt{5}-1}{4}$ (c) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\dfrac{\sqrt{2+\sqrt{2}}}{4}$

(15) If $\cos\alpha = -0.6$ and $\pi < \alpha < \frac{3\pi}{2}$, then $\tan\frac{\alpha}{4}$ is equal to ...

(a) $\dfrac{1-\sqrt{5}}{2}$ (b) $\dfrac{\sqrt{5}-1}{2}$ (c) $\dfrac{\sqrt{5}}{2}$ (d) $\dfrac{\sqrt{5}+1}{2}$

(16) If $0 < \theta < \frac{\pi}{2}$ is an acute angle and $2x \cdot \sin^2\frac{\theta}{2} + 1 = x$, then $\tan\theta$ is ...

(a) $\sqrt{x^2-1}$ (b) $\sqrt{x^2+1}$ (c) $\sqrt{x^2-2}$ (d) $\sqrt{x^2-\frac{1}{2}}$

(17) If $\tan x = \frac{b}{a}$, then the value of $a\cos 2x + b\sin 2x$ is ...

(a) $a - b$ (b) a (c) b (d) $a + b$

(18) The value of $\cos 6^\circ \cdot \sin 24^\circ \cdot \cos 72^\circ$ is ...

(a) $\frac{-1}{8}$ (b) $\frac{-1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{4}$

(19) The maximum value of the expression $\sin^6\theta + \cos^6\theta$ is ...

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{5}{8}$ (d) $\frac{13}{8}$

(20) If $\cos A = \frac{3}{4}$, then the value of $32\sin\frac{A}{2}\ \sin\frac{5A}{2}$ is equal to ...

(a) -11 (b) $-\sqrt{11}$ (c) $\sqrt{11}$ (d) 11

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Summary

We studied following points in this chapter :

1. $sin2\alpha = 2sin\alpha\ cos\alpha$

2. $cos2\alpha = cos^2\alpha - sin^2\alpha = 2cos^2\alpha - 1 = 1 - 2sin^2\alpha$

3. $1 + cos2\alpha = 2cos^2\alpha$ and $1 - cos2\alpha = 2sin^2\alpha$

4. $sin2\alpha = \frac{2tan\alpha}{1 + tan^2\alpha}$

5. $cos2\alpha = \frac{1 - tan^2\alpha}{1 + tan^2\alpha}$

6. $tan2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha}$ $\qquad \alpha \in R - \left[\left\{(2k-1)\frac{\pi}{2}\right\} \cup \left\{(2k-1)\frac{\pi}{4}\right\}\right]\ k \in Z$

7. $cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha}$ $\qquad \alpha \in R - \left\{\frac{k\pi}{2} \mid k \in Z\right\}$

8. $sin3\alpha = 3sin\alpha - 4sin^3\alpha$

9. $cos3\alpha = 4cos^3\alpha - 3cos\alpha$

10. $tan3\alpha = \frac{3tan\alpha - tan^3\alpha}{1 - 3tan^2\alpha}$ $\qquad \alpha \in R - \left\{(2k-1)\frac{\pi}{6},\ k \in Z\right\}$

11. $cot3\alpha = \frac{cot^3\alpha - 3cot\alpha}{3cot^2\alpha - 1}$ $\qquad \alpha \in R - \left\{\frac{k\pi}{3} \mid k \in Z\right\}$

12. $sin^2\frac{\alpha}{2} = \frac{1 - cos\alpha}{2}$

13. $cos^2\frac{\alpha}{2} = \frac{1 + cos\alpha}{2}$

14. $tan^2\frac{\alpha}{2} = \frac{1 - cos\alpha}{1 + cos\alpha}$ $\qquad \alpha \in R - \left\{(2k-1)\pi \mid k \in Z\right\}$

15. $sin18^{\circ} = \frac{\sqrt{5} - 1}{4}$, $cos18^{\circ} = \sqrt{\frac{10 + 2\sqrt{5}}{16}}$

16. $sin36^{\circ} = \sqrt{\frac{10 - 2\sqrt{5}}{16}}$, $cos36^{\circ} = \frac{\sqrt{5} + 1}{4}$

17. $sin22\frac{1}{2}^{\circ} = \frac{\sqrt{2 - \sqrt{2}}}{2}$, $cos22\frac{1}{2}^{\circ} = \frac{\sqrt{2 + \sqrt{2}}}{2}$, $tan22\frac{1}{2}^{\circ} = \sqrt{2} - 1$

Chapter 6

TRIGONOMETRIC EQUATIONS AND PROPERTIES OF A TRIANGLE

If equations are trains threading the landscape of numbers, then no train stops at pi.

– Richard Preston

Pure mathematics is in its way, the poetry of logical ideas.

– Albert Einstein

6.1 Introduction

In the previous semester and in chapters 4, 5 we have studied about trigonometric functions, their graphs and their properties like zeros, range, periodic nature, identities. Trigonometry is useful in land surveying. We know that by using trigonometry we can find the height of a hill without actually measuring it. In 1852, **Radhanath Sikdar**, an *Indian mathematician* and a surveyor from Bengal, was the first to identify Mount Everest as the world's highest peak, using trigonometric calculations. Trigonometry is useful in modern navigation such as satellite systems, astronomy, aviation, oceanography.

In this chapter we will learn how to solve trigonometric equations and properties of a triangle using trigonometry.

6.2 Trigonometric Equations

A trigonometric equation is an equation containing trigonometric functions, e.g. $sin^2x - 4cosx = 1$ is a trigonometric equation.

A trigonometric equation that holds true for all values of the variable in its domain is called a trigonometric identity, e.g. $cos2\theta = 2cos^2\theta - 1$ is a trigonometric identity.

There are other equations, which are true only for some proper subsets of domain of functions involved. We will learn some techniques for solving such trigonometric equations, as well as how to obtain the complete set of solutions of an equation based on a single solution of that equation. The equations $sinx = \frac{1}{2}$ has not only the solution $x = \frac{\pi}{6}$ but also $x = \frac{5\pi}{6}$, $x = 2\pi + \frac{\pi}{6}$, $x = 3\pi - \frac{\pi}{6}$ etc. are also solutions of $sinx = \frac{1}{2}$. Thus, we can say that $x = \frac{\pi}{6}$ is a solution of $sinx = \frac{1}{2}$ but it is not the complete solution of the equation. **A general solution to an equation is the set of all possible solutions of that equation.** Note that some trigonometric equations may not have any solution, e.g. $sinx = \pi$. Due to periodic nature of trigonometric functions, if a trigonometric equation has a solution it may have infinitely many solutions. The set of all such solution is known as the **general solution**.

Look at the graph of $y = sinx$. Observe any of the horizontal line $y = k$ where k varies from -1 to 1. We can see that the graph of $y = k$ intersects the graph of $y = sinx$ in infinitely many points (figure 6.1). This means that if we take $a \in [-1, 1]$, then there are infinitely many real numbers x such that $sinx = a$. For a solution of a trigonometric equation, we need a unique real number α such that $sin\alpha = a$. For that we have to restrict the domain suitably. If we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, etc. then we get a unique number α such that $sin\alpha = a$. We assume that the restricted domain for $y = sinx$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In this domain any horizontal line $y = k$, $k \in [-1, 1]$ intersects the graph of $y = sinx$ only at one point (figure 6.2).

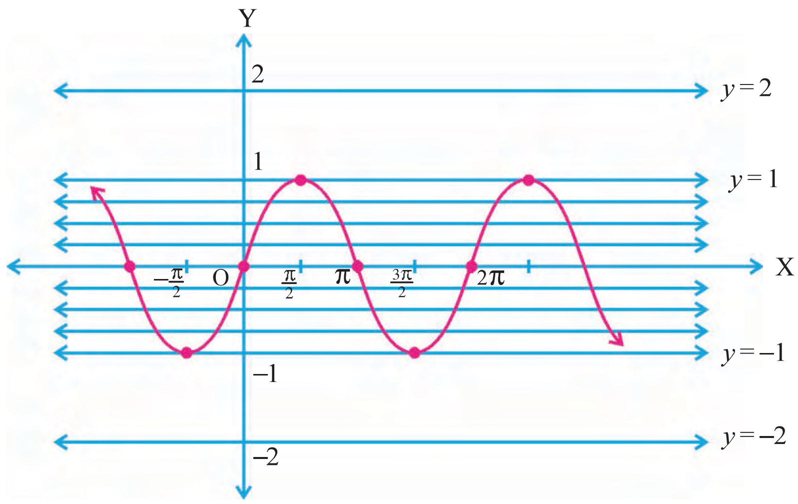

Figure 6.1

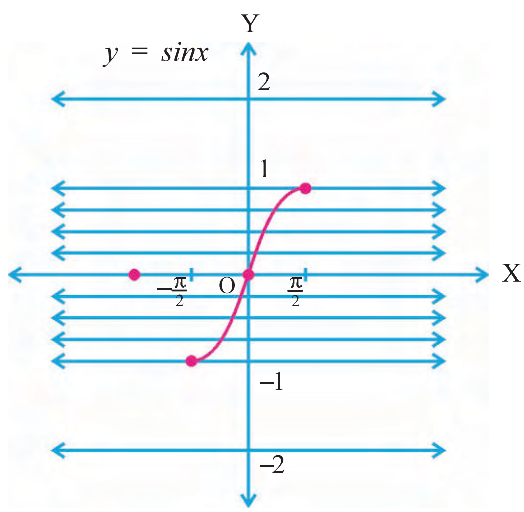

Figure 6.2

Similar situation arises for the function $y = cosx$. (figure 6.3)

We take the restricted domain $[0, \pi]$ for $y = cosx$. (figure 6.4)

Note that any horizontal line $y = a$ where $|a| > 1$ will not intersect the graph of $y = sinx$ or $y = cosx$. Thus $sinx = a$ or $cosx = a$ where $|a| > 1$ has no solution.

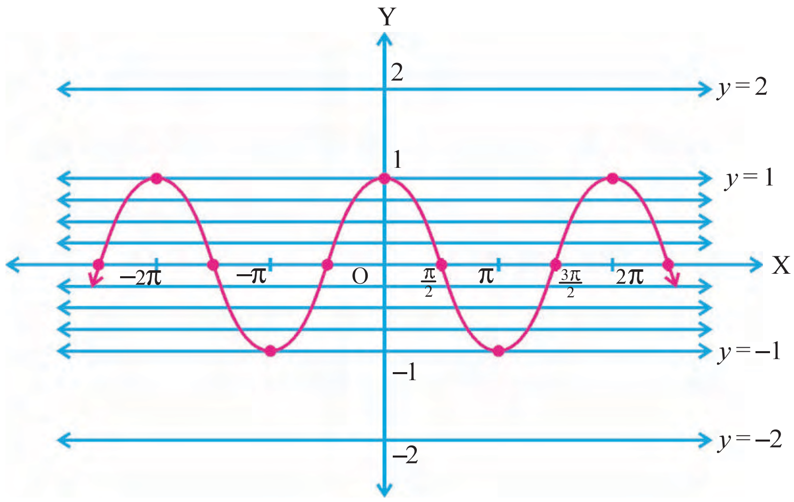

Figure 6.3

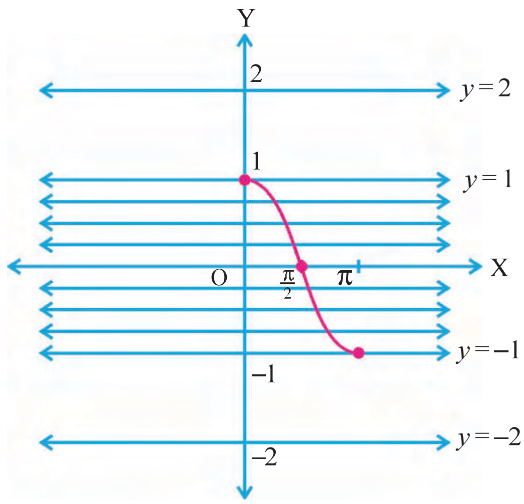

Figure 6.4

For the function $y = tanx$, if we draw any horizontal line in the plane it will intersect the graph of $y = tanx$ at infinitely many points (figure 6.5). This means that if we take any $a \in$ R, then there are infinitely many real number x such that $tanx = a$. we need a unique value α such that $tan\alpha = a$. So we have to restrict domain suitably. We take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as restricted domain of $y = tanx$. (figure 6.6). We shall discuss this in more detail when we study the concept of inverse trigonometric functions in the third semester in 12th standard.

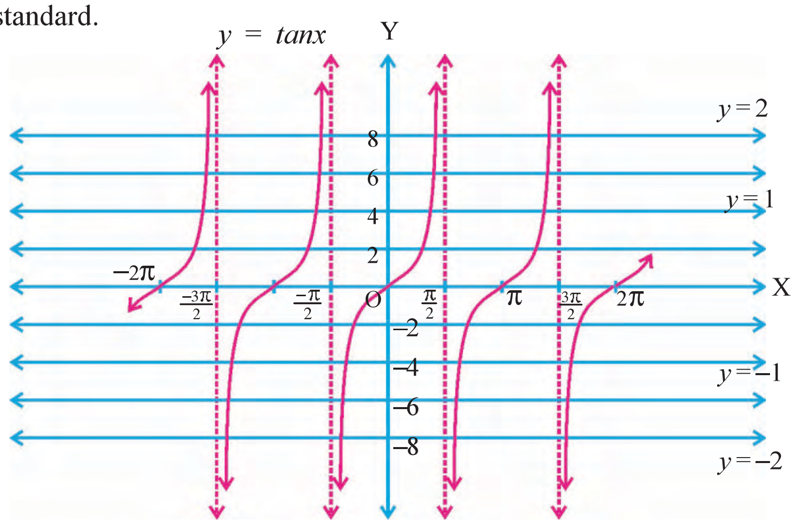

Figure 6.5

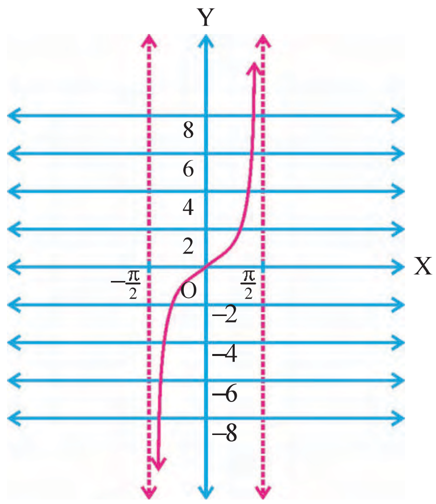

Figure 6.6

Thus, for any $a \in [-1, 1]$ there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that, $a = \sin\alpha$.

Also, for any $a \in [-1, 1]$ there is a unique $\alpha \in [0, \pi]$, such that, $a = \cos\alpha$.

Finally, for any $a \in$ R there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that, $a = \tan\alpha$.

We know the set of zeros of sine, cosine and tangent functions. That actually means that we already know the general solutions of the equations $\sin\theta = 0$, $\cos\theta = 0$, $\tan\theta = 0$.

$\sin\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

$\cos\theta = 0 \Leftrightarrow \theta = (2k + 1)\frac{\pi}{2}, k \in Z$

$\tan\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

We shall now solve the equations, $\sin\theta = a, -1 \le a \le 1$, $\cos\theta = a, -1 \le a \le 1$ and $\tan\theta = a, a \in$ R.

6.3 General Solution of $\sin\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that, $a = \sin\alpha$.

Now, $\sin\theta = a = \sin\alpha$

$\therefore \quad \sin\theta - \sin\alpha = 0$

$\Leftrightarrow \quad 2\cos\frac{\theta+\alpha}{2}\ \sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad \cos\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = (2n + 1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = n\pi, n \in Z$ **(Why ?)**

$\Leftrightarrow \quad \theta = (2n + 1)\pi - \alpha$ or $\theta = 2n\pi + \alpha, n \in Z$

$\Leftrightarrow \quad \theta = (2n + 1)\pi + (-1)^{2n+1}\alpha$ or $\theta = 2n\pi + (-1)^{2n}\alpha, n \in Z$

Therefore, the general solution is given by $\theta = k\pi + (-1)^k\alpha, k \in Z$.

(We have replaced $2n + 1$, $2n$ by k because any integer is of the form either $2n + 1$ or $2n$)

Thus, $\sin\theta = \sin\alpha \Leftrightarrow \theta = k\pi + (-1)^k\alpha, k \in Z$

Hence, the solution set of $sin\theta = a, -1 \le a \le 1$ is given by $\{k\pi + (-1)^k\alpha \mid k \in Z\}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin\theta = a = sin\alpha$.

(We may take any $\alpha \in R$ such that $a = sin\alpha$. The solution remains same. This convention of taking $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is only for the uniformity of the form of the solution set.)

General Solution of $cos\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in [0, \pi]$ such that, $a = cos\alpha$.

Now, $cos\theta = a = cos\alpha$

$$\therefore \quad cos\theta - cos\alpha = 0 \Leftrightarrow -2sin\frac{\theta+\alpha}{2}\, sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow sin\frac{\theta+\alpha}{2} = 0 \text{ or } sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow \frac{\theta+\alpha}{2} = k\pi \text{ or } \frac{\theta-\alpha}{2} = k\pi,\ k \in Z$$

$$\Leftrightarrow \theta = 2k\pi - \alpha \text{ or } \theta = 2k\pi + \alpha,\ k \in Z$$

Therefore the general solution is given by $\theta = 2k\pi \pm \alpha$, $k \in Z$.

Thus, $cos\theta = cos\alpha \Leftrightarrow \theta = 2k\pi \pm \alpha,\ k \in Z$

Hence, **the solution set of $cos\theta = a, -1 \le a \le 1$ is given by $\{2k\pi \pm \alpha \mid k \in Z\}$ where $\alpha \in [0, \pi]$ and $cos\theta = a = cos\alpha$.**

General Solution of $tan\theta = a$, where $a \in R$

Here $a \in R$. Therefore, there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that, $a = tan\alpha$.

Now, $tan\theta = a = tan\alpha$

$$\therefore \quad tan\theta - tan\alpha = 0 \Leftrightarrow \frac{sin\theta}{cos\theta} - \frac{sin\alpha}{cos\alpha} = 0$$

$$\Leftrightarrow \frac{sin\theta\, cos\alpha - cos\theta\, sin\alpha}{cos\theta\, cos\alpha} = 0$$

$$\Leftrightarrow \frac{sin(\theta-\alpha)}{cos\theta\, cos\alpha} = 0$$

$$\Leftrightarrow sin(\theta - \alpha) = 0$$

$$\Leftrightarrow \theta - \alpha = k\pi,\ k \in Z$$

$$\Leftrightarrow \theta = k\pi + \alpha,\ k \in Z$$

Thus, $tan\theta = tan\alpha \Leftrightarrow \theta = k\pi + \alpha,\ k \in Z$

Hence, **the solution set of $tan\theta = a, a \in R$ is given by $\{k\pi + \alpha \mid k \in Z\}$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\theta = a = tan\alpha$.**

By the word 'solve' we shall mean to obtain the general solution set of the given equation.

Example 1 : Solve : (1) $2sin2\theta - 1 = 0$ (2) $sin^2\theta - sin\theta - 2 = 0$

Solution : (1) $2sin2\theta - 1 = 0$

$\therefore \quad sin2\theta = \frac{1}{2} = sin\left(\frac{\pi}{6}\right)$

We know that general solution of $sin\theta = sin\alpha$ is $k\pi + (-1)^k\alpha$, $k \in Z$.

$\therefore \quad 2\theta = k\pi + (-1)^k \frac{\pi}{6}, k \in Z$

$\therefore \quad \theta = \frac{k\pi}{2} + (-1)^k \frac{\pi}{12}, k \in Z$

Hence, the required solution set is $\left\{\frac{k\pi}{2} + (-1)^k \frac{\pi}{12} \;\middle|\; k \in Z\right\}$.

(2) $sin^2\theta - sin\theta - 2 = 0$

$\therefore \quad (sin\theta + 1)(sin\theta - 2) = 0$

$\therefore \quad sin\theta = -1$ or $sin\theta = 2$

But $sin\theta = 2$ is not possible. **(Why ?)**

So, $sin\theta = -1 = sin\left(-\frac{\pi}{2}\right)$

$\therefore \quad \theta = k\pi + (-1)^k \left(-\frac{\pi}{2}\right), k \in Z$

Hence, the required solution set is $\left\{k\pi + (-1)^{k+1} \frac{\pi}{2} \;\middle|\; k \in Z\right\}$.

Example 2 : Solve : (1) $2cos5\theta + \sqrt{3} = 0$ (2) $2cos^2\theta - \sqrt{3}cos\theta = 0$

Solution : (1) $2cos5\theta + \sqrt{3} = 0$

$\therefore \quad cos5\theta = -\frac{\sqrt{3}}{2} = cos\left(\pi - \frac{\pi}{6}\right) = cos\left(\frac{5\pi}{6}\right)$ $\quad$ **$\left(\frac{5\pi}{6} \in [0, \pi]\right)$**

We know that general solution of $cos\theta = cos\alpha$ is $\theta = 2k\pi \pm \alpha$, $k \in Z$.

$\therefore \quad 5\theta = 2k\pi \pm \frac{5\pi}{6}, k \in Z$

$\therefore \quad \theta = \frac{2k\pi}{5} \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{\frac{2k\pi}{5} \pm \frac{\pi}{6} \;\middle|\; k \in Z\right\}$.

(2) $2cos^2\theta - \sqrt{3}cos\theta = 0$

$\therefore \quad cos\theta(2cos\theta - \sqrt{3}) = 0$

$\therefore \quad cos\theta = 0$ or $cos\theta = \frac{\sqrt{3}}{2} = cos\left(\frac{\pi}{6}\right)$

$\therefore \quad \theta = (2k + 1)\frac{\pi}{2}, k \in Z$ or $\theta = 2k\pi \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{(2k + 1)\frac{\pi}{2} \;\middle|\; k \in Z\right\} \cup \left\{2k\pi \pm \frac{\pi}{6} \;\middle|\; k \in Z\right\}$.

Example 3 : Solve : (1) $sin5x - sin3x - sinx = 0$ (2) $cosx + cos2x + cos3x = 0$

Solution : (1) $sin5x - sin3x - sinx = 0$

$\therefore \quad 2cos4x\ sinx - sinx = 0$

$\therefore \quad sinx(2cos4x - 1) = 0$

$\therefore \quad sinx = 0$ or $cos4x = \frac{1}{2} = cos\left(\frac{\pi}{3}\right)$

$\therefore \quad x = k\pi, k \in Z$ or $4x = 2k\pi \pm \frac{\pi}{3}, k \in Z$

$\therefore \quad x = k\pi, k \in Z$ or $x = \frac{k\pi}{2} \pm \frac{\pi}{12}, k \in Z$

Hence, the required solution set is $\{k\pi \mid k \in Z\} \cup \left\{\frac{k\pi}{2} \pm \frac{\pi}{12} \,\middle|\, k \in Z\right\}$.

(2) $\quad cosx + cos2x + cos3x = 0$

$\therefore \quad cos3x + cosx + cos2x = 0$

$\therefore \quad 2cos2x\ cosx + cos2x = 0$

$\therefore \quad cos2x\ (2cosx + 1) = 0$

$\therefore \quad cos2x = 0$ or $cosx = -\frac{1}{2} = cos\frac{2\pi}{3}$ $\quad \left(\frac{2\pi}{3} \in [0, \pi]\right)$

$\therefore \quad 2x = (2k + 1)\frac{\pi}{2}, k \in Z$ or $x = 2k\pi \pm \frac{2\pi}{3}, k \in Z$

$\therefore \quad x = (2k + 1)\frac{\pi}{4}, k \in Z$ or $x = 2k\pi \pm \frac{2\pi}{3}, k \in Z$

Hence, the required solution set is $\left\{(2k + 1)\frac{\pi}{4} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi \pm \frac{2\pi}{3} \,\middle|\, k \in Z\right\}$.

Example 4 : Solve : (1) $tan^2\theta + (1 - \sqrt{3})tan\theta - \sqrt{3} = 0$

(2) $tan\theta + tan4\theta + tan7\theta = tan\theta\ tan4\theta\ tan7\theta$

Solution : (1) $tan^2\theta + (1 - \sqrt{3})tan\theta - \sqrt{3} = 0$

$\therefore \quad tan^2\theta + tan\theta - \sqrt{3}tan\theta - \sqrt{3} = 0$

$\therefore \quad tan\theta(tan\theta + 1) - \sqrt{3}(tan\theta + 1) = 0$

$\therefore \quad (tan\theta + 1)(tan\theta - \sqrt{3}) = 0$

$\therefore \quad tan\theta = -1$ or $tan\theta = \sqrt{3}$

$\therefore \quad tan\theta = tan\left(-\frac{\pi}{4}\right)$ or $tan\theta = tan\frac{\pi}{3}$

$\therefore \quad \theta = k\pi - \frac{\pi}{4}, k \in Z$ or $\theta = k\pi + \frac{\pi}{3}, k \in Z$

Hence, the required solution set is $\left\{k\pi - \frac{\pi}{4} \,\middle|\, k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{3} \,\middle|\, k \in Z\right\}$.

(2) $\quad tan\theta + tan4\theta + tan7\theta = tan\theta\ tan4\theta\ tan7\theta$

$\therefore \quad tan\theta + tan4\theta = -tan7\theta + tan\theta\ tan4\theta\ tan7\theta$

$\therefore \quad tan\theta + tan4\theta = -tan7\theta\ (1 - tan\theta\ tan4\theta)$ $\quad$ **(i)**

First we prove that $1 - tan\theta\ tan4\theta \neq 0$.

If $1 - tan\theta\ tan4\theta = 0$ then by (i) we have $tan\theta + tan4\theta = 0$.

Thus, $tan\theta\ tan4\theta = 1$ and $tan4\theta = -tan\theta$ which gives $tan^2\theta = -1$ which is not possible in R.

Now, by (i) we have $\frac{tan\theta + tan4\theta}{1 - tan\theta\, tan4\theta} = -tan7\theta$

$\therefore$ $tan(\theta + 4\theta) = -tan7\theta$

$\therefore$ $tan5\theta = tan(-7\theta)$

$\therefore$ $5\theta = k\pi - 7\theta,\ k \in Z$

$\therefore$ $\theta = \frac{k\pi}{12},\ k \in Z$

Also $tan\theta$, $tan4\theta$, $tan7\theta$, should be defined.

$\therefore$ $\theta \neq (2m + 1)\frac{\pi}{2},\ 4\theta \neq (2m + 1)\frac{\pi}{2},\ 7\theta \neq (2m + 1)\frac{\pi}{2},\ k \in Z$

$\therefore$ If $\theta = \frac{k\pi}{12},\ k \in Z$ then $k \neq 6, 18, 30, ...$

$4\theta = \frac{k\pi}{3} \neq (2m + 1)\frac{\pi}{2}$ for any $k \in Z - \{6, 18, ...\}$

$7\theta = \frac{7k\pi}{12} \neq (2m + 1)\frac{\pi}{2}$ for any $k \in Z - \{6, 18, ...\}$

$\therefore$ $k \neq 6, 18, ...$

$\therefore$ $k \neq 12n + 6,\ n \in Z$

$\therefore$ k is not odd multiple of 6.

$\therefore$ The solution set is $\left\{\frac{k\pi}{12} \,\middle|\, k \in Z \text{ where } k \neq 12n + 6\right\},\quad n \in Z$

Example 5 : Solve : (1) $4sin\theta = cosec\theta$ (2) $sec\theta + tan\theta = 2 - \sqrt{3}$

Solution : (1) $4sin\theta = cosec\theta$

$\therefore$ $4sin\theta = \frac{1}{sin\,\theta}$

$\therefore$ $4sin^2\theta = 1$

$\therefore$ $sin\theta = \pm\frac{1}{2}$

$\therefore$ $sin\theta = sin\left(\frac{\pi}{6}\right)$ or $sin\theta = sin\left(-\frac{\pi}{6}\right)$

$\therefore$ $\theta = k\pi + (-1)^k\frac{\pi}{6},\ k \in Z$ or $\theta = k\pi + (-1)^k\left(-\frac{\pi}{6}\right),\ k \in Z$

$\therefore$ $\theta = k\pi + (-1)^k\frac{\pi}{6},\ k \in Z$ or $\theta = k\pi + (-1)^{k+1}\frac{\pi}{6},\ k \in Z$

$\therefore$ $\theta = k\pi \pm \frac{\pi}{6},\ k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{6} \,\middle|\, k \in Z\right\}$.

(2) $sec\theta + tan\theta = 2 - \sqrt{3}$ **(i)**

Since $sec^2\theta - tan^2\theta = 1$, we have $sec\theta - tan\theta = \frac{1}{2 - \sqrt{3}} = \frac{2 + \sqrt{3}}{(2 - \sqrt{3})(2 + \sqrt{3})} = 2 + \sqrt{3}$

$\therefore$ $sec\theta - tan\theta = 2 + \sqrt{3}$ **(ii)**

Solving (i) and (ii) we get, $sec\theta = 2$ and $tan\theta = -\sqrt{3}$

Note that the above is a simultaneous system of trigonometric equations.

Since $cos\theta = \frac{1}{2} > 0$ and $tan\theta = -\sqrt{3} < 0$, P($\theta$) is in the fourth quadrant.

$\therefore \quad cos\theta = cos\left(-\frac{\pi}{3}\right)$ and $tan\theta = tan\left(-\frac{\pi}{3}\right)$

$\therefore \quad \theta = 2k\pi - \frac{\pi}{3}, k \in Z$ **(P(θ) is in fourth quadrant.)**

Hence, required solution set is $\left\{2k\pi - \frac{\pi}{3} \;\middle|\; k \in Z\right\}$.

6.4 The General Solution of $acosx + bsinx = c, a, b, c \in R$ and $a^2 + b^2 \neq 0$

For the given real numbers a and b, we can find $r > 0$ and $\alpha \in [0, 2\pi)$ such that $a = rcos\alpha$ and $b = rsin\alpha$. **(chapter 4)**

$\therefore \quad a^2 + b^2 = r^2 cos^2\alpha + r^2 sin^2\alpha = r^2$

$\therefore \quad r = \sqrt{a^2 + b^2}$ **($r > 0$)**

Now, $acosx + bsinx = c$

$\therefore \quad rcos\alpha\, cosx + rsin\alpha\, sinx = c$

$\therefore \quad rcos(x - \alpha) = c$

$\therefore \quad cos(x - \alpha) = \frac{c}{r}$ **(i)**

The last equation will have a solution if and only if $\left|\frac{c}{r}\right| \leq 1$, that is if and only if $c^2 \leq r^2$, that is if and only if $c^2 \leq a^2 + b^2$.

If $cos(x - \alpha) = cos\beta$, where $cos\beta = \frac{c}{r}$, $\beta \in [0, \pi]$, then the general solution of (i) is $x - \alpha = 2k\pi \pm \beta, k \in Z$ where $\alpha \in [0, 2\pi)$ such that $a = rcos\alpha$ and $b = rsin\alpha$.

Thus, if $c^2 \leq a^2 + b^2$, the general solution of $acosx + bsinx = c$ is $x = 2k\pi + \alpha \pm \beta, k \in Z$, where $\alpha \in [0, 2\pi)$ such that $a = rcos\alpha$ and $b = rsin\alpha$ and $cos\beta = \frac{c}{r}, \beta \in [0, \pi], r = \sqrt{a^2 + b^2}$.

If $c^2 > a^2 + b^2$, the equation has no solution. In this case the solution set is $\emptyset$.

Example 6 : Solve : $\sqrt{3}cosx + sinx = \sqrt{2}$

Solution : Method 1 : Here $a = \sqrt{3}, b = 1, c = \sqrt{2}$.

$\therefore \quad r^2 = a^2 + b^2 = 3 + 1 = 4$

Hence, $r = 2$. Here $c^2 \leq a^2 + b^2$. So the given equation has a non-empty solution.

$a = rcos\alpha$ and $b = rsin\alpha$ gives $cos\alpha = \frac{\sqrt{3}}{2}$ and $sin\alpha = \frac{1}{2}$. Therefore $\alpha = \frac{\pi}{6}$

Now, $cos\beta = \frac{c}{r} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore \quad \beta = \frac{\pi}{4}$

Hence, required solution set is $\{2k\pi + \alpha \pm \beta \mid k \in Z\} = \left\{2k\pi + \frac{\pi}{6} \pm \frac{\pi}{4} \;\middle|\; k \in Z\right\}$.

Method 2 : $\sqrt{3}\cos x + \sin x = \sqrt{2}$

$\therefore \quad \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$\therefore \quad x - \frac{\pi}{6} = 2k\pi \pm \frac{\pi}{4}, k \in Z$

Hence, the required solution set is $\left\{2k\pi + \frac{\pi}{6} \pm \frac{\pi}{4} \,\middle|\, k \in Z\right\}$.

$\therefore$ The required solution set is $\left\{2k\pi + \frac{5\pi}{12} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi - \frac{\pi}{12} \,\middle|\, k \in Z\right\}$.

Example 7 : Solve : $3\cos\theta + 4\sin\theta = 6$

Solution : Here $a = 3$, $b = 4$, $c = 6$.

$\therefore \quad r^2 = a^2 + b^2 = 25.\ c^2 = 36.$ So, $c^2 > a^2 + b^2$

Hence, the solution set is $\emptyset$.

Exercise 6.1

Solve the following equations :

1. $2\cos 2\theta + \sqrt{2} = 0$
2. $2\cos^2\theta + \sqrt{3}\cos\theta = 0$
3. $2\cos\theta + \sec\theta = 3$
4. $4\sin^2\theta - 8\cos\theta + 1 = 0$
5. $\sqrt{2}\,\mathrm{cosec}\,3\theta - 2 = 0$
6. $2\sin^2\theta - \sin\theta = 0$
7. $2\sin\theta + \mathrm{cosec}\,\theta = 3$
8. $\sin 2\theta + \cos\theta = 0$
9. $\sin 7\theta = \sin\theta + \sin 3\theta$
10. $\cos^2\theta - \cos\theta = 0$
11. $\tan 2\theta - \sqrt{3} = 0$
12. $\sqrt{3}\cot\theta - \cot^2\theta = 0$
13. $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$
14. $\cos\theta + \sin\theta = 1$
15. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$
16. $2\cos\theta + \sin\theta = 3$
17. $3 - \cot^2 5\theta = 0$
18. $\mathrm{cosec}^2 2\theta - 2 = 0$
19. $\sqrt{2} + \sec 4\theta = 0$
20. $\tan 3\theta + \cot\theta = 0$

*

6.5 Properties of a triangle

The literal meaning of the word trigonometry is "the science of measurement of (the parts of) a triangle." A triangle has three angles and three sides. Measures of angles and sides are not independent of each other. In this article we shall get the exact relationship between the parts of a triangle.

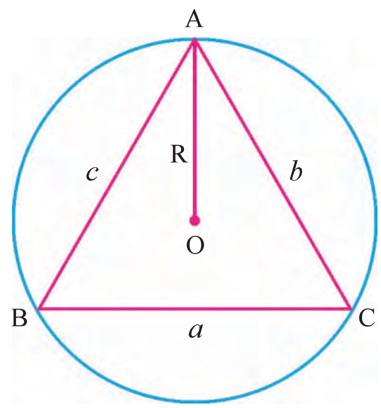

Figure 6.7

We will use following notation in relation to a triangle :

$m\angle BAC = A$, $m\angle ABC = B$, $m\angle BCA = C$

$A + B + C = \pi$

(A, B, C are taken in radian measures.)

$AB = c$, $BC = a$, $CA = b$

The radius of the circumcircle of the triangle, that is, circumradius = R

***sine* Rule :**

In ΔABC we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

We shall prove here that $\frac{a}{\sin A} = 2R$. The other two can be proved similarly.

There are three possibilities for A :

(1) $0 < A < \frac{\pi}{2}$ (2) $A = \frac{\pi}{2}$ (3) $\frac{\pi}{2} < A < \pi$

Case 1 : $0 < A < \frac{\pi}{2}$

Suppose O is the circumcentre of ΔABC. Let $\overrightarrow{BO}$ intersect the circumcircle at D. Here BD = 2OB = 2R and

D = $m\angle BDC = m\angle CAB = A$ **(i)**

(Angles in the same segment)

Now in ΔBCD, $m\angle BCD = \frac{\pi}{2}$ **(Angle in a semicircle)**

$\therefore \quad \sin D = \frac{BC}{BD} = \frac{a}{2R}$

$\therefore \quad \sin A = \frac{a}{2R}$ **(by (i))**

$\therefore \quad \frac{a}{\sin A} = 2R$

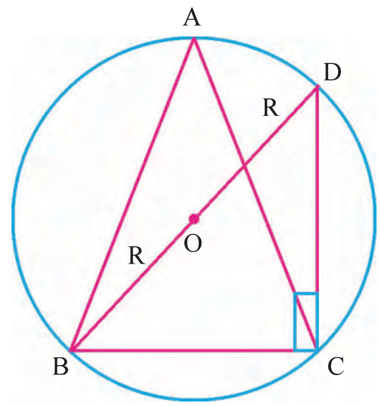

Figure 6.8

Case 2 : ΔABC is right angled and $A = \frac{\pi}{2}$

$\therefore \quad \overline{BC}$ is a diameter of the circumcircle.

$\therefore \quad BC = 2R$

Now, $a = BC = 2R = 2R\sin\frac{\pi}{2} = 2R\sin A$

$\therefore \quad \frac{a}{\sin A} = 2R$

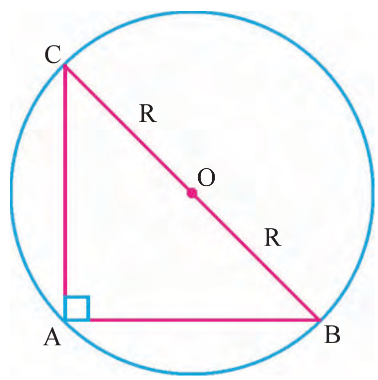

Figure 6.9

Case 3 : $\frac{\pi}{2} < A < \pi$

As $\angle BAC$ is obtuse, so vertex A is on the minor arc BC. Now take any Point A' on the major arc BC.

Here, $m\angle BA'C = (\pi - A) < \frac{\pi}{2}$ **$\left(\frac{\pi}{2} < A < \pi\right)$**

$\therefore$ By case (1) applied to $\Delta BA'C$, we get

$BC = a = 2R\sin A' = 2R\sin(\pi - A) = 2R\sin A$

$\therefore \quad \frac{a}{\sin A} = 2R$

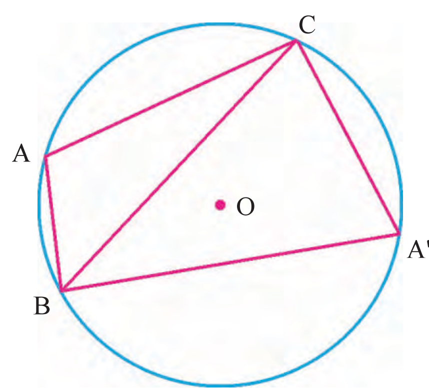

Figure 6.10

Thus, in each case, $\frac{a}{\sin A} = 2R$

Similarly, we can prove that $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

cosine Rule :

In ΔABC, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We shall prove that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

As shown in the figure 6.11, without loss of generality we take vertex A as the origin and $\overrightarrow{AB}$ in the positive direction of the X-axis. Since AB = c, the coordinates of B are $(c, 0)$. Now AC = b and $m\angle CAB = A$. So vertex C is $(b\cos A, b\sin A)$.

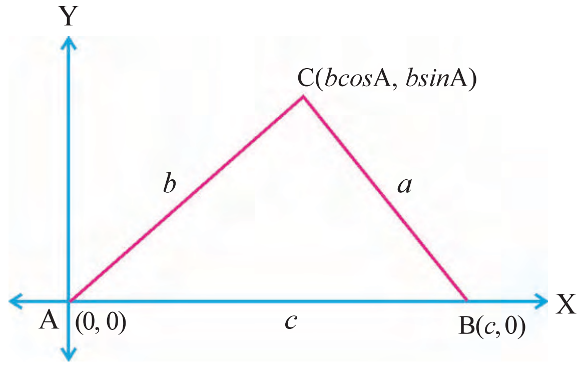

Figure 6.11

Now, $a = BC$

$\therefore \quad a^2 = BC^2$

$= (b\cos A - c)^2 + (b\sin A - 0)^2$

$= b^2\cos^2 A - 2bc \cos A + c^2 + b^2\sin^2 A$

$= b^2(\cos^2 A + \sin^2 A) - 2bc \cos A + c^2$

$\therefore \quad a^2 = b^2 - 2bc \cos A + c^2$

$\therefore \quad 2bc \cos A = b^2 + c^2 - a^2$

$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

In the same way, we can prove the results,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note :

(1) The above proof will not change even if $\angle BAC$ is a right angle or an obtuse angle.

(2) If the lengths of the three sides of a triangle are known, we can find the measure of all the angles using cosine rule. Similarly, if two sides and the included angle are given, then by *cosine* rule we can find the remaining sides and remaining angles.

Important Formula :

We shall obtain an important result by the use of sine and cosine rules.

Projection Formula :

$$a = b\cos C + c\cos B, \ b = c\cos A + a\cos C, \ c = a\cos B + b\cos A$$

We shall prove $a = b\cos C + c\cos B$

We prove the result using cosine rule. (Try to prove it using sine rule)

$$b\cos C + c\cos B = b\frac{a^2+b^2-c^2}{2ab} + c\frac{c^2+a^2-b^2}{2ca}$$

$$= \frac{a^2+b^2-c^2}{2a} + \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{a^2+b^2-c^2+c^2+a^2-b^2}{2a} = \frac{2a^2}{2a} = a$$

Thus, $a = b\cos C + c\cos B$

Similarly, the other two projection formulae can also be proved.

Example 8 : For ΔABC prove that,

(1) $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

(2) $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right) = 0$

Solution : (1)

$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$

$$= a\left(\frac{b}{2R} - \frac{c}{2R}\right) + b\left(\frac{c}{2R} - \frac{a}{2R}\right) + c\left(\frac{a}{2R} - \frac{b}{2R}\right)$$

$$= \frac{a(b-c)+b(c-a)+c(a-b)}{2R} = 0$$

(2) $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) = a\sin\left(\frac{\pi-(B+C)}{2}\right)\sin\left(\frac{B-C}{2}\right)$ **$(A + B + C = \pi)$**

$$= a\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= \frac{a}{2}(\sin B - \sin C)$$

$$= \frac{a}{2}\left(\frac{b}{2R} - \frac{c}{2R}\right)$$

$$= \frac{1}{4R}(ab - ac) \qquad \text{(i)}$$

Similarly, $b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) = \frac{1}{4R}(bc - ab)$ **(ii)**

$c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right) = \frac{1}{4R}(ac - bc)$ **(iii)**

Adding (i), (ii) and (iii) we get

$$\text{L.H.S.} = a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)$$

$$= \frac{1}{4R}(ab - ac + bc - ab + ac - bc) = 0 = \text{R.H.S.}$$

Example 9 : In any ΔABC, prove that

(1) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(2) $\frac{\tan C}{\tan A} = \frac{b^2+c^2-a^2}{a^2+b^2-c^2}$

Solution :

(1) L.H.S. $= \dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$

$$= \frac{b^2+c^2-a^2}{2bc} \times \frac{1}{a} + \frac{c^2+a^2-b^2}{2ca} \times \frac{1}{b} + \frac{a^2+b^2-c^2}{2ab} \times \frac{1}{c} \quad \textbf{(cosine rule)}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc} = \text{R.H.S.}$$

(2) L.H.S. $= \dfrac{\tan C}{\tan A} = \dfrac{\sin C \cos A}{\cos C \sin A}$

$$= \frac{\frac{c}{2R}\left(\frac{b^2+c^2-a^2}{2bc}\right)}{\frac{a}{2R}\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$= \frac{b^2+c^2-a^2}{a^2+b^2-c^2} = \text{R.H.S.}$$

Example 10 : In ΔABC, prove that

$(a + b)\cos C + (b + c)\cos A + (c + a)\cos B = a + b + c$

Solution : L.H.S. $= (a + b)\cos C + (b + c)\cos A + (c + a)\cos B$

$= a\cos C + b\cos C + b\cos A + c\cos A + c\cos B + a\cos B$

$= b\cos C + c\cos B + c\cos A + a\cos C + a\cos B + b\cos A$

$= a + b + c =$ R.H.S.

Exercise 6.2

For ΔABC, prove **(1 to 9)** :

1. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$
2. $a^2(\cos^2B - \cos^2C) + b^2(\cos^2C - \cos^2A) + c^2(\cos^2A - \cos^2B) = 0$
3. $\dfrac{a^2\sin(B-C)}{\sin A} + \dfrac{b^2\sin(C-A)}{\sin B} + \dfrac{c^2\sin(A-B)}{\sin C} = 0$
4. $a^3\sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B) = 0$
5. $a\sin\left(\frac{A}{2}+C\right) = (b + c)\sin\frac{A}{2}$
6. $a\cos\left(\frac{B-C}{2}\right) = (b + c)\sin\frac{A}{2}$
7. $\sin\left(\frac{A-B}{2}\right) = \dfrac{a-b}{c}\cos\frac{C}{2}$
8. $\tan\left(\frac{A}{2}+B\right) = \dfrac{c+b}{c-b}\tan\frac{A}{2}$
9. $\dfrac{1+\cos A\cos(B-C)}{1+\cos C\cos(A-B)} = \dfrac{b^2+c^2}{b^2+a^2}$
10. Prove : $\sin^2A + \sin^2B = \sin^2C \Rightarrow \Delta$ABC is right angled at C.

11. Prove : $(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B) \Rightarrow \Delta ABC$ is either isosceles or right angled.

12. Prove : $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$

13. Prove : $\left(\frac{b^2 - c^2}{a^2}\right)\sin 2A + \left(\frac{c^2 - a^2}{b^2}\right)\sin 2B + \left(\frac{a^2 - b^2}{c^2}\right)\sin 2C = 0$

14. Prove : $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = c + a - b$

15. Prove : $4\left(bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{C}{2}\right) = (a + b + c)^2$

16. Show that a triangle having sides equal to 3, 5, 7 is an obtuse angled triangle and determine the measure of the obtuse angle.

17. If the angles of a triangle are in the ratio 1 : 2 : 3, find the ratio of sides opposite to these angles.

18. The measures of angles A, B, C of a ΔABC are in A.P. and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, find A.

19. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, prove that a^2, b^2, c^2 are in A.P.

20. In a ΔABC, $a = 2b$ and $|A - B| = \frac{\pi}{3}$. Find C.

*

Miscellaneous Problems :

Example 11 : Solve $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$, where $\alpha \neq k\pi$, $k \in Z$

Solution : $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$, where $\alpha \neq k\pi$, $k \in Z$

$\therefore \quad \sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$

$\therefore \quad 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha\sin^2 x - 4\sin^3\alpha$

$\therefore \quad 3\sin\alpha = 4\sin\alpha\sin^2 x$

$\therefore \quad \sin^2 x = \frac{3}{4}$ **(Since $\alpha \neq k\pi$, $\sin\alpha \neq 0$)**

$\therefore \quad \sin x = \pm\frac{\sqrt{3}}{2} = \sin\left(\pm\frac{\pi}{3}\right)$

$\therefore \quad x = k\pi + (-1)^k\frac{\pi}{3}, k \in Z$ or $x = k\pi + (-1)^k\left(-\frac{\pi}{3}\right), k \in Z$

$\therefore \quad x = k\pi \pm \frac{\pi}{3}, k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{3} \;\middle|\; k \in Z\right\}$.

Example 12 : Solve : $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$

Solution : $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$

$\therefore \quad \frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta} = 4$

$\therefore \quad \frac{(1 + \tan\theta)^2 + (1 - \tan\theta)^2}{(1 - \tan\theta)(1 + \tan\theta)} = 4$

$\therefore \quad \dfrac{2 + 2tan^2\theta}{1 - tan^2\theta} = 4$

$\therefore \quad 2 + 2tan^2\theta = 4 - 4tan^2\theta$

$\therefore \quad 6tan^2\theta = 2$

$\therefore \quad tan^2\theta = \frac{1}{3}$

$\therefore \quad tan\theta = \pm\frac{1}{\sqrt{3}} = tan\left(\pm\frac{\pi}{6}\right)$

$\therefore \quad \theta = k\pi \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{6} \;\middle|\; k \in Z\right\}$.

Example 13 : If $\dfrac{sin\,A}{4} = \dfrac{sin\,B}{5} = \dfrac{sin\,C}{6}$, show that $\dfrac{cos A}{12} = \dfrac{cos B}{9} = \dfrac{cos C}{2}$ and hence find the value of $cos A + cos B + cos C$.

Solution : We have $\dfrac{sin\,A}{4} = \dfrac{sin\,B}{5} = \dfrac{sin\,C}{6}$

$\therefore \quad \dfrac{\frac{a}{2R}}{4} = \dfrac{\frac{b}{2R}}{5} = \dfrac{\frac{c}{2R}}{6}$

$\therefore \quad \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say), where $k > 0$

$\therefore \quad a = 4k,\ b = 5k,\ c = 6k$

Now, $cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{25k^2 + 36k^2 - 16k^2}{2 \cdot 5k \cdot 6k} = \dfrac{45k^2}{60k^2} = \dfrac{3}{4}$

$\therefore \quad \dfrac{cos A}{12} = \dfrac{1}{16}$

Similarly, $\dfrac{cos B}{9} = \dfrac{1}{16}$ and $\dfrac{cos C}{2} = \dfrac{1}{16}$

Hence, $\dfrac{cos A}{12} = \dfrac{cos B}{9} = \dfrac{cos C}{2}$

Also, $cos A + cos B + cos C = \frac{12}{16} + \frac{9}{16} + \frac{2}{16} = \frac{23}{16}$

Exercise 6

Solve **(1 to 10)** :

1. $2(sec^2\theta + sin^2\theta) = 5$

2. $2 - cos x = 2tan\frac{x}{2}$

3. $4sin\theta\, sin2\theta\, sin4\theta = sin3\theta$

4. $sin^2\theta - cos\theta = \frac{1}{4}$

5. $\sqrt{3}tan3\theta + \sqrt{3}tan2\theta + tan3\theta\, tan2\theta = 1$

6. $cosec x = 1 + cot x$

7. $sin^8x + cos^8x = \frac{17}{32}$

8. $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

9. $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

10. $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

For ΔABC, prove **(11 to 14)** :

11. $a\cos A + b\cos B + c\cos C = 4R\sin A \sin B \sin C = \frac{abc}{2R^2}$

12. $a(\cos C - \cos B) = 2(b - c)\cos^2\frac{A}{2}$

13. $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc$

14. $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} \Rightarrow \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

15. Prove : *cosine* rule using sine rule.

16. Prove : $(a - b)^2 \cos^2\frac{C}{2} + (a + b)^2 \sin^2\frac{C}{2} = c^2$

17. Prove : $abc(\cot A + \cot B + \cot C) = R(a^2 + b^2 + c^2)$

18. If length of the sides of a triangle are 4, 5 and 6, prove that the largest measure of an angle is twice that of the angle with smallest measure.

19. If length of the sides of a triangle are m, n, $\sqrt{m^2 + mn + n^2}$, prove that the largest measure of an angle of the triangle is $\frac{2\pi}{3}$.

20. If length of the two sides of a triangle are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and if the included angle between them has measure $\frac{\pi}{3}$, then show that the perimeter of the triangle is $2\sqrt{3} + \sqrt{6}$.

21. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is ... ☐

(a) $\emptyset$ (b) $\left\{\frac{\pi}{4}\right\}$

(c) $\left\{k\pi + \frac{\pi}{4} \mid k \in Z\right\}$ (d) $\left\{2k\pi + \frac{\pi}{4} \mid k \in Z\right\}$

(2) Number of ordered pairs (a, x) satisfying the equation $\sec^2 (a + 2)x + a^2 - 1 = 0$; $-\pi < x < \pi$ is ... ☐

(a) 2 (b) 1 (c) 3 (d) infinite

(3) The general solution of the equation $\sin^{50}x - \cos^{50}x = 1$ is ... ☐

(a) $2k\pi + \frac{\pi}{2}$, $k \in Z$ (b) $2k\pi + \frac{\pi}{3}$, $k \in Z$

(c) $k\pi + \frac{\pi}{3}$, $k \in Z$ (d) $k\pi + \frac{\pi}{2}$, $k \in Z$

(4) The number of solutions of the equation $3\sin^2x - 7\sin x + 2 = 0$, in the interval $[0, 5\pi]$ is ... ☐

(a) 0 (b) 5 (c) 6 (d) 10

(5) The real roots of the equation $cos^7x + sin^4x = 1$, in the interval $(-\pi, \pi)$, are ...

(a) $0, \frac{\pi}{3}, -\frac{\pi}{3}$ (b) $0, \frac{\pi}{4}, -\frac{\pi}{4}$ (c) $0, \frac{\pi}{2}, -\frac{\pi}{2}$ (d) $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$

(6) The number of points of intersection of $2y = 1$ and $y = sinx$, $-2\pi < x \le 2\pi$ is ...

(a) 2 (b) 4 (c) 3 (d) 1

(7) The general solution of $sin\theta + cos\theta = 2$ is ...

(a) $k\pi, k \in Z$ (b) $2k\pi + \frac{\pi}{2}, k \in Z$

(c) $\emptyset$ (d) $(2k + 1)\frac{\pi}{2}, k \in Z$

(8) The general solution of $cos2\theta = cos^2\theta - sin^2\theta$ is ...

(a) R (b) $k\pi, k \in Z$

(c) $\emptyset$ (d) $(2k + 1)\frac{\pi}{2}, k \in Z$

(9) In a ΔABC, if $\frac{cos A}{a} = \frac{cos B}{b} = \frac{cos C}{c}$ and $a = 2$, then the area of the triangle is ...

(a) 1 (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

(10) In a ΔABC, $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, numbers of such triangles are ...

(a) 1 (b) 0 (c) 2 (d) infinite

(11) The perimeter of ΔABC is 6 times the arithmetic mean of the sines of its angles. If a is 1, then A is ...

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(12) In a ΔABC, $a = 2b$ and A = 3B, then A = ...

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

(13) If A, B, C in a ΔABC are in A.P. and the sides a, b, c are in G.P., then a^2, b^2, c^2 are in ...

(a) G.P. (b) A.P.

(c) $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in A.P. (d) no relation

(14) In ΔABC, A = $\frac{\pi}{4}$, $c = \frac{\pi}{3}$, then $a + c\sqrt{2}$ = ...

(a) b (b) $\sqrt{3}b$ (c) $\sqrt{2}b$ (d) $2b$

(15) In a ΔABC, $2acsin\frac{1}{2}(A - B + C)$ = ...

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 + a^2$ (d) $c^2 - a^2 - b^2$

*

Summary

We studied following points in this chapter :

1. $sin\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

2. $cos\theta = 0 \Leftrightarrow \theta = (2k + 1)\frac{\pi}{2}\ k \in Z$

3. $tan\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

4. Solution set of $sin\theta = a, -1 \le a \le 1$ is given by $\{k\pi + (-1)^k\alpha \mid k \in Z\}$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin\theta = a = sin\alpha$.

5. Solution set of $cos\theta = a, -1 \le a \le 1$ is given by $\{2k\pi \pm \alpha \mid k \in Z\}$, where $\alpha \in [0, \pi]$ and $cos\theta = a = cos\alpha$.

6. Solution set of $tan\theta = a, a \in R$ is given by $\{k\pi + \alpha \mid k \in Z\}$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\theta = a = tan\alpha$.

7. If $c^2 \le a^2 + b^2$, the general solution of $acosx + bsinx = c$ is

   $x = 2k\pi + \alpha \pm \beta, k \in Z$, where $\alpha \in [0, 2\pi)$ such that $a = rcos\alpha$ and $b = rsin\alpha$ and

   $cos\beta = \frac{c}{r}, \beta \in [0, \pi], r = \sqrt{a^2 + b^2}$

   If $c^2 > a^2 + b^2$, the solution set is $\emptyset$.

8. The *sine* rule is : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

9. The *cosine* rule is :

   $cosA = \frac{b^2 + c^2 - a^2}{2bc}$, $cosB = \frac{c^2 + a^2 - b^2}{2ca}$ and $cosC = \frac{a^2 + b^2 - c^2}{2ab}$

10. Projection Formula :

    $a = bcosC + ccosB, b = ccosA + acosC, c = acosB + bcosA$

—❖—

Aryabhata gave an accurate approximation for π. He wrote in the *Aryabhatiya* the following :

Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.

This gives $\pi = \frac{62832}{20000} = 3.1416$ which is a surprisingly accurate value. In fact $\pi = 3.14159265$ correct to 8 places.

He gave a table of *sines* calculating the approximate values at intervals of $90^o/24 = 3^o\ 45'$. In order to do this he used a formula for $sin(n + 1)x - sin\ nx$ in terms of $sin\ nx$ and $sin(n - 1)x$. He also introduced the versine (versin = 1 − *cosine*) into trigonometry.

Aryabhata gives the radius of the planetary orbits in terms of the radius of the Earth/Sun orbit as essentially their periods of rotation around the Sun. He believes that the Moon and planets shine by reflected sunlight. Incredibly he believes that the orbits of the planets are ellipses. He correctly explains the causes of eclipses of the Sun and the Moon. The Indian belief up to that time was that eclipses were caused by a demon called Rahu. His value for the length of the year at 365 days 6 hours 12 minutes 30 seconds is an overestimate since the true value is less than 365 days 6 hours.

Chapter 7

SEQUENCES AND SERIES

7.1 Introduction

The word 'sequence' used in the English language and in mathematics has the same sense. That is the sequence emphasises on the order of occurrence. When we talk about a sequence of events, it clearly indicates the order of occurrence of the events. For example, India won the ICC World Cup-2011. As we know that Indian team played a sequence of matches and won certain number of them and finally won the final match. Here, we can see the sequence of events taking place in a definite order. Similarly, in mathematics, when we talk about a sequence of numbers, it clearly indicates the first number, the second number, the third number and so on. Historically, ***Aryabhata*** was the first mathematician to give the formula for the sum of the squares of first n natural numbers, the sum of cubes of first n natural numbers etc. This is found in his work ***Aryabhatiyam***. Such kind of work is also observed in the work of famous Italian mathematician ***Fibonacci*** **(1175-1250). The numbers of Fibonacci sequence are also known as Fibonacci numbers** and they are applied in many fields of knowledge.

Now, let us discuss about sequences mathematically. Observe the sequence of even numbers 2, 4, 6, ..., we can easily see that the sequence is 2(1), 2(2), 2(3), ..., so we can generalise that nth even number must be $2(n)$. So we can think of a function $f : N \rightarrow R$, $f(n) = 2n$. Similarly the sequence 1, 4, 9, 16, ... can be written as $f : N \rightarrow R$, $f(n) = n^2$. So we define sequence as a function whose domain is N or $\{1, 2, 3, ..., n\}$.

Sequence : A function $f : N \rightarrow R$ or $f : \{1, 2, 3, ..., n\} \rightarrow R$ is called a sequence. $f : \{1, 2, 3, ..., n\} \rightarrow R$ is called a finite sequence. Here $n \in N$.

For instance, $f : N \rightarrow R$, $f(n) = 3n - 1$.

Taking $n = 1, 2, 3, ...$ we get $f(1) = 2$, $f(2) = 5$, $f(3) = 8$, ... 2, 5, 8, ... are called respectively first, the second, the third, ... term of the sequence. $f(n)$ is called the nth term or a general term.

$f(n)$ is also denoted by a_n or t_n or T_n or u_n etc.

$\{f(n)\}$ or $\{a_n\}$ or $\{t_n\}$ indicates the sequence having the nth term as $f(n)$ or a_n or t_n respectively.

According as codomain of the function is N, Z or R, the sequence is called a sequence of natural numbers, a sequence of integers or sequence of real numbers respectively.

The nth term of a sequence may be in the form of formula, but it is not necessary that every sequence is defined by means of some formula. For example, the sequence of prime numbers 2, 3, 5, 7, 11, 13, ... There is no formula to get the nth prime number, so the sequence is not expressed by defining a rule.

Let us see one interesting sequence, $f(n) = (n - 1) \cdot (n - 2) \cdot (n - 3) + (2n - 1)$. Obviously $f(1) = 1$, $f(2) = 3$, $f(3) = 5$. We may be tempted to say that $f(4) = 7$, but it is not so, it is 13. Thus **by using a few terms only we can not guess the general term of a sequence.**

Example 1 : Find first five terms of the sequence : $f : N \rightarrow R$, $f(n) = 2n^2 - 4$.

Solution : Here $f(n) = 2n^2 - 4$

$\therefore$ $f(1) = 2(1)^2 - 4 = -2$, $f(2) = 2(2)^2 - 4 = 4$,

$f(3) = 2(3)^2 - 4 = 14$, $f(4) = 2(4)^2 - 4 = 28$, $f(5) = 2(5)^2 - 4 = 46$.

Thus, the first five terms are -2, 4, 14, 28 and 46.

Example 2 : For $f : N \rightarrow R$, $f(n) = n(-1)^n$, find the difference between 17th and 16th terms.

Solution : Here $f(n) = n(-1)^n$

$\therefore$ $f(16) = 16(-1)^{16} = 16$ and $f(17) = 17(-1)^{17} = -17$

Now, $f(17) - f(16) = (-17) - (16) = -33$

$\therefore$ The difference $= | f(17) - f(16) | = 33$

Example 3 : $f : N \rightarrow R$, $f(n) = 8 - n^3$. Find the first four terms of the sequence.

Solution : $f(1) = 8 - (1)^3 = 7$, $f(2) = 8 - (2)^3 = 0$, $f(3) = 8 - (3)^3 = -19$ and

$f(4) = 8 - (4)^3 = -56$.

$\therefore$ The first four terms are 7, 0, -19 and -56.

Example 4 : Let the sequence $f : N \rightarrow R$ be defined by $f(1) = 1$ and $f(n) = f(n - 1) - 1$ for $n \geq 2$. Find the first five terms of the sequence.

Solution : Here $f(1) = 1$

Now $f(n) = f(n - 1) - 1$, for $n \geq 2$

$\therefore$ $f(2) = f(2 - 1) - 1 = f(1) - 1 = 1 - 1 = 0$

$f(3) = f(2) - 1 = -1$, $f(4) = f(3) - 1 = -2$, $f(5) = f(4) - 1 = -3$

$\therefore$ The first five terms are 1, 0, -1, -2 and -3.

Example 5 : If $f : N \rightarrow R$, $f(n) = cos\frac{n\pi}{2}$, find the first six terms of the sequence f.

Solution : Here $f(n) = cos\frac{n\pi}{2}$

$\therefore \quad f(1) = cos\frac{\pi}{2} = 0, \quad f(2) = cos\pi = -1, \quad f(3) = cos\frac{3\pi}{2} = 0$

$f(4) = cos2\pi = 1, \quad f(5) = cos\frac{5\pi}{2} = 0, \quad f(6) = cos3\pi = -1$

So the first six terms are 0, −1, 0, 1, 0 and −1.

Example 6 : What will be the 10th term of the sequence defined by

$f(n) = (n - 1)(n + 2)(n - 3)$?

Solution : Here, $f(n) = (n - 1)(n + 2)(n - 3)$

$\therefore \quad f(10) = (10 - 1)(10 + 2)(10 - 3) = 9 \cdot 12 \cdot 7 = 756$

$\therefore$ Hence the 10th term is 756.

7.2 Series :

Let $a_1, a_2, a_3, ..., a_n, ...$ be a given sequence. Let us think of the sequence formed by using the terms of the given sequence as follows :

$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, ... a_1 + a_2 + a_3 + ... + a_n, ...$ Such a new sequence is called a **series** derived from sequence $\{a_n\}$.

Usually, S_n denotes the sum of the first n terms of a sequence. So the sequence $S_1, S_2, S_3, ..., S_n$ becomes the series corresponding to the given original sequence.

Hence **every series is a sequence and *n*th term of the series is the sum of the first *n* terms of its corresponding sequence.**

For instance, take the sequence of odd natural numbers. i.e. 1, 3, 5, 7, 9, ...

$\therefore \quad S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 3 = 4$

$S_3 = a_1 + a_2 + a_3 = 1 + 3 + 5 = 9$

$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 3 + 5 + 7 = 16$

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We get the sequence 1, 4, 9, 16, ... which is the sequence of squares of natural numbers. i.e. $S_n = n^2$. It is called the **series** derived from the sequence $f(n) = 2n - 1$.

Let us obtain nth term a_n of a sequence from the sum of first n terms S_n of the same sequence. We can derive the formula for a_n as follows, if we are given the formula of S_n :

$S_1 = a_1$

$S_2 = a_1 + a_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$

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$S_n = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n = S_{n-1} + a_n$

We observe that $S_n = S_{n-1} + a_n$ for $n = 2, 3, 4, ...$

$\therefore \quad S_n - S_{n-1} = a_n \quad \forall n \geq 2$ and $S_1 = a_1$

This gives the formula for a_n, when the sum of first n terms S_n is given.

Example 7 : For the sequence $\{a_n\}$, $S_n = n^3 - 2n$, find the first four terms and 8th term of $\{a_n\}$.

Solution : $S_n = n^3 - 2n$

$\therefore \quad S_1 = (1)^3 - 2(1) = 1 - 2 = -1, \qquad S_2 = (2)^3 - 2(2) = 8 - 4 = 4,$

$S_3 = (3)^3 - 2(3) = 27 - 6 = 21, \qquad S_4 = (4)^3 - 2(4) = 64 - 8 = 56$

So, $a_1 = S_1 = -1, \quad a_2 = S_2 - S_1 = 4 - (-1) = 5, \quad a_3 = S_3 - S_2 = 21 - 4 = 17,$

$a_4 = S_4 - S_3 = 56 - 21 = 35.$

$\therefore$ The first four terms of $\{a_n\}$ are -1, 5, 17 and 35.

The 8th term, $a_8 = S_8 - S_7$

$= [(8)^3 - 2(8)] - [(7)^3 - 2(7)]$

$= [512 - 16] - [343 - 14] = 167$

Example 8 : From the formula for the series, $S_n = 4^n - 1$, obtain the formula for the corresponding sequence.

Solution : $a_1 = S_1 = 4^1 - 1 = 3, \quad S_n = 4^n - 1$

$\therefore \quad S_{n-1} = 4^{n-1} - 1$

$a_n = S_n - S_{n-1}, \forall n \geq 2 = (4^n - 1) - (4^{n-1} - 1)$

$= 4^n - 4^{n-1}$

$= 4^{n-1}(4 - 1)$

$= 3 \cdot 4^{n-1}, \quad \forall n \geq 2$

Taking $n = 1$, $3 \cdot 4^{1-1} = 3 = a_1$

$\therefore \quad a_n = 3 \cdot 4^{n-1}, \forall n \geq 1$

Exercise 7.1

1. Write the first five terms of the following sequence :

(1) $f(n) = 3n + 1$ (2) $f(n) = \frac{n-(-1)^n}{2}$ (3) $f(n) = n$th prime number

2. The Fibonacci sequence is defined by,

$a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$, find a_3, a_4, a_5, a_6.

3. Obtain a_2, a_3, a_4 for the following sequences :

(1) $a_1 = -3$ and $a_n = 2a_{n-1} + 1, \forall n > 1.$

(2) $a_1 = \frac{1}{2}$ and $a_n = 3a_{n-1} + (-1)^n, \forall n \geq 2.$

4. Find the first three terms and tenth term of the sequence $\{a_n\}$:

(1) $S_n = n^2 - 1$ (2) $S_n = \frac{n(n+1)}{2}$

5. From the following formula for the series S_n, obtain the formula for corresponding sequence :

(1) $S_n = \frac{a(r^n - 1)}{r-1}, r \neq 1, a \neq 0$ (2) $S_n = 4\{1 - (-3)^{n-1}\}$

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7.3 Arithmetic Progression (A.P.)

Observe the sequence 1, 3, 5, 7, ... Here each term (after the first) is obtained by adding the same number 2 to its preceding term. The difference between two consecutive terms is a non-zero constant. Such a sequence is called an **arithmetic progression**. We define it as follows :

Arithmetic Progression : A sequence $f : N \rightarrow R$, $f(n) = an + b$, $a, b \in R$, $a \neq 0$ is called an arithmetic progression (A.P.). Thus an A.P. is a linear function of n, where $n \in N$.

For example, the sequence $f(n) = 3n - 4$, $n \in N$ is an A.P., its terms are $-1, 2, 5, 8, 11, ...$ Here difference between any two consecutive terms is 3, a constant.

In the above discussion, we can observe that the difference between any two consecutive terms is a non-zero constant and f is a linear function of n, $n \in N$. Now we shall combine these two properties in the following theorem.

Theorem 1 : Difference between any two successive terms in an A.P. is a non-zero constant.

Proof : Suppose $\{f(n)\} = \{an + b\}$ is an A.P., $a, b \in R$, $a \neq 0$.

For any $k \in N$, $f(k + 1) - f(k) = [a(k + 1) + b] - (ak + b)$

$= ak + a + b - ak - b$

$= a$, a non-zero constant

Thus, the difference of between any two successive terms $f(k + 1)$ and $f(k)$ is a non-zero constant. We call it the **common difference** of the A.P. and usually denote it by 'd'. Now, onwards the common difference will be termed as difference. Here we take $d = f(k + 1) - f(k)$ which may be positive or negative.

The converse of above theorem is also true. Suppose the first term of a sequence $\{f(n)\}$ is 'a' and the difference $f(k + 1) - f(k) = d$, $d \neq 0$ for all $k \in N$. Then it is clear that the sequence is an A.P. In general we conclude that **nth term of the A.P. as $f(n) = a + (n - 1)d$, $d \neq 0$** and it is a linear function of n. We shall prove our conclusion by the method of mathematical induction.

Theorem 2 : If the first term of a sequence $\{f(n)\}$ is a and if the difference of two successive terms is $d \neq 0$, then $f(n) = a + (n - 1)d$, $\forall n \in N$ and so it is an A.P.

Proof : Let the statement $P(n) : f(n) = a + (n - 1)d$, $\forall n \in N$

(1) For $n = 1$, $f(1) = a$, the first term and

$a + (n - 1)d = a + (1 - 1)d = a$

$\therefore$ P(1) is true.

(2) Let $P(k) : f(k) = a + (k - 1)d$ be true for some $k \in N$. **(i)**

Then we shall prove that $P(k + 1)$ is true.

$f(k + 1) = f(k) + d$ **($f(k + 1) - f(k) = d$)**

$= [a + (k - 1)d] + d$ **(from (i))**

$\therefore$ $f(k + 1) = a + kd$

$= a + [(k + 1) - 1]d$

Thus $P(k)$ is true. $\Rightarrow$ $P(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Here, $f(n) = a + (n - 1)d = dn + (a - d)$ is a linear function of n (as $d \neq 0$), so f is an A.P.

We conclude from these two theorems that if 'a' is the first term and 'd' is the common difference of an A.P., then the A.P. can be written as $a, a + d, a + 2d, ..., a + (n - 1)d, ...$

Thus the formula for nth term of an A.P. is $f(n) = a + (n - 1)d$. a_n is also used for the last term of a finite A.P. having domain $\{1, 2, 3, ..., n\}$.

If we denote nth term by t_n, then $t_n = a + (n - 1)d$, where a is the first term and d is the common difference.

Note : a, b, c are consecutive terms in A.P. $\Leftrightarrow b - a = c - b$

$\Leftrightarrow 2b = a + c$

Example 9 : For an A.P. 3, 8, 13, 18, ... find the 17th and 40th terms.

Solution : $a = 3$, $d = 5$

nth term of the A.P. is $t_n = a + (n - 1)d$

$= 3 + (n - 1)5$

$= 5n - 2$

Taking $n = 17$, $t_{17} = 5(17) - 2 = 83$ and

taking $n = 40$, $t_{40} = 5(40) - 2 = 198$.

$\therefore$ 17th term is 83 and 40th term is 198.

Example 10 : Which term of the A.P. 3, 14, 25, 36, ... will be 121 less than its 37th term ?

Solution : Here $a = 3$, $d = 11$, given $m = 37$

mth term, $t_m = a + (m - 1)d$

$\therefore \quad t_{37} = 3 + (37 - 1)11$

$= 3 + 396 = 399$

Let t_n be the term 121 less then t_{37}.

$\therefore \quad t_n = t_{37} - 121 = 399 - 121 = 278$

$\therefore \quad a + (n - 1)d = 278$

$\therefore \quad 3 + (n - 1)11 = 278$

$\therefore \quad (n - 1)11 = 278 - 3 = 275$

$\therefore \quad n - 1 = 25$

$\therefore \quad n = 26$

Thus, the 26th term is 121 less than its 37th term.

Note : Order of the term 121 less is $\frac{121}{11} = 11$ less than 37th term (here $d = 11$).

So $37 - 11 = 26$th term is the required term.

Example 11 : If the 11th term of an A.P. is zero, then prove that its 31st term is double than the 21st term.

Solution : $t_n = a + (n - 1)d$

$\therefore \quad t_{11} = a + 10d$

$\therefore \quad 0 = a + 10d \qquad$ **(i)**

Now, $2 \cdot t_{21} = 2(a + 20d)$

$= 2a + 40d$

$= (a + 30d) + (a + 10d)$

$= t_{31} + 0 = t_{31}$ **(from (i))**

Thus, 31st term of the A.P. is double than the 21st term.

Example 12 : If the pth term of an A.P. is q and the qth term is p, $p \neq q$, then find the nth term of the A.P.

Solution : Here t_p is $a + (p - 1)d = q$ **(i)**

and t_q is $a + (q - 1)d = p$ **(ii)**

Solving (i) and (ii), we get

$(p - q)d = q - p$

$\therefore$ As $p \neq q$, $d = -1$ and $a = p + q - 1$

Now the nth term $t_n = a + (n - 1)d$

$= p + q - 1 + (n - 1)(-1)$

$= p + q - n$

Arithmetic Series :

The series corresponding to an A.P. is called an Arithmetic Series.

The nth term of the arithmetic series corresponding to the A.P.

$a, a + d, a + 2d, ..., a + (n - 1)d$ is

$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 1)d]$

Now we shall prove the expression for the sum of first n terms of the A.P.,

i.e. $S_n = \frac{n}{2}[2a + (n - 1)d]$ by the principle of mathematical induction.

Theorem 3 : If first term of an A.P. is a and d is the common difference, then the sum of first n terms is $S_n = \frac{n}{2}[2a + (n - 1)d]$, $\forall n \in N$.

Proof : Let the statement P(n) : $S_n = \frac{n}{2}[2a + (n - 1)d]$, $\forall n \in N$.

(1) For $n = 1$, $S_1 = \frac{1}{2}[2a + (1 - 1)d] = a$, i.e. the sum of the first term is the first term 'a' itself.

$\therefore$ P(1) is true.

(2) Let P(k) : $S_k = \frac{k}{2}[2a + (k - 1)d]$ be true for some $k \in N$. **(i)**

Let $n = k + 1$

$S_{k+1} = S_k + (k + 1)$th term

$= \frac{k}{2}[2a + (k - 1)d] + a + [(k + 1) - 1]d$ **(from (i))**

$= \frac{1}{2}[2ak + k(k - 1)d + 2a + 2kd]$

$= \frac{1}{2}[2a(k + 1) + k(k - 1 + 2)d]$

$$= \frac{1}{2}[2a(k + 1) + kd(k + 1)]$$

$$= \frac{k+1}{2}[2a + \{(k + 1) - 1\}d]$$

Thus, P(k) is true. $\Rightarrow$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction P(n) is true. $\forall n \in$ N.

Note : Formula for S_n of an A.P. of finite term is

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + \{a + (n - 1)d\}] = \frac{n}{2}(a + l)$$

where a is the first term and l is the last term, i.e. $l = t_n = a + (n - 1)d$.

Thus, the formula of S_n for A.P. $= \frac{\text{number of terms}}{2}$ [first term + last term]

Example 13 : Find the sum of the first fifteen terms of A.P. 15, 11, 7, 3, ...

Solution : Here $a = 15$, $d = 11 - 15 = -4$ and $n = 15$

Now, $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore$ $S_{15} = \frac{15}{2}[2(15) + (15 - 1)(-4)]$

$= \frac{15}{2}[30 - 56] = \frac{15}{2}[-26] = -195$

$\therefore$ The sum of the first 15 terms is -195.

Example 14 : The sum of n terms of two AP.s are in the ratio $(3n + 6) : (5n - 13)$, $\forall n \in$ N. Find the ratio of their 11th terms.

Solution : Suppose, the first term and common difference of one A.P. are a_1 and d_1 and the same for the other A.P. are a_2 and d_2 respectively.

According to the given condition,

$$\frac{\text{Sum of the first } n \text{ terms of first A.P.}}{\text{Sum of the first } n \text{ terms of second A.P.}} = \frac{3n+6}{5n-13}$$

$$\therefore \quad \frac{\frac{n}{2}[2a_1 + (n - 1)d_1]}{\frac{n}{2}[2a_2 + (n - 1)d_2]} = \frac{3n+6}{5n-13}$$

$$\therefore \quad \frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2} = \frac{3n+6}{5n-13} \qquad \text{(i)}$$

$\therefore$ Let t_n and t_n' be the nth terms of given A.P.s.

Now, $\frac{t_{11}}{t'_{11}} = \frac{a_1 + 10d_1}{a_2 + 10d_2}$

$= \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$

$= \frac{2a_1 + (21 - 1)d_1}{2a_2 + (21 - 1)d_2}$

So, substituting $n = 21$ is (i), we have

$$\frac{t_{11}}{t'_{11}} = \frac{3(21)+6}{5(21)-13} = \frac{69}{92} = \frac{3}{4}$$

$\therefore$ The ratio of the 11th terms of the two A.P.s is 3 : 4.

Note : Sometimes we need to assume some consecutive terms of an A.P.

If **three** or **five** or **seven** consecutive terms are given, then we assume the middle term as 'a' and preceding terms decreasing by 'd' and succeeding terms increasing by 'd'.

So we assume,

The 3 consecutive terms in A.P. : $a - d, a, a + d$

The 5 consecutive terms in A.P. : $a - 2d, a - d, a, a + d, a + 2d$

The 7 consecutive terms in A.P. : $a - 3d, a - 2d, a - d, a, a + d, a + 2d, a + 3d$

If **four** or **six** consecutive terms are given, then there are two middle terms, so we assume them as $a - d$ and $a + d$. Here the difference between consecutive terms is taken as '$2d$', so preceding term is decreased by '$2d$' and succeeding term is increased by '$2d$'. So we assume,

The 4 consecutive terms in A.P. : $a - 3d, a - d, a + d, a + 3d$

The 6 consecutive terms in A.P. : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$

Example 15 : The sum and the product of three consecutive terms of an A.P. are 24 and 312 respectively. Find the three terms.

Solution : Suppose the three consecutive terms of the A.P. are $a - d, a, a + d$.

According to the given conditions,

$(a - d) + a + (a + d) = 24$ and $(a - d) \cdot a \cdot (a + d) = 312$

Thus, $3a = 24$.

So $a = 8$ and $(8 - d) \cdot 8 \cdot (8 + d) = 312$

$\therefore$ $64 - d^2 = 39$

$\therefore$ $d^2 = 25$

$\therefore$ $d = 5$ or $d = -5$

If $a = 8$ and $d = 5$, then the required terms are 3, 8, 13 and if $a = 8$ and $d = -5$, then they are 13, 8, 3.

Thus the required terms are 3, 8, 13.

Example 16 : The sum of four consecutive terms of an A.P. is 24 and the product of first and last terms is -45. Find the terms.

Solution : Suppose the four consecutive terms of the A.P. are

$a - 3d, a - d, a + d, a + 3d$.

Their sum $(a - 3d) + (a - d) + (a + d) + (a + 3d) = 24$

$\therefore$ $4a = 24$. So $a = 6$.

Also $(a - 3d)(a + 3d) = -45$

$\therefore$ $(6 - 3d)(6 + 3d) = -45$

$\therefore$ $36 - 9d^2 = -45$

$\therefore$ $9d^2 = 81$