3.6 ELECTROMAGNETIC INDUCTION. MAXWELL'S EQUATIONS

3.288 Obviously, from Lenz's law, the induced current and hence the induced e.m.f. in the loop is anticlockwise.

From Faraday's law of electromagnetic indcution,

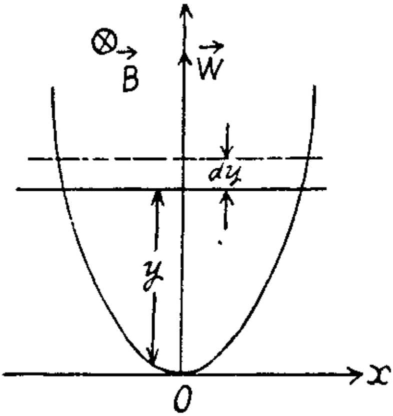
$$\xi_{in} = \left|\frac{d\Phi}{dt}\right|$$

Here, $d\Phi = \vec{B}\cdot d\vec{S} = -2B\,x\,dy,$

and from $y = ax^2, x = \sqrt{\frac{y}{a}}$

Hence, $\xi_{in} = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt}$

$$= By\sqrt{\frac{8w}{a}}, \text{ using } \frac{dy}{dt} = \sqrt{2\,wy}$$



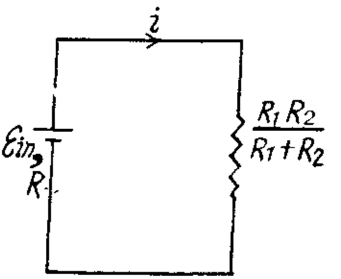
3.289 Let us assume, $\vec{B}$ is directed into the plane of the loop. Then the motional e.m.f.

$$\xi_{in} = \left|\int -(\vec{v}\times\vec{B})\cdot d\vec{l}\right| = vBl$$

and directed in the same of $(\vec{v}\times\vec{B})$ (Fig.)

So, $$i = \frac{\xi_{in}}{R + \frac{R_1 R_2}{R_1 + R_2}} = \frac{Bvl}{R + R_u}$$

As R_1 and R_2 are in parallel connections.



3.290 (a) As the metal disc rotates, any free electron also rotates with it with same angular velocity ω, and that's why an electron must have an acceleration $\omega^2 r$ directed towards the disc's centre, where r is separation of the electron from the centre of the disc. We know from Newton's second law that if a particle has some acceleration then there must be a net effecetive force on it in the direction of acceleration. We also know that a charged particle can be influenced by two fields electric and magnetic. In our problem magnetic field is absent hence we reach at the conslusion that there is an electric field near any electron and is directed opposite to the acceleration of the electron.

If E be the electric field strength at a distance r from the centre of the disc, we have from Newton's second law.

$$F_n = m\,w_n$$

$$eE = m\,r\,\omega^2, \text{ or, } E = \frac{m\,\omega^2 r}{e},$$

and the potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a \vec{E}\cdot d\vec{r} = \int_0^a \frac{m\,\omega^2 r}{e}dr, \text{ as } \vec{E}\uparrow\downarrow d\vec{r}$$

Thus $$\varphi_{cen} - \varphi_{rim} = \Delta\varphi = \frac{m\,\omega^2}{e}\frac{a^2}{2} = 3\cdot0\ \text{nV}$$

(b) When field $\vec{B}$ is present, by definition, of motional e.m.f. :

$$\varphi_1 - \varphi_2 = \int_1^2 -(\vec{v}\times\vec{B})\cdot d\vec{r}$$

Hence the sought potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a -v\,B\,dr = \int_0^a -\omega\,r\,B\,dr\,,\ (\text{as } v = \omega\,r)$$

Thus $$\varphi_{rim} - \varphi_{cen} = \varphi = \frac{1}{2}\omega\,B\,a^2 = 20\ \text{mV}$$

(In general $\omega < \frac{eB}{m}$ so we can neglect the effect discussed in (1) here).

3.291 By definition,

$$\vec{E} = -(\vec{v}\times\vec{B})$$

So, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_A^C -(\vec{v}\times\vec{B})\cdot d\vec{r} = \int_0^d -v\,B\,dr$$

But, $v = \omega r$, where r is the perpendicular distance of the point from A.

Hence, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_0^d -\omega\,B\,r\,dr = -\frac{1}{2}\omega\,B\,d^2 = -10\ \text{mV}$$

This result can be generalized to a wire AC of arbitary planar shape. We have

$$\int_A^C \vec{E}\cdot d\vec{r} = -\int_A^C (\vec{v}\times\vec{B})\cdot dr = -\int_A^C ((\omega\times\vec{r})\times\vec{B})\cdot d\vec{r}$$

$$= -\int_A^C (\vec{B}\cdot\vec{r}\,\vec{\omega} - \vec{B}\cdot\vec{\omega}\,\vec{r})\cdot d\vec{r}$$

$$= -\frac{1}{2}B\omega d^2,$$

d being AC and $\vec{r}$ being measured from A.

3.292 Flux at any moment of time,

$$|\Phi_t| = |\vec{B}\cdot d\vec{S}| = B\left(\frac{1}{2}R^2\varphi\right)$$

where φ is the sector angle, enclosed by the field.

Now, magnitude of induced e.m.f. is given by,

$$\xi_{in} = \left|\frac{d\Phi_t}{dt}\right| = \left|\frac{BR^2}{2}\frac{d\varphi}{dt}\right| = \frac{BR^2}{2}\omega,$$

where ω is the angular velocity of the disc. But as it starts rotating from rest at $t = 0$ with an angular acceleration β its angular velocity $\omega(t) = \beta t$. So,

$$\xi_{in} = \frac{BR^2}{2}\beta t.$$

According to Lenz law the first half cycle current in the loop is in anticlockwise sense, and in subsequent half cycle it is in clockwise sense.

Thus in general, $\xi_{in} = (-1)^n \frac{BR^2}{2}\beta t$, where n in number of half revolutions.

The plot $\xi_{in}(t)$, where $t_n = \sqrt{2\pi n/\beta}$ is shown in the answer sheet.

3.293 Field, due to the current carrying wire in the region, right to it, is directed into the plane of the paper and its magnitude is given by,

$B = \frac{\mu_0}{2\pi}\frac{i}{r}$ where r is the perpendicular distance from the wire.

As B is same along the length of the rod thus motional e.m.f.

$$\xi_{in} = \left|-\int_1^2 (\vec{v}\times\vec{B})\cdot d\vec{l}\right| = vBl$$

and it is directed in the sense of $(\vec{v}\times\vec{B})$

So, current (induced) in the loop,

$$i_{in} = \frac{\xi_{in}}{R} = \frac{1}{2}\frac{\mu_0 I v i}{\pi R r}$$

3.294 Field, due to the current carrying wire, at a perpendicular distance x from it is given by,

$$B(x) = \frac{\mu_0}{2\pi}\frac{i}{x}$$

Motional e.m.f is given by $\left|\int -(\vec{v}\times\vec{B})\cdot d\vec{l}\right|$

There will be no induced e.m.f. in the segments (2) and (4) as, $\vec{v}\uparrow\uparrow d\vec{l}$ and magnitude of e.m.f. induced in 1 and 3, will be

$$\xi_1 = v\left(\frac{\mu_0}{2\pi}\frac{i}{x}\right)a \text{ and } \xi_2 = v\left(\frac{\mu_0}{2\pi}\frac{i}{(a+x)}\right)a,$$

respectively, and their sense will be in the direction of $(\vec{v}\times\vec{B})$.

So, e.m.f., induced in the network $= \xi_1 - \xi_2$ [as $\xi_1 > \xi_2$]

$$= \frac{a v \mu_0 i}{2\pi}\left[\frac{1}{x} - \frac{1}{a+x}\right] = \frac{v a^2 \mu_0 i}{2\pi x(a+x)}$$

3.295 As the rod rotates, an emf.

$$\frac{d}{dt}\frac{1}{2}a^2\,\theta\cdot B = \frac{1}{2}a^2 B\,\omega$$

is induced in it. The net current in the conductor is then $\dfrac{\xi(t)-\frac{1}{2}a^2 B\,\omega}{R}$

A magnetic force will then act on the conductor of magnitude BI per unit length. Its direction will be normal to B and the rod and its torque will be

$$\int_0^a \left(\frac{\xi(t)-\frac{1}{2}a^2 B\,\omega}{R}\right) dx\,B\,x$$

Obviously both magnetic and mechanical torque acting on the C.M. of the rod must be equal but opposite in sense. Then
for equilibrium at constant ω

$$\frac{\xi(t)-\frac{1}{2}a^2 B\,\omega}{R}\cdot\frac{B\,a^2}{2} = \frac{1}{2}mga\sin\omega t$$

or, $\quad \xi(t) = \frac{1}{2}a^2 B\,\omega + \frac{mgR}{aB}\sin\omega t = \frac{1}{2\,aB}(a^3 B^2\omega + 2\,mgR\sin\omega t)$

(The answer given in the book is incorrect dimensionally.)

3.296 From Lenz's law, the current through the connector is directed form A to B. Here $\xi_{in} = vBl$ between A and B
where v is the velocity of the rod at any moment.

For the rod, from $F_x = mw_x$

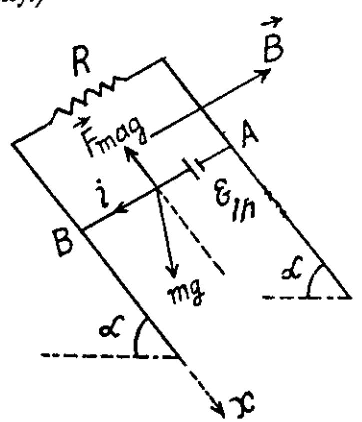
or, $\quad mg\sin\alpha - i\,l\,B = mw$

For steady state, acceleration of the rod must be equal to zero.

Hence, $\quad mg\sin\alpha = i\,l\,B \qquad (1)$

But, $\quad i = \frac{\xi_{in}}{R} = \frac{v\,B\,l}{R}$

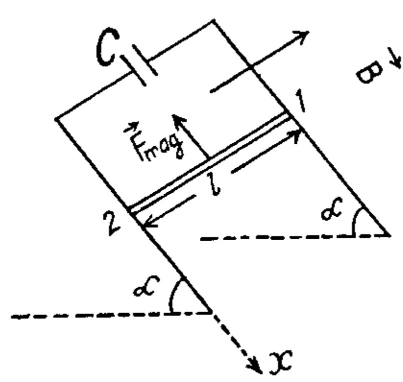
From (1) and (2) $\quad v = \frac{mg\sin\alpha\,R}{B^2 l^2}$



3.297 From Lenz's law, the current through the copper bar is directed from 1 to 2 or in other words, the induced crrent in the circuit is in clockwise sense.

Potential difference across the capacitor plates,

$$\frac{q}{C} = \xi_{in} \quad \text{or,} \quad q = C\,\xi_{in}$$



Hence, the induced current in the loop,

$$i = \frac{dq}{dt} = C\frac{d\xi_{in}}{dt}$$

But the variation of magnetic flux through the loop is caused by the movement of the bar. So, the induced e.m.f. $\xi_{in} = B\,lv$

and, $$\frac{d\xi_{in}}{dt} = Bl\frac{dv}{dt} = Blw$$

Hence, $$i = C\frac{d\xi}{dt} = C\,B\,l\,w$$

Now, the forces acting on the bars are the weight and the Ampere's force, where $F_{amp} = i\,l\,B\,(CB\,l\,w)\;B = C\,l^2 B^2 w.$

From Newton's second law, for the rod, $F_x = mw_x$

or, $$mg\sin\alpha - C\,l^2 B^2 w = mw$$

Hence $$w = \frac{mg\sin\alpha}{C\,l^2 B^2 + m} = \frac{g\sin\alpha}{1 + \dfrac{l^2 B^2 C}{m}}$$

3.298 Flux of $\vec{B}$, at an arbitrary moment of time t :

$$\Phi_t = \vec{B}\cdot\vec{S} = B\frac{\pi a^2}{2}\cos\omega t,$$

From Faraday's law, induced e.m.f., $\xi_{in} = -\dfrac{d\Phi}{dt}$

$$= -\frac{d\left(B\pi\dfrac{a^2}{2}\cos\omega t\right)}{dt} = \frac{B\pi a^2\omega}{2}\sin\omega t.$$

and induced current, $$i_{in} = \frac{\xi_{in}}{R} = \frac{B\pi a^2}{2R}\omega\sin\omega t.$$

Now thermal power, generated in the circuit, at the moment $t = t$:

$$P(t) = \xi_{in}\times i_{in} = \left(\frac{B\pi a^2\omega}{2}\right)^2\frac{1}{R}\sin^2\omega t$$

and mean thermal power generated,

$$\langle P\rangle = \frac{\left[\dfrac{B\pi a^2\omega}{2}\right]^2\dfrac{1}{R}\displaystyle\int_0^T\sin^2\omega t\,dt}{\displaystyle\int_0^T dt} = \frac{1}{2R}\left(\frac{\pi\omega a^2 B}{2}\right)^2$$

Note : The claculation of ξ_{in} which can also be checked by using motional emf is correct even though the conductor is not a closed semicircle , for the flux linked to the rectangular part containing the resistance R is not changing. The answer given in the book is off by a factor 1/4.

3.299 The flux through the coil changes sign. Initially it is BS per turn.

Finally it is $-BS$ per turn. Now if flux is Φ at an intermediate state then the current at that moment will be

$$i = \frac{-N\,\frac{d\,\Phi}{dt}}{R}$$

So charge that flows during a sudden turning of the coil is

$$q = \int i\,dt = -\frac{N}{R}[\,\Phi - (-\,\Phi)\,] = 2\,N\,BS\,/R$$

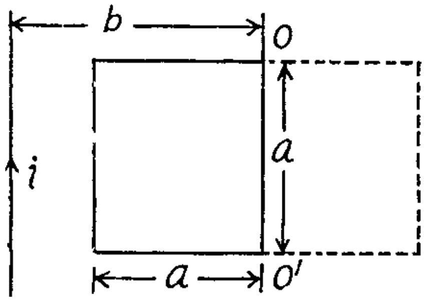
Hence, $$B = \frac{1}{2}\frac{qR}{N\,S} = 0{\cdot}5\text{ T on putting the values.}$$

3.300 According to Ohm's law and Faraday's law of induction, the current i_0 appearing in the frame, during its rotation, is determined by the formula,

$$i_0 = -\frac{d\,\Phi}{dt} = -\frac{L\,d\,i_0}{dt}$$

Hence, the required amount of electricity (charge) is,

$$q = \int i_0\,dt = -\frac{1}{R}\int (d\,\Phi + L\,di_0) = -\frac{1}{R}\,(\,\Delta\,\Phi + L\,\Delta i_0\,)$$



Since the frame has been stopped after rotation, the current in it vanishes, and hence $\Delta\,i_0 = 0$. It remains for us to find the increment of the flux $\Delta\,\Phi$ through the frame ($\Delta\,\Phi = \Phi_2 - \Phi_1$).

Let us choose the normal $\vec{n}$ to the plane of the frame, for instance, so that in the final position, $\vec{n}$ is directed behind the plane of the figure (along $\vec{B}$).

Then it can be easily seen that in the final position, $\Phi_2 > 0$, while in the initial position, $\Phi_1 < 0$ (the normal is opposite to $\vec{B}$), and $\Delta\,\Phi$ turns out to be simply equal to the fulx through the surface bounded by the final and initial positions of the frame :

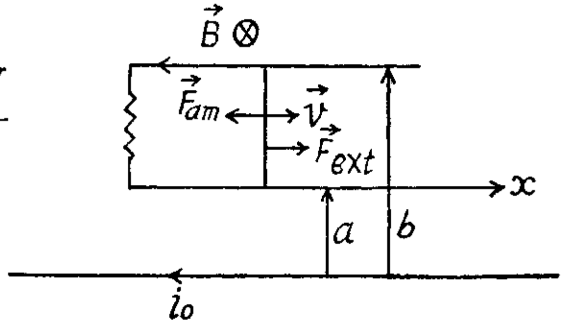
$$\Delta\,\Phi = \Phi_2 + |\,\Phi_1| = \int_{b-a}^{b+a} B\,a\,dr,$$

where B is a function of r, whose form can be easily found with the help of the theorem of circulation. Finally omitting the minus sign, we obtain,

$$q = \frac{\Delta\Phi}{R} = \frac{\mu_0\,a\,i}{2\,\pi\,R}\ln\frac{b+a}{b-a}$$

3.301 As $\vec{B}$, due to the straight current carrying wire, varies along the rod (connector) and enters linerarly so, to make the calculations simple, $\vec{B}$ is made constant by taking its average value in the range $[a, b]$.

$$\langle B\rangle = \frac{\int_a^b B\,dr}{\int_a^b dr} = \frac{\int_a^b \frac{\mu_0}{2\pi}\frac{i_0}{r}\,dr}{\int_a^b dr}$$



or, $$\langle B\rangle = \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}$$

(a) The flux of $\vec{B}$ changes through the loop due to the movement of the connector. According to Lenz's law, the current in the loop will be anticlockwise. The magnitude of motional e.m.f.,

$$\xi_{in} = v\langle B\rangle(b-a)$$

$$= \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}(b-a)\frac{dx}{dt} = \frac{\mu_0}{2\pi}i_0\ln\frac{b}{a}v$$

So, induced current

$$i_{in} = \frac{\xi_{in}}{R} = \frac{\mu_0}{2\pi}\frac{i_0 v}{R}\ln\frac{b}{a}$$

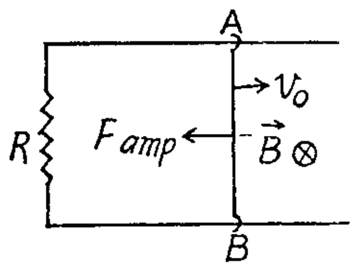
(b) The force required to maintain the constant velocity of the connector must be the magnitude equal to that of Ampere's acting on the connector, but in opposite direction.

So, $$F_{ext} = i_{in}l\langle B\rangle = \left(\frac{\mu_0}{2\pi}\frac{i_0}{R}v\ln\frac{b}{a}\right)(b-a)\left(\frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}\right)$$

$$= \frac{v}{R}\left(\frac{\mu_0}{2\pi}i_0\ln\frac{b}{a}\right)^2$$, and will be directed as shown in the (Fig.)

3.302 (a) The flux through the loop changes due to the movement of the rod AB. According to Lenz's law current should be anticlockwise in sense as we have assumed $\vec{B}$ is directed into the plane of the loop. The motion e.m.f $\xi_{in}(t) = B\,l\,v$

and induced current $i_{in} = \dfrac{vBl}{R}$



From Newton's law in projection form $F_x = mw_x$

$$-F_{amp} = m\frac{v\,dv}{dx}$$

But $$F_{amp} = i_{in}lB = \frac{vB^2l^2}{R}$$

So, $$-\frac{vB^2l^2}{R} = mv\frac{dv}{dx}$$

or, $$\int_0^x dx = -\frac{mR}{B^2 l^2}\int_{v_0}^{0} dv \quad \text{or,} \quad x = \frac{m R v_0}{B^2 l^2}$$

(b) From equation of energy conservation; $E_f - E_i$ + Heat liberated = $A_{cell} + A_{ext}$

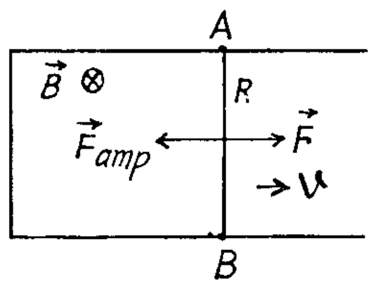
$$\left[0 - \frac{1}{2} m v_0^2\right] + \text{Heat liberated} = 0 + 0$$

So, heat liberated = $\frac{1}{2} m v_0^2$

3.303 With the help of the calculation, done in the previous problem, Ampere's force on the connector,

$\vec{F}_{amp} = \frac{v B^2 l^2}{R}$ directed towards left.

Now from Newton's second law,



$$F - F_{amp} = m\frac{dv}{dt}$$

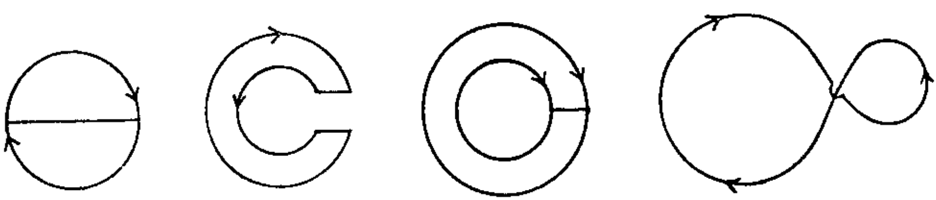
So, $$F = \frac{v B^2 l^2}{R} + m\frac{dv}{dt}$$

or, $$\int_0^t dt = m\int_0^v \frac{dv}{F - \frac{v B^2 l^2}{R}}$$

or, $$\frac{t}{m} = -\frac{R}{B^2 l^2}\ln\left(\frac{F - \frac{v B^2 l^2}{R}}{F}\right)$$

Thus $$v = \left(1 - e^{\frac{-t B^2 l^2}{Rm}}\right)\frac{RF}{B^2 l^2}$$

3.304 According to Lenz, the sense of induced e.m.f. is such that it opposes the cause of change of flux. In our problem, magnetic field is directed away from the reader and is diminishing.



(a) (b) (c) (d)

So, in figure (a), in the round conductor, it is clockwise and there is no current in the connector

In figure (b) in the outside conductor, clockwise.

In figure (c) in both the conductor, clockwise; and there is no current in the connector to obey the charge conservation.

In figure (d) in the left side of the figure, clockwise.

3.305 The loops are connected in such a way that if the current is clockwise in one, it is anticlockwise in the other. Hence the e.m.f. in loop b opposes the e.m.f. in loop a.

e.m.f. in loop $a = \frac{d}{dt}(a^2 B) = a^2 \frac{d}{dt}(B_0 \sin \omega t)$

Similarly, e.m.f. in loop $b = b^2 B_0 \omega \cos \omega t$.

Hence, net e.m.f. in the circuit $= (a^2 - b^2) B_0 \omega \cos \omega t$, as both the e.m.f's are in opposite sense, and resistance of the circuit $= 4(a+b)\rho$

Therefore, the amplitude of the current

$$= \frac{(a^2 - b^2) B_0 \omega}{4(a+b)\rho} = 0{\cdot}5 \text{ A}.$$

3.306 The flat shape is made up of concentric loops, having different radii, varying from 0 to a.

Let us consider an elementary loop of radius r, then e.m.f. induced due to this loop

$$= \frac{-d(\vec{B} \cdot \vec{S})}{dt} = \pi r^2 B_0 \omega \cos \omega t.$$

and the total induced e.m.f.,

$$\xi_{ind} = \int_0^a (\pi r^2 B_0 \omega \cos \omega t)\, dN, \qquad (1)$$

where $\pi r^2 \omega \cos \omega t$ is the contribution of one turn of radius r and dN is the number of turns in the interval $[r, r+dr]$.

So,
$$dN = \left(\frac{N}{a}\right) dr \qquad (2)$$

From (1) and (2), $\xi = \int_0^a -(\pi r^2 B_0 \omega \cos \omega t) \frac{N}{a} dr = \frac{\pi B_0 \omega \cos \omega t N a^2}{3}$

Maximum value of e.m.f. amplitude $\xi_{max} = \frac{1}{3} \pi B_0 \omega N a^2$

3.307 The flux through the loop changes due to the variation in $\vec{B}$ with time and also due to the movement of the connector.

So, $\xi_{in} = \left| \frac{d(\vec{B} \cdot \vec{S})}{dt} \right| = \left| \frac{d(BS)}{dt} \right|$ as $\vec{S}$ and $\vec{B}$ are colliniear

But, B, after t sec. of beginning of motion $= Bt$, and S becomes $= l \frac{1}{2} w t^2$, as connector starts moving from rest with a constant acceleration w.

So,
$$\xi_{ind} = \frac{3}{2} B l w t^2$$

3.308 We use $B = \mu_0 n I$

Then, from the law of electromagnetic induction

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

So, for $r < a$

$$E_\varphi 2\pi r = -\pi r^2 \mu_0 n \dot{I} \text{ or, } E_\varphi = -\frac{1}{2}\mu_0 n r \dot{I}. \text{ (where } \dot{I} = dI/dt)$$

For $r > a$

$$E_\varphi 2\pi r = -\pi a^2 \mu_0 n \dot{I} \text{ or, } E_\varphi = -\mu_0 n \dot{I} a^2/2r$$

The meaning of minus sign can be deduced from Lenz's law.

3.309 The e.m.f. induced in the turn is $\mu_0 n \dot{I} \pi \frac{d^2}{4}$

The resistance is $\frac{\pi d}{S}\rho$.

So, the current is $\frac{\mu_0 n \dot{I} S d}{4\rho} = 2\,\text{m A}$, where ρ is the resistivity of copper.

3.310 The changing magnetic field will induce an e.m.f. in the ring, which is obviously equal, in the two parts by symmetry (the e.m.f. induced by electromagnetic induction does not depend on resistance). The current, that will flow due to this, will be different in the two parts. This will cause an acceleration of charge, leading to the setting up of an electric field E which has opposite sign in the two parts. Thus,

$$\frac{\xi}{2} - \pi a E = r I \text{ and, } \frac{\xi}{2} + \pi a E = \eta r I,$$

where ξ is the total induced e.m.f. From this,

$$\xi = (\eta + 1) rI,$$

and

$$E = \frac{1}{2\pi a}(\eta - 1) rI = \frac{1}{2\pi a}\frac{\eta - 1}{\eta + 1}\xi$$

But by Faraday's law, $\xi = \pi a^2 b$

so,

$$E = \frac{1}{2} ab \frac{\eta - 1}{\eta + 1}$$

3.311 Go to the rotating frame with an instantaneous angular velocity $\vec{\omega}(t)$. In this frame, a Coriolis force, $2m\vec{v}' \times \vec{\omega}(t)$

acts which must be balanced by the magnetic force, $e\vec{v} \times \vec{B}(t)$

Thus,

$$\vec{\omega}(t) = -\frac{e}{2m}\vec{B}(t).$$

(It is assumed that $\vec{\omega}$ is small and varies slowly, so ω^2 and $\dot{\omega}$ can be neglected.)

3.312 The solenoid has an inductance,

$$L = \mu_0 n^2 \pi b^2 l,$$

where n = number of turns of the solenoid per unit length. When the solenoid is connected to the source an e.m.f. is set up, which, because of the inductance and resistance, rises slowly, according to the equation,

$$RI + L\dot{I} = V$$

This has the well known solution,

$$I = \frac{V}{R}(1 - e^{-tR/L}).$$

Corresponding to this current, an e.m.f. is induced in the ring. Its magnetic field $B = \mu_0 n I$ in the solenoid, produces a force per unit length, $\frac{dF}{dl} = B\,i = \mu_0^2 n^2 \pi a^2 I\dot{I}/r$

$$= \frac{\mu_0^2 \pi a^2 V^2}{r}\left(\frac{n^2}{RL}\right) e^{-tR/L}(1 - e^{-tR/L}),$$

acting on each segment of the ring. This force is zero initially and zero for large t. Its maximum value is for some finite t. The maximum value of

$$e^{-tR/L}(1 - e^{-tR/L}) = \frac{1}{4} - \left(\frac{1}{2} - e^{-tR/L}\right)^2 \text{ is } \frac{1}{4}.$$

So $$\frac{dF_{max}}{dl} = \frac{\mu_0^2 \pi a^2 V^2}{r}\frac{n^2}{4RL} = \frac{\mu_0 a^2 V^2}{4 r R l b^2}$$

3.313 The amount of heat generated in the loop during a small time interval dt,

$$dQ = \xi^2/R\,dt, \text{ but, } \xi = -\frac{d\Phi}{dt} = 2at - a\tau,$$

So, $$dQ = \frac{(2at - a\tau)^2}{R}dt$$

and hence, the amount of heat, generated in the loop during the time interval 0 to τ.

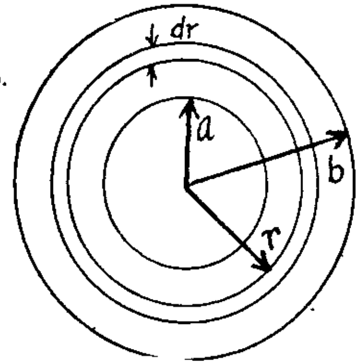
$$Q = \int_0^\tau \frac{(2at - a\tau)^2}{R}dt = \frac{1}{3}\frac{a^2\tau^3}{R}$$

3.314 Take an elementary ring of radius r and width dr.

The e.m.f. induced in this elementary ring is $\pi r^2 \beta$.

Now the conductance of this ring is.

$$d\left(\frac{1}{R}\right) = \frac{h\,dr}{\rho\,2\pi r} \text{ so } dI = \frac{h\,r\,dr}{2\rho}\beta$$



Integrating we get the total current,

$$I = \int_a^b \frac{hr\,dr}{2\rho}\beta = \frac{h\beta(b^2 - a^2)}{4\rho}$$

3.315 Given $L = \mu_0 n^2 V = \mu_0 n^2 l_0 \pi R^2$, where R is the radius of the solenoid.

Thus,
$$n = \sqrt{\frac{L}{\mu_0 l_0 \pi}} \frac{1}{R}.$$

So, length of the wire required is,
$$l = n l_0 2 \pi R = \sqrt{\frac{4 \pi L l_0}{\mu_0}} = 0{\cdot}10 \text{ km}.$$

3.316 From the previous problem, we know that,

l' = length of the wire needed $= \sqrt{\frac{L l 4 \pi}{\mu_0}}$, where l = length of solenoid here.

Now, $R = \frac{\rho_0 l'}{S}$, (where S = area of crossection of the wire. Also $m = \rho S l'$)

Thus,
$$l' = \frac{R S}{\rho_0} = \frac{R m}{\rho \rho_0 l'} \quad or \quad l' = \sqrt{\frac{Rm}{\rho\rho_0}}$$

where ρ_0 = resistivity of copper and ρ = its density.

Equating,
$$\frac{Rm}{\rho\rho_0} = \frac{Ll}{\mu_0 / 4 \pi}$$

or,
$$L = \frac{\mu_0}{4 \pi} \frac{m R}{\rho\rho_0 l}$$

3.317 The current at time t is given by,
$$I(t) = \frac{V}{R}(1 - e^{-tR/L})$$

The steady state value is, $I_0 = \frac{V}{R}$

and
$$\frac{I(t)}{I_0} = \eta = 1 - e^{-tR/L} \text{ or }, \; e^{-tR/L} = 1 - \eta$$

or,
$$t_o \frac{R}{L} = \ln \frac{1}{1 - \eta} \quad \text{or,} \quad t_o = \frac{L}{R} \ln \frac{1}{1 - \eta} = 1{\cdot}49 \text{ s}$$

3.318 The time constant τ is given by
$$\tau = \frac{L}{R} = \frac{L}{\rho_0 \frac{l_0}{S}},$$

where, ρ_0 = resistivity, l_0 = length of the winding wire, S = cross section of the wire.

But
$$m = l \rho_0 S$$

So eliminating S, $\tau = \dfrac{L}{\dfrac{\rho_0 l_0}{m/\rho l_0}} = \dfrac{mL}{\rho\rho_0 l_0^2}$

From problem 3.315 $l_0 = \sqrt{\frac{4\pi l L}{\mu_0}}$

(note the interchange of l and l_0 because of difference in notation here.)

Thus, $$\tau = \frac{mL}{\rho\rho_0 \frac{4\pi}{\mu_0} L\, l} = \mu_0 4\pi \frac{m}{\rho\,\rho_0\, l} = 0{\cdot}7\ \text{ms},$$

3.319 Between the cables, where $a < r < b$, the magnetic field $\vec{H}$ satisfies

$$H_\varphi 2\pi r = I \quad \text{or,} \quad H_\varphi = \frac{I}{2\pi r}$$

So $$B_\varphi = \frac{\mu\,\mu_0 I}{2\pi r}$$

The associated flux per unit length is, $\Phi = \int_{r=a}^{r=b} \frac{\mu\,\mu_0 I}{2\pi r} \times 1 \times dr = \frac{\mu\,\mu_0 I}{2\pi} \ln \frac{b}{a}$

Hence, the inductance per unit length $L_1 = \frac{\Phi}{I} = \frac{\mu\mu_0}{2\pi} \ln \eta$, where $\eta = \frac{b}{a}$

We get $L_1 = 0{\cdot}26\ \frac{\mu H}{m}$

3.320 Within the solenoid, $H_\varphi \cdot 2\pi r = NI$ or $H_\varphi = \frac{NI}{2\pi r}$, $B_\varphi = \mu\mu_0 \frac{NI}{2\pi r}$

and the flux, $\Phi = N\Phi_1, = N \frac{\mu\mu_0}{2\pi} NI \int_b^{a+b} \frac{a\,dr}{r}$

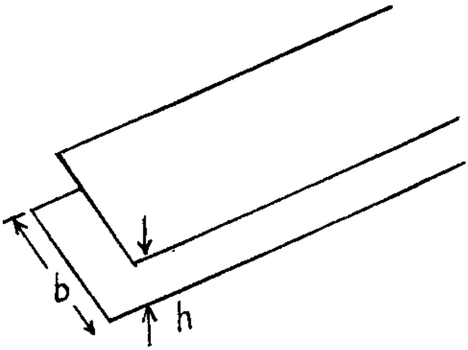
Finally, $$L = \frac{\mu\,\mu_0}{2\pi} N^2 a \ln\left(1 + \frac{a}{b}\right)$$

3.321 Neglecting end effects the magnetic field B, between the plates, which is mainly parallel to the plates, is $B = \mu_0 \frac{I}{b}$

(For a derivation see 3.229 b)

Thus, the associated flux per unit length of the plates is,

$$\Phi = \mu_0 \frac{I}{b} \times h \times 1 = \left(\mu_0 \frac{h}{b}\right) \times I.$$



So, L_1 = inductance per unit length $= \mu_0 \frac{h}{b} = 25$ nH/m.

3.322 For a single current carrying wire, $B_\varphi = \frac{\mu_0 I}{2\pi r}$ $(r > a)$. For the double line cable, with current, flowing in opposite directions, in the two conductors,

$B_\varphi \approx \frac{\mu_0 I}{\pi r}$, between the cables, by superposition. The associated flux is,

$$\Phi = \int_a^{d-a} \frac{\mu_0 I}{\pi} \frac{dr \times 1}{r} \approx \frac{\mu_0 I}{\pi} \ln \frac{d}{a} = \frac{\mu_0}{\pi} \ln \eta \times I, \text{ per unit length}$$

Hence,
$$L_1 = \frac{\mu_0}{\pi} \ln \eta$$

is the inductance per unit length.

3.323 In a superconductor there is no resistance, Hence,

$$L \frac{dI}{dt} = + \frac{d\Phi}{dt}$$

So integrating,
$$I = \frac{\Delta\Phi}{L} = \frac{\pi a^2 B}{L}$$

because
$$\Delta\Phi = \Phi_f - \Phi_i, \Phi_f = \pi a^2 B, \Phi_i = 0$$

Also, the work done is, $A = \int \xi I \, dt = \int I \, dt \frac{d\Phi}{dt} = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\pi^2 a^4 B^2}{L}$

3.324 In a solenoid, the inductance $L = \mu\mu_0 n^2 V = \mu \mu_0 \frac{N^2 S}{l}$,

where S = area of cross section of the solenoid, l = its length, $V = Sl$, $N = nl$ = total number of turns.

When the length of the solenoid is increased, for example, by pulling it, its inductance will decrease. If the current remains unchanged, the flux, linked to the solenoid, will also decrease. An induced e.m.f. will then come into play, which by Lenz's law will try to oppose the decrease of flux, for example, by increasing the current. In the superconducting state the flux will not change and so,

$$\frac{I}{l} = \text{constant}$$

Hence,
$$\frac{I}{l} = \frac{I_0}{l_0}, \quad \text{or,} \quad I = I_0 \frac{l}{l_0} = I_0 (1 + \eta)$$

3.325 The flux linked to the ring can not change on transition to the superconduction state, for reasons, similar to that given above. Thus a current I must be induced in the ring, where,

$$I = \frac{\Phi}{L} = \frac{\pi a^2 B}{\mu_0 a \left(\ln \frac{8a}{b} - 2\right)} = \frac{\pi a B}{\mu_0 \left(\ln \frac{8a}{b} - 2\right)}$$

3.326 We write the equation of the circuit as,

$$Ri + \frac{L}{\eta}\frac{di}{dt} = \xi,$$

for $t \geq 0$. The current at $t = 0$ just after inductance is changed, is

$i = \eta\frac{\xi}{R}$, so that the flux through the inductance is unchanged.

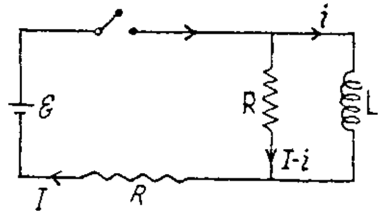
We look for a solution of the above equation in the form

$$i = A + Be^{-t/C}$$

Substituting $C = \frac{L}{\eta R}$, $B = \eta - 1, A = \frac{\xi}{R}$

Thus, $$i = \frac{\xi}{R}\left(1 + (\eta - 1)\, e^{-\eta R t/L}\right)$$

3.327 Clearly, $L\frac{di}{dt} = R\,(I - i) = \xi - RI$



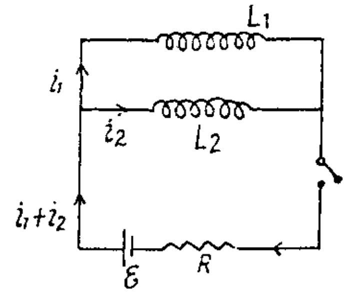
So, $2L\frac{di}{dt} = \xi - Ri$

This equation has the solution (as in 3.312)

$i = \frac{\xi}{R}(1 - e^{-tR/2L})$

3.328 The equations are,

$$L_1\frac{di_1}{dt} = L_2\frac{di_2}{dt} = \xi - R\,(i_1 + i_2)$$



Then, $\frac{d}{dt}(L_1 i_1 - L_2 i_2) = 0$

or, $L_1 i_1 - L_2 i_2 =$ constant

But initially at $t = 0$, $i_1 = i_2 = 0$

so constant must be zero and at all times,

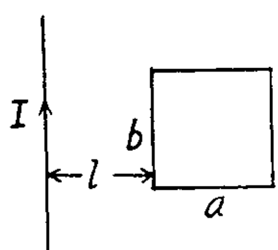
$$L_1 i_1 = L_2 i_2$$

In the final steady atate, current must obviously be $i_1 + i_2 = \frac{\xi}{R}$. Thus in steady state,

$$i_1 = \frac{\xi L_2}{R\,(L_1 + L_2)} \text{ and } i_2 = \frac{\xi L_1}{R\,(L_1 + L_2)}$$

3.329 Here, $B = \frac{\mu_0 I}{2\pi r}$ at a distance r from the wire. The flux through the frame is obtained as,

$$\Phi_{12} = \int_l^{a+l} \frac{\mu_0 I}{2\pi r}\, b\,dr = \frac{\mu_0 b}{2\pi} I \ln\left(1 + \frac{a}{l}\right)$$



Thus, $L_{12} = \frac{\Phi_{12}}{I} = \frac{\mu_0 b}{2\pi}\ln\left(1 + \frac{a}{l}\right)$

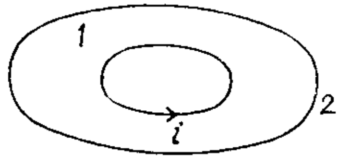
3.330 Here also, $B = \frac{\mu_0 I}{2\pi r}$ and $\Phi = \mu_0 \mu \frac{I}{2\pi} \int_a^b \frac{h\,dr}{r} N \cdot$

Thus, $$L_{12} = \frac{\mu \mu_0 h N}{2\pi} \ln \frac{b}{a}$$

3.331 The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current i flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.

Magnetic induction at the centre of the loop, : $B = \frac{\mu_0 i}{2b}$

So, flux throug loop 1, : $\Phi_{12} = \pi a^2 \frac{\mu_0 i}{2b}$



and from reciprocity theorem,

$$\varphi_{12} = \Phi_{21},\ \Phi_{21} = \frac{\mu_0 \pi a^2 i}{2b}$$

So, $L_{12} = \frac{\Phi_{21}}{i} = \frac{1}{2} \mu_0 \pi a^2 / b$

3.332 Let $\vec{p}_m$ be the magnetic moment of the magnet M. Then the magnetic field due to this magnet is,

$$\frac{\mu_0}{4\pi} \left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3} \right].$$

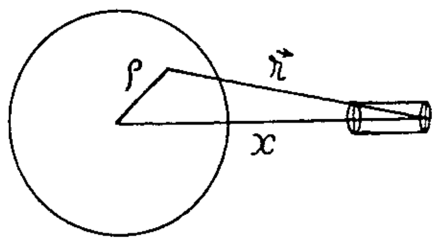
The flux associated with this, when the magnet is along the axis at a distance x from the centre, is

$$\Phi = \frac{\mu_0}{4\pi} \int \left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3} \right] \cdot d\vec{S} = \Phi_1 - \Phi_2.$$

$$\text{where,}\ \Phi_2 = \frac{\mu_0}{4\pi} p_m \int_0^a \frac{2\pi \rho\, d\rho}{(x^2+\rho^2)^{3/2}} = \frac{\mu_0 p_m}{2} \left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}} \right)$$

$$\text{and}\ \Phi_1 = \frac{3\mu_0 p_m x^2}{4\pi} \int_0^a \frac{2\pi \rho\, d\rho}{(x^2+\rho^2)^{5/2}}$$

$$= \frac{\mu_0 p_m x^2}{2} \left(\frac{1}{x^3} - \frac{1}{(x^2+a^2)^{3/2}} \right)$$



So, $$\Phi = \frac{-\mu_0 p_m a^2}{2(x^2+a^2)^{3/2}}$$

When the flux changes, an e.m.f. $-N\,\frac{d\Phi}{dt}$ is induced and a current $-\frac{N}{R}\frac{d\Phi}{dt}$ flows. The total charge q, flowing, as the magnet is removed to infinity from $x = 0$ is,

$$q = \frac{N}{R}\Phi\,(x = 0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

or,
$$p_m = \frac{2aqR}{N\mu_0}$$

3.333 If a current I flows in one of the coils, the magnetic field at the centre of the other coil is,

$$B = \frac{\mu_0 a^2 I}{2(l^2 + a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2l^3}, \text{ as } l >> a.$$

The flux associated with the second coil is then approximately $\mu_0 \pi a^4 I/2l^3$

Hence,
$$L_{12} = \frac{\mu_0 \pi a^4}{2l^3}$$

3.334 When the current in one of the loop is $I_1 = \alpha t$, an e.m.f. $L_{12}\frac{dI_1}{dt} = L_{12}\alpha$, is induced in the other loop. Then if the current in the other loop is I_2 we must have,

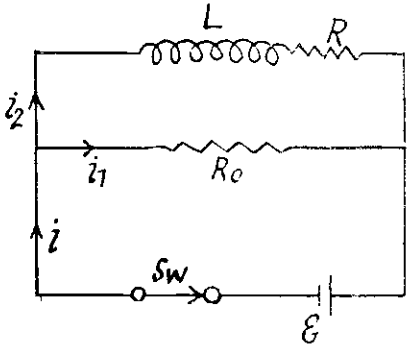
$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This familiar equation has the solution,

$I_2 = \frac{L_{12}\alpha}{R}\left(1 - e^{\frac{-tR}{L_2}}\right)$ which is the required current

3.335 Initially, after a steady current is set up, the current is flowing as shown.

In steady condition $i_{20} = \frac{\xi}{R}$, $i_{10} = \frac{\xi}{R_0}$.



When the switch is disconnected, the current through R_0 changes from i_{10} to the right, to i_{20} to the left. (The current in the inductance cannot change suddenly.). We then have the equation,

$$L\,\frac{di_2}{dt} + (R + R_0)\,i_2 = 0.$$

This equation has the solution $i_2 = i_{20}\,e^{-t(R + R_0)/L}$

The heat dissipated in the coil is,

$$Q = \int_0^\infty i_2^2 R\,dt = i_{20}^2 R\int_0^\infty e^{-2t(R + R_0)/L}\,dt$$

$$= R\,i_{20}^2 \times \frac{L}{2(R + R_0)} = \frac{L\xi^2}{2R(R + R_0)} = 3\,\mu\text{J}$$

3.336 To find the magnetic field energy we recall that the flux varies linearly with current. Thus, when the flux is Φ for current i, we can write $\phi = A\,i$. The total energy inclosed in the field, when the current is I, is

$$W = \int \xi i\,dt = \int N\,\frac{d\Phi}{dt}\,i\,dt$$

$$= \int N\,d\Phi\,i = \int_0^I N A\,i\,di = \frac{1}{2} N A I^2 = \frac{1}{2} N \Phi I$$

The characteristic factor $\frac{1}{2}$ appears in this way.

3.337 We apply circulation theorem,

$$H \cdot 2\pi b = N I, \quad \text{or,} \quad H = N I / 2\pi b.$$

Thus the total energy,

$$W = \frac{1}{2} B H \cdot 2\pi b \cdot \pi a^2 = \pi^2 a^2\, b\, B H.$$

Given N, I, b we know H, and can find out B from the $B-H$ curve. Then W can be calculated.

3.338 From $\oint \vec{H} \cdot d\vec{r} = N I$,

$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b = N I, (d >> b)$$

Also, $$B = \mu\,\mu_0 H. \text{ Thus, } H = \frac{N I}{\pi d + \mu b}.$$

Since B is continuous across the gap, B is given by,

$B = \mu\,\mu_0 \dfrac{N I}{\pi d + \mu b}$, both in the magnetic and the gap.

(a) $$\frac{W_{gap}}{W_{magnetic}} = \frac{\dfrac{B^2}{2\mu_0} \times S \times b}{\dfrac{B^2}{2\mu\mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d}.$$

(b) The flux is $N \int \vec{B} \cdot d\vec{S} = N \mu\mu_0 \dfrac{N I}{\pi d + \mu b} \cdot S = \mu_0 \dfrac{S N^2 I}{b + \dfrac{\pi d}{\mu}}$,

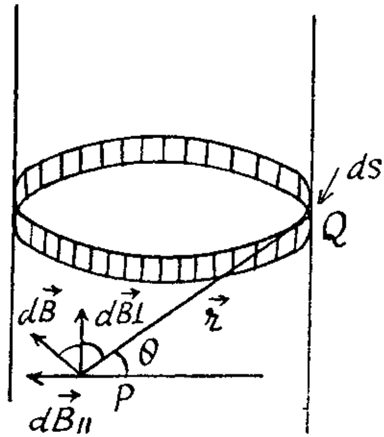
So, $$L = \frac{\mu_0 S N^2}{b + \dfrac{\pi d}{\mu}}.$$

Energy wise; total energy

$$= \frac{B^2}{2\mu_0}\left(\frac{\pi d}{\mu} + b\right) S = \frac{1}{2} \frac{\mu_0 N^2 S}{b + \dfrac{\pi d}{\mu}} \cdot I^2 = \frac{1}{2} L I^2$$

The L, found in the one way, agrees with that, found in the other way. Note that, in calculating the flux, we do not consider the field in the gap, since it is not linked to the winding. But the total energy includes that of the gap.

3.339 When the cylinder with a linear charge density λ rotates with a circular frequency ω, a surface current density (charge / length × time) of $i = \frac{\lambda\omega}{2\pi}$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i(\vec{e}\times\vec{r})}{r^3}dS$ where $\vec{e}$ is the direction of the current. In magnitude, $|\vec{e}\times\vec{r}| = r$, since $\vec{e}$ is normal to $\vec{r}$ and the direction of $d\vec{B}$ is as shown.

It's component, $d\vec{B}_{\parallel}$ cancels out by cylindrical symmetry. The component that survives is,

$$|\vec{B}_{\perp}| = \frac{\mu_0}{4\pi}\int\frac{i\,dS}{r^2}\cos\theta = \frac{\mu_0 i}{4\pi}\int d\Omega = \mu_0 i,$$

where we have used $\frac{dS\cos\theta}{r^2} = d\Omega$ and $\int d\Omega = 4\pi$, the total solid angle around any point.

The magnetic field vanishes outside the cylinder by similar argument.

The total energy per unit length of the cylinder is,

$$W_1 = \frac{1}{2\mu_0}\mu_0^2\left(\frac{\lambda\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi}a^2\lambda^2\omega^2$$

3.340 $w_E = \frac{1}{2}\varepsilon_0 E^2$, for the electric field,

$w_B = \frac{1}{2\mu_0}B^2$ for the magnetic field.

Thus,
$$\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2,$$

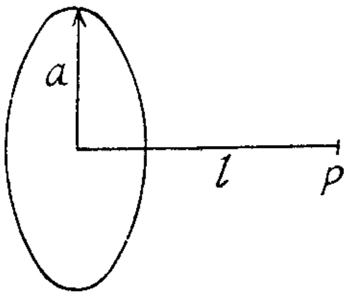
when
$$E = \frac{B}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8 \text{ V/m}$$

3.341 The electric field at P is,

$$E_p = \frac{ql}{4\pi\varepsilon_0(a^2+l^2)^{3/2}}$$

To get the magnetic field, note that the rotating ring constitutes a current $i = q\,\omega/2\,\pi$, and the corresponding magnetic field at P is,

$$B_p = \frac{\mu_0\, a^2\, i}{2\,(a^2 + l^2)^{3/2}}\,.$$



Thus, $$\frac{w_E}{w_M} = \frac{\varepsilon_0\,\mu_0\,E^2}{B^2} = \varepsilon_0\,\mu_0 \left(\frac{q l \times 2}{4\,\pi\,\varepsilon_0\,\mu_0\,a^2 i}\right)^2$$

$$= \frac{1}{\varepsilon_0\,\mu_0}\left(\frac{l}{a^2\,\omega}\right)^2$$

or, $$\frac{w_M}{w_E} = \varepsilon_0\,\mu_0\,\omega^2\,a^4/l^2$$

3.342 The total energy of the magnetic field is,

$$\frac{1}{2}\int (\vec{B}\cdot\vec{H})\,dV = \frac{1}{2}\int \vec{B}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{J}\right) dV$$

$$= \frac{1}{2\,\mu_0}\int \vec{B}\cdot\vec{B}\,dV - \frac{1}{2}\int \vec{J}\cdot\vec{B}\,dV.$$

The second term can be interpreted as the energy of magnetization, and has the density

$$-\frac{1}{2}\vec{J}\cdot\vec{B}.$$

3.343 (a) In series, the current I flows through both coils, and the total e.m.f. induced. when the current changes is,

$$-2L\,\frac{dI}{dt} = -L'\,\frac{dI}{dt}$$

or, $$L' = 2L$$

(b) In parallel, the current flowing through either coil is, $\frac{I}{2}$ and the e.m.f. induced is

$$-L\left(\frac{1}{2}\,\frac{dI}{dt}\right).$$

Equating this to $-L'\dfrac{dI}{dt}$, we find $L' = \dfrac{1}{2}L$.

3.344 We use $L_1 = \mu_0\,n_1^{\,2}\,V, L_2 = \mu_0\,n_2^{\,2}\,V$

So, $$L_{12} = \mu_0\,n_1\,n_2\,V = \sqrt{L_1\,L_2}$$

3.345 The interaction energy is

$$\frac{1}{2\,\mu_0}\int \left|\vec{B}_1 + \vec{B}_2\right|^2 dV - \frac{1}{2\,\mu_0}\int \left|\vec{B}_1\right|^2 dV - \frac{1}{2\,\mu_0}\int \left|\vec{B}_2\right|^2 dV$$

$$= \frac{1}{\mu_0}\int \vec{B}_1\cdot\vec{B}_2\,dV$$

Here, if $\vec{B}_1$ is the magnetic field produced by the first of the current carrying loops, and $\vec{B}_2$, that of the second one, then the magnetic field due to both the loops will be $\vec{B}_1 + \vec{B}_2$.

3.346 We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and makes an angle θ with the direction of the magnetic field, due to the bigger loop. This magnetic field is,

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude,

$$|W| = \frac{\mu_0 I_1 I_2}{2b} \pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

3.347 (a) There is a radial outward conduction current. Let Q be the instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \quad \text{or,} \quad \vec{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}.$$

On the other hand $\vec{j}_d = \frac{\partial \vec{D}}{\partial t} = \frac{d}{dt}\left(\frac{Q}{4\pi r^2}\hat{r}\right) = -\vec{j}$

(b) At the given moment, $\vec{E} = \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\hat{r}$

and by Ohm's law, $\vec{j} = \frac{\vec{E}}{\rho} = \frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

Then, $$\vec{j}_d = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$$

and $$\oint \vec{j}_d \cdot d\vec{S} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho}\int \frac{dS\cos\theta}{r^2} = -\frac{q}{\varepsilon_0\varepsilon\rho}.$$

The surface integral must be $-ve$ because $\vec{j}_d$, being opposite of $\vec{j}$, is inward.

3.348 Here also we see that neglecting edge effects, $\vec{j}_d = -\vec{j}$. Thus Maxwell's equations reduce to, div $\vec{B} = 0$, Curl $\vec{H} = 0$, $\vec{B} = \mu\vec{H}$

A general solution of this equation is $\vec{B} =$ constant $= \vec{B}_0 \cdot \vec{B}_0$ can be thought of as an extraneous magnetic field. If it is zero, $\vec{B} = 0$.

3.349 Given $I = I_m \sin\omega t$. We see that

$$j = \frac{I_m}{S}\sin\omega t = -j_d = -\frac{\partial D}{\partial t}$$

or, $D = \frac{I_m}{\omega S}\cos\omega t$, so, $E_m = \frac{I_m}{\varepsilon_0 \omega S}$ is the amplitude of the electric field and is 7 V/cm

3.350 The electric field between the plates can be written as,

$$E = Re\,\frac{V_m}{d}\,e^{i\omega t}, \text{ instead of } \frac{V_m}{d}\cos\omega t.$$

This gives rise to a conduction current,

$$j_c = \sigma E = \text{Re}\,\frac{\sigma}{d}\,V_m\,e^{i\omega t}$$

and a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \text{Re}\,\varepsilon_0\,\varepsilon\,i\,\omega\,\frac{V_m}{d}\,e^{i\omega t}$$

The total current is,

$$j_T = \frac{V_m}{d}\sqrt{\sigma^2 + (\varepsilon_0\,\varepsilon\,\omega)^2}\,\cos(\omega t + \alpha)$$

where, $\tan\alpha = \dfrac{\sigma}{\varepsilon_0\,\varepsilon\,\omega}$ on taking the real part of the resultant.

The corresponding magnetic field is obtained by using circulation theorem,

$$H\cdot 2\pi r = \pi r^2 j_T$$

or, $H = H_m\cos(\omega t + \alpha)$, where, $H_m = \dfrac{r V_m}{2d}\sqrt{\sigma^2 + (\varepsilon_0\,\varepsilon\;\omega)^2}$

3.351 Inside the solenoid, there is a magnetic field,

$$B = \mu_0\,n\,I_m\sin\omega t.$$

Since this varies in time there is an associated electric field. This is obtained by using,

$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$

For $r < R,\ 2\pi r E = -\dot{B}\cdot\pi r^2$, or, $E = -\dfrac{\dot{B}\,r}{2}$

For $r > R,\ E = -\dfrac{\dot{B}\,R^2}{2r}$

The associated displacement current density is,

$$j_d = \varepsilon_0\frac{\partial E}{\partial t} = \begin{cases} -\varepsilon_0\,\ddot{B}\;r/2 \\ -\varepsilon_0\,\ddot{B}\;R^2/2\,r \end{cases}$$

The answer, given in the book, is dimensionally incorrect without the factor ε_0.

3.352 In the non-relativistic limit.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

(a) On a straight line coinciding with the charge path,

$$\vec{j_d} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{q}{4\pi}\left[\frac{-\vec{V}}{r^3} - \frac{3\,\vec{r}\,\dot{r}}{r^4}\right], \left(\text{using, } \frac{d\vec{r}}{dt} = -\vec{v}.\right)$$

But in this case, $\dot{r} = -v$ and $v\frac{\vec{r}}{r} = \vec{v}$, so, $j_d = \frac{2q\vec{v}}{4\pi r^3}$

(b) In this case, $\dot{r} = 0$, as, $\vec{r} \perp \vec{v}$. Thus,

$$j_d = -\frac{q\vec{v}}{4\pi r^3}$$

3.353 We have, $E_p = \frac{qx}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$

then $$j_d = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{qv}{4\pi(a^2 + x^2)^{5/2}}(a^2 - 2x^2)$$

This is maximum, when $x = x_m = 0$, and minimum at some other value. The maximum displacement current density is

$$(j_d)_{\max} = \frac{qv}{4\pi a^3}$$

To check this we calculate $\frac{\partial j_d}{\partial x}$;

$$\frac{\partial j_d}{\partial x} = \frac{qv}{4\pi}[(-4x(a^2 + x^2) - 5x(a^2 - 2x^2)]$$

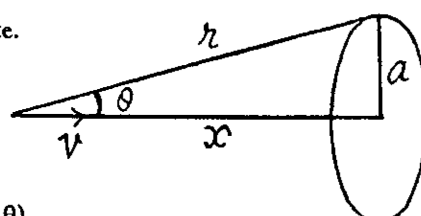
This vanishes for $x = 0$ and for $x = \sqrt{\frac{3}{2}}\,a$. The latter is easily shown to be a smaller local minimum (negative maximum).

3.354 We use Maxwell's equations in the form,

$$\oint \vec{B} \cdot d\vec{r} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S},$$

when the conduction current vanishes at the site.

We know that,



$$\int \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\int \frac{d\vec{S} \cdot \hat{r}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{4\pi\varepsilon_0} 2\pi(1 - \cos\theta),$$

where, $2\pi(1 - \cos\theta)$ is the solid angle, formed by the disc like surface, at the charge.

Thus, $$\oint \vec{B} \cdot d\vec{r} = 2\pi a B = \frac{1}{2}\mu_0 q \cdot \sin\theta \cdot \dot{\theta}$$

On the other hand, $x = a\cot\theta$

differentiating and using $\frac{dx}{dt} = -v$,

$$v = a\,\mathrm{cosec}^2\,\theta\,\dot{\theta}$$

Thus, $$B = \frac{\mu_0 q v r \sin\theta}{4\pi r^3}$$

This can be written as, $\vec{B} = \dfrac{\mu_0 q (\vec{v} \times \vec{r})}{4 \pi r^3}$

and $\vec{H} = \dfrac{q}{4\pi} \dfrac{\vec{v} \times \vec{r}}{r^3}$ (The sense has to be checked independently.)

3.355 (a) If $\vec{B} = \vec{B}(t)$, then,

Curl $\vec{E} = \dfrac{-\partial \vec{B}}{\partial t} \neq 0.$

So, $\vec{E}$ cannot vanish.

(b) Here also, curl $\vec{E} \neq 0$, so $\vec{E}$ cannot be uniform.

(c) Suppose for instance, $\vec{E} = \vec{a} f(t)$

where $\vec{a}$ is spatially and temporally fixed vector. Then $-\dfrac{\partial \vec{B}}{dt} =$ curl $\vec{E} = 0$. Generally speaking this contradicts the other equation curl $\vec{H} = \dfrac{\partial \vec{D}}{\partial t} \neq 0$ for in this case the left hand side is time independent but RHS. depends on time. The only exception is when $f(t)$ is linear function. Then a uniform field $\vec{E}$ can be time dependent.

3.356 From the equation Curl $\vec{H} - \dfrac{\partial \vec{D}}{\partial t} = \vec{j}$

We get on taking divergence of both sides

$$-\frac{\partial}{\partial t} \text{div } \vec{D} = \text{div } \vec{j}$$

But div $\vec{D} = \rho$ and hence div $\vec{j} + \dfrac{\partial \rho}{\partial t} = 0$

3.357 From $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

we get on taking divergence

$$0 = -\frac{\partial}{\partial t} \text{div } \vec{B}$$

This is compatible with div $\vec{B} = 0$

3.358 A rotating magnetic field can be represented by,

$B_x = B_0 \cos \omega t$; $B_y = B_0 \sin \omega t$ and $B_z = B_{zo}$

Then curl, $\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$.

So, $-(\text{Curl } \vec{E})_x = -\omega B_0 \sin \omega t = -\omega B_y$

$-(\text{Curl } \vec{E})_y = \omega B_0 \cos \omega t = \omega B_x$ and $-(\text{Curl } \vec{E})_z = 0$

Hence, Curl $\vec{E} = -\vec{\omega} \times \vec{B}$,

where, $\vec{\omega} = \vec{e}_3 \omega$.

3.359 Consider a particle with charge e, moving with velocity $\vec{v}$, in frame K. It experiences a force $\vec{F} = e\vec{v} \times \vec{B}$

In the frame K', moving with velocity $\vec{v}$, relative to K, the particle is at rest. This means that there must be an electric field $\vec{E}$ in K', so that the particle experinces a force,

$$\vec{F}' = e\vec{E}' = \vec{F} = e\,\vec{v} \times \vec{B}$$

Thus,
$$\vec{E}' = \vec{v} \times \vec{B}$$

3.360 Within the plate, there will appear a $(\vec{v} \times \vec{B})$ force, which will cause charges inside the plate to drift, until a countervailing electric field is set up. This electric field is related to B, by $E = e\,B$, since v & B are mutually perpendicular, and E is perpendicular to both. The charge density $\pm\,\sigma$, on the force of the plate, producing this electric field, is given by

$$E = \frac{\sigma}{\varepsilon_0} \text{ or } \sigma = \varepsilon_0\, v\, B = 0{\cdot}40\text{ pC/m}^2$$

3.361 Choose $\vec{\omega} \uparrow\uparrow \vec{B}$ along the z–axis, and choose $\vec{r}$, as the cylindrical polar radius vector of a reference point (perpendicular distance from the axis). This point has the velocity,

$$\vec{v} = \vec{\omega} \times \vec{r},$$

and experiences a $(\vec{v} \times \vec{B})$ force, which must be counterbalanced by an electric field,

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -(\vec{\omega} \cdot \vec{B})\,\vec{r}.$$

There must appear a space charge density,

$$\rho = \varepsilon_0 \operatorname{div} \vec{E} = -2\,\varepsilon_0\,\vec{\omega} \cdot \vec{B} = -8\text{ pC/m}^3$$

Since the cylinder, as a whole is electrically neutral, the surface of the cylinder must acquire a positive charge of surface density,

$$\sigma = +\frac{2\,\varepsilon_0\,(\vec{\omega} \cdot \vec{B})\,\pi\,a^2}{2\,\pi\,a} = \varepsilon_0\,a\,\vec{\omega} \cdot \vec{B} = +{\cdot}2\text{ pC/m}^2$$

3.362 In the reference frame K', moving with the particle,

$$\vec{E}' = \vec{E} + \vec{v}_0 \times \vec{B} = \frac{q\,\vec{r}}{4\,\pi\,\varepsilon_0\,r^3}$$

$$\vec{B}' = \vec{B} - \vec{v}_0 \times \vec{E}/c^2 = 0.$$

Here, $\vec{v}_0$ = velocity of K', relative to the K frame, in which the particle has velocity $\vec{v}$.

Clearly, $\vec{v}_0 = \vec{v}$. From the second equation,

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2} = \varepsilon_0\,\mu_0 \times \frac{q}{4\,\pi\,\varepsilon_0}\,\frac{\vec{v} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi}\,\frac{q\,(\vec{v} \times \vec{r})}{r^3}$$

3.363 Suppose, there is only electric field $\vec{E}$, in K. Then in K', considering nonrelativistic velocity $\vec{v}$, $\vec{E'} = \vec{E}$, $\vec{B} = -\frac{\vec{v} \times \vec{E}}{c^2}$,

So, $$\vec{E'} \cdot \vec{B'} = 0$$

In the relativistic case,

$$\left.\begin{array}{l} \vec{E'}_{\parallel} = \vec{E}_{\parallel} \\ E'_{\perp} = \dfrac{\vec{E}\perp}{\sqrt{1 - v^2/c^2}} \end{array}\right\} \begin{array}{l} \vec{B'}_{\parallel} = \vec{B}_{\parallel} = 0 \\ \vec{B'}_{\perp} = \dfrac{-\vec{v} \times \vec{E}/c^2}{\sqrt{1 - v^2/c^2}} \end{array}$$

Now, $$\vec{E'} \cdot \vec{B'} = \vec{E'}_{\parallel} \cdot \vec{B'}_{\parallel} + \vec{E'}_{\perp} \cdot \vec{B'}_{\perp} = 0, \text{ since}$$

$$E'_{\perp} \cdot B'_{\perp} = -\vec{E}_{\perp} \cdot (\vec{v} \times \vec{E})/(1 - v^2/c^2) = -\vec{E}_{\perp} \cdot (\vec{v} \times \vec{E}_{\perp}) \Big/ \left(1 - \frac{v^2}{c^2}\right) = 0$$

3.364 In K, $\vec{B} = b\,\dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2}$, $b =$ constant.

In $$K',\ \vec{E'} = \vec{v} \times \vec{B} = bv\,\frac{y\hat{j} - x\hat{i}}{x^2 + y^2} = bv\,\frac{\vec{r}}{r^2}$$

The electric field is radial ($\vec{r} = x\hat{i} + y\hat{j}$).

3.365 In K, $\vec{E} = a\,\dfrac{\vec{r}}{r^2}$, $\vec{r} = (x\hat{i} + y\hat{j}\ )$

In $$K', \vec{B'} = -\frac{\vec{v} \times \vec{E}}{c^2} = \frac{a\,\vec{r} \times \vec{v}}{c^2 r^2}$$

The magnetic lines are circular.

3.366 In the non relativistic limit, we neglect v^2/c^2 and write,

$$\left.\begin{array}{l} \vec{E'}_{\parallel} = \vec{E}_{\parallel} \\ \vec{E}_{\perp} = \vec{E}_{\perp} + \vec{v} \times \vec{B} \end{array}\right\} \begin{array}{l} \vec{B'}_{\parallel} = \vec{B}_{\parallel} \\ \vec{B'}_{\perp} = \vec{B}_{\perp} - \vec{v} \times \vec{E}/c^2 \end{array}$$

These two equations can be combined to give,

$$\vec{E'} = \vec{E} + \vec{v} \times \vec{B},\ \vec{B'} = \vec{B} - \vec{v} \times \vec{E}/c^2$$

3.367 Choose $\vec{E}$ in the direction of the z–axis, $\vec{E} = (0, 0, E)$. The frame K' is moving with velocity $\vec{v} = (v \sin\alpha, 0, v\cos\alpha)$, in the $x - z$ plane. Then in the frame K',

$$\vec{E'}_{\parallel} = \vec{E}_{\parallel}\ B'_{\parallel} = 0$$

$$\vec{E'}_{\perp} = \frac{\vec{E}_{\perp}}{\sqrt{1 - v^2/c^2}}\ \vec{B'}_{\perp} = \frac{-\vec{v} \times \vec{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

The vector along $\vec{v}$ is $\vec{e} = (\sin\alpha, 0, \cos\alpha)$ and the perpendicular vector in the $x - z$ plane is,

$$\vec{f} = (-\cos\alpha, 0, \sin\alpha),$$

(a) Thus using $\vec{E} = E\cos\alpha\,\vec{e} + E\sin\alpha\,\vec{f}$,

$$E'_{\parallel} = E\cos\alpha \text{ and } E'_{\parallel} = \frac{E\sin\alpha}{\sqrt{1-v^2/c^2}},$$

So
$$E' = E\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \text{ and } \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-v^2/c^2}}$$

(b) $B'_{\parallel} = 0,\ \vec{B'}_{\perp} = \dfrac{\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}$

$$B' = \frac{\beta E\sin\alpha}{c\sqrt{1-\beta^2}}$$

3.368 Choose $\vec{B}$ in the z direction, and the velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$ in the $x-z$ plane, then in the K' frame,

$$\vec{E'}_{\parallel} = \vec{E}_{\parallel} = 0 \qquad \vec{B'}_{\parallel} = \vec{B}_{\parallel}$$
$$\vec{E'}_{\perp} = \frac{\vec{v}\times\vec{B}}{\sqrt{1-v^2/c^2}} \qquad \vec{B'}_{\perp} = \frac{\vec{B}_{\perp}}{\sqrt{1-v^2/c^2}}$$

We find similarly, $E' = \dfrac{c\,\beta\,B\sin\alpha}{\sqrt{1-\beta^2}}$

$$B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \quad \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$$

3.369 (a) We see that, $\vec{E'}\cdot\vec{B'} = \vec{E'}_{\parallel}\cdot\vec{B'}_{\parallel} + \vec{E'}_{\perp}\cdot\vec{B'}_{\perp}$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{(\vec{E}_{\perp}+\vec{v}\times\vec{B})\cdot\left(\vec{B}_{\perp} - \dfrac{\vec{v}\times\vec{E}}{c^2}\right)}{1-\dfrac{v^2}{c^2}}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B})\cdot(\vec{v}\times\vec{E})/c^2}{1-v^2/c^2}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B}_{\perp})\cdot(\vec{v}\times\vec{E}_{\perp})/c^2}{1-\dfrac{v^2}{c^2}}$$

But,
$$\vec{A}\times\vec{B}\cdot\vec{C}\times\vec{D} = A\cdot C\,B\cdot D - A\cdot D\,B\cdot C,$$

so,
$$\vec{E'}\cdot\vec{B'} = \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \vec{E}_{\perp}\cdot\vec{B}_{\perp}\frac{\left(1-\dfrac{v^2}{c^2}\right)}{1-\dfrac{v^2}{c^2}} = \vec{E}\cdot\vec{B}$$

(b) $E'^2 - c^2B'^2 = E'^2_{\parallel} - c^2B'^2_{\parallel} + E'^2_{\perp} - c^2B'^2_{\perp}$

$$= E_\parallel^2 - c^2 B_\parallel^2 + \frac{1}{1-\frac{v^2}{c^2}}\left[(\vec{E}_\perp + \vec{v}\times\vec{B})^2 - c^2\left(\vec{B}_\perp - \frac{\vec{v}\times\vec{E}}{c^2}\right)^2\right]$$

$$= E_\parallel^2 - c^2 B_\parallel^2 + \frac{1}{1-\frac{v^2}{c^2}}\left[E_\perp^2 - c^2 B_\perp^2 + (\vec{v}\times\vec{B}_\perp)^2 - \frac{1}{c^2}(\vec{v}\times E_1)^2\right]$$

$$= E_\parallel^2 - c^2 B_\parallel^2 + \frac{1}{1-\frac{v^2}{c^2}}[E_\perp^2 - c^2 B_\perp^2]\left(1-\frac{v^2}{c^2}\right) = E^2 - c^2 B^2,$$

since, $(\vec{v}\times\vec{A}_\perp)^2 = v^2 A_\perp^2$

3.370 In this case, $\vec{E}\cdot\vec{B} = 0$, as the fields are mutually perpendicular. Also,

$$E^2 - c^2 B^2 = -20\times 10^8 \left(\frac{\text{V}}{\text{m}}\right)^2 \text{ is } -ve.$$

Thus, we can find a frame, in which $E' = 0$, and

$$B' = \frac{1}{c}\sqrt{c^2B^2 - E^2} = B\sqrt{1-\frac{E^2}{c^2B^2}} = 0{\cdot}20\sqrt{1-\left(\frac{4\times 10^4}{3\times 10^8\times 2\times 10^{-4}}\right)^2} = 0{\cdot}15 \text{ mT}$$

3.371 Suppose the charge q moves in the positive direction of the x–axis of the frame K. Let us go over to the moving frame K', at whose origin the charge is at rest. We take the x and x' axes of the two frames to be coincident, and the y & y' axes, to be parallel.

In the K' frame, $\vec{E}' = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q\vec{r}'}{r'^3}$,

and this has the following components,

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{qx'}{r'^3},\ E_y' = \frac{1}{4\pi\varepsilon_0}\frac{qy'}{r'^3}.$$

Now let us go back to the frame K. At the moment, when the origins of the two frames coincide, we take $t = 0$. Then,

$$x = r\cos\theta = x'\sqrt{1-\frac{v^2}{c^2}},\ y = r\sin\theta = y'$$

Also, $$E_x = E_x',\ E_y = E_y'\Big/\sqrt{1-v^2/c^2}$$

From these equations, $r'^2 = \dfrac{r^2\,(1-\beta^2\sin^2\theta)}{1-\beta^2}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3(1-\beta^2\sin^2\theta)^{3/2}}\left[(1-\beta^2)^{3/2}\left(x'\,\hat{i} + \frac{y'}{\sqrt{1-\beta^2}}\hat{j}\right)\right]$$

$$= \frac{q\vec{r}(1-\beta^2)}{4\pi\varepsilon_0 r^3(1-\beta^2\sin^2\theta)^{3/2}}$$