Chapter – 2

Basic Algebra

Ex 2.1

Question 1.

Classify each element of $\left\{\sqrt{7}, \frac{-1}{4}, 0, 3.14, 4, \frac{22}{7}\right\}$ as a member of ....

$\mathbb{N}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{Z}$

Solution:

$\frac{-1}{4}$ is a rational number (*i.e.*) $\frac{-1}{4} \in Q$,

3.14 ∈ Q

0, 4 are integers and 0 ∈ Z, 4 ∈ N, Z, Q

$\frac{22}{7} \in \mathbb{Q}$

Question 2.

Prove that √3 is an irrational number.

(Hint: Follow the method that we have used to prove √2 ∉ Q.

Solution:

Suppose that √3 is rational P

$$\text{Then} \quad \sqrt{3} = \frac{p}{q} \text{ (where } p \text{ and } q \text{ are integers which are co-prime)}$$

$$\Rightarrow \quad p = \sqrt{3}\, q \Rightarrow p^2 = 3q^2 \quad \ldots .(1)$$

$$\frac{p^2}{3} = q^2 \Rightarrow 3 \text{ is a factor of } p$$

$$\text{So let } p = 3c$$

$$\text{substituting } p = 3c \text{ in (1) we get}$$

$$(3c)^2 = 3q^2 \Rightarrow 9c^2 = 3q^2$$

$$\Rightarrow c^2 = \frac{3q^2}{9} = \frac{q^2}{3}$$

⇒ 3 is a factor of q also
so 3 is a factor of p and q which is a contradiction.
⇒ $\sqrt{3}$ is not a rational number
⇒ $\sqrt{3}$ is an irrational number

Question 3.
Are there two distinct irrational numbers such that their difference is a rational number? Justify.

Solution:
Let the two irrational numbers be $2 + \sqrt{5}$ and $4 + \sqrt{5}$
Their difference = $(2 + \sqrt{5}) - (4 + \sqrt{5})$
$= 2 + \sqrt{5} - 4 - \sqrt{5}$
$= 2 - 4 = -2$
which is a rational number.

Question 4.
Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number?

Solution:
Let the two irrational numbers be $3 + \sqrt{5}$ and $7 - \sqrt{5}$
Their sum = $3 + \sqrt{5} + 7 - \sqrt{5} = 3 + 7 = 10$
which is a rational number
Consider the two irrational numbers $2 + \sqrt{3}$, $2 - \sqrt{3}$
Their product = $(2 + \sqrt{3})(2 - \sqrt{3})$
$= 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$
which is a rational number.

Question 5.
Find a positive number smaller than $\frac{1}{2^{1000}}$. Justify.

Solution:

We know $\frac{1}{2} = 0.5$

$$\frac{1}{2^2} = (0.5)^2 = 0.25$$

$$\frac{1}{2^3} = (0.5)^3 = 0.125$$

$$\frac{1}{2^4} = (0.5)^4 = 0.0625$$

$$\vdots$$

$$\frac{1}{2^{1000}} = (0.5)^{1000} \simeq 0$$

There will not be a positive number smaller than 0.

So there will not be a +ve number smaller than $\frac{1}{2^{1000}}$.

Ex 2.2

Question 1.
Solve for x.

(i) |3 – x| < 7

Solution:
$-7 < 3 - x < 7$
$-7 - 3 < -x < 7 - 3$
$-10 < -x < 4$ $10 > x > -4$
$-4 < x < 10$
∴ The solution set is $x \in (-4, 10)$

(ii) |4x – 5| ≥ -2

Solution:

$$4x - 5 \le -2 \text{ or } 4x - 5 \ge -2$$

$$4x \le 2 + 5 \,(= 7) \qquad \Big| \qquad 4x \ge -2 + 5 \,(= 3)$$

$$\Rightarrow \quad x \le \frac{7}{4} \quad \ldots.(1) \qquad \Big| \qquad \Rightarrow x \ge \frac{3}{4} \qquad \ldots.(2)$$

$$\text{From (1) and (2)} \Rightarrow \frac{3}{4} \le x \le \frac{7}{4}$$

(iii)

$$\left|3 - \frac{3}{4}x\right| \le \frac{1}{4}$$

Solution:

$$\frac{-1}{4} \le 3 - \frac{3}{4}x \le \frac{1}{4}$$

$$-\frac{1}{4} - 3 \le -\frac{3}{4}x \qquad \Big| \qquad -\frac{3}{4} \le \frac{1}{4}$$

$$-\frac{13}{4} \le \frac{-3}{4}x \qquad \Big| \qquad -\frac{3}{4}x \le \frac{1}{4} - 3\left(= -\frac{11}{4}\right)$$

$$\Rightarrow \quad 3x \le 13 \qquad \Big| \qquad -\frac{3x}{4} \le \frac{-11}{4} \Rightarrow 3x \ge 11$$

$$\Rightarrow \quad x \le \frac{13}{3} \quad \ldots.(1) \qquad \Big| \qquad \Rightarrow x \ge \frac{11}{3} \quad \ldots.(2)$$

$$\text{from (1) and (2)} \Rightarrow \frac{11}{3} \le x \le \frac{13}{3}$$

(iv) |x| – 10 < -3

Solution:

|x| < -3 + 10 (= 7)
|x| < 7 ⇒ -7 < x < 7

Question 2.

Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.

Solution:

$$\frac{1}{|2x-1|} < 6$$

$$\Rightarrow \quad |2x-1| > \frac{1}{6}$$

$$\Rightarrow \quad 2x-1 > -\frac{1}{6} \qquad \text{(or)} \quad 2x-1 < \frac{1}{6}$$

$$2x > -\frac{1}{6} + 1 \left(= \frac{5}{6}\right) \qquad 2x < \frac{1}{6} + 1 \left(= \frac{7}{6}\right)$$

$$\Rightarrow \quad x > \frac{5}{12} \quad \ldots.(1) \qquad \Rightarrow x < \frac{7}{12} \quad \ldots.(2)$$

$$\text{From (1) and (2)} \Rightarrow \frac{5}{12} < x < \frac{7}{12}$$

$$\text{(i.e.)} \ x \in \left(-\infty, \frac{5}{12}\right) \cup \left(\frac{7}{12}, \infty\right)$$

Question 3.

Solve $-3|x| + 5 \leq -2$ and graph the solution set in a number line.

Solution:

$-3\,|x| + 5 \leq -2$

$-3\,|x| \leq -2 - 5$

$-3\,|x| \leq -7$

$3\,|x| \geq 7$

$|x| \geq 7/3$

$$\Rightarrow |x| \geq \frac{7}{3}$$

$$\Rightarrow \quad x \leq \frac{-7}{3} \text{ and } x \geq \frac{7}{3}$$

$$(i.e.) \ \frac{-7}{3} \leq x \leq \frac{7}{3} \quad (i.e.) \ x \in \left(-\infty, \frac{-7}{3}\right] \cup \left[\frac{7}{3}, \infty\right]$$

Question 4.
Solve 2|x + 1| – 6 ≤ 7 and graph the solution set in a number line.

Solution:

$$2|x+1|-6 \le 7$$
$$\Rightarrow \quad 2|x+1| \le 7+6\ (=13)$$
$$\Rightarrow \quad |x+1| \le \frac{13}{2}$$
$$\Rightarrow \quad x+1 > \frac{-13}{2} \quad \text{(or)} \quad x+1 < \frac{13}{2}$$

$$x+1 > \frac{-13}{2}$$
$$\Rightarrow \quad x > \frac{-13}{2} - 1\ \left(= \frac{-15}{2}\right) \quad \ldots.(1)$$

$$x+1 < \frac{13}{2}$$
$$\Rightarrow x < \frac{13}{2} - 1\ \left(= \frac{11}{2}\right) \quad \ldots.(2)$$

From (1) and (2) $\frac{-15}{2} \le x \le \frac{11}{2}$

Question 5.
Solve $\frac{1}{5}|10x-2| < 1$.

Solution:

$$\frac{1}{5}|10x-2| < 1 \quad \Rightarrow \quad |10x-2| < 1 \times 5\ (=5)$$
$$\Rightarrow \quad 10x-2 > -5 \quad \text{(or)} \quad 10x-2 < 5$$

$$10x > -5+2\ (=-3)$$
$$\Rightarrow \quad x > \frac{-3}{10}$$

$$10x-2 < 5$$
$$\Rightarrow 10x < 5+2\ (=7)$$
$$\Rightarrow \quad x < \frac{7}{10}$$

Combining we get $\frac{-3}{10} < x < \frac{7}{10}$

Question 6.
Solve |5x – 12| < -2

Solution:

$$5x - 12 > -2 \quad \text{(or)} \quad 5x - 12 < 2$$

$$\Rightarrow \quad 5x > -2 + 12\ (= 10) \qquad \Big| \qquad 5x < 2 + 12\ (= 14)$$

$$\Rightarrow \quad x > \frac{10}{5}\ (= 2) \qquad \Big| \qquad \Rightarrow x < \frac{14}{5}$$

$$x > 2$$

$$\text{so } 2 < x < \frac{14}{5}$$

Ex 2.3

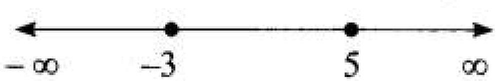
Question 1.
Represent the following inequalities in the interval notation:

Solution:

⇒ x ∈ [-1, 4)
[] closed interval, end points are included
() → open interval
end points are excluded

(ii) x ≤ 5 and x ≥ -3[i] x ≤ 5 and x ≥ -3

Solution:



x ∈ [-3, 5)

(iii) $x < -1$ or $x < 3$

Solution:

$x \in (-\infty, -1)$ or $x \in (-\infty, 3)$

(iv) $-2x > 0$ or $3x - 4 < 11$

Solution:
$-2x > 0 \Rightarrow 2x < 0 \Rightarrow x < 0$
$x \in (-\infty, 0)$
$3x - 4 < 11$
$\Rightarrow 3x - 4 + 4 < 11 + 4$
$3x < 15 \Rightarrow \frac{3x}{3} < \frac{15}{3}$
$(i.e.) \quad x < 5$
$x \in (-\infty, 5)$

Question 2.
Solve $23x < 100$ when
(i) x is a natural number,
(ii) x is an integer.

Solution:

$23x < 100$
$\Rightarrow \quad \frac{23x}{23} < \frac{100}{23}$
(i.e.,) $x > 4.3$
(i) $x = 1, 2, 3, 4$ ($x \in N$)
(ii) $x = \ldots. -3, -2, -1, 0, 1, 2, 3, 4$ ($x \in Z$)

Question 3.
Solve $-2x \geq 9$ when
(i) x is a real number,
(ii) x is an integer,

(iii) x is a natural number.

Solution:

-2x > 9 ⇒ 2x ≤ -9

$$\frac{2x}{2} \leq -\frac{9}{2}$$

$$x \leq -\frac{9}{2} \ (= -4.5)$$

(ii) x = ... -3, -2, -1, 0, 1, 2, 3, 4

(iii) x = 1, 2, 3, 4

Question 4.

Solve (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

Solution:

$$\begin{aligned} \frac{3(x-2)}{5} &\leq \frac{5(2-x)}{3} \\ \Rightarrow \quad 9(x-2) &\leq 25(2-x) \\ 9x - 18 &\leq 50 - 25x \\ 9x + 25x &\leq 50 + 18 \\ 34x &\leq 68 \\ \frac{34x}{34} &\leq \frac{68}{34} \end{aligned}$$

$$x \leq 2 \ (i.e.)\ x \in (-\infty, 2]$$

(ii)

$$\frac{5-x}{3} < \frac{x}{2} - 4$$

Solution:

$$\frac{5-x}{3} < \frac{x-8}{2}$$

$$\begin{aligned}
\Rightarrow \quad 2(5-x) &< 3(x-8) \\
10-2x &< 3x-24 \\
10+24 &< 3x+2x \\
34 &< 5x
\end{aligned}$$

$$\Rightarrow \quad 5x > 34 \Rightarrow \frac{5x}{5} > \frac{34}{5}$$

$$x > \frac{34}{5} \; (i.e.) \; x \in \left(\frac{34}{5}, \infty\right)$$

Question 5.
To secure an A grade one must obtain an average of 90 marks or more in 5 subjects each of a maximum of 100 marks. If one scored 84, 87, 95, 91 in the first four subjects, what is the minimum mark one scored in the fifth subject to get an A grade in the course?

Solution:
Given, to secure A grade in 5 subjects required average mark of 90 or more.
The marks scored in the first four subjects are 84, 87, 95, 91
Let the marks scored in the fifth subject be.
Then by the given data, we have

$$\frac{84+87+95+91+x}{5} \geq 90$$

$$\frac{357+x}{5} \geq 90$$

Multiplying both sides by 5, we get
$357 + x \geq 450$
$x \geq 450 - 357$
$x \geq 93$
∴ The person must obtain a minimum of 93 marks to get A grade in the course.

Question 6.
A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?

Solution:

12% solution of acid in 600 l ⇒ $600 \times \frac{12}{100} = 72$ l of acid

15% of 600 l ⇒ $600 \times \frac{15}{100} = 90$ l

18% of 600 l ⇒ $600 \times \frac{18}{100} = 108$ l

Let x litres of 18% acid solution be added

Given, $(600 + x) \frac{15}{100} \geq 72 + \frac{30x}{100}$

$(600 + x)15 \geq 7200 + 30x$

$9000 + 15x \geq 7200 + 30x$

$1800 \geq 15x$

$x \leq 120$

Let x litres of 18% acid solution be added

Given, $(600 + x) \frac{18}{100} \leq 72 + \frac{30}{100}x$

$(600 + x)\ 18 \leq 7200 + 30x$

$10800 + 18 \leq 7200 + 30x$

$3600 \leq 12x$

$x > 300$

The solution is $120 \leq x > 300$

Question 7.

Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

Solution:

Let the two numbers be x and x + 2

$x + x + 2 < 40$

$\Rightarrow 2x < 38$

$\frac{2x}{2} < \frac{38}{2}$

$\Rightarrow x < 19$ and $x > 10$

so $x = 11 \Rightarrow x + 2 = 13$
$x = 13 \Rightarrow x + 2 = 15$
$x = 15 \Rightarrow x + 2 = 17$
When $x = 17 \Rightarrow x + 2 = 19$
So the possible pairs are (11, 13), (13, 15), (15, 17), (17, 19)

Question 8.
A model rocket is launched from the ground. The height h of the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t;\ 0 \le r \le 20$. At what time the rocket is 495 feet above the ground?

Solution:
Given $h(t) = -5t^2 + 100t,\ 0 \le t \le 20$.
Let the time be 't' sec when the rocket is 495 feet above the ground.
$\therefore h(t) = 495$ for time 't' sec
$-5t^2 + 100t = 495$
$5t^2 - 100t + 495 = 0$
$t^2 - 20t + 99 = 0$
$t^2 - 11t - 9t + 99 = 0$
$t(t - 11) - 9(t - 11) = 0$
$(t - 9)(t - 11) = 0$
$t - 9 = 0$ or $t - 11 = 0$
$t = 9$ or $t = 11$
$\therefore$ At $t = 11$ or 9 seconds, the rocket is 495 feet above the ground.

Question 9.
A Plumber can be paid according to the following schemes: In the first scheme he will be paid Rs. 500 plus Rs.70 per hour, and in the second scheme he will be paid Rs. 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?

Solution:
Let the number of hours to complete the job = x
Wages for the first scheme = Rs. 500 + Rs. 70 per hour = $500 + 70x$
Wages for the second scheme = Rs. 120 per hour = $120x$
Let us find the value of x for which the first scheme gives better wages.
$500 + 70x > 120x$
$500 > 120x - 70x$
$500 > 50x$

$500/50 > x$
$x < 10$
∴ The value of x so that the first scheme gives better wages is = 1, 2, 3, 4, 5, 6, 7, 8, 9

Question 10.
A and B are working on similar jobs but their annual salaries differ by more than Rs 6000. If B earns Rs. 27000 per month, then what are the possibilities of A's salary per month?

Solution:
Let A's salary be x, B's salary is Rs. 27,000
Given their difference in salary is more than Rs. 6,000
Assume A's salary is more than B's salary.
$\therefore x - 27{,}000 > 6000$
$\therefore x > 6000 + 27000$
$x > 33000$

Assume B's salary is more than A's salary.
$\therefore 27{,}000 - x > 6000$
$\therefore 27000 - 6000 > x$
$x < 21000$

The possibilities of A's salary are greater than Rs. 33,000 or less than Rs. 21,000.

Ex 2.4

Question 1.
Construct a quadratic equation with roots 7 and -3.

Solution:
Let the given roots be $\alpha = 7$ and $\beta = -3$
Sum of the roots $\alpha + \beta = 7 + (-3)$
$\alpha + \beta = 7 - 3 = 4$
Product of the roots $\alpha\beta = (7)(-3)$
$\alpha\beta = -21$
The required quadratic equation is

x^2 – (sum of two roots) x + Product of the roots = 0
$x^2 - 4x - 21 = 0$

Question 2.
A quadratic polynomial has one of its zeros $1 + \sqrt{5}$ and it satisfies p(1) = 2. Find the quadratic polynomial.

Solution:
Given $\alpha = 1 + \sqrt{5}$ So, $\beta = 1 - \sqrt{5}$

$$\alpha + \beta = 2;\ \alpha\beta = 1^2 - (-\sqrt{5})^2 = 1 - 5 = -4$$

The quadratic polynomial is
$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$
$p(x) = k(x^2 - 2x - 4)$
$p(1) = k(1 - 2 - 4) = -5k$
Given p (1) = 2

$$\Rightarrow -5k = 2$$
$$\Rightarrow \quad k = \frac{-2}{5}$$
$$\therefore p(x) = \frac{-2}{5}(x^2 - 2x - 4)$$

Question 3.
If α and β are the roots of the quadratic equation $x^2 + \sqrt{2}x + 3 = 0$, form a quadratic polynomial with zeroes 1/α, 1/β.

Solution:
α and β are the roots of the equation $x^2 + \sqrt{2}x + 3 = 0$

$$\Rightarrow \quad \alpha + \beta = -\frac{\sqrt{2}}{1} = -\sqrt{2} \text{ and } \alpha\beta = \frac{3}{1} = 3$$

(*i.e.*) $\alpha + \beta = -\sqrt{2}$ and $\alpha\beta = 3$

The quadratic equation with roots

$$\frac{1}{\alpha} \text{ and } \frac{1}{\beta} \text{ is } x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\frac{1}{\alpha} \times \frac{1}{\beta}\right) = 0$$

$$\text{Now } \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-\sqrt{2}}{3}$$

$$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{3}$$

So the required quadratic equation is

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) x + \left(\frac{1}{\alpha} \times \frac{1}{\beta}\right) = 0$$

$$(i.e,) \ x^2 - \left(\frac{-\sqrt{2}}{3}\right) x + \frac{1}{3} = 0$$

$$x^2 + \frac{\sqrt{2}}{3} x + \frac{1}{3} = 0 \Rightarrow (\div \text{ by } 3)$$

$$3x^2 + \sqrt{2}\, x + 1 = 0$$

Question 4.
If one root of $k(x - 1)^2 = 5x - 7$ is double the other root, show that $k = 2$ or -25.

Solution:
$k(x - 1)^2 = 5x - 7$
(i.e.,) $k(x^2 - 2x + 1) - 5x + 7 = 0$
$x^2 (k) + x(-2k - 5) + k + 1 = 0$
$kx^2 - x(2k + 5) + (k + 7) = 0$

Here it is given that one root is double the other.
So let the roots to α and 2α

$$\text{Sum of the roots} = \alpha + 2\alpha = 3\alpha = \frac{2k+5}{k} \Rightarrow \alpha = \frac{2k+5}{3k} \quad \ldots . (1)$$

$$\text{Product of the roots} = \alpha(2\alpha) = 2\alpha^2 = \frac{k+7}{k}$$

$$\Rightarrow \quad \alpha^2 = \frac{k+7}{2k} \quad \ldots . (2)$$

Substituting α value from (1) in (2)

$$\text{we get } \left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{2k}$$

$$\frac{4k^2 + 25 + 20k}{9k^2} = \frac{k+7}{2k}$$

$2(4k^2 + 25 + 20k) = 9k\,(k + 7)$

$2(4k^2 + 25 + 20k) = 9k^2 + 63k$
$8k^2 + 50 + 40k - 9k^2 - 63k = 0$
$-k^2 - 23k + 50 = 0$
$k^2 + 23k - 5o = 0$
$(k + 25)(k - 2) = 0$
$k = -25$ or 2

Question 5.
If the difference of the roots of the equation $2x^2 - (a + 1)x + a - 1 = 0$ is equal to their product then prove that $a = 2$.

Solution:

$$2x^2 - (a+1)x + (a-1) = 0$$

Let α and β be the roots $\Rightarrow \alpha + \beta = \frac{a+1}{2}$ and $\alpha\beta = \frac{a-1}{2}$

$$\Rightarrow \quad (\alpha + \beta)^2 = (\alpha\beta)^2$$

$$(i.e.,)\ (\alpha + \beta)^2 - 4\alpha\beta = (\alpha\beta)^2$$

$$\Rightarrow \quad \left(\frac{a+1}{2}\right)^2 - 4\left(\frac{a-1}{2}\right) = \left(\frac{a-1}{2}\right)^2$$

$$\frac{a^2+2a+1}{4} - 2(a-1) = \frac{a^2-2a+1}{4}$$

$$a^2 + 2a + 1 - 8(a-1) = a^2 - 2a + 1$$

$$a^2 + 2a + 1 - 8a + 8 - a^2 + 2a - 1 = 0$$

$$-4a + 8 = 0 \Rightarrow 4a = 8$$

$$a = \frac{8}{4} = 2$$

Question 6.
Find the condition that one of the roots of $ax^2 + bx + c$ may be
(i) negative of the other
(ii) thrice the other
(iii) reciprocal of the other.

Solution:
(i) Let the roots be α and $-\beta$
Sum of the roots $= -b/a = 0 \Rightarrow b = 0$

(ii) Let the roots be α, 3α

Sum of the roots $= 4\alpha = -b/a \Rightarrow \alpha = \frac{-b}{4a}$

Product of the roots $= 3\alpha^2 = \frac{c}{a} \Rightarrow 3\left(\frac{-b}{4a}\right)^2 = c/a$

$3\left(\frac{b^2}{16a^2}\right) = c/a \Rightarrow 3b^2 \times a = 16a^2 c$

$(\div \text{ by } a)\ 3b^2 = 16ac$

(iii) Let the roots be $\alpha, \frac{1}{\alpha}$

Product of the roots $= \alpha \times \frac{1}{\alpha} = \frac{c}{a} \Rightarrow 1 = \frac{c}{a} \Rightarrow c = a$

Question 7.
If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots that $ae = 2(b + f)$.

Solution:

Let α be the common root

then $\alpha^2 - a\alpha + b = 0$ ....(1)

$\alpha + \beta = a$

$\alpha\beta + b \Rightarrow \beta = \frac{b}{\alpha} = \frac{2b}{e}$

we are given that

$x^2 - ex + f = 0$ has equal roots.

So the roots will be α, β

Product of the roots

$\alpha \times \alpha = f$

$\alpha^2 = f$

substituting α and α^2, values in (1) we get

$f - a\left(\frac{e}{2}\right) + b = 0$

$f - \frac{ae}{2} + b = 0$

$\frac{ae}{2} = b + f \Rightarrow ae = 2(b + f)$

Now Sum of the roots $= \alpha + \alpha = e \Rightarrow 2\alpha = e \Rightarrow \alpha = \frac{e}{2}$

Question 8.
Discuss the nature of roots of

(i) $-x^2 + 3x + 1 = 0$

(ii) $4x^2 - x - 2 = 0$

(iii) $9x^2 + 5x = 0$

Solution:

(i) $-x^2 + 3x + 1 = 0$
$x^2 - 3x - 1 = 0$ ———- (1)
Compare this equation with the equation
$ax^2 + bx + c = 0$ ——– (2)
we have $a = 1, b = -3, c = -1$
Discriminant $= b^2 - 4ac$
$b^2 - 4ac = (-3)^2 - 4 \times 1 \times -1$
$= 9 + 4 = 13$
$b^2 - 4ac = 13 > 0$
∴ The two roots are real and distinct.

(ii) $4x^2 - x - 2 = 0$
$4x^2 - x - 2 = 0$ ——(3)
Compare this equation with the equation
$ax^2 + bx + c = 0$ (4)
we have $a = 4$, $b = -1$, $c = -2$
Discriminant $= b^2 - 4ac$
$b^2 - 4ac = (-1)^2 - 4 (4) (-2)$
$= 1 + 32$
$= 33$
$b^2 - 4ac = 33 > 0$
∴ The two roots are real and distinct.

(iii) $9x^2 + 5x = 0$
$9x^2 + 5x = 0$ ——- (5)
Compare this equation with the equation
$ax^2 + bx + c = 0$ ——– (6)
we have $a = 9, b = 5$, $c = 0$
Discriminant $= b^2 - 4ac$
$b^2 - 4ac = 5^2 - 4 \times 9 \times 0$
$b^2 - 4ac = 25 > 0$
∴ The two roots are real and distinct.

Question 9.
Without sketching the graphs find whether the graphs of the following functions will intersect the x- axis and if so in how many points.
(i) $y = x^2 + x + 2$

(ii) $y = x^2 - 3x - 1$
(iii) $y = x^2 + 6x + 9$

Solution:

(i) $y = x^2 + x + 2$

$$y = (x + \frac{1}{2})^2 - (\frac{1}{2})^{2+2} = (x + \frac{1}{2})^2 - \frac{1}{4} + 2$$

$$= (x + \frac{1}{2})^2 + \frac{7}{4}$$

Then $D = b^2 - 4ac = (1)^2 - 4(1)(2) = 1 - 8 = -7$

$\therefore$ D is –ve. The parabola will never intersect x axis.

The minimum point is $(-\frac{1}{2}, \frac{7}{4})$

(ii) $y = x^2 - 3x - 7 = (x - \frac{3}{2})^2 - (\frac{3}{2})^2 - 7$

$$= (x - \frac{3}{2})^2 - \frac{9}{4} - 7 = (x - \frac{3}{2})^2 - \frac{37}{4}$$

Then $D = (-3)^2 - 4(1)(-7) = 9 + 28 = 37$

$\therefore$ D is +ve. The parabola intersect x axis at two point.

The minimum point is $(\frac{3}{2}, -\frac{37}{4})$

(iii) $y = x^2 + 6x + 9$

$= (x + 3)^2$

Then $D = (6)^2 - 4(1)(9) = 36 - 36 = 0$

$\therefore$ D is zero. The parabola intersect x axis at $(-3, 4)$.

The minimum point is $(-3, 0)$

Question 10.
Write $f(x) = x^2 + 5x + 4$ in completed square form.

Solution:

$$x^2 + 5x + 4 = x^2 + 2\left(\frac{5}{2}\right)x + 4 + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2$$

$$= x^2 + 2\left(\frac{5}{2}\right)x + \frac{25}{4} + 4 - \frac{25}{4}$$

$$= \left(x+\frac{5}{2}\right)^2 - \frac{9}{4} = \left(x+\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$y = \left(x+\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

Ex 2.5

Question 1.
Solve $2x^2 + x - 15 \leq 0$.

Solution:
To find the solution of the inequality
$ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$ (for $a > 0$)
First we have to solve the quadratic equation $ax^2 + bx + c = 0$
Let the roots be a and P (where a < P)
So for the inequality $ax^2 + bx + c \geq 0$ the roots lie outside α and β
(i.e.,) $x \leq \alpha$ and $x \geq \beta$

So for the inequality $ax^2 + bx + c \leq 0$. The roots lie between α and β
(i.e.,) $x > \alpha$ and $x < \beta$ (i.e.) $a \leq x \leq \beta$

$$x = \frac{-1 \pm \sqrt{1-4(2)(-15)}}{2(2)} = \frac{-1 \pm \sqrt{121}}{4}$$

$$x = \frac{-1 \pm 11}{4}; \ x = \frac{-1+11}{4}, \frac{-1-11}{4}$$

$$x = \frac{5}{2}, \quad x = -3$$

Here $\alpha = -3$ and $\beta = \frac{5}{2}$
(Note that $\alpha < \beta$)

So for the inequality $2x^2 + x - 15 \leq 0$
x lies between -3 and $\frac{5}{2}$

(i.e.) $x \in \left[-3, \frac{5}{2}\right]$ or $-3 \leq x \leq \frac{5}{2}$

Question 2.
Solve $-x^2 + 3x - 2 \geq 0$

Solution:
$-x^2 + 3x - 2 \geq 0 \Rightarrow x^2 - 3x + 2 \leq 0$
$(x - 1)(x - 2) \leq 0$
$[(x - 1)(x - 2) = 0$
$\Rightarrow x = 1$ or 2.
Here $\alpha = 1$ and $\beta = 2$. Note that $\alpha < \beta]$
So for the inequality $(x - 1)(x - 2) \leq 2$
x lies between 1 and 2
(i.e.) $x \geq 1$ and $x \leq 2$ or $x \in [1, 2]$ or $1 \leq x \leq 2$

Ex 2.6

Question 1.
Find the zeros of the polynomial function $f(x) = 4x^2 - 25$

Solution:

$$\begin{aligned} 4x^2 - 25 &= 0 \\ 4x^2 &= 25 \\ x^2 &= \frac{25}{4} \Rightarrow x = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2} \\ x &= -\frac{5}{2}, x = +\frac{5}{2} \end{aligned}$$

$\therefore$ The zeros of the polynomial are $x = \pm\frac{5}{2}$

Question 2.
If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of equation.

Solution:
$x = -2$ is one root
So applying synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -17 & -22 \\ & & -2 & +6 & 22 \\ \hline & 1 & -3 & -11 & \boxed{0} \end{array}$$

So the other factor is $x^2 - 3x - 11 = 0$

$$x = \frac{3 \pm \sqrt{9+44}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{53}}{2}$$

So the roots are $-2, \frac{3 \pm \sqrt{53}}{2}$

Question 3.
Find the real roots of $x^4 = 16$

Solution:
The given equation is $x^4 = 16$
$\Rightarrow x^4 - 16 = 0$
$\Rightarrow (x^2)^2 - 4^2 = 0$
$\Rightarrow (x^2 - 4)(x^2 + 4) = 0$
$(x^2 - 2^2)(x^2 + 4) = 0$
$(x + 2)(x - 2)(x^2 + 4) = 0$
$x + 2 = 0$ or $x - 2 = 0$ or $x^2 + 4 = 0$
$x = -2$ or $x = 2$
$x^2 + 4 = 0$
$\Rightarrow x^2 = -4$
$\Rightarrow x = \pm\sqrt{4}$
which is imaginary. Therefore, the real roots of the given equation are -2, 2.

Question 4.
Solve $(2x + 1)^2 - (3x + 2)^2 = 0$

Solution:

$$(2x+1)^2 - (3x+2)^2 = 0$$

$$\Rightarrow (2x+1)^2 = (3x+2)^2$$

$$\Rightarrow \quad 2x + 1 = \pm (3x + 2)^2$$

$$2x + 1 = +(3x + 2) \quad \text{or} \quad 2x + 1 = -(3x + 2)$$

$$\begin{array}{rcl|rcl} \text{when } 2x + 1 &=& 3x + 2 & \text{when } 2x + 1 &=& -3x - 2 \\ \Rightarrow \quad 2x - 3x &=& 2 - 1 = 1 & \Rightarrow 2x + 3x &=& -2 - 1 = -3 \\ -x &=& 1 \text{ or } x = -1 & 5x &=& -3 \text{ or } x = -\frac{-3}{5} \end{array}$$

$$\therefore \; x = -1 \text{ or } \frac{-3}{5}$$

Ex 2.7

Question 1.
Factorize: $x^4 + 1$. (Hint: Try completing the square.)

Solution:

$$\begin{aligned} x^4 + 1 &= x^4 + 1 + 2x^2 - 2x^2 \\ &= (x^2 + 1)^2 - (\sqrt{2}x)^2 \\ &= (x^2 + 1 - \sqrt{2}\,x)(x^2 + 1 + \sqrt{2}\,x) \end{aligned}$$

Question 2.
If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$, then find the value of a.

Solution:
Let $3x^3 + 8x^2 + 8x + a = (x^2 + x + 1)(3x + a)$.
Equating coefficient of x
$8 = a + 3$
$8 - 3 = a$
$a = 5$

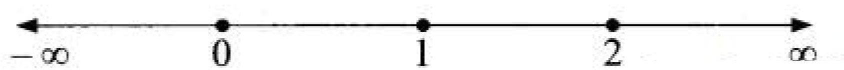
Ex 2.8

Question 1.
Find all values of x for which x3(x−1)(x−2) > 0

Solution:

$$\frac{x^3(x-1)}{(x-2)} > 0$$

Now we have to find the signs of
x^3, x – 1 and x – 2 as follows
$x^3 = 0$; x – 1 = 0 ⇒ x = 1; x – 2 = 0 ⇒ x = 2
Plotting the points in a number line and finding intervals



Intervals	x^3	$x-1$	$x^3(x-1)$	$x-2$	$\frac{x^3(x-1)}{x-2}$
$(-\infty, 0)$	–	–	+	–	–ve
(0, 1)	+	–	–	–	+ve
(1, 2)	+	+	+	–	–ve
$(2, \infty)$	+	+	+	+	+ve

So the solution set = $(0, 1) \cup (2, \infty)$

Question 2.
Find all values of x that satisfies the inequality $\frac{2x-3}{(x-2)(x-4)} < 0$

Solution:

$$\frac{2x-3}{(x-2)(x-4)} < 0$$

$$2x-3 = 0 \Rightarrow x = \frac{3}{2}$$

$$x-2 = 0 \Rightarrow x = 2$$

$$x-4 = 0 \Rightarrow x = 4$$

Plotting the points 3/2, 2 and 4 on the number line and taking the intervals.

Intervals	$2x-3$	$x-2$	$x-4$	$(x-2)(x-4)$	$\frac{2x-3}{(x-2)(x-4)}$
$(-\infty, 3/2)$	$-$	$-$	$-$	$+$	$-ve$
$(3/2, 2)$	$+$	$-$	$-$	$+$	$+ve$
$(2, 4)$	$+$	$+$	$-$	$-$	$-ve$
$(4, \infty)$	$+$	$+$	$+$	$+$	$+ve$

The solution for $\frac{2x-3}{(x-2)(x-4)} < 0$ is in the interval $(-\infty, \frac{3}{2}) \cup (2, 4)$

Question 3.

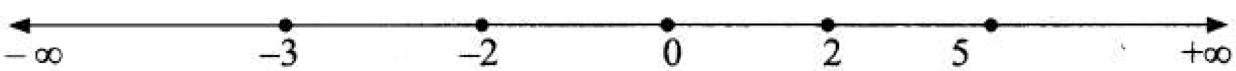
Solve $\frac{x^2-4}{x^2-2x-15} \leq 0$

Solution:

$$\frac{x^2-4}{x^2-2x-15} \leq 0 \quad \Rightarrow \quad \frac{(x-2)(x+2)}{(x+3)(x-5)} \leq 0$$

$x-2=0 \Rightarrow x=2; \quad x+2=0 \Rightarrow x=-2$

$x+3=0 \Rightarrow x=-3; \quad x-5=0 \Rightarrow x=5$



The intervals are $(-\infty, -3)$, $(-3, -2)$, $[-2, 2]$, $(2, 5)$, $(5, \infty)$

Intervals	$(x-2)$	$(x+2)$	$(x+3)$	$(x-5)$	$\frac{(x-2)(x+2)}{(x+3)(x-5)}$
$(-\infty, -3)$	$-$	$-$	$-$	$-$	$\frac{(-)(-)}{(-)(-)}$ +ve
$(-3, -2)$	$-$	$-$	$+$	$-$	$\frac{(-)(-)}{(+)(-)}$ −ve
$(-2, 2)$	$-$	$+$	$+$	$-$	$\frac{(-)(+)}{(+)(-)}$ +ve
$(2, 5)$	$+$	$+$	$+$	$-$	$\frac{(+)(+)}{(+)(-)}$ −ve
$(5, \infty)$	$+$	$+$	$+$	$+$	$\frac{(+)(+)}{(+)(+)}$ +ve

So the solution for the inequality $\frac{x^2-4}{x^2-2x-15} \le 0$ in the interval $(-3, -2) \cup (2, 5)$

Ex 2.9

Question 1.

$$\frac{1}{x^2-a^2}$$

Solution:

Factorizing the denominator

$\text{Dr} = x^2 - a^2 = (x-a)(x+a)$

Let $\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)}$

Equating the numerator we get

$1 = A(x + a) + B(x - a)$

This equation is true for any value of x

$x - a = 0$

$\Rightarrow x = a$

$x + a = 0$

$\Rightarrow x = -a$

To find A and B

$$\text{Put } x = a$$

$$1 = A(2a) + B(0)$$

$$\Rightarrow \quad A = \frac{1}{2}a$$

$$\text{Put } x = -a$$

$$1 = A(0) + B(-2a)$$

$$\Rightarrow \quad B = -\frac{1}{2}a$$

$$\therefore \frac{1}{x^2 - a^2} = \frac{\frac{1}{2}a}{(x-a)} + \frac{-\frac{1}{2}a}{x+a}$$

$$= \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}$$

Question 2.

$$\frac{3x+1}{(x-2)(x+1)}$$

Solution:

$$\text{Let } \frac{3x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$(i.e.,) \quad \frac{3x+1}{(x-2)(x+1)} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

Equating numerator parts

$$3x + 1 = A(x + 1) + B(x - 2)$$

This equation is true for any value of x.

To find A and B

$$\text{Put } x = -1$$
$$-3 + 1 = A(0) + B(-1 -2)$$
$$-3B = -2 \Rightarrow B = 2/3$$
$$\text{Put } x = 2$$
$$3(2) + 1 = A(2 + 1) + B(0)$$
$$3A = 7 \Rightarrow A = 7/3$$
$$\text{Hence } \frac{3x+1}{(x-2)(x+1)} = \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$$

Question 3.

$$\frac{x}{(x^2+1)(x-1)(x+2)}$$

Solution:

$$\text{Let } \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$(i.e.,) \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{A(x+2)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+2)}{(x-1)(x+2)(x^2+1)}$$

Equating numerator on both sides

$x = A(x + 2)(x^2 + 1) + B(x - 1)(x^2 + 1) + (Cx + D)(x - 1)(x + 2)$

This equations is true for any value of x to find A, B, C and D.

$$\text{Put } x = 1$$
$$1 = A(3)(2) + B(0) + (0)$$
$$6A = 1 \Rightarrow A = 1/6$$
$$\text{Put } x = -2$$
$$-2 = A(0) + C(0) + B(-3)(5)$$
$$\Rightarrow -15B = -2 \Rightarrow B = 2/15$$
$$\text{Put } x = 0$$
$$\Rightarrow 2A - B - 2D = 0$$
$$(i.e.,) \frac{2}{6} - \frac{2}{15} - 2D = 0$$
$$\Rightarrow 2D = \frac{2}{6} - \frac{2}{15} = \frac{10-4}{30} = \frac{6}{30} = \frac{1}{5}$$

$$\Rightarrow \quad D = \frac{1}{5\times 2} = \frac{1}{10}$$

$$D = \frac{1}{10}$$

Equating co-efficient of x^3

$$A + B + C = 0$$

$$\frac{1}{6} + \frac{2}{15} + C = 0 \Rightarrow C = \frac{-1}{6} - \frac{2}{15} = \frac{-5-4}{30}$$

$$C = \frac{-9}{30} = \frac{-3}{10}$$

$$\therefore \frac{x}{(x^2+1)(x-1)(x+2)} = \frac{1}{6(x-1)} + \frac{2}{15(x+2)} + \frac{\frac{-3}{10}x + \frac{1}{10}}{x^2+1}$$

$$= \frac{1}{6(x-1)} + \frac{2}{15(x+2)} + \frac{1-3x}{10(x^2+1)}$$

Question 4.

$$\frac{x}{(x-1)^3}$$

Solution:

$$\text{Let } \frac{x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\therefore \frac{x}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

Equating numerator on both sides

$$x = A(x-1)^2 + B(x-1) + C$$

Put $x = 1$

$$1 = A(0) + B(0) + C \Rightarrow C = 1$$

Equating co-eff-of $x^2 \Rightarrow A = 0$

Put $x = 0$

$$A - B + C = 0$$

$$0 - B + 1 = 0 \Rightarrow B = 1$$

$$\therefore \frac{x}{(x-1)^3} = \frac{0}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} = \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

Question 5.

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

Solution:

$$\text{Let } \frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\text{So } \frac{1}{x^4-1} = \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)}{x^4-1}$$

Equating numerator on both sides we get

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

Put $x = 1$

$$1 = A(2)(2) \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4}$$

$$\text{Put } x = -1 \Rightarrow B(-4) = 1 \Rightarrow B = \frac{-1}{4}$$

$$\text{Put } x = 0 \Rightarrow A - B - D = 1$$

$$(ie.,)\ \frac{1}{4} + \frac{1}{4} - D = 1 \Rightarrow D = \frac{1}{4} + \frac{1}{4} - 1 = \frac{-1}{2}$$

Equating co-eff-of x^3

$$A + B + C = 0$$

$$\frac{1}{4} - \frac{1}{4} + C = 0 \Rightarrow C = 0$$

$$\therefore \frac{1}{x^4 - 1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{0 - \frac{1}{2}}{x^2 + 1}$$

$$(ie.) \frac{1}{x^4 - 1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$$

Question 6.

$$\frac{(x-1)^2}{x^3 + x}$$

Solution:

$$x^3 + x = x(x^2 + 1)$$

$$\therefore \frac{(x-1)^2}{x^3 + x} = \frac{(x-1)^2}{x(x^2+1)}$$

$$\text{Let } \frac{(x-1)^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$\therefore \frac{(x-1)^2}{x(x^2+1)} = \frac{A(x^2+1) + (Bx + C)(x)}{x(x^2+1)}$$

Equating numerator on both sides

$(x – 2)^2 = A(x^2 + 1) + (Bx + c)(x)$

Put x = 0

1 = A

Equating co-eff of x^2

1 = A + B

(i.e.,) 1 + B = 1 ⇒ B = 0

put x = 1

A(2) + B + C = 0 (i.e.,) 2A + B + C = 0

$2 + 0 + C = 0 \Rightarrow C = -2$

$$\therefore \frac{(x-1)^2}{x(x^2+1)} = \frac{1}{x} + \frac{0x-2}{x^2+1}$$

$$= \frac{1}{x} - \frac{2}{x^2+1}$$

Question 7.

$$\frac{x^2+x+1}{x^2-5x+6}$$

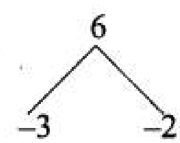
Solution:

Since numerator and denominator are of same degree we have divide the numerator by the denominator

$$\begin{array}{r|l} x^2-5x+6 \,) & x^2+x+1 \;\; (1 \\ & \underline{x^2-5x+6} \\ & \quad 6x-5 \end{array}$$

$$\therefore \frac{x^2+x+1}{x^2-5x+6} = 1 + \frac{6x-5}{x^2-5x+6} \quad ....(1)$$

$$\text{Now } \frac{6x-5}{x^2-5x+6} = \frac{6x-5}{(x-2)(x-3)}$$



$$\text{Let } \frac{6x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$6x-5 = A(x-3) + B(x-2)$$

Put $x = 3$

$$18-5 = A(0) + B(3-2)$$

$$B = 13$$

Put $x = 2$

$$12-5 = A(-1) + B(0)$$

$$\Rightarrow -A = 7$$

$$A = -7$$

$$\therefore \frac{6x-5}{(x-2)(x-3)} = \frac{-7}{x-2} + \frac{13}{x-3}$$

Substituting the value in ....(1)

$$\frac{x^2+x+1}{x^2-5x+6} = 1+\frac{-7}{x-2}+\frac{13}{x-3}$$
$$= 1-\frac{7}{x-2}+\frac{13}{x-3}$$

Question 8.

$$\frac{x^3+2x+1}{x^2+5x+6}$$

Solution:

Numerator is of greater degree than the denominator
So dividing Numerator by the denominator

$$\begin{array}{r|l} & x-5 \\ \hline x^2+5x+6 & x^3+0x^2+2x+1 \\ & x^3+5x^2+6x \\ & (-)\quad(-)\quad(-) \\ \hline & -5x^2-4x+1 \\ & -5x^2-25x-30 \\ \hline & 21x+31 \end{array}$$

$$\text{So } \frac{x^3+2x+1}{x^2+5x+6} = (x-5)+\frac{21x+31}{x^2+5x+6} \quad .... (1)$$

$$\text{Now } \frac{21x+31}{x^2+5x+6} = \frac{21x+31}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}$$
$$= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

Equating Numerator on both sides

⇒ 21x + 31 = A(x + 3) + B(x + 2)
Put x = -3
-63 + 31 = B(-1)
B = 32
Put x = -2
-42 + 31 = A(1) + B(0)

A = -11

$$\therefore \quad \frac{21x+31}{(x+2)(x+3)} = \frac{-11}{x+2} + \frac{32}{x+3} \qquad \text{.... (2)}$$

Substituting (2) in (1) we get

$$\frac{x^3+2x+1}{x^2+5x+6} = (x-5) + \frac{-11}{x+2} + \frac{32}{x+3}$$

$$= (x-5) + \frac{32}{x+3} - \frac{11}{x+2}$$

Question 9.

$$\frac{x+12}{(x+1)^2(x-2)}$$

Solution:

$$\text{Let } \frac{x+12}{(x+1)^2(x-2)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(x-2)(x+1) + C(x-2)}{(x-2)(x+1)^2}$$

equating Numerator on both sides

$$x+12 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

$$\text{Put } x = 2$$

$$2+12 = A(9) + 0 + 0$$

$$9A = 14 \Rightarrow A = \frac{14}{9}$$

$$\text{Put } x = -1$$

$$-1+12 = A(0) + B(0) + C(-3)$$

$$-3C = 11$$

$$C = \frac{-11}{3}$$

$$\text{put } x = 0 \Rightarrow A - 2B - 2C = 12$$

$$\frac{14}{9} - 2B + \frac{22}{3} = 12$$

$$-2B = 12 - \frac{14}{9} - \frac{22}{3}$$
$$= \frac{108 - 14 - 66}{9}$$
$$= \frac{28}{9}$$
$$\Rightarrow B = -\frac{28}{9 \times 2} = \frac{-14}{9}$$
$$\therefore \frac{x+12}{(x+1)^2(x-2)} = \frac{14}{9(x-2)} - \frac{14}{9(x+1)} - \frac{11}{3(x+1)^2}$$

Question 10.

$$\frac{6x^2 - x + 1}{x^3 + x^2 + x + 1}$$

Solution:

Factorizing $x^3 + x^2 + x + 1$

We get $x^3 + x^2 + x + 1 = (x + 1)\ (x^2 + 1)$

$$\therefore \frac{6x^2 - x + 1}{x^3 + x^2 + x + 1} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 1}$$
$$= \frac{A(x^2+1) + (Bx + C)(x+1)}{(x+1)(x^2+1)}$$

Equating Numerator on both sides we get

$6x^2 – x + 1 = A(x^2 + 1) + (Bx + c)(x + 1)$

$6 + 1 + 1 = A(2) + 0 \Rightarrow 2A = 8 \Rightarrow A = 4$

Equating co-eff of x^2

$6 = A + B$

(i.e.,) $4 + B = 6 \Rightarrow B = 6 – 4 = 2$

put $x = 0$

$1 = A+ C$

$4 + C = 1 \Rightarrow C = 1 – 4 = -3$

$$\therefore \frac{6x^2 - x + 1}{x^3 + x^2 + x + 1} = \frac{4}{x+1} + \frac{2x - 3}{x^2 + 1}$$

Question 11.

$$\frac{2x^2+5x-11}{x^2+2x-3}$$

Solution:

Since Numerator and are of same degree divide Numerator by the denominator

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2+2x-3 & 2x^2+5x-11 \\ & 2x^2+4x-6 \\ \hline & \qquad x-5 \end{array}$$

$$\therefore \frac{2x^2+5x-11}{x^2+2x-3} = 2+\frac{x-5}{x^2+2x-3} \qquad ....(1)$$

$$\text{Now } \frac{x-5}{x^2+2x-3} = \frac{x-5}{(x-1)(x+3)}$$

$$= \frac{A}{x-1}+\frac{B}{x+3} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

equating Numerator on both sides we get

x – 5 = A(x + 3) + B(x – 1)

Put x = -3

-3 -5 = A(0) + B(-4)

-4B = -8 ⇒ B = 2

Put x = 1

1 – 5 = A(4) + B(0)

4A = -4 ⇒ A = -1

$$\therefore \frac{x-5}{x^2+2x-3} = \frac{-1}{x-1}+\frac{2}{x+3} \qquad ....(2)$$

Substituting (2) in (1) we get

$$\frac{2x^2+5x-11}{x^2+2x-3} = 2+\left(\frac{-1}{x-1}+\frac{2}{x+3}\right)$$

$$= 2+\frac{2}{x+3}-\frac{1}{x-1}$$

Question 12.

$$\frac{7+x}{(1+x)(1+x^2)}$$

Solution:

$$\text{Let } \frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x}+\frac{(Bx+C)}{(1+x^2)} = \frac{A(x^2+1)+(Bx+C)(1+x)}{(1+x)(1+x^2)}$$

Equating Numerator on both sides we get

$$7+x = A(1+x^2)+(Bx+C)(1+x)$$

$$\text{put } x = -1$$

$$7-1 = A(2)+0 \Rightarrow 2A=6 \Rightarrow A=3$$

Equating co-eff of x^2

$$A+B = 0$$

$$3+B = 0$$

$$\Rightarrow B = -3$$

$$\text{put } x = 0$$

$$A+C = 7$$

$$3+C = 7$$

$$\Rightarrow C = 7-3=4$$

$$\therefore \frac{7+x}{(1+x)(1+x^2)} = \frac{3}{1+x}+\frac{-3x+4}{1+x^2}$$

$$= \frac{3}{1+x}+\frac{4-3x}{1+x^2}$$

Ex 2.10

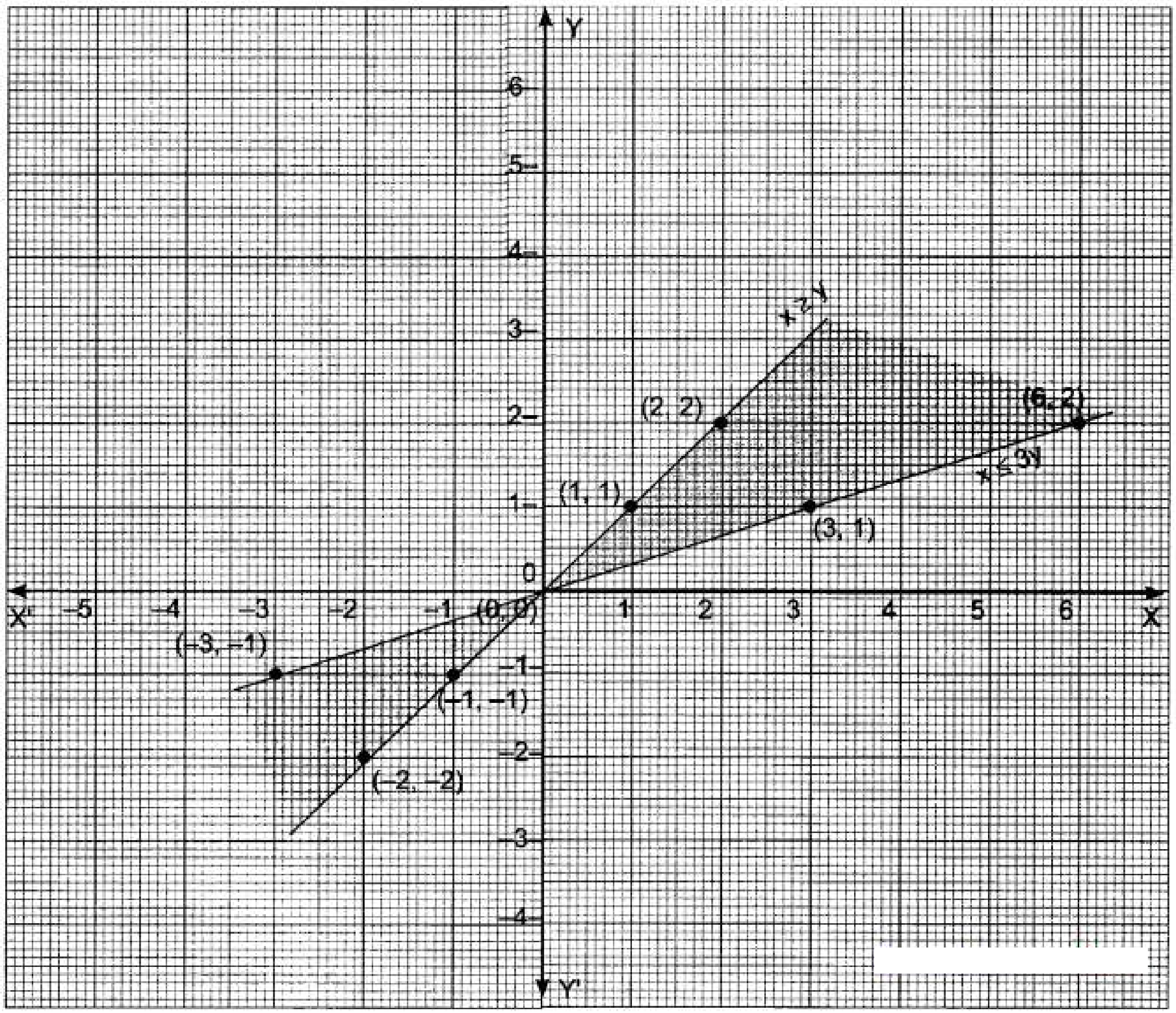
Question 1.
$x \leq 3y, x \geq y$

Solution:
Given in equation are $x \leq 3y, x \geq y$
Suppose $x = 3y \Rightarrow x3 = = y$

x	0	3	6	–3
y	0	1	2	–1

If x = y

x	1	2	–1	–2
y	1	2	–1	–2



Question 2.
$y \geq 2x$, $-2x + 3y \leq 6$

Solution:
Suppose $y = 2x$

x	1	–1	2	–2
y	2	–2	4	–4

$-2x + 3y = 6$

$-2x = 6 - 3y$

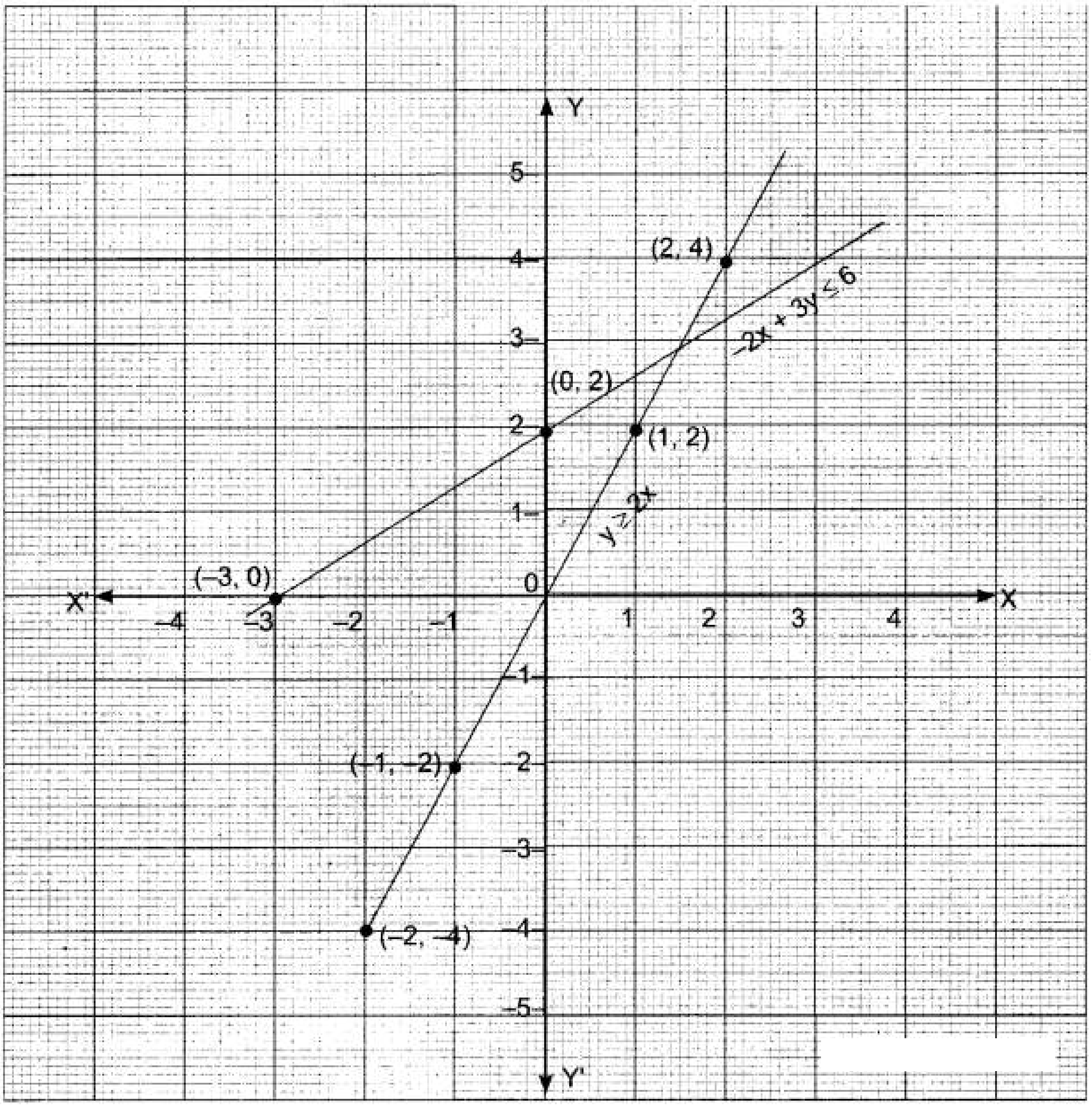
$$x = \frac{6 - 3y}{-2}$$

$$x = -\frac{6}{2} - \frac{3y}{-2}$$

$$x = -3 + \frac{3y}{2}$$

$$3 + x = \frac{3y}{2}$$

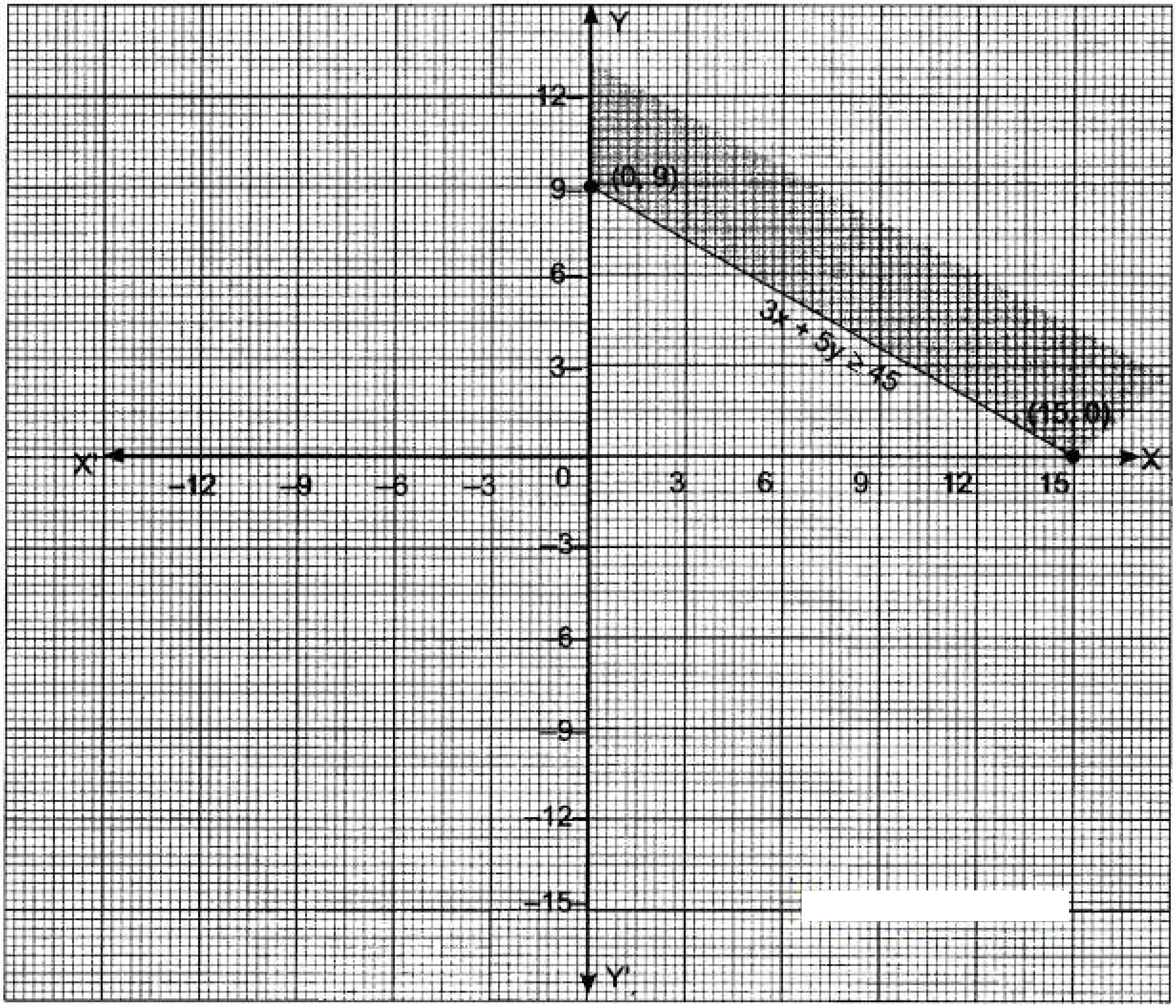
x	0	–3
y	2	0



Question 3.
$3x + 5y \geq 45, x \geq 0, y \geq 0$.

Solution:
If $3x + 5y = 45$

x	0	15
y	9	0



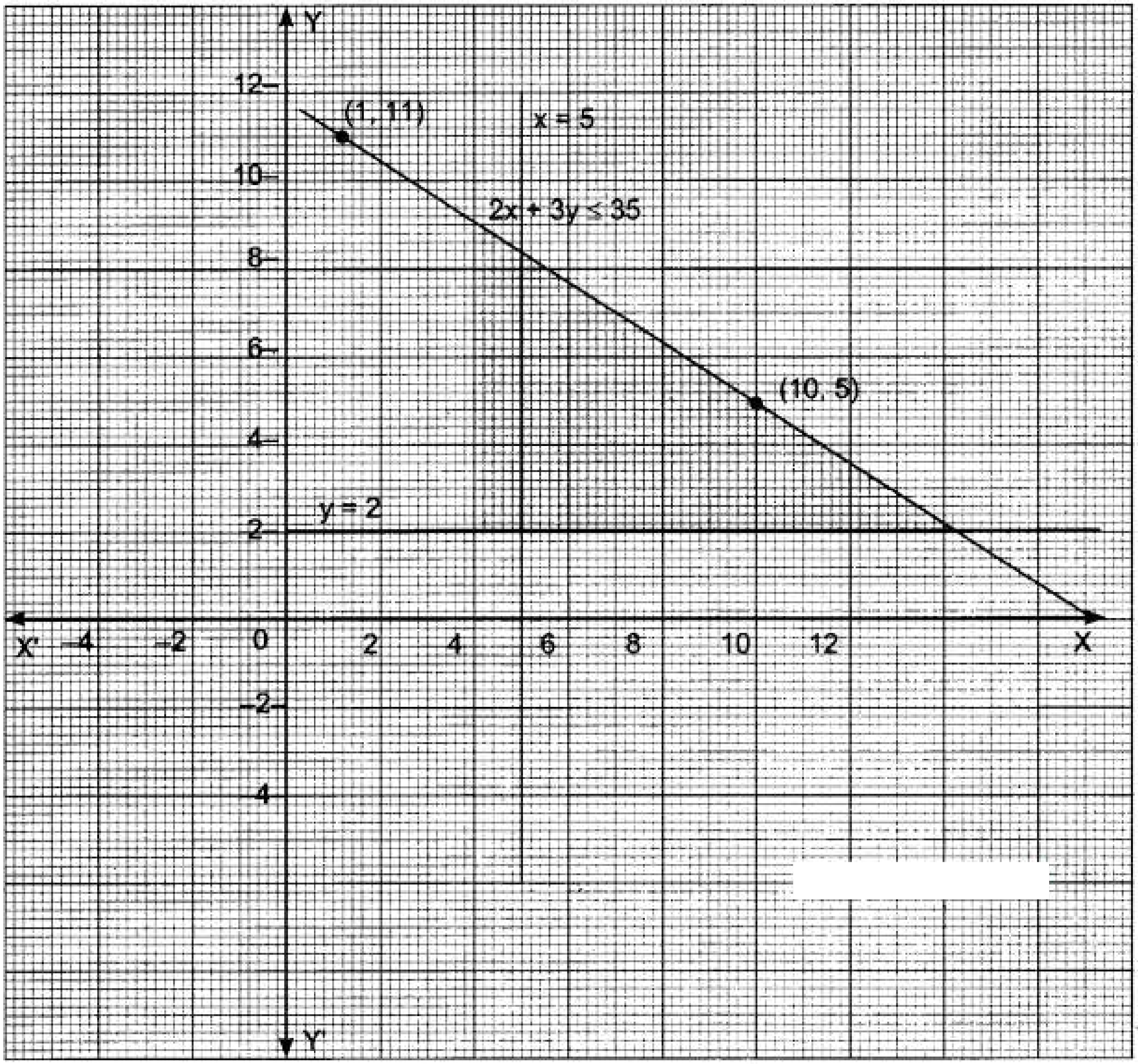
x ≥ 0 is nothing but the positive portion of Y-axis and y ≥ 0 is the positive portion of X-axis.
Shaded region is the required portions.

Question 4.
2x + 3y ≤ 35, y ≥ 2, x ≥ 5

Solution:
If 2x + 3y = 35 then

x	1	10
y	11	5

y = 2 is a line parallel to X-axis at a distance 2 units
x = 5 is a line parallel to Y-axis at a distance of 5 units



The required region is below 2x + 3y = 35, above y = 2 and to the right of x = 5

Question 5.
2x + 3y ≤ 6, x + 4y ≤ 4, x ≥ 0, y ≥ 0.

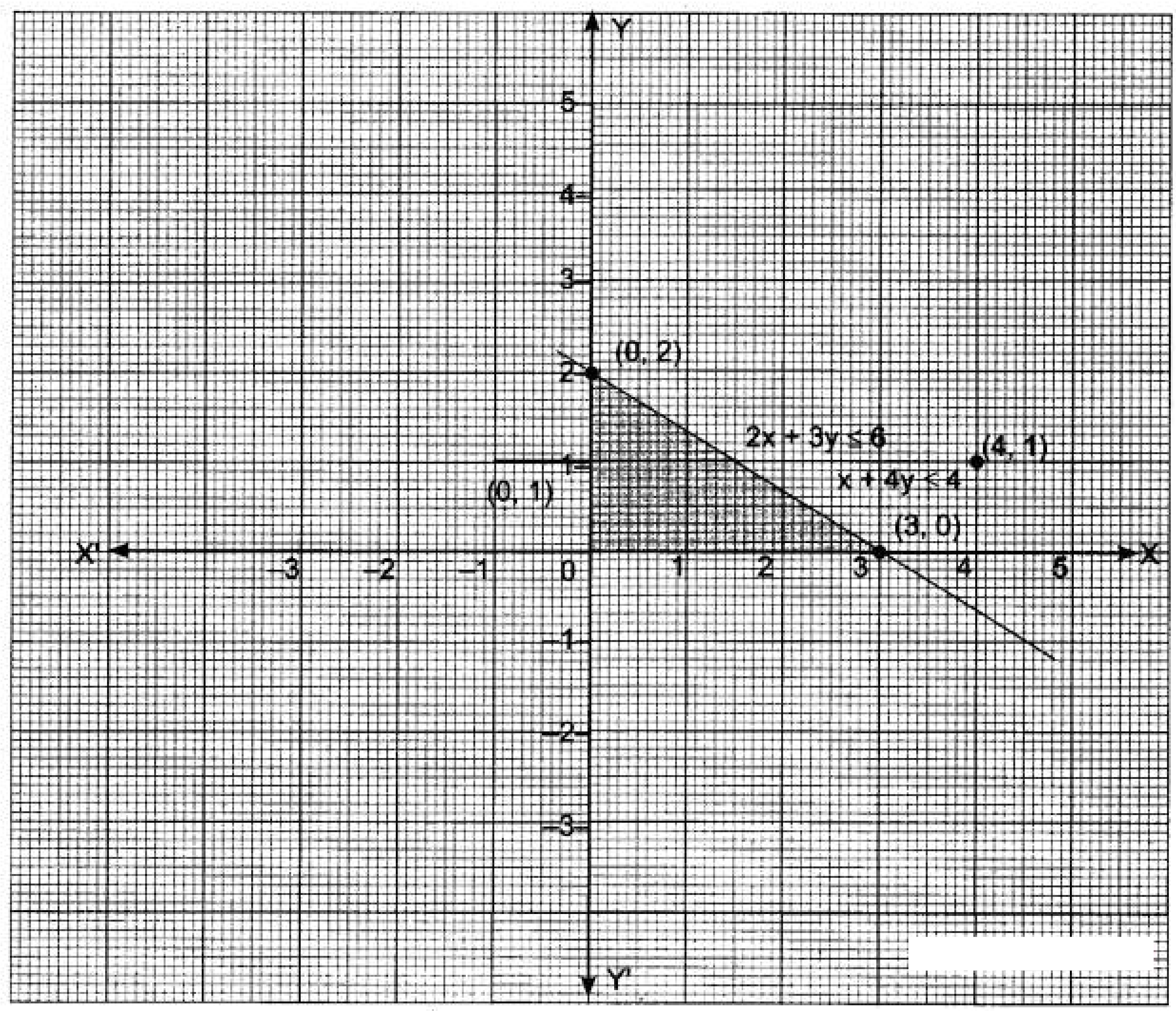
Solution:
If 2x + 3y = 6

x	0	3
y	2	0

x + 4y = 4

x	0	4
y	1	0

$x \geq 0$, $y \geq 0$ represents the area in the 1 quadrant.



The required region is below $2x + 3y = 6$ and below $x + 4y = 4$ bounded by x-axis and y-axis.

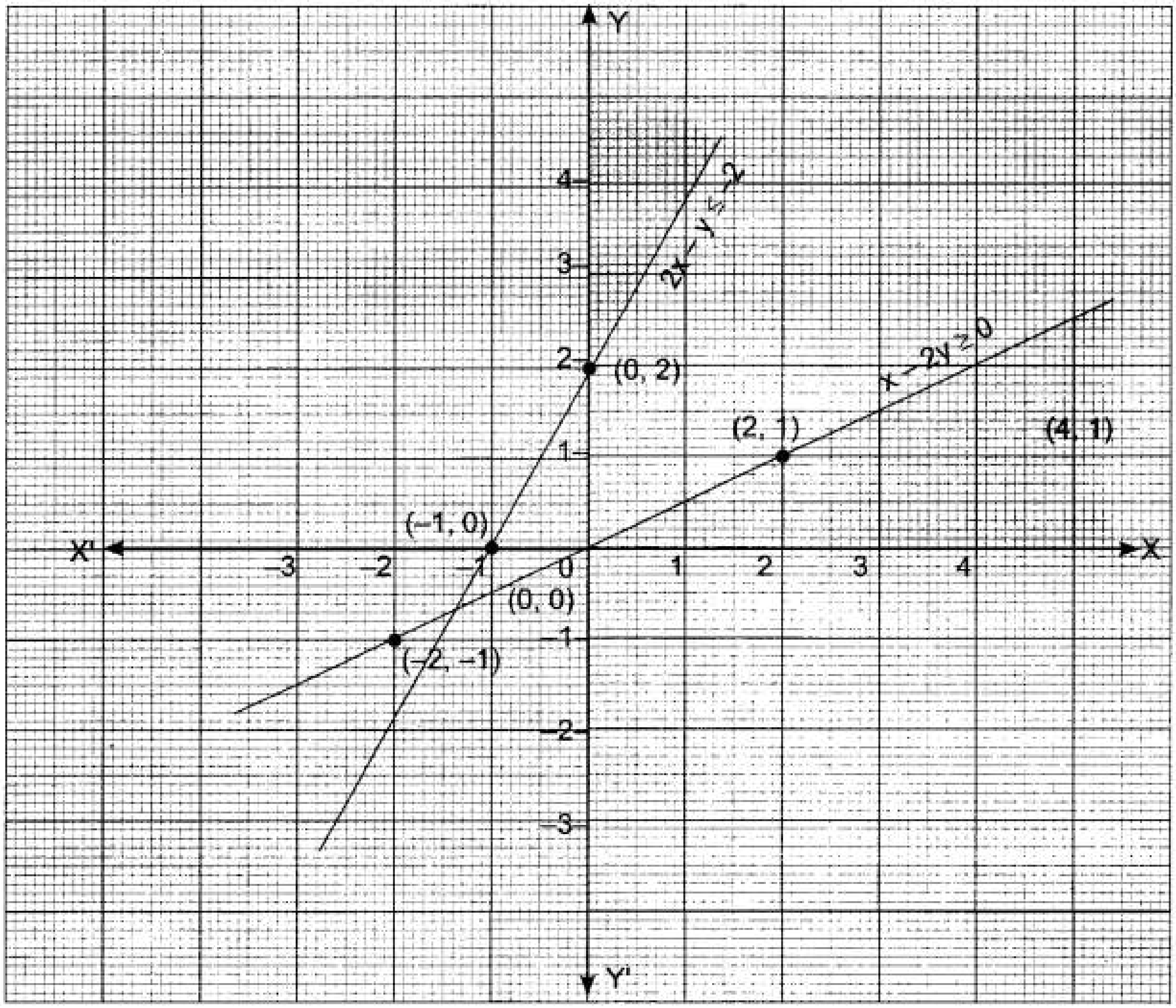
Question 6.
$x - 2y \geq 0$, $2x - y \leq -2$, $x \geq 0$, $y \geq 0$

Solution:
If $x - 2y = 0$

x	0	2	–2
y	0	1	–1

2x – y = -2

x	0	–1
y	2	0



x ≥ 0, y ≥ 0 represents the portion in the 1 quadrant only.

Question 7.
2x + y ≥ 8, x + 2y ≥ 8, x + y ≤ 6.

Solution:
2x + y = 8

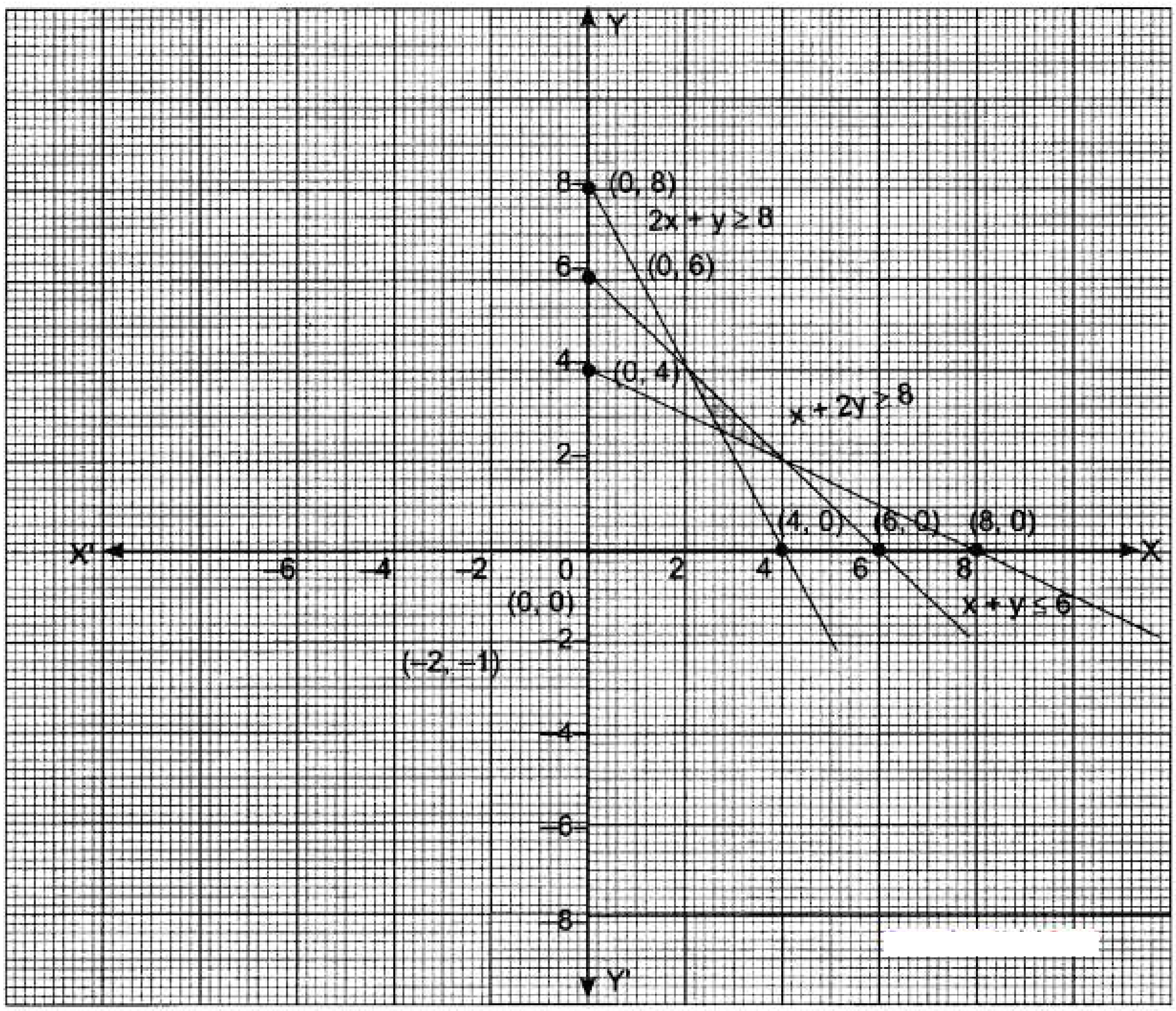
x	0	4
y	8	0

$x + 2y = 8$

x	0	8
y	4	0

$x + y = 6$

x	0	6
y	6	0



Ex 2.11

Question 1.

(i) $(125)^{\frac{2}{3}}$ (ii) $16^{\frac{-3}{4}}$ (iii) $(-1000)^{\frac{-2}{3}}$ (iv) $\left(3^{-6}\right)^{\frac{1}{3}}$ (v) $\dfrac{27^{\frac{-2}{3}}}{27^{\frac{-1}{3}}}$

Solution:

(i)

$$(5^3)^{2/3} = 5^{3\times\frac{2}{3}} = 5^2 = 25$$

(ii)

$$16^{\frac{-3}{4}} = \frac{1}{16^{\frac{3}{4}}}\ \frac{1}{\left(2^4\right)^{3/4}} = \frac{1}{2^{4\times\frac{3}{4}}} = \frac{1}{2^3} = \frac{1}{8}$$

(iii)

$$\left[(-10)^3\right]^{-\frac{2}{3}} = (-10)^{3\times-\frac{2}{3}} = (-10)^{-2} = \frac{1}{(-10)^2} = \frac{1}{100}$$

(iv)

$$3^{-6\times\frac{1}{3}} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

(v)

$$(27)^{-\frac{2}{3}+\frac{1}{3}} = (27)^{-\frac{1}{3}} = \left(3^3\right)^{\frac{-1}{3}} = 3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$

Question 2.

Evaluate $\left[\left((256)^{\frac{-1}{2}}\right)^{\frac{-1}{4}}\right]^3$

Solution:

$$256 = 16^2 = (2^4)^2 = 2^8$$

$$\therefore\ 256^{-1/2} = (2^8)^{-1/2} = 2^{-4}$$

$$\left[(256)^{-1/2}\right]^{-1/4} = 2^{-4\times\frac{-1}{4}} = 2^1$$

$$\left\{\left[(256)^{-1/2}\right]^{-1/4}\right\}^3 = (2^1)^3 = 2^3 = 8$$

Question 3.

If $(x^{1/2} + x^{-1/2})^2 = \frac{9}{2}$, then find the value of $(x^{1/2} - x^{-1/2})$ for $x > 1$.

Solution:

Given $(x^{1/2} + x^{-1/2})^2 = \frac{9}{2} \Rightarrow \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = \frac{9}{2}$

$(i.e.,)\quad x + \frac{1}{x} + 2\sqrt{x}\frac{1}{\sqrt{x}} = \frac{9}{2} \Rightarrow x + \frac{1}{x} + 2 = \frac{9}{2} \Rightarrow x + \frac{1}{x} = \frac{9}{2} - 2 = \frac{9-4}{2} = \frac{5}{2}$

Now $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = x + \frac{1}{x} - 2\sqrt{x}\frac{1}{\sqrt{x}} = x + \frac{1}{x} - 2 = \frac{5}{2} - 2 = \frac{5-4}{2} = \frac{1}{2}$

$\Rightarrow\quad \sqrt{x} - \frac{1}{\sqrt{x}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Question 4.

Simplify and hence find the value of n : $3^{2n}9^2m^{-n}/3^{3n} = 27$

Solution:

$$\frac{3^{2n}9^2 3^{-n}}{3^{3n}} = \frac{3^{2n}(3^2)^2 3^{-n}}{3^{3n}} = 3^{2n}3^4 3^{-n}3^{-3n}$$

$$= 3^{2n+4-n-3n} = 3^{4-2n} = 27 \Rightarrow \text{given } 27 = 3^3$$

$\Rightarrow\quad 3^{4-2n} = 3^3 \Rightarrow 4 - 2n = 3$

$4 - 3 = 2n \Rightarrow 2n = 1 \Rightarrow n = \frac{1}{2}$

Question 5.
Find the radius of the spherical tank whose volume is $32\pi/3$ units.

Solution:

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3 = \frac{32\pi}{3}$$

$$\Rightarrow r^3 = \frac{32\pi}{3} \times \frac{3}{4\pi} = 8 = 2^3$$

$$\Rightarrow r = 2 \text{ units}$$

Question 6.

Simplify by rationalising the denominator $\frac{7+\sqrt{6}}{3-\sqrt{2}}$

Solution:

$$\frac{7+\sqrt{6}}{3-\sqrt{2}} = \frac{7+\sqrt{6}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{21+7\sqrt{2}+3\sqrt{6}+\sqrt{12}}{3^2-\sqrt{2}^2}$$

$$= \frac{21+7\sqrt{2}+3\sqrt{6}+2\sqrt{3}}{9-2} = \frac{21+7\sqrt{2}+3\sqrt{6}+2\sqrt{3}}{7}$$

Question 7.

Simplify $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$

Solution:

$$\frac{1}{3-\sqrt{8}} = \frac{1}{3-\sqrt{8}} \times \frac{3+\sqrt{8}}{3+\sqrt{8}} = \frac{3+\sqrt{8}}{3^2-8} = \frac{3+\sqrt{8}}{1} = 3+\sqrt{8}$$

$$\frac{1}{\sqrt{8}-\sqrt{7}} = \frac{1}{\sqrt{8}-\sqrt{7}} \times \frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}+\sqrt{7}} = \frac{\sqrt{8}+\sqrt{7}}{8-7} = \sqrt{8}+\sqrt{7}$$

$$\frac{1}{\sqrt{7}-\sqrt{6}}=\frac{1}{\sqrt{7}-\sqrt{6}}\times\frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}}=\frac{\sqrt{7}+\sqrt{6}}{7-6}=\sqrt{7}+\sqrt{6}$$

$$\frac{1}{\sqrt{6}-\sqrt{5}}=\frac{1}{\sqrt{6}-\sqrt{5}}\times\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}=\frac{\sqrt{6}+\sqrt{5}}{6-5}=\sqrt{6}+\sqrt{5}$$

$$\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\sqrt{5}+2}{5-4}=\frac{\sqrt{5}+2}{1}=\sqrt{5}+2$$

$$\therefore\quad \frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$

$$= 3+\sqrt{8}-\left(\sqrt{8}+\sqrt{7}\right)+\left(\sqrt{7}+\sqrt{6}\right)-\left(\sqrt{6}+\sqrt{5}\right)+\left(\sqrt{5}+2\right)$$

$$= 3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2 \;=\; 5$$

Question 8.

If $x = \sqrt{2}+\sqrt{3}$ **find** $x^2+1 \,/\, x^2-2$

Solution:

$x=\sqrt{3}+\sqrt{2} \;\Rightarrow\; \frac{1}{x}=\sqrt{3}-\sqrt{2}$

$$\text{Now } x^2+1 \quad = \quad \frac{x+\frac{1}{x}}{x}$$

$$= \quad \frac{\sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2}}{x}=\frac{2\sqrt{3}}{x}$$

$$= \quad \frac{x-2/x}{x}=\frac{\left(\sqrt{3}+\sqrt{2}\right)-2\left(\sqrt{3}-\sqrt{2}\right)}{x}$$

$$= \quad \sqrt{3}+\sqrt{2}-2\sqrt{3}+2\sqrt{2} = \frac{3\sqrt{2}-\sqrt{3}}{x}$$

$$\text{So } \frac{x^2+1}{x^2-2} = \frac{x+\frac{1}{x}}{x} \Big/ \frac{x-\frac{2}{x}}{x}$$

$$= \frac{2\sqrt{3}}{x} \Big/ \frac{3\sqrt{2}-\sqrt{3}}{x} = \frac{2\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3\sqrt{2}-\sqrt{3}} \times \frac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}+\sqrt{3}} = \frac{6\sqrt{6}+2\times 3}{18-3}$$

$$= \frac{6\sqrt{6}+6}{15} = \frac{6(\sqrt{6}+1)}{15} = \frac{2(\sqrt{6}+1)}{5}$$

Ex 2.12

Question 1.
Let $b > 0$ and $b \neq 1$. Express $y = b^x$ in logarithmic form. Also, state the domain and range of the logarithmic function.

Solution:
Given $y = b^x$
By the definition of logarithm $\log_b y = x$
The domain of a logarithmic function is the set of positive real numbers and the range is the set of real numbers.

Question 2.
Compute $\log_9 27 - \log_{27} 9$.

Solution:
Let $\log_9 27 = x \Rightarrow 27 = 9^x \Rightarrow 3^3 = (3^2)^x = 3^{2x}$
$\Rightarrow 2x = 3 \Rightarrow x = 3/2$
Let $\log_{27} 9 = x$

$$9 = 27^x$$

$$3^2 = (3^3)^x \Rightarrow 3^2 = 3^{3x}$$

$$3x = 2 \Rightarrow x = 2/3$$

$$\therefore \log_9 27 - \log_{27} 9 = \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6}$$

Question 3.
Solve $\log_8 x + \log_4 x + \log_2 x = 11$

Solution:

$$\log_8 x = \frac{1}{\log_x 8} = \frac{1}{\log_x 2^3} = \frac{1}{3\log_x 2}$$

$$\log_4 x = \frac{1}{\log_x 4} = \frac{1}{\log_x 2^2} = \frac{1}{2\log_x 2}$$

$$\log_2 x = \frac{1}{\log_x 2}$$

$$\therefore \log_8 x + \log_4 x + \log_2 x = 11$$

$$\Rightarrow \frac{1}{3\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{\log_x 2} = 11$$

$$\Rightarrow \frac{1}{\log_x 2}\left[\frac{1}{3} + \frac{1}{2} + 1\right] = 11$$

$$\Rightarrow \frac{1}{\log_x 2}\left[\frac{2+3+6}{6}\right] = 11$$

$$\Rightarrow \frac{1}{\log_x 2} = 11 \times \frac{6}{11} = 6$$

$$(i.e.,) = 6$$

$$x = 2^6 = 64$$

Question 4.
Solve $\log_4 2^{8x} = 2^{\log_2 8}$

Solution:

Let $\log_4 2^{8x} = A \Rightarrow 2^{8x} = 4^A$

$2^{8x} = (2^2)^A = 2^{2A}$

$$\Rightarrow 2A = 8x \Rightarrow A = \frac{8x}{2} = 4x$$

LHS = $4x$

RHS = $2^{\log_2 8} = 2^3 = 8$ $\qquad\left[\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3\times 1 = 3\right]$

$\therefore\ 4x = 8$

$x = 8/4 = 2$

Question 5.

If $a^2 + b^2 = 7ab$ show that $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

Solution:

$$a^2 + b^2 = 7ab$$

$$\therefore\ a^2 + b^2 + 2ab = 7ab + 2ab = 9ab$$

$$(i.e.,)\ (a+b)^2 = 9ab$$

$$\therefore\ a + b = \sqrt{9ab} = 3\sqrt{ab}$$

$$\Rightarrow \frac{a+b}{3} = \sqrt{ab} = (ab)^{\frac{1}{2}}$$

Taking log on both sides we get

$$\log\frac{a+b}{3} = \log(ab)^{1/2} = \frac{1}{2}\log ab = \frac{1}{2}[\log a + \log b]$$

Question 6.

Prove that $\log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = 0$

Solution:

$[\log a + \log b + \log c = \log abc]$

$$\text{LHS} = \log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = \log\frac{a^2}{bc}\times\frac{b^2}{ca}\times\frac{c^2}{ab}$$

$$= \log\frac{a^2b^2c^2}{a^2b^2c^2} = \log 1 = 0 = \text{RHS}$$

Another method

$$\log \frac{a^2}{bc} = \log a^2 - \log bc = 2 \log a - \log b - \log c \quad \dots (1)$$

$$\log \frac{b^2}{ca} = \log b^2 - \log ca = 2 \log b - \log c - \log a \quad \dots. (2)$$

$$\log \frac{c^2}{ab} = \log c^2 - \log ab = 2 \log c - \log a - \log b \quad \dots. (3)$$

$$(1) + (2) + (3) = 2 \log a - \log b - \log c + 2 \log b - \log c - \log a + 2 \log c - \log a - \log b = 0 = \text{RHS}$$

Question 7.

Prove that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$

Solution:

LHS = log 2 + 16 [log 16 – log 15] + 12 [log 25 – log 24] + 7 [log 81 – log 80]

$$\left\{\begin{array}{l} Now\ \ 16 = 2^4;\ 15 = 3 \times 5 \\ \quad 25 = 5^2;\ 24 = 2^3 \times 3 \\ \quad 81 = 3^4;\ 80 = 2^4 \times 5 \end{array}\right\}$$

$= \log 2 + 16 [\log 2^4 - \log 3 \times 5] + 12 [\log 5^2 - \log 2^3 \times 3] + 7[\log 3^4 - \log 2^4 \times 5]$

$= \log 2 + 16 [4\log 2 - \log 3 - \log 5] + 12 [2 \log 5 - 3 \log 2 - \log 3] + 7 [4 \log 3 - 4 \log 2 - \log 5]$

$= \log 2 + 64 \log 2 - 16 \log 3 - 16 \log 5 + 24 \log 5 - 36 \log 2 - 12 \log 3 + 28 \log 3 - 28 \log 2 - 7 \log 5$

$= \log 2 [1 + 64 - 36 - 28] + \log 3 [-16 - 12 + 28] + \log 5 [-16 + 24 - 7]$

$= \log 2 (1) + \log 3 (0) + \log 5(1)$

$= \log 2 + \log 5 = \log 2 \times 5 = \log 10 = 1 = \text{RHS}$

Question 8.

Prove that $\log_{a^2} a \ \log_{b^2} b \ \log_{c^2} c = \frac{1}{8}$

Solution:

Let $\log_{a^2} a = x$

$\Rightarrow \quad a = (a^2)^x = a^{2x}$

$\Rightarrow \quad a^{2x} = a^1 \Rightarrow 2x = 1 \Rightarrow x = 1/2$

$\log_{b^2} b = 1/2$ and $\log_{c^2} c = 1/2$

$$\text{LHS} = \log_{a^2} a \; \log_{b^2} b \; \log_{c^2} c = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8} = \text{RHS}$$

Question 9.

Prove that $\log a + \log a^2 + \log a^3 + \ldots. + \log a^n = \frac{n(n+1)}{2} \log a$

Solution:

$$\begin{aligned}\text{LHS} &= \log a + \log a^2 + \log a^3 + \ldots. + \log a^n \\ &= \log a + 2\log a + 3\log a + \ldots. + n\log a\end{aligned}$$

$$= \log a\,(1 + 2 + 3 + \ldots.n)$$

$$= \log a\left[\frac{n(n+1)}{2}\right] = \frac{n(n+1)}{2} \log a = \text{RHS}$$

Question 10.

If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ **then prove that** $xyz = 1$

Solution:

Let $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$ (say)

Now $\frac{\log x}{y-z} = k \Rightarrow \log x = k(y-z) \Rightarrow x = e^{k(y-z)}$

$$\frac{\log y}{z-x} = k \Rightarrow \log y = k(z-x) \Rightarrow y = e^{k(z-x)}$$

$$\frac{\log z}{x-y} = k \Rightarrow \log z = k(x-y) \Rightarrow z = e^{k(x-y)}$$

Now LHS $= xyz = e^{k(y-z)}\, e^{k(z-x)}\, e^{k(x-y)}$

$= e^{k(y-z+z-x+x-y)}$

$= e^{k(0)} = e^0 = 1 = \text{RHS}$

Question 11.

Solve $\log_2 x - 3\log_{1/2} x = 6$

Solution:

$$\begin{aligned}\log_2 x &= \log_{1/2} x \,.\, \log_2 1/2\\ &= \log_{1/2} x\ \log_2 1-2\\ &= \log_{1/2} x\ (-1) = -\log_{1/2} x\end{aligned}$$

$\therefore\ \log_2 x - 3\log_{1/2} x = 6 \Rightarrow -\log_{1/2} x - 3\log_{1/2} x = 6$

$(i.e.,)\quad -\log_{1/2} x - \log_{1/2} x^3 = 0$

$\Rightarrow\quad -\log_{1/2}(x \times x^3) = 6$

$\Rightarrow\quad \log_{1/2} x^4 = -6$

$\Rightarrow\quad x^4 = \left(\frac{1}{2}\right)^{-6} = \frac{1}{2^{-6}} = 2^6 = 64$

$\Rightarrow\quad x = (64)^{1/4} = \left[(64)^{1/2}\right]^{1/2} = (8)^{1/2} = 2\sqrt{2}$

Question 12.

Solve $\log_{5-x}(x^2 - 6x + 65) = 2$

Solution:

Given $\log_{5-x}(x^2 - 6x + 65) = 2$

By the definition of logarithm
$x^2 - 6x + 65 = (5 - x)^2$
$x^2 - 6x + 65 = 25 - 10x + x^2$
$10x - 6x + 65 - 25 = 0$
$4x + 40 = 0$
$4x = - 40$
$x = -40/4 = -10$
$x = -10$

Ex 2.13

Question 1.
If $|x + 2| \leq 9$, then x belongs to
(a) $(-\infty, -7)$
(b) $[-11, 7]$
(c) $(-\infty, -7) \cup [11, \infty)$
(d) $(-11, 7)$

Solution:
(b) $[-11, 7]$
Hint:
Given $|x + 2| \leq 9$
$- 9 \leq (x + 2) \leq 9$
$- 9 - 2 \leq x \leq 9 - 2$
$- 11 \leq x \leq 7$
$\therefore x \in [-11, 7]$

Question 2.
Given that x, y and b are real numbers $x < y$, $b \geq 0$, then ……..
(a) $xb < yb$ (b) $xb > yb$
(c) $xb \leq vb$
(d) $xlb \geq ylb$

Solution:
(a) $xb < yb$

Hint:

$$x < y,\ (b > 0) \Rightarrow \frac{x}{b} < \frac{y}{b}$$
$$(i.e.,)\ xb < yb$$

Question 3.

If $\frac{|x-2|}{x-2} \geq 0$ then x belongs to ....

(a) [2, ∞]
(b) (2, ∞)
(c) (-∞, 2)
(d) (-2, ∞)

Solution:

(b) (2, ∞)

Hint:

$$\frac{|x-2|}{x-2} \geq 0$$
$$\therefore x > 2$$
$$x \in (2, \infty)$$

Question 4.

The solution of 5x – 1 < 24 and 5x + 1 > -24 is …….

(a) (4, 5)
(b) (-5, -4)
(c) (-5, 5)
(d) (-5, 4)

Solution:

(c) (-5, 5)

Hint:

$$5x - 1 < 24$$
$$\Rightarrow 5x < 25$$
$$\Rightarrow x < 5$$
$$\therefore x \in (-5, 5)$$

$$5x + 1 > -24$$
$$5x > -24 - 1$$
$$5x > -25$$
$$x > -5$$

Question 5.
The solution set of the following inequality $|x - 1| \geq |x - 3|$ is ....
(a) [0, 2]
(b) $(2, \infty)$
(c) (0, 2)
(d) $(-\infty, 2)$

Solution:
(b) $(2, \infty)$

Question 6.
The value of $\log_{\sqrt{2}} 512$ is ....
(a) 16
(b) 18
(c) 9
(d) 12

Solution:
(b) 18
Hint:

$$\log_{\sqrt{2}} 512 = x \Rightarrow 512 = \sqrt{2}^{x}$$

$$2^9 = \left(2^{\frac{1}{2}}\right)^x = 2^{\frac{x}{2}}$$

$$\Rightarrow \frac{x}{2} = 9 \Rightarrow x = 18$$

Question 7.
The value of $\log_3 \frac{1}{81}$ is ....
(a) -2
(b) -8
(c) -4
(d) -9

Solution:
(c) -4

Hint:

$$\log_3 \frac{1}{81} = x \Rightarrow \frac{1}{81} = 3^x$$

$$3^{-4} = 3^x \Rightarrow x = -4$$

Question 8.

If $\log_{\sqrt{x}} 0.25 = 4$ then the value of x is ....

(a) 0.5
(b) 2.5
(c) 1.5
(d) 1.25

Solution:

(a) 0.5

Hint:

$$\Rightarrow \quad 0.25 = \left(\sqrt{x}\right)^4 = x^2$$

$$(0.5)^2 = x^2 \Rightarrow x = 0.5$$

Question 9.

The value of $\log_a b \log_b c \log_c a$ is ....

(a) 2
(b) 1
(c) 3
(d) 4

Solution:

(b) 1

Hint:

$$\Rightarrow \quad \log_b a \log_c b \log_a c = \log_a{}^a = 1$$

Question 10.

If 3 is the logarithm of 343, then the base is ……

(a) 5
(b) 7

(c) 6
(d) 9

Solution:
(b) 7
Hint.
Given $\log_x 343 = 3$
$343 = x^3$
$7 \times 7 \times 7 = x^3$
$7^3 = x^3$
$x = 7$
Base of the logarithm $x = 7$

Question 11.
Find a so that the sum and product of the roots of the equation $2x^2 + (a - 3)x + 3a - 5 = 0$ are equal is ……..
(a) 1
(b) 2
(c) 0
(d) 4

Solution:
(b) 2
Hint:

$$\text{Sum} = \frac{-(a-3)}{2} = \frac{3-a}{2}$$

$$\text{Product} = \frac{3a-5}{2}$$

$$\text{Given sum} = \text{product}$$

$$\Rightarrow \frac{3-a}{2} = \frac{3a-5}{2}$$

$$\Rightarrow 3 - a = 3a - 5$$

$$\Rightarrow -a - 3a = -5 - 3$$

$$-4a = -8$$

$$4a = 8$$

$$a = 2$$

Question 12.
If a and b are the roots of the equation $x^2 - kx + 16 = 0$ and satisfy $a^2 + b^2 = 32$, then the value of k is ……
(a) 10
(b) -8
(c) (-8, 8)

(d) 6

Solution:
(c) -8, 8
Hint:
Given a and b are the roots of $x^2 - kx + 16 = 0$ satisfying $a^2 + b^2 = 32$

Sum of the roots $a + b = \frac{-(-k)}{1}$

$a + b = k$

Product of the roots $ab = \frac{16}{1}$

$ab = 16$

$a^2 + b^2 = (a + b)^2 - 2ab$

$32 = k^2 - 2 \times 16$

$32 = k^2 - 32$

$k^2 = 32 + 32 = 64$

$k = \pm 8$

Question 13.
The number of solutions of $x^2 + |x - 1| = 1$ is ....
(a) 1
(b) 0
(c) 2
(d) 3

Solution:
(c) 2

$x^2 + x - 1 = 1$	$x^2 - (x - 1) = 1$
$x^2 + x - 1 - 1 = 0$	$x^2 - x + 1 - 1 = 0$
$x^2 + x - 2 = 0$	$x^2 - x = 0$
$(x + 1)(x - 1) = 0$	$x(x - 1) = 0$
$x = 1, -2$	$x = 0, 1$

We have two solutions 0, 1

Question 14.
The equations whose roots are numerically equal but opposite in sign to the roots of $3x^2 - 5x - 7 = 0$ is ……
(a) $3x^2 - 5x - 7 = 0$
(b) $3x^2 + 5x - 7 = 0$
(c) $3x^2 - 5x + 7 = 0$
(d) $3x^2 + x - 7 = 0$

Solution:
(b) $3x^2 + 5x - 7 = 0$
Hint:
$\alpha + \beta = 5/3$, $\alpha\beta = -7/3$
With roots $-\alpha, -\beta$
Sum $= -\alpha - \beta = -(\alpha + \beta) = -5/3$
Product $= (-\alpha)(-\beta) = \alpha\beta = -7/3$
Equation is $x^2 - (-5/3)x + (-7/3) = 0$
(*i.e.*,) $3x^2 + 5x - 7 = 0$

Question 15.
If 8 and 2 are the roots of $x^2 + ax + c = 0$ and 3, 3 are the roots of $x^2 + ax + b = 0$, then the roots of the equation $x^2 + ax + b = 0$ are …….
(a) 1, 2
(b) -1, 1
(c) 9, 1
(d) -1, 2

Solution:
(c) 9, 1
Hint:
Sum $= 8 + 2 = 10 = -a \Rightarrow a = -10$
Product $= 3 \times 3 = 9 = b \Rightarrow b = 9$
Now the equation $x^2 + ax + b = 0$
$\Rightarrow x^2 - 10x + 9 = 0$
$\Rightarrow (x - 9)(x - 1) = 0$
$x = 1$ or 9

Question 16.
If a and b are the real roots of the equation $x^2 - kx + c = 0$, then the distance between the points (a, 0) and (b, 0) is ……..

(a) $\sqrt{k^2-4c}$ (b) $\sqrt{4k^2-c}$ (c) $\sqrt{4c-k^2}$ (d) $\sqrt{k-8c}$

Solution:

(a) $\sqrt{k^2-4c}$

Hint:
$a + b = k$, $ab = c$

Distance between $(a, 0)$ and $(b, 0)$ is $\sqrt{(a-b)^2} = \sqrt{(a+b)^2+4ab} = \sqrt{k^2-4c}$

Question 17.
If $\frac{kx}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}$ then the value of k is …………… .

(a) 1
(b) 2
(c) 3
(d) 4

Solution:
(c) 3

$$\frac{kx}{(x+2)(x-1)} = \frac{2(x-1)+1(x+2)}{(x+2)(x-1)}$$
$$= \frac{3x}{(x+2)(x-1)} \Rightarrow kx = 3x \Rightarrow k = 3$$

Question 18.
If $\frac{1-2x}{3+2x-x^2} = \frac{A}{3-x} + \frac{B}{x+1}$ then the value of A + B is …………… .

(a) -1/2
(b) -2/3
(c) 1/2
(d) 2/3

Solution:
(a) -1/2
Hint:

$$\frac{1-2x}{3+2x-x^2} = \frac{A(x+1)+B(3-x)}{(3-x)(x+1)}$$

$$\begin{aligned}
&\text{Put } x = -1\\
&\quad 3 = A(0) + B(4) \Rightarrow B = 3/4\\
&\text{put } x = 3\\
&\quad 1-6 = A(4) + B(0) \Rightarrow 4A = -5 \Rightarrow A = -5/4
\end{aligned}$$

$$\therefore A + B = \frac{-5}{4} + \frac{3}{4} = \frac{-2}{4} = \frac{-1}{2}$$

Question 19.
The number of real roots of $(x + 3)^4 + (x + 5)^4 = 16$ is ……
(a) 4
(b) 2
(c) 3
(d) 0

Solution:
(a) 4
Hint:
The given equation is $(x + 3)^4 + (x + 5)^4 = 16$
Since it is a fourth degree equation it has four roots.
∴ Number of roots = 4

Question 20.
The value of $\log_3 11 . \log_{11} 13 . \log_{13} 15 . \log_{15} 27 . \log_{27} 81$ is …….
(a) 1
(b) 2
(c) 3
(d) 4

Solution:
(d) 4
Hint.
$\log_3 11 . \log_{11} 13 . \log_{13} 15 . \log_{15} 27 . \log_{27} 81$

$= \log_3 13 \,.\, \log_{13} 15 \,.\, \log_{15} 27 \,.\, \log_{27} 81$
$= \log_3 15 \,.\, \log_{15} 27 \,.\, \log_{27} 81$
$= \log_3 27 \,.\, \log_{27} 81$
$= \log_3 81$
$= \log_3 3^4$
$= 4 \log_3 3$
$= 4 \times 1$
$= 4$

$$\log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4 \times 1 = 4$$