PARABOLA [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. Consider a circle with its center lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

a. $(p/2, p)$ or $(p/2, -p)$ **b.** $(p/2, -p/2)$

c. $(-p/2, p)$ **d.** $(-p/2, -p/2)$

(IIT-JEE 1995)

2. The curve described parametrically by $x = t^2 + t + 1$, and $y = t^2 - t + 1$ represents

a. a pair of straight lines **b.** an ellipse

c. a parabola **d.** a hyperbola

(IIT-JEE 1999)

3. If $x + y = k$ is normal to $y^2 = 12x$, then k is

a. 3 **b.** 9 **c.** −9 **d.** −3

(IIT-JEE 2000)

4. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is

a. 1/8 **b.** 8 **c.** 4 **d.** 1/4

(IIT-JEE 2000)

5. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

a. $\sqrt{3}y = 3x + 1$ **b.** $\sqrt{3}y = -(x + 3)$

c. $\sqrt{3}y = x + 3$ **d.** $\sqrt{3}y = -(3x + 1)$

(IIT-JEE 2001)

6. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

a. $x = -1$ **b.** $x = 1$ **c.** $x = -3/2$ **d.** $x = 3/2$

(IIT-JEE 2001)

7. The locus of the midpoint of the segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directix

a. $y = 0$ **b.** $x = -a$

c. $x = 0$ **d.** none of these

(IIT-JEE 2002)

8. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$. Then the possible value of the slope of this chord is

a. {−1, 1} **b.** {−2, 2} **c.** {−2, 1/2} **d.** {2, −1/2}

(IIT-JEE 2003)

9. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is

a. $\pi/6$ **b.** $\pi/4$ **c.** $\pi/3$ **d.** $\pi/2$

(IIT-JEE 2004)

10. Tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

a. (−6, −11) **b.** (−9, −13)

c. (−10, −15) **d.** (−6, −7)

(IIT-JEE 2005)

11. The axis of a parabola is along the line $y = x$ and the distance of its vertex and focus from the origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively. If vertex and focus both lie in the first quadrant, then the equation of the parabola is

a. $(x + y)^2 = (x - y - 2)$

b. $(x - y)^2 = (x + y - 2)$

c. $(x - y)^2 = 4(x + y - 2)$

d. $(x - y)^2 = 8(x + y - 2)$ **(IIT-JEE 2006)**

12. Consider the two curves $C_1: y^2 = 4x$, $C_2: x^2 + y^2 - 6x + 1 = 0$. Then

a. C_1 and C_2 touch each other only at one point

b. C_1 and C_2 touch each other exactly at two points

c. C_1 and C_2 intersect (but do not touch) at exactly two points

d. C_1 and C_2 neither intersect nor touch each other

(IIT-JEE 2008)

13. Let (x, y) by any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is

a. $x^2 = y$ **b.** $y^2 = 2x$

c. $y^2 = x$ **d.** $x^2 = 2y$ **(IIT-JEE 2011)**

14. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is

a. 3 **b.** 6 **c.** 9 **d.** 15

(JEE Advanced 2014)

Multiple Correct Answers Type

1. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x - 2)^2$ is/are

a. $y = 4(x - 1)$ **b.** $y = 0$

c. $y = -4(x - 1)$ **d.** $y = -30x - 50$

(IIT-JEE 2006)

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the endpoints of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are

a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

(IIT-JEE 2008)

3. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of triangle PTN is a parabola whose

a. vertex is $(2a/3, 0)$ b. directrix is $x = 0$

c. latus rectum is $2a/3$ d. focus is $(a, 0)$

(IIT-JEE 2009)

4. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be

a. $-1/r$ b. $1/r$ c. $2/r$ d. $-2/r$

(IIT-JEE 2010)

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by

a. $y - x + 3 = 0$ b. $y + 3x - 33 = 0$

c. $y + x - 15 = 0$ d. $y - 2x + 12 = 0$

(IIT-JEE 2011)

6. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P?

a. $(4, 2\sqrt{2})$ b. $(9, 3\sqrt{2})$

c. $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ d. $(1, \sqrt{2})$

(JEE Advanced 2015)

Linked Comprehension Type

For Problems 1 and 2

$ABCD$ is a square of side length 2. C_1 is a circle inscribed in the square and C_2 is a circle circumscribing the square. P and Q are any two points on C_1 and C_2, respectively. Also, R is fixed point and L is a fixed line in the same plane. A circle C touches line L and circle C_1 externally. S is a point which is equidistant from given point R and fixed line L. Point R coincides with B.

(IITJEE 2006)

1. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

a. ellipse b. hyperbola

c. parabola d. parts of straight line

2. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

a. $\frac{1}{2}$ sq. units b. $\frac{2}{3}$ sq. units

c. 1 sq. units d. 2 sq. units

For Problems 3–5

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. **(IIT-JEE 2007)**

3. The ratio of the areas of the triangles PQS and PQR is

a. $1 : \sqrt{2}$ b. $1 : 2$ c. $1 : 4$ d. $1 : 8$

4. The radius of the circumcircle of triangle PRS is

a. 5 b. $3\sqrt{3}$ c. $3\sqrt{2}$ d. $2\sqrt{3}$

5. The radius of the incircle of triangle PQR is

a. 4 b. 3 c. 8/3 d. 2

For Problems 6 and 7

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$. **(JEE Advanced 2013)**

6. The length of chord PQ is

a. $7a$ b. $5a$ c. $2a$ d. $3a$

7. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

a. $2\sqrt{7}/3$ b. $-2\sqrt{7}/3$ c. $2\sqrt{5}/3$ d. $-2\sqrt{5}/3$

For Problems 8 and 9

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, $Q(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$. **(JEE Advanced 2014)**

8. The value of r is

a. $-\frac{1}{t}$ b. $\frac{t^2+1}{t}$

c. $\frac{1}{t}$ d. $\frac{t^2-1}{t}$

9. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

a. $\frac{(t^2+1)^2}{2t^3}$ b. $\frac{a(t^2+1)^2}{2t^3}$

c. $\frac{a(t^2+1)^2}{t^3}$ d. $\frac{a(t^2+2)^2}{t^3}$

Matching Column Type

1. Three normals are drawn from (3,0) to the parabola $y^2 = 4x$ which meet the parabola at points P, Q, and R.

Column I	Column II
(a) Area of ΔPQR	(p) 2
(b) Radius of circumcircle of ΔPQR	(q) $\frac{5}{2}$
(c) Centroid of ΔPQR	(r) $\left(\frac{5}{2}, 0\right)$
(d) Circumcenter of ΔPQR	(s) $\left(\frac{2}{3}, 0\right)$

(IIT-JEE 2006)

2. A line L: $y = mx + 3$ meets the y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$, at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of triangle EFG has a local maximum.

Column I	Column II
(a) $m =$	(p) 1/2
(b) Maximum area of ΔEFG is	(q) 4
(c) $y_0 =$	(r) 2
(d) $y_1 =$	(s) 1

(JEE Advanced 2013)

Integer Answer Type

1. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the endpoints of its latus rectum and the point $P(1/2, 2)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the endpoints of the latus rectum. Then Δ_1/Δ_2 is________.
(IIT-JEE 2011)

2. Let S be the focus of the parabola $y^2 = 8x$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of triangle PQS is________.
(IIT-JEE 2012)

3. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is
(JEE Advanced 2015)

4. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is
(JEE Advanced 2015)

Assertion-Reasoning Type

1. **Statement 1:** The curve $y = -x^2/2 + x + 1$ is symmetric with respect to the line $x = 1$.
Statement 2: A parabola is symmetric about its axis.
   a. Both the statements are true and Statement 1 is the correct explanation of Statement 2.
   b. Both the statements are true but Statement 1 is not the correct explanation of Statement 2.
   c. Statement 1 is true and Statement 2 is false.
   d. Statement 1 is false and Statement 2 is true.
(IIT-JEE 2007)

Fill in the Blanks Type

1. The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is____________.
(IIT-JEE 1994)

Subjective Type

1. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2$. **(IIT-JEE 1981)**

2. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola, find the slope of AB.
(IIT-JEE 1982)

3. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(1, 2)$. **(IIT-JEE 1984)**

4. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than 1/2. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. **(IIT-JEE 1991)**

5. Through the vertex O of the parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. **(IIT-JEE 1994)**

6. Show that the locus of the point which divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola.
(IIT-JEE 1995)

7. Points A, B, and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B, and C, taken in pairs, intersect at points P, Q, and R. Determine the ratio of the areas of triangles ABC and PQR. **(IIT-JEE 1996)**

8. From a point A, common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle, and the chord of contact of the parabola. **(IIT-JEE 1996)**

9. The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of point P is a hyperbola. **(IIT-JEE 1998)**

10. Let C_1 and C_2 be, respectively, the parabola $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q be any

point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line $y = x$. Prove that P_1 lies on C_2, Q_1 lies on C_1, and $PQ \geq \min\{PP_1, QQ_1\}$. Hence, or otherwise, determine points P_0 and Q_0 on the parabolas C_1 and C_2, respectively, such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2.

(IIT-JEE 2000)

11. Normals are drawn from a point P with slopes m_1, m_2, and m_3 to the parabola $y^2 = 4x$. If the locus of P with $m_1m_2 = \alpha$ is a part of the parabola itself, then find α.

(IIT-JEE 2003)

12. Tangent is drawn to the parabola $y^2 - 2y - 4x + 5 = 0$ at a point P which cuts the directrix at point Q. A point R is such that it divides QP externally in the ratio 1/2: 1. Find the locus of point R. **(IIT-JEE 2004)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. a.	**2.** c.	**3.** b.	**4.** c.
5. c.	**6.** d.	**7.** c.	**8.** a.
9. c.	**10.** d.	**11.** d.	**12.** b.
13. c.	**14.** d.		

Multiple Correct Answers Type

1. a., b.	**2.** b., c.	**3.** a., d.	**4.** c., d.
5. a., b., d.	**6.** a., d.		

Linked Comprehension Type

1. c.	**2.** c.	**3.** c.	**4.** b.
5. d.	**6.** b.	**7.** d.	**8.** d.
9. b.			

Matching Column Type

1. (a) – (p); (b) – (q); (c) – (s); (d) – (r)

2. (a) – (s); (b) – (p); (c) – (q); (d) – (r)

Integer Answer Type

1. 2	**2.** 4	**3.** 4	**4.** 2

Assertion–Reasoning Type

1. a.

Fill in the Blanks Type

1. $(-1, 0)$

Subjective Type

2. $\pm\sqrt{2}$

3. $x + y - 3 = 0$

4. 3/4

5. $y^2 = 2(x - 4)$

6. (2/9, 8/9)

7. 2 : 1

8. $15a^2/4$

11. 2

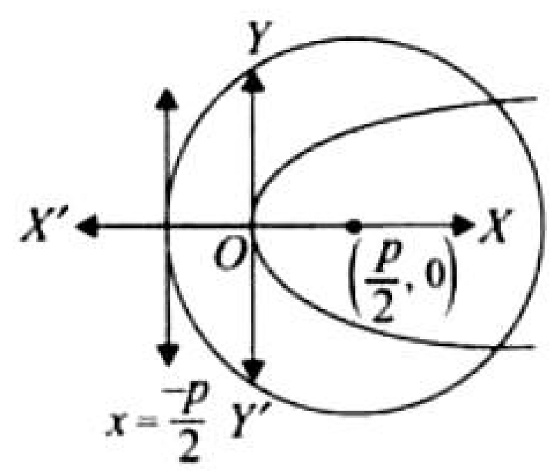
12. $(x + 1)(y - 1)^2 + 4 = 0$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. **a.** The focus of the parabola $y^2 = 2px$ is $(p/2, 0)$ and directrix is $x = -p/2$.



Center of circle $\equiv \left(\frac{p}{2}, 0\right)$

and Radius $= \frac{p}{2} + \frac{p}{2} = p$

Thus, the equation of the circle is

$\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$

or $4x^2 + 4y^2 - 4px - 3p^2 = 0$

Solving this circle with the given parabola, we have (eliminating y)

$4x^2 + 8px - 4px - 3p^2 = 0$

or $4x^2 + 4px - 3p^2 = 0$

or $(2x + 3p)(2x - p) = 0$

or $x = \frac{-3p}{2}, \frac{p}{2}$

or $y^2 = -3p^2$ (not possible)

$\therefore y^2 = 2p \cdot \frac{p}{2}$ or $y = \pm p$

Therefore, the required points are $(p/2, p)$, and $(p/2, -p)$.

2. **c.** We have $x = t^2 + t + 1$ and $y = t^2 - t + 1$

$\frac{x+y}{2} = t^2 + 1, \frac{x-y}{2} = t$

Eliminating t, we get

$2(x + y) = (x - y)^2 + 4$

Since the second-degree terms form a perfect square, it represents a parabola (also, $\Delta \neq 0$).

3. **b.** $y = mx + c$ is normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

$y = -x + k$ is normal to $y^2 = 12x$, so

$m = -1$, $c = k$, and $a = 3$. Therefore,

$c = k = -2(3)(-1) - 3(-1)^3 = 9$

4. **c.** $y^2 = kx - 8$

or $y^2 = k\left(x - \frac{8}{k}\right)$

The directrix of the parabola is

$x = \frac{8}{k} - \frac{k}{4}$

Now, $x = 1$ coincides with

$x - \frac{8}{k} = \frac{-k}{4}$

$\therefore \frac{8}{k} - \frac{k}{4} = 1$ or $k = 4$

5. **c.** The equation of tangent to the parabola $y^2 = 4x$ having slope m is

$y = mx + \frac{1}{m}$ (i)

The equation of tangent to the circle $(x - 3)^2 + y^2 = 9$ having slope m is

$y = m(x - 3) \pm 3\sqrt{1 + m^2}$ (ii)

Equations (i) and (ii) are identical. So,

$\frac{1}{m} = -3m \pm 3\sqrt{1 + m^2}$

or $1 + 3m^2 = \pm 3m\sqrt{1 + m^2}$

or $1 + 6m^2 + 9m^4 = 9(m^2 + m^4)$

or $3m^2 = 1$

or $m = \pm \frac{1}{\sqrt{3}}$

Hence, the equation of common tangent is $\sqrt{3}y = x + 3$ (as tangent is lying above the x-axis).

6. **d.** $y^2 + 4y + 4x + 2 = 0$

$y^2 + 4y + 4 = -4x + 2$

$(y + 2)^2 = -4\left(x - \frac{1}{2}\right)$

It is of the form $Y^2 = -4AX$ whose directrix is given by $X = A$.

Therefore, the required equation is

$x - \frac{1}{2} = 1$

or $x = \frac{3}{2}$

7. **c.** If (h, k) is the midpoint of the line joining focus $(a, 0)$ and $Q(at^2, 2at)$ on the parabola, then

$h = \frac{a + at^2}{2}, k = at$

Eliminating t, we get

$2h = a + a\left(\frac{k^2}{a^2}\right)$

or $k^2 = a(2h - a)$

or $k^2 = 2a(h - \frac{a}{2})$

Therefore, the locus of (h, k) is

$y^2 = 2a(x - \frac{a}{2})$

whose directrix is

$(x - \frac{a}{2}) = -\frac{a}{2}$

or $x = 0$

8. **a.** For the parabola $y^2 = 16x$, focus is $(4, 0)$. Let m be the slope of focal chord. Then its equation is

$y = m(x - 4)$ (i)

Given that the above line is a tangent to the circle $(x - 6)^2 + y^2 = 2$ for which the center is $C(6, 0)$ and radius r is $\sqrt{2}$.

Therefore, the length of perpendicular from $(6, 0)$ to (i) is r.

Therefore,

$$\frac{|6m - 4m|}{\sqrt{m^2 + 1}} = \sqrt{2}$$

or $2m^2 = m^2 + 1$

or $m^2 = 1$

or $m = \pm 1$

9. **c.** Equation of tangent to parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

The above tangent passes through (1, 4). So,

$$4 = m + \frac{1}{m}$$

or $m^2 - 4m + 1 = 0$

Now, angle between the tangents is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{16 - 4}}{1 + 1} = \sqrt{3}$$

or $\theta = \frac{\pi}{3}$

Alternative Method:

The combined equation of tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is

$(y^2 - 4x)(4^2 - 4 \times 1) = [y \times 4 - 2(x + 1)]^2$ [Using $SS_1 = T^2$]

or $12(y^2 - 4x) = 4(2y - x - 1)^2$

or $3(y^2 - 4x) = 4y^2 + x^2 + 1 - 4xy + 2x - 4y$

or $x^2 + y^2 - 4xy + 14x - 4y + 1 = 0$

Now, we know that the angle between the two lines given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right)$

Therefore, the required angle is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{2^2 - 1 \times 1}}{1 + 1}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

10. **d.** The given curve is

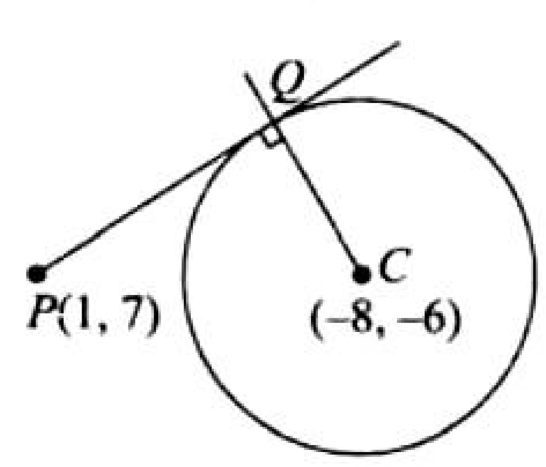
$y = x^2 + 6$

The equation of tangent at (1, 7) is

$\frac{1}{2}(y + 7) = x(1) + 6$

or $2x - y + 5 = 0$ (i)

According to the question, (i) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at Q [center of circle is (−8, −6)].



Then the equation of CQ which is perpendicular to (i) and passing through (−8, −6) is

$$y + 6 = -\frac{1}{2}(x + 8)$$

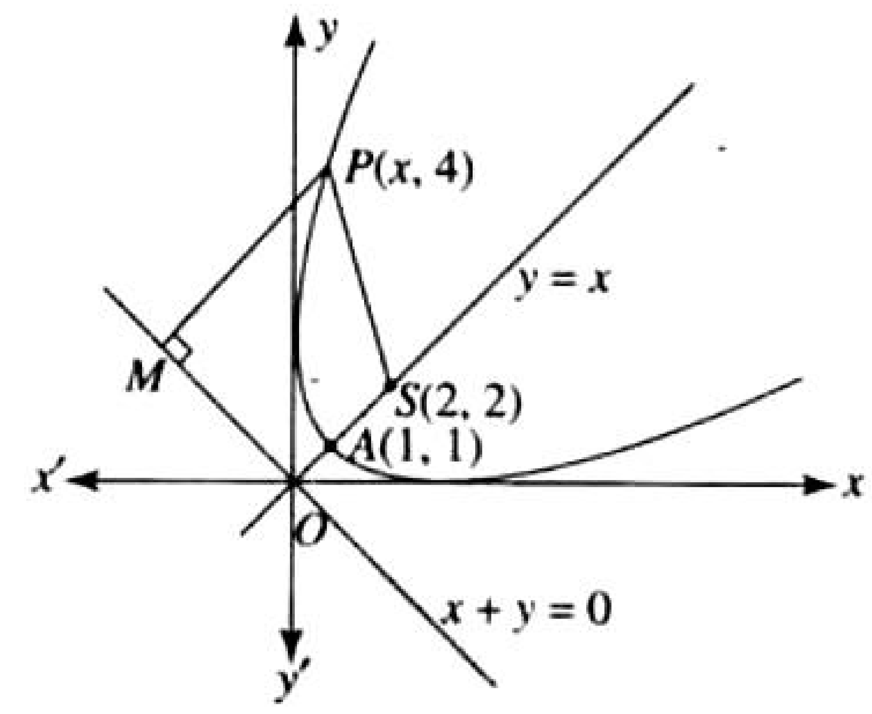
or $x + 2y + 20 = 0$ (ii)

Now, Q is the point of intersection of (i) and (ii), i.e.,

$x = -6, y = -7$

Therefore, the required point is (−6, −7).

11. **d.** Axis of parabola is $y = x$.



Since vertex is at distance $\sqrt{2}$ from (0, 0), vertex is $A(1, 1)$.

Also, focus is at distance $2\sqrt{2}$ from (0, 0), focus is $S(2, 2)$.

Distance $SA = \sqrt{2}$

So, directrix is at distance $\sqrt{2}$ from origin, which is $x + y = 0$.

Hence, equation of the parabola is

$$\sqrt{(x-2)^2 + (y-2)^2} = \frac{|x + y|}{\sqrt{2}} \quad (\because SP = PM)$$

or $x^2 + y^2 - 2xy = 8(x + y - 2)$

or $(x - y)^2 = 8(x + y - 2)$

12. **b.** Solving the curves, we have

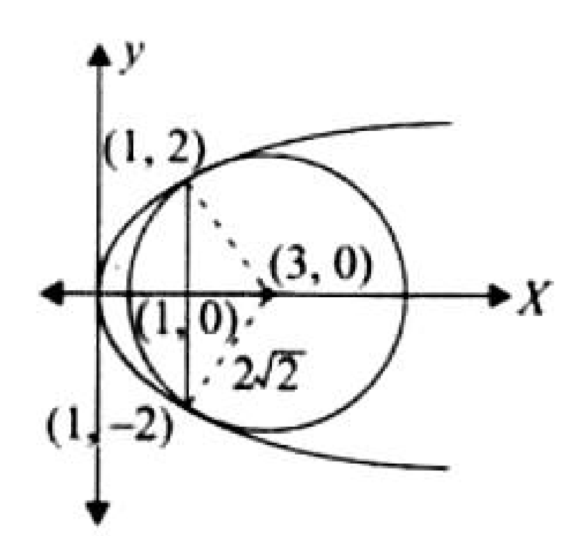
$x^2 + 4x - 6x + 1 = 0$

or $(x - 1)^2 = 0$

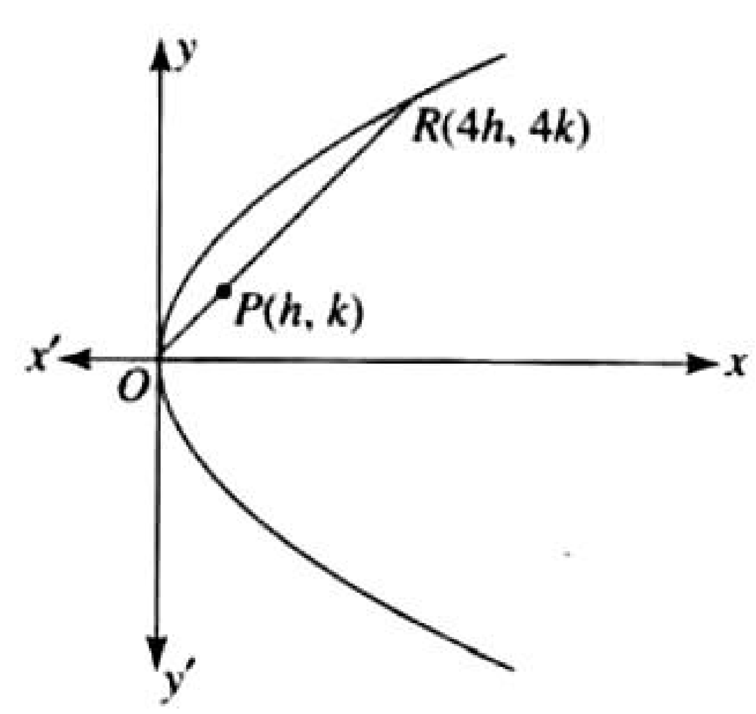
or $x = 1$

Hence, the two curves touch each other. For $x = 1, y = \pm 2$.

The circle and the parabola touch each other at (1, 2) and (1, −2) as shown in the figure.



13. **c.**



Point $P(h, k)$ divides the line segment OR in ratio 1 : 3 .

So, coordinates of point R are $(4h, 4k)$.

(Using ratio $OP : PR = 1 : 3$)

Point R lies on the parabola

$\therefore \quad (4k)^2 = 4 \times 4h$ or $k^2 = h$

Hence, locus is $y^2 = x$.

14. d. Equation of tangent to parabola $y^2 = 8x$ having slope m is

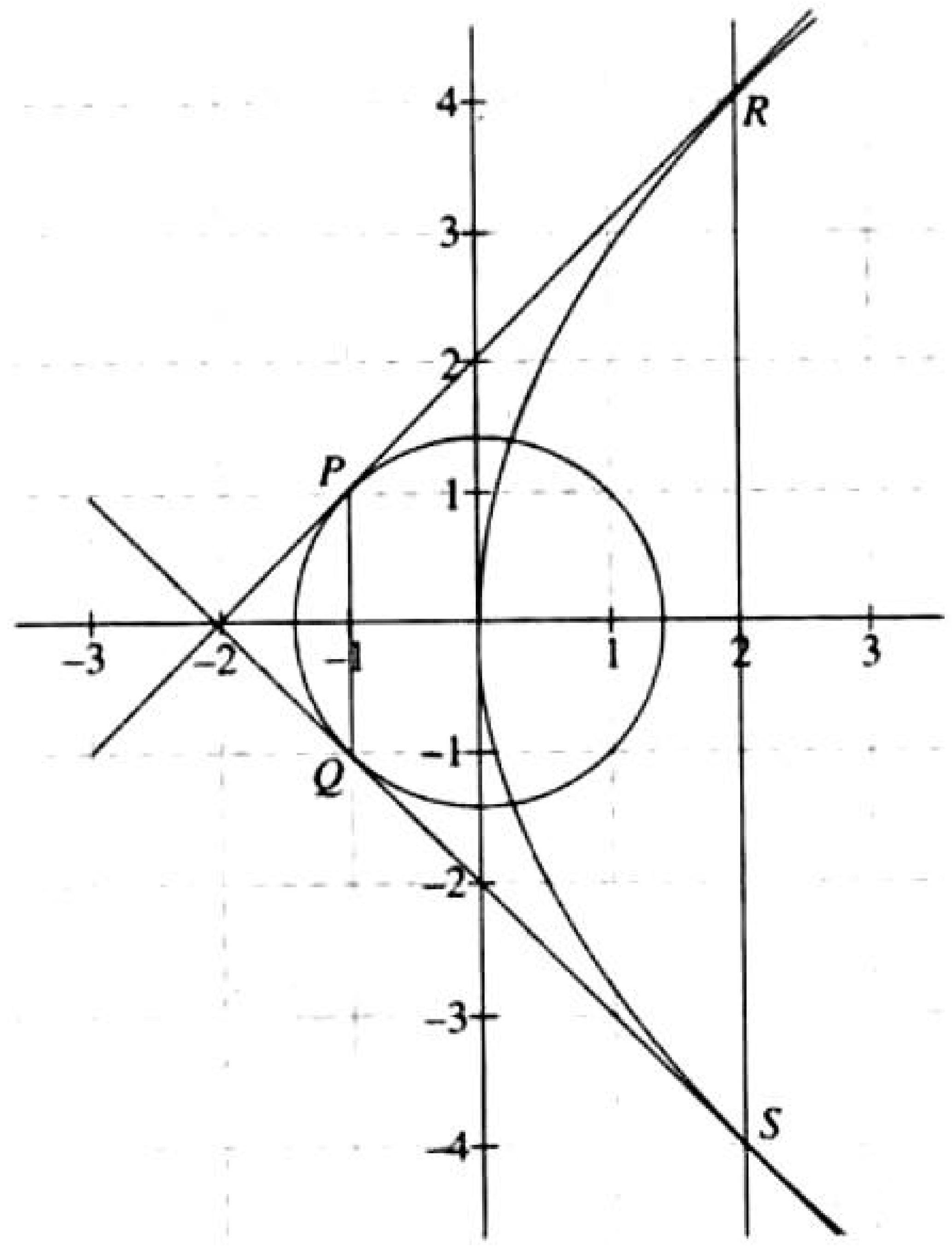
$$y = mx + \frac{2}{m}$$

It is also tangent to circle $x^2 + y^2 = 2$

$$\Rightarrow \quad \frac{\frac{2}{m}}{\sqrt{1+m^2}} = \pm\sqrt{2}$$

$\Rightarrow \quad m = \pm 1$

So, equation of tangents are $y = \pm(x + 2)$.



Chord of contact PQ : $(-2)x + y(0) = 2$ or $x = -1$

Chord of contact RS : $y \times 0 = 4(x - 2)$ or $x = 2$

Line $x = -1$ meets the circle at points $P(-1, 1)$ and $Q(-1, -1)$.

Line $x = 2$ meets the parabola at points $R(2, 4)$ and $S(2, -4)$.

Area of trapezium $PQRS$

$$= \frac{1}{2}(PQ + RS) \times (\text{Height}) = \frac{1}{2}(10) \times (3) = 15 \text{ sq. units}$$

Multiple Correct Answers Type

1. a., b. If $y = mx + c$ is tangent to $y = x^2$, then $x^2 - mx - c = 0$ has equal roots. So,

$m^2 + 4c = 0$

or $c = -\frac{m^2}{4}$

So, the tangent to $y = x^2$ is

$$y = mx - \frac{m^2}{4}$$

Since this is also tangent to $y = -(x - 2)^2$,

$$mx - \frac{m^2}{4} - x^2 + 4x - 4 = 0$$

or $x^2 + (m - 4)x + \left(4 - \frac{m^2}{4}\right) = 0$

has equal roots. So,

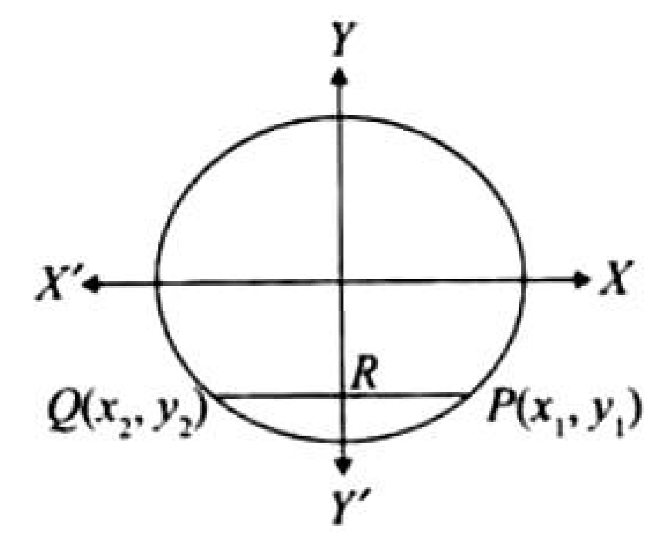
$$(m - 4)^2 - 4\left(4 - \frac{m^2}{4}\right) = 0$$

or $m^2 - 8m + 16 + m^2 - 16 = 0$

or $m = 0, 4$

So, $y = 0$ or $y = 4x - 4$ is the tangent.

2. b., c.



$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$b^2 = a^2(1 - e^2)$

$\therefore 1 = 4(1 - e^2)$

or $e = \frac{\sqrt{3}}{2}$

$$\therefore P \equiv \left(ae, -\frac{b^2}{a}\right) \equiv \left(\sqrt{3}, -\frac{1}{2}\right) \text{ and } Q \equiv \left(-ae, -\frac{b^2}{a}\right) \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$$

$(\because y_1, y_2 < 0)$

The midpoint of PQ is $R \equiv (0, -1/2)$.

$PQ = 2\sqrt{3}$ = Length of latus rectum

Therefore, two parabolas are possible with vertex

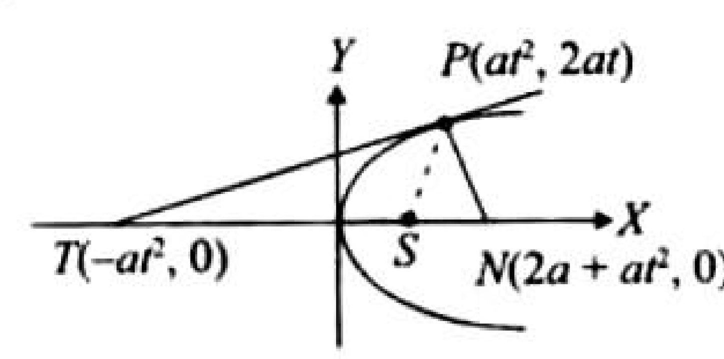
$$\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \text{ or } \left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

Hence, the equations of the parabolas are

$$x^2 = 2\sqrt{3}\left(y + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) \text{ or } x^2 = 2\sqrt{3}\left(y - \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ or $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. a., d.



Tangent at point $P(at^2, 2at)$ is $ty = x + at^2$.

It meets the x-axis at $(-at^2, 0)$.

Normal at point P is $y = -tx + 2at + at^3$.

It meets the x-axis at $(2a + at^2, 0)$.

Let the centroid of triangle PNT be $G \equiv (h, k)$. Then,

$$h = \frac{2a + at^2}{3} \text{ and } k = \frac{2at}{3}$$

Eliminating t, we get

$$\therefore \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$$

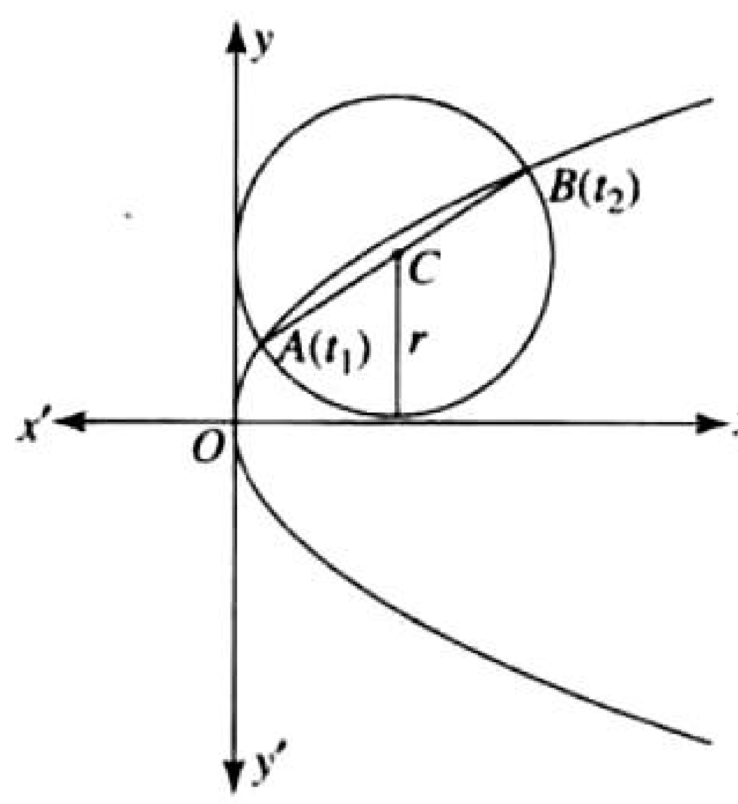
So, the required parabola is

$$\frac{9y^2}{4a^2} = \frac{(3x-2a)}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$$

or $y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

Vertex $\equiv \left(\frac{2a}{3}, 0\right)$; focus $\equiv \left(\frac{2a}{3} + \frac{a}{3}, 0\right) \equiv (a, 0)$

4. c., d.



We have points $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ on the parabola $y^2 = 4x$.

For circle on AB as diameter center is $C\left(\frac{t_1^2 + t_2^2}{2}, (t_1 + t_2)\right)$.

Since circle is touching the x-axis, we have $r = |t_1 + t_2|$

or $\quad t_1 + t_2 = \pm r$

Also slope of AB, $m = \frac{2t_1 - 2t_2}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm\frac{2}{r}$

5. a., b., d. $y^2 = 4x$

The equation of normal is $y = mx - 2m - m^3$.

It passes through (9, 6). So,

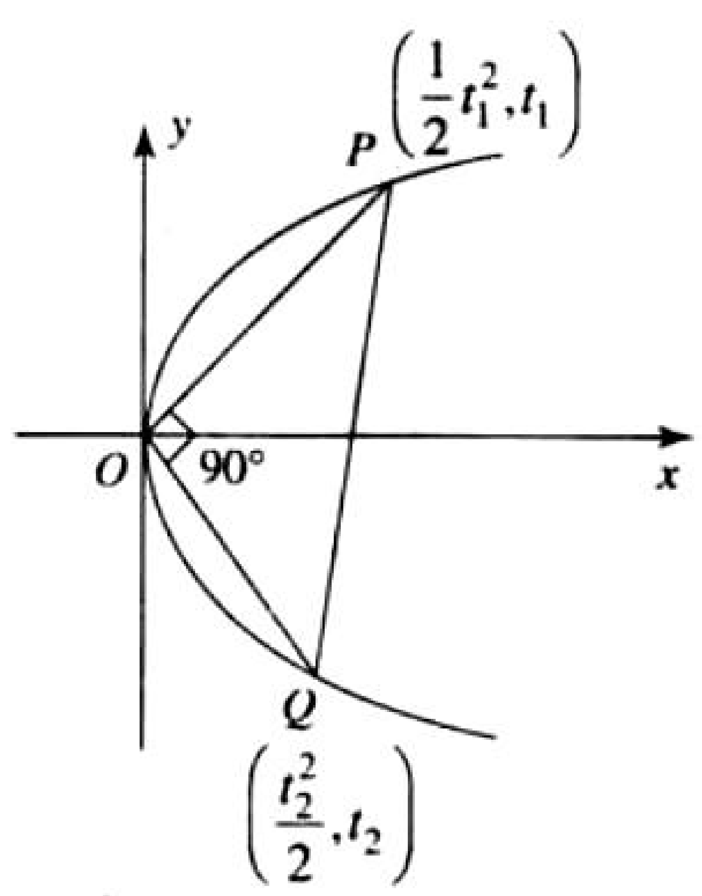
$m^3 - 7m + 6 = 0$

or $(m-1)(m-2)(m+3) = 0$

or $m = 1, 2, -3$

$\therefore$ normals are $y - x + 3 = 0$, $y + 3x - 33 = 0$, $y - 2x + 12 = 0$

6. a., d.



$OP \perp OQ$

$\Rightarrow \quad t_1 t_2 = -4$

$\Rightarrow \quad$ Area of $\Delta OPQ = \frac{1}{2} OP \cdot OQ$

$\Rightarrow \quad \left|\frac{1}{2}\sqrt{\frac{t_1^4}{4} + t_1^2}\sqrt{\frac{t_2^4}{4} + t_2^2}\right| = 3\sqrt{2}$

$\Rightarrow \quad \frac{1}{2} \cdot 4 \cdot \sqrt{\frac{(t_1^2 + 4)}{4}\frac{(t_2^2 + 4)}{4}} = 3\sqrt{2}$

$\Rightarrow \quad 4 \cdot \frac{(16 + 4(t_1^2 + t_2^2) + 16)}{16} = 9 \times 2$

$\Rightarrow \quad 8 + t_1^2 + t_2^2 = 18$

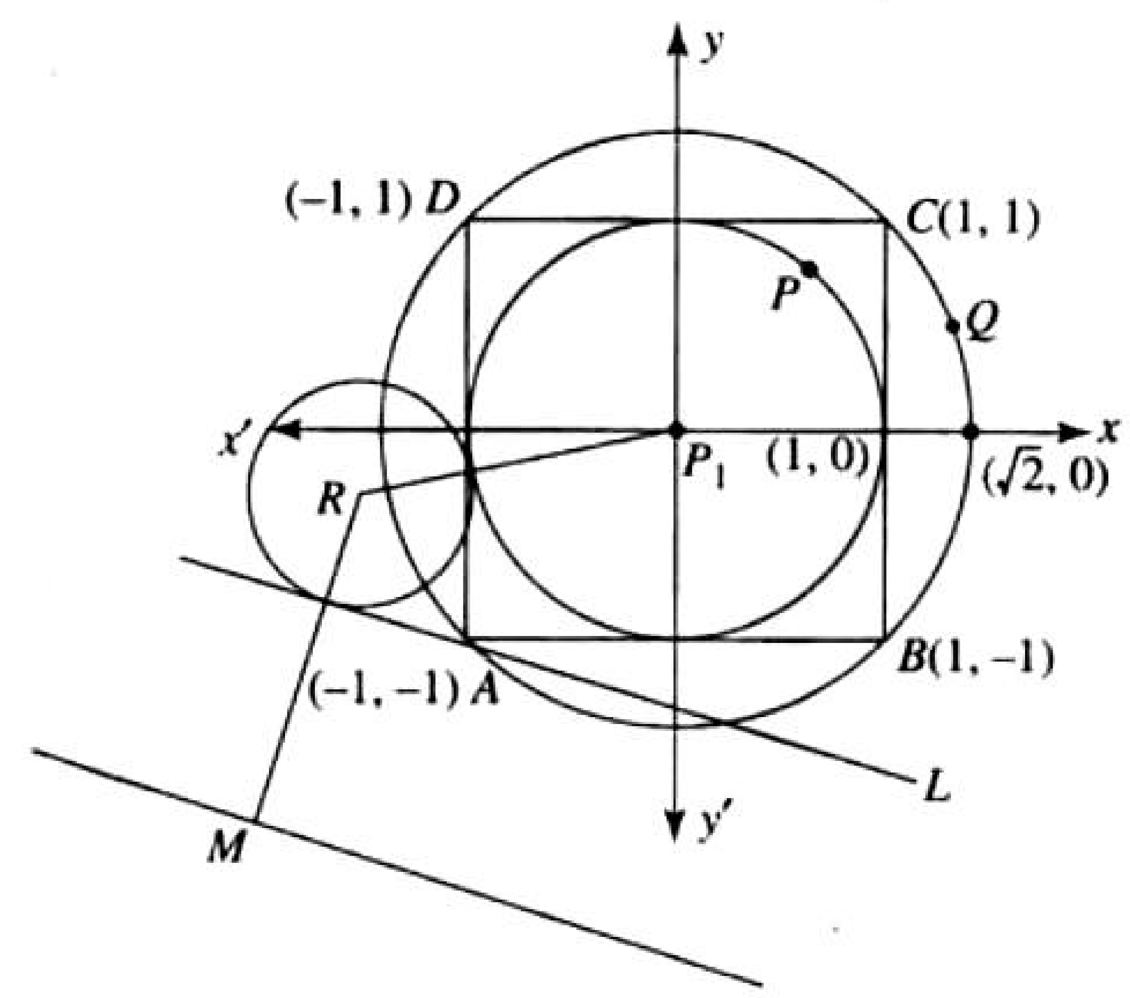
$\Rightarrow \quad t_1^2 + t_2^2 - 10 = 0$

$\Rightarrow \quad t_1^4 - 10t_1^2 + 16 = 0$

$\Rightarrow \quad t_1^2 = 2, 8$

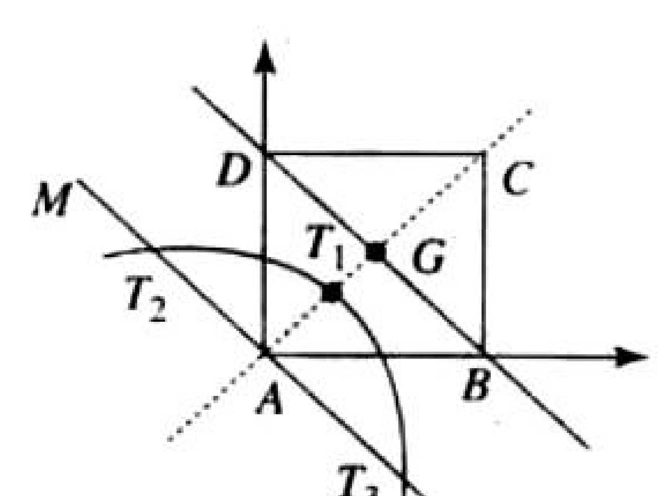
Linked Comprehension Type

1. c.



Consider square $ABCD$ with coordinates as shown in the figure.
Clearly circle inscribed in square is $C_1: x^2 + y^2 = 1$
Let R be the centre of the required circle.
Now, draw a line parallel to L at a distance of r_1 (radius of C_1) from it.
Now, $RP_1 = RM$, which means that R lies on a parabola.

2. c.



$\because \quad AG = \sqrt{2}$

$\therefore \quad AT_1 = T_1G = \frac{1}{\sqrt{2}}$ [as A is the focus, T_1 is the vertex and BD is the directrix of parabola]

Also, T_2T_3 is latus rectum.

$\therefore \quad T_2T_3 = 4 \times \frac{1}{\sqrt{2}}$

$\therefore \quad$ Area of $\Delta T_1T_2T_3 = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 1$

For solutions 3–5

Sol. Solving the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$, we get

$x^2 + 8x - 9 = 0$

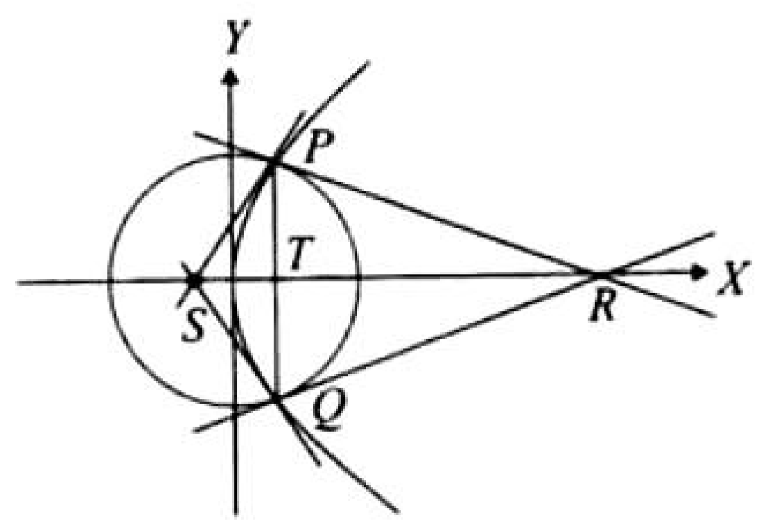
or $x = 1, -9$

or $x = 1$ ($x = -9$ is not possible)

or $y^2 = 8$

or $y = \pm 2\sqrt{2}$

Hence, the points of intersection are $P(1, 2\sqrt{2})$ and $Q(1, -2\sqrt{2})$.



Tangent to the parabola at point P is

$2\sqrt{2}y = 4(x + 1)$

It meets the x-axis at $S(-1, 0)$.

Tangent to the circle at point P is $(1)x + 2\sqrt{2}\, y = 9$.

It meets the x-axis at $R(9, 0)$.

3. c. $\dfrac{\text{Ar}(\Delta PQS)}{\text{Ar}(\Delta PQR)} = \dfrac{\frac{1}{2}PQ \times ST}{\frac{1}{2}PQ \times TR} = \dfrac{ST}{TR} = \dfrac{2}{8} = \dfrac{1}{4}$

4. b. For ΔPRS,

$\text{Ar}(\Delta PRS) = \Delta = \frac{1}{2} \times SR \times PT = \frac{1}{2} \times 10 \times 2\sqrt{2}$

$\therefore \Delta = 10\sqrt{2},\ a = PS = 2\sqrt{3}$

$b = PR = 6\sqrt{2},\ c = SR = 10$

$\therefore$ Radius of circumcircle $= R = \dfrac{abc}{2\Delta}$

$$= \frac{2\sqrt{3} \times 6\sqrt{2} \times 10}{4 \times 10\sqrt{2}} = 3\sqrt{3}$$

5. d. Radius of incircle of triangle PQR is

$\dfrac{\text{Area of } \Delta PQR}{\text{Semi-perimeter of } \Delta PQR} = \dfrac{\Delta}{s}$

We have $a = PR = 6\sqrt{2}$, $b = QR = PR = 6\sqrt{2}$, and $c = PQ = 4\sqrt{2}$. Also,

$\Delta = \frac{1}{2} \times PQ \times TR = 16\sqrt{2}$

$\therefore s = \dfrac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$

$\therefore r = \dfrac{16\sqrt{2}}{8\sqrt{2}} = 2$

6. b. Since PQ is focal chord, let $P \equiv (at^2, 2at)$ and $Q \equiv (a/t^2, -2a/t)$.

We know that point of intersection of tangents at $P(t_1)$ and $Q(t_2)$ is $(a\, t_1 t_2, a(t_1 + t_2))$

$\therefore$ The point of intersection of tangents at P and Q is

$\left(-a, a\left(t - \frac{1}{t}\right)\right)$

As the point of intersection lies on $y = 2x + a$, we have

$a\left(t - \frac{1}{t}\right) = -2a + a$

or $t - \frac{1}{t} = -1$

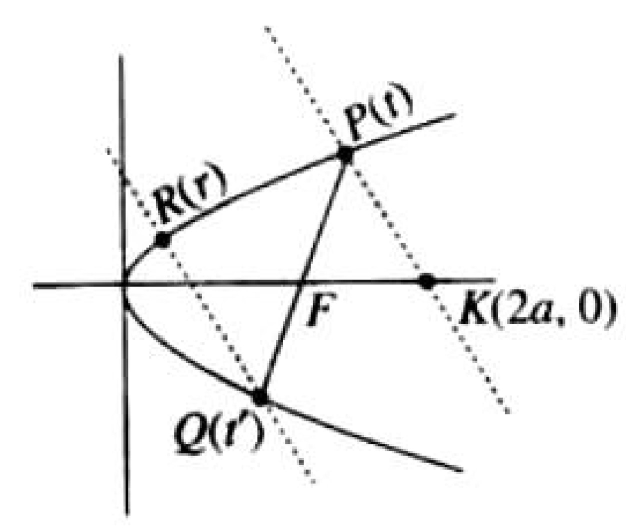
or $\left(t + \frac{1}{t}\right)^2 = 5$

$\therefore$ Length of focal chord $PQ = a\left(t + \frac{1}{t}\right)^2 = 5a$

7. d. Angle made by chord PQ at vertex $(0, 0)$ is given by

$\tan\theta = \left|\dfrac{m_{OP} - m_{OQ}}{1 + m_{OP} \cdot m_{OQ}}\right| \dfrac{(2/t) + 2t}{1 - 4} = \dfrac{2\{(1/t) + t\}}{-3} = \dfrac{-2\sqrt{5}}{3}$

8. d.



Slope (PK) = Slope (QR)

$$\frac{2at - 0}{at^2 - 2a} = \frac{-\frac{2a}{t} - 2ar}{\frac{a}{t^2} - ar^2}$$

$$\Rightarrow \quad \frac{t}{t^2 - 2} = -\left(\frac{\frac{1}{t} + r}{\frac{1}{t^2} - r^2}\right)$$

$$\Rightarrow \quad \frac{1}{t^2 - 2} = \frac{1 + rt}{t^2 r^2 - 1}$$

$\Rightarrow \quad t^2r^2 - 1 = t^2 + rt^3 - 2 - 2rt$

$\Rightarrow \quad t^2r^2 + (2 - t^2)tr + (1 - t^2) = 0$

$\Rightarrow \quad tr = -1$ or $tr = t^2 - 1$

$\Rightarrow \quad r = -1/t$ or $r = \dfrac{t^2 - 1}{t}$

But for $r = -1/t$, points Q and R are coincident.

$\therefore \quad r = \dfrac{t^2 - 1}{t}$

9. b. Tangent at P : $ty = x + at^2$ or $y = \dfrac{x}{t} + at$

Normal at S : $y + \dfrac{x}{t} = \dfrac{2a}{t} + \dfrac{a}{t^3}$

Solving, $2y = at + \dfrac{2a}{t} + \dfrac{a}{t^3}$

$\Rightarrow \quad y = \dfrac{a(t^2 + 1)^2}{2t^3}$

Matching Column Type

1. **(a) – (p); (b) – (q); (c) – (s); (d) – (r).**

   The equation of normal to $y^2 = 4x$ is

   $y = mx - 2m - m^3$

   As it passes through (3, 0), we get $m = 0, 1, -1$.

   Then three points on the parabola are given by $(m^2, -2m)$, for $m = 0, 1, -1$. So,

   $P \equiv (0, 0)$, $Q \equiv (1, -2)$, $R \equiv (1, 2)$

   $\therefore$ Area of $\Delta PQR = \frac{1}{2} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2$ sq. units

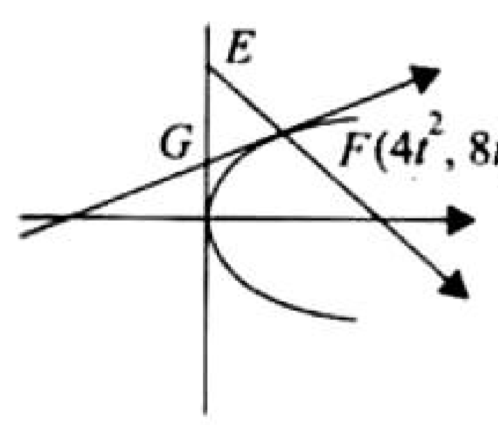
   $R = \frac{abc}{4\Delta} = \frac{\sqrt{5} \times \sqrt{5} \times 4}{4 \times 2} = \frac{5}{2}$ (where a, b, c are the sides of ΔPQR)

   Centroid of $\Delta PQR \equiv \left(\frac{2}{3}, 0\right)$,

   Circumcenter $\equiv \left(\frac{5}{2}, 0\right)$

2. **(a) – (s); (b) – (p); (c) – (q); (d) – (r)**

   Let point F on the parabola be $(4t^2, 8t)$.



   Tangent at this point is $ty = x + 4t^2$.

   It meets the y-axis at $(0, 4t)$.

   Then the area of triangle EFG is $A(t) = 2t^2(3 - 4t) = 6t^2 - 8t^3$.

   Differentiating w.r.t. t, we get

   $A'(t) = 12t - 24t^2$

   For $A'(t) = 0$, $t = 1/2$, which is a point of maxima. So, point F is (1, 4).

   Slope of $EF = 1$

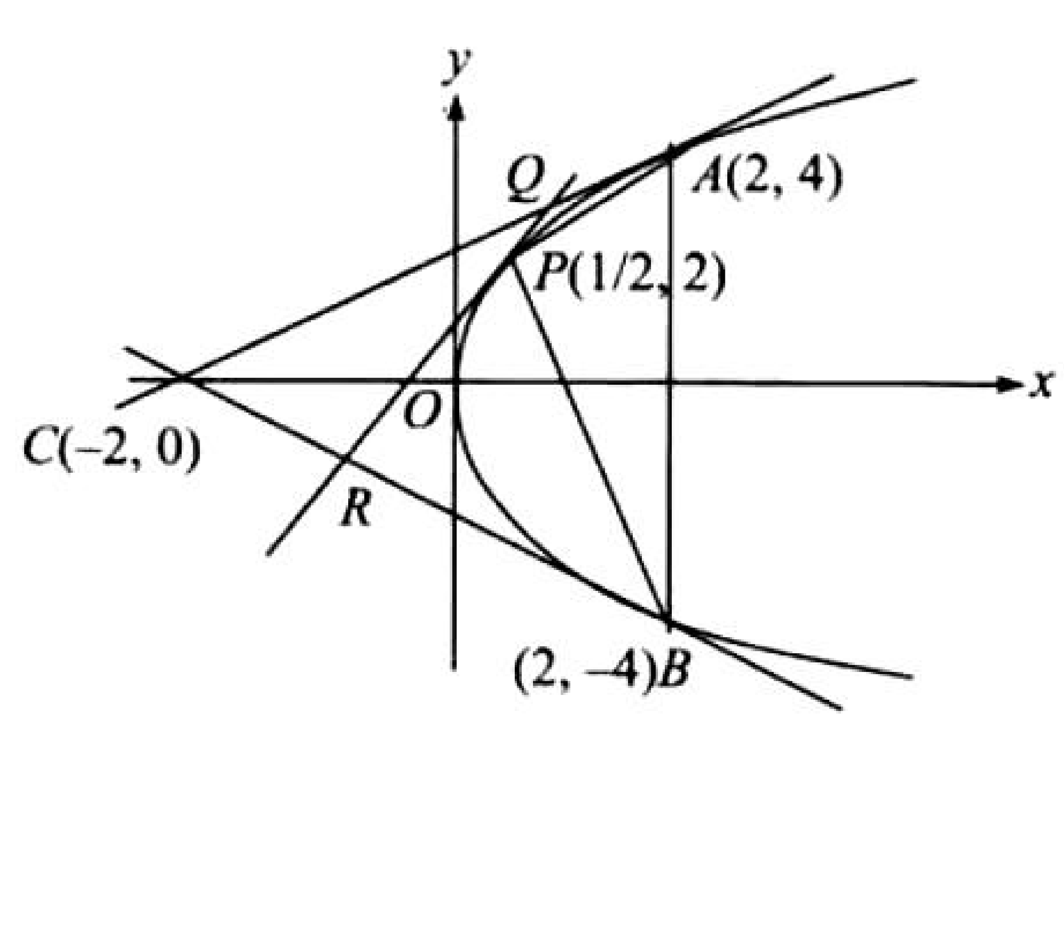
   $\therefore m = 1$ or $A(t)|_{\text{max.}} = \frac{1}{2}$ sq. units

   $y_0 = 4$

   and $y_1 = 2$

Integer Answer Type

1. **(2)**



   $y^2 = 8x$

   Tangents at the endpoints of latus rectum meet the directrix on the x-axis at (–2, 0).

   Δ_1 = Area of ΔABQ

   $= \frac{1}{2}(8)\left(2 - \frac{1}{2}\right) = 6$

   Δ_2 = Area of ΔCQR

   Now, tangent at point $A(2, 4)$ is $4y = 4(x + 2)$ or $y = x + 2$.
   Tangent at point $(2, -4)$ is $-4y = 4(x + 2)$ or $y = -x - 2$.

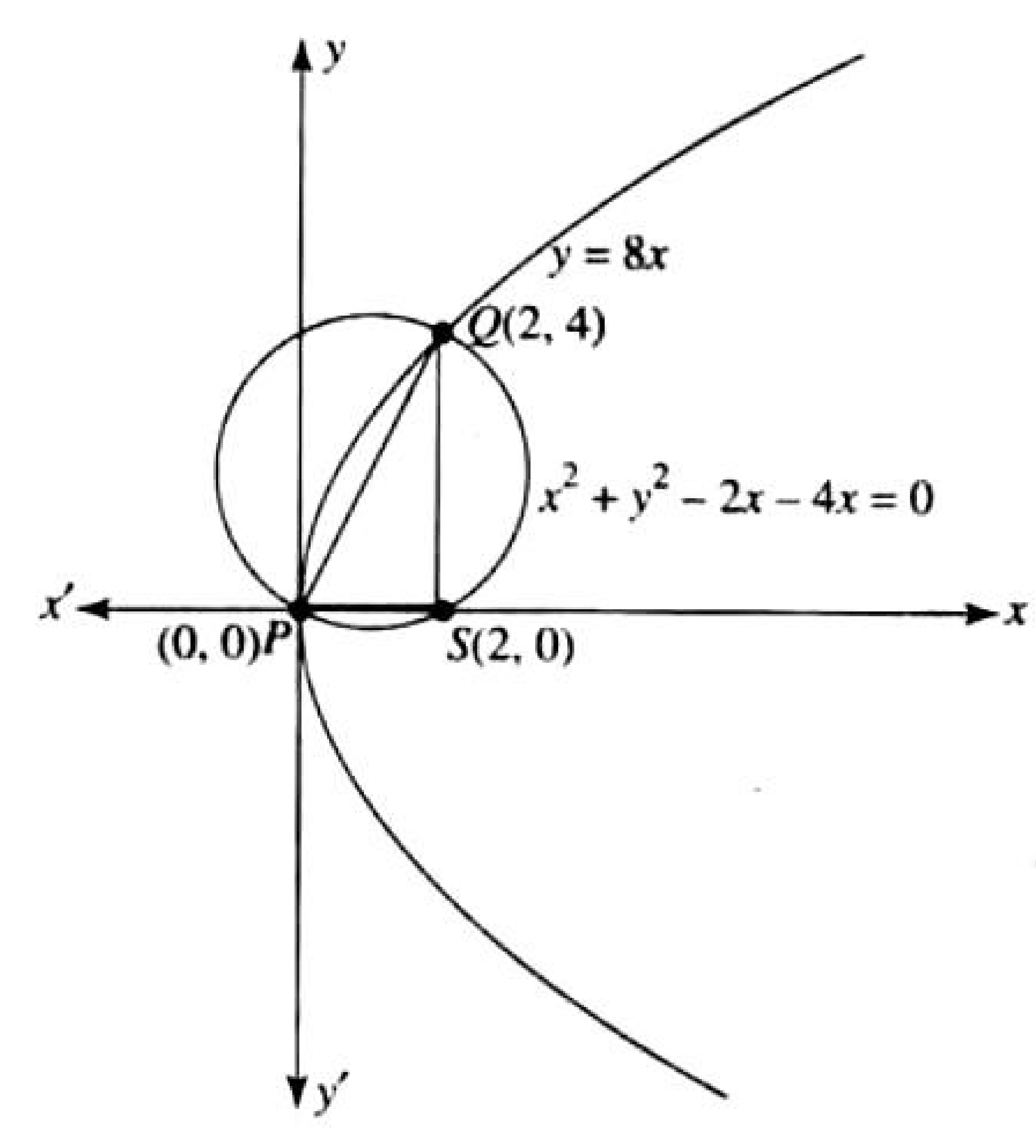
   Also, tangent at point $P(1/2, 2)$ is $2y = 4(x + 1/2)$ or $y = 2x + 1$

   Solving for Q and R, we get $Q(1, 3)$ and $R(-1, -1)$.

   Hence, area of ΔCQR is $\frac{1}{2}\left\|\begin{matrix} -1 & -1 & 1 \\ 1 & 3 & 1 \\ -2 & 0 & 1 \end{matrix}\right\| = \frac{1}{2}(-3 + 2 + 6 + 1) = 3$

   Hence, the ratio of area is $6/3 = 2$.

2. **(4)**



   We have parabola $y^2 = 8x$ with circle $x^2 + y^2 - 2x - 4y = 0$

   Solving parabola $(2t^2, 4t)$ with circle, we get $4t^4 + 16t^2 - 4t^2 - 16t = 0$

   Or $\quad t^4 + 3t^2 - 4t = 0$

   $\Rightarrow \quad t = 0, 1$

   So, the points P and Q are (0, 0) and (2, 4), respectively which are also diametrically opposite points on the circle. The focus is $S(2, 0)$.

   Area of $\Delta PQS = \frac{1}{2} \times 2 \times 4 = 4$ sq. units

3. **(4)**

   let $P(t^2, 2t)$ be a point on the curve $y^2 = 4x$ and $Q(h, k)$ be it's image in $x + y + 4 = 0$.

   $\therefore \quad \frac{h - t^2}{1} = \frac{k - 2t}{1} = -\frac{2(t^2 + 2t + 4)}{2}$

   $\Rightarrow \quad h = -(2t + 4)$ and $k = -(t^2 + 4)$

   Now $y = -5$ intersect this locus.

$\therefore \quad k = -5$, so $t = \pm 1$

Hence, $h = -2, -6$

So, points of intersection are $A(-2, -5)$ and $B(-6, -5)$.

Hence, $AB = 4$.

4. (2)

End points of latus rectum of parabola $y^2 = 4x$ are $(1, \pm 2)$.

Equation of normals at points $(1, \pm 2)$ are

$y = -x + 3$ and $y = x - 3$

or $\quad x + y - 3 = 0$ and $x - y - 3 = 0$

These lines are tangent to circle. $(x-3)^2 + (y+2)^2 = r^2$

$$\therefore \quad \left|\frac{3 \pm 2 - 3}{\sqrt{1+1}}\right| = r$$

or $\quad r^2 = 2$

Assertion-Reasoning Type

1. **a.** The given curve is

$$y = -\frac{x^2}{2} + x + 1$$

or $(x-1)^2 = -2\left(y - \frac{3}{2}\right)$

which is a parabola. So, it should be symmetric with respect to its axis $x - 1 = 0$.

Therefore, both the statements are true and statement 2 is the correct explanation of statement 1.

Fill in the Blanks Type

1. Tangents at the extremities of the focal chord intersect on the directrix and tangents at the end of latus rectum intersect at the foot of directrix, $(-1, 0)$.

Subjective Type

1. The equation of a normal to the parabola $y^2 = 4x$ in its slope form is given by

$y = mx - 2am - am^3$

Therefore, the equation of normal to $y^2 = 4x$ is

$y = mx - 2m - m^3$ (i)

Since the normal drawn at three different points on the parabola passes through (h, k), it must satisfy (i). Therefore,

$k = mh - 2m - m^3$

or $m^3 - (h-2)m + k = 0$

This cubic equation in m has three different roots, say m_1, m_2, and m_3. Therefore,

$m_1 + m_2 + m_3 = 0$ (ii)

$m_1m_2 + m_2m_3 + m_3m_1 = -(h-2)$ (iii)

Now, $(m_1 + m_2 + m_3)^2 = 0$ [Squaring (ii)]

or $m_1^2 + m_2^2 + m_3^2 = -2(m_1m_2 + m_2m_3 + m_3m_1)$

or $m_1^2 + m_2^2 + m_3^2 = 2(h-2)$ [Using (iii)]

Since the LHS of this equation is the sum of perfect squares, it is positive. So,

$h - 2 > 0$

or $h > 2$

2. Normal at point $A(at_1^2, 2at_1)$ meets the parabola at point $B(at_2^2, 2at_2)$. So,

$t_2 = -t_1 - \frac{2}{t_1}$ (i)

Also, AB subtends right angle at the vertex. So,

$t_1t_2 = -4$ (ii)

Eliminating t_2 from (i) and (ii), we get

$-\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$

or $\frac{2}{t_1} = t_1$

$\therefore$ Slope of AB = Slope of normal at $A = \pm\sqrt{2}$

3. The equation of normal to the parabola $x^2 = 4y$ having slope m is

$x = my - 2m - m^3$

Since it passes through the point $(1, 2)$, we have

$1 = 2m - 2m - m^3$

or $m = -1$

Hence, the equation of normal is $x = -y + 2 + 1$ or $x + y - 3 = 0$.

4. Using the result of problem 1, we have $h \geq 2a$, if three normals can be drawn to $y^2 = 4ax$ from (h, k).

Hence, for the given question,

$c \geq 2 \times \frac{1}{4}$ or $c \geq \frac{1}{2}$

Alternative Method:

We know that from any point, normal to $y^2 = 4ax$ is given by $y = mx - 2am - am^3$

For $y^2 = x$, $a = 1/4$. So, normal is

$$y = mx - \frac{m}{2} - \frac{m^3}{4}$$

This normal passes through $(c, 0)$. Therefore,

$$mc - \frac{m}{2} - \frac{m^3}{4} = 0 \quad \text{(i)}$$

$$\therefore m\left[c - \frac{1}{2} - \frac{m^2}{4}\right] = 0$$

$$\therefore m = 0 \text{ or } m^2 = 4\left(c - \frac{1}{2}\right)$$

$m = 0$ shows normal is $y = 0$, i.e., the x-axis is always a normal. Also,

$m^2 \geq 0$

or $4\left(c - \frac{1}{2}\right) \geq 0$

or $c \geq 1/2$

At $c = 1/2$, from (i), $m = 0$.

Therefore, for other real value of m, $c > 1/2$.

Now, for other two normals to be perpendicular to each other, we must have $m_1 \cdot m_2 = -1$.

Therefore, the product of the roots of the equation

$\frac{m^2}{4} + \frac{1}{2} - c = 0$ is -1. So,

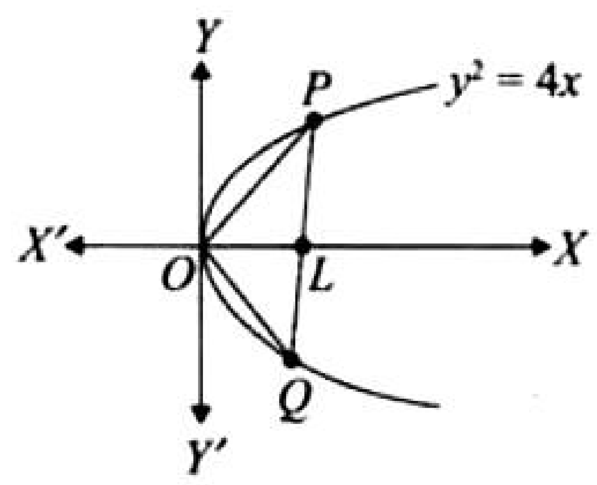
$$\frac{\{(1/2) - c\}}{1/4} = -1$$

or $\frac{1}{2} - c = -\frac{1}{4}$

or $c = \frac{3}{4}$

5. The chord joining $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ subtends right angle at the origin. Therefore,

$t_1 t_2 = -4$



Also, slope of chord $PQ = \frac{2}{t_1 + t_2}$

So, the equation of chord PQ is

$$y - 2t_1 = \frac{2}{t_1 + t_2}(x - t_1^2)$$

or $(t_1 + t_2)y - 2t_1^2 - 2t_1 t_2 = 2(x - t_1^2)$

or $(t_1 + t_2)y - 2t_1^2 + 8 = 2(x - t_1^2)$

or $(t_1 + t_2)y + 8 = 2x$

or $2(x - 4) = (t_1 + t_2)y$

which always passes through point (4, 0).

If (h, k) is the midpoint of PQ, then

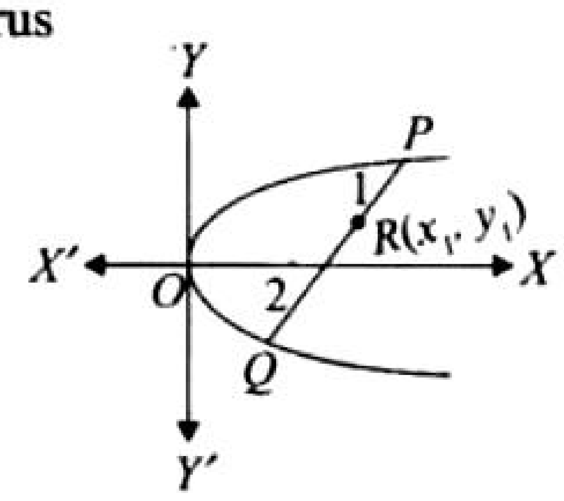
$$h = \frac{t_1^2 + t_2^2}{2} \text{ and } k = t_1 + t_2$$

$$\text{or } h = \frac{(t_1 + t_2)^2 - 2(t_1 t_2)}{2}$$

$$= \frac{k^2 + 8}{2}$$

or $y^2 = 2(x - 4)$, which is required lorus

6. Let $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ be the ends of the chord PQ of the parabola $y^2 = 4x$. Therefore, (i)



Slope of chord $PQ = \frac{2}{t_2 + t_1} = 2$

or $t_2 + t_1 = 1$ (ii)

If $R(x_1, y_1)$ is a point dividing PQ internally in the ratio 1 : 2, then

$$x_1 = \frac{t_2^2 + 2t_1^2}{1 + 2}$$

$$\text{and } y_1 = \frac{1(2t_2) + 2(2t_1)}{1 + 2}$$

$$\therefore t_2^2 + 2t_1^2 = 3x_1 \quad \text{(iii)}$$

$$\text{and } t_2 + 2t_1 = \frac{(3y_1)}{2} \quad \text{(iv)}$$

From (ii) and (iv), we get

$$t_1 = \frac{3}{2}y_1 - 1, \; t_2 = 2 - \frac{3}{2}y_1$$

Substituting in (iii), we get

$$\left(2 - \frac{3}{2}y_1\right)^2 + 2\left(\frac{3}{2}y_1 - 1\right)^2 = 3x_1$$

$$\left(y_1 - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x_1 - \frac{2}{9}\right)$$

Therefore, the locus of the point $R(x_1, y_1)$ is

$$\left(y - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x - \frac{2}{9}\right)$$

which is a parabola having vertex at (2/9, 8/9).

7. Let the three points on the parabola $y^2 = 4ax$ be

$A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, and $C(at_3^2, 2at_3)$.

Then, the equations of tangents at A, B, and C are, respectively,

$$y = \frac{x}{t_1} + at_1 \quad \text{(i)}$$

$$y = \frac{x}{t_2} + at_2 \quad \text{(ii)}$$

$$\text{and } y = \frac{x}{t_3} + at_3 \quad \text{(iii)}$$

Solving the above equations pairwise, we get the points $P(at_1t_2, a(t_1 + t_2))$, $Q(at_2t_3, a(t_2 + t_3))$, and $R(at_3t_1, a(t_3 + t_1))$. Now,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left\|\begin{matrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{matrix}\right\|$$

$$= a^2 \left\|\begin{matrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{matrix}\right\|$$

$$= \left|a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right| \quad \text{(iv)}$$

$$\text{Also, area of } \Delta PQR = \frac{1}{2}\left\|\begin{matrix} 1 & a_1t_2 & a(t_1 + t_2) \\ 1 & a_2t_3 & a(t_2 + t_3) \\ 1 & a_3t_1 & a(t_3 + t_1) \end{matrix}\right\|$$

$$= \frac{a^2}{2}\left\|\begin{matrix} 1 & t_1t_2 & t_1 + t_2 \\ 1 & t_2t_3 & t_2 + t_3 \\ 1 & t_3t_1 & t_3 + t_1 \end{matrix}\right\|$$

$$= \frac{a^2}{2}\left\|\begin{matrix} 0 & (t_1 - t_3)t_2 & t_1 - t_3 \\ 0 & (t_2 - t_1)t_3 & t_2 - t_1 \\ 1 & t_3t_1 & t_3 + t_1 \end{matrix}\right\|$$

Expanding along C_1, we get

$$\text{Area of } \Delta PQR = \left|\frac{a^2}{2}(t_1 - t_3)(t_2 - t_1)(t_2 - t_3)\right| \quad \text{(v)}$$

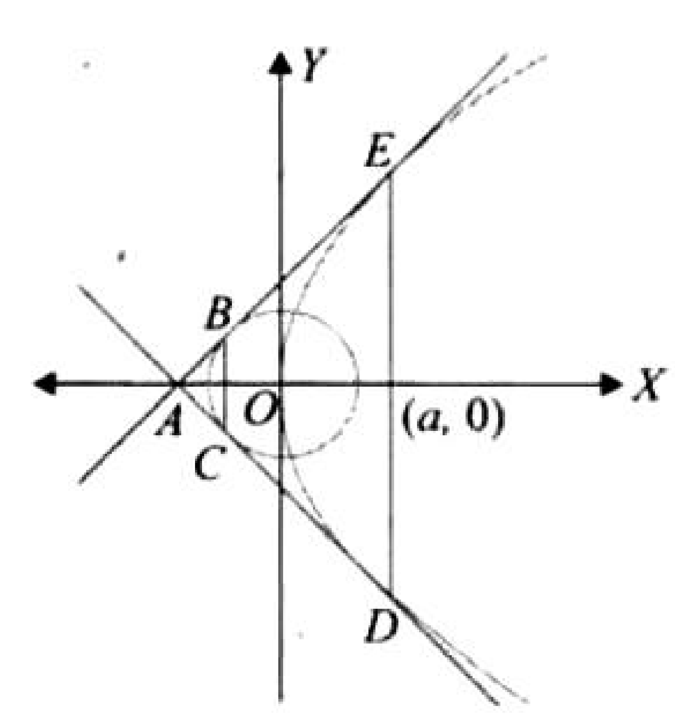
From (iv) and (v), we get

$$\frac{\text{Ar}(\Delta ABC)}{\text{Ar}(\Delta PQR)} = \frac{a^2 \,|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|}{(a^2/2)\,|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|} = \frac{2}{1}$$

Therefore, the required ratio is 2 : 1.

8. The equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}$$



This line will touch the circle

$x^2 + y^2 = \frac{a^2}{2}$

if $\frac{a}{m} = \pm \frac{a}{\sqrt{2}} \sqrt{m^2+1}$ $\quad [c = \pm r\sqrt{1+m^2}]$

or $\frac{a^2}{m^2} = \frac{a^2}{2}(m^2+1)$

or $2 = m^4 + m^2$

or $m^4 + m^2 - 2 = 0$

or $(m^2+2)(m^2-1) = 0$

or $m = 1, -1$

Thus, the two tangents (common one) are $y = x + a$ and $y = -x - a$.

These two intersect each other at $(-a, 0)$.

The chord of contact of the circle w.r.t. $A(-a, 0)$ is

$(-a)x + (0)y = \frac{a^2}{2}$

or $x = -\frac{a}{2}$

and the chord of contact of the parabola w.r.t. $A(-a, 0)$ is

$(0)y = 2a(x - a)$

or $x = a$

Note that DE is the latus rectum of the parabola $y^2 = 4ax$. Therefore, its length is $4a$.

The chords of contact are clearly parallel to each other. So, the required quadrilateral is a trapezium.

$\text{Ar(trap } BCDE) = \frac{1}{2}(BC + DE) \times KL$

$= \frac{1}{2}(a + 4a)\left(\frac{3a}{2}\right)$

$= \frac{15a^2}{4}$

9. The point of intersection of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$R(h, k) \equiv (at_1t_2, a(t_1 + t_2))$

$\therefore t_1 + t_2 = \frac{k}{a}$ and $t_1t_2 = \frac{h}{a}$

Now, $\tan 45° = \left|\frac{(1/t_1) - (1/t_2)}{1 + (1/t_1t_2)}\right|$

or $1 = \left|\frac{t_2 - t_1}{t_1t_2 + 1}\right|$

$= \left|\frac{\sqrt{(t_1+t_2)^2 - 4t_1t_2}}{t_1t_2 + 1}\right|$

or $1 = \left|\frac{(k^2/a^2) - 4(h/a)}{((h/a)+1)^2}\right|$

or $k^2 - 4ah = (h + a)^2$

or $x^2 - y^2 + 6ay + a^2 = 0$

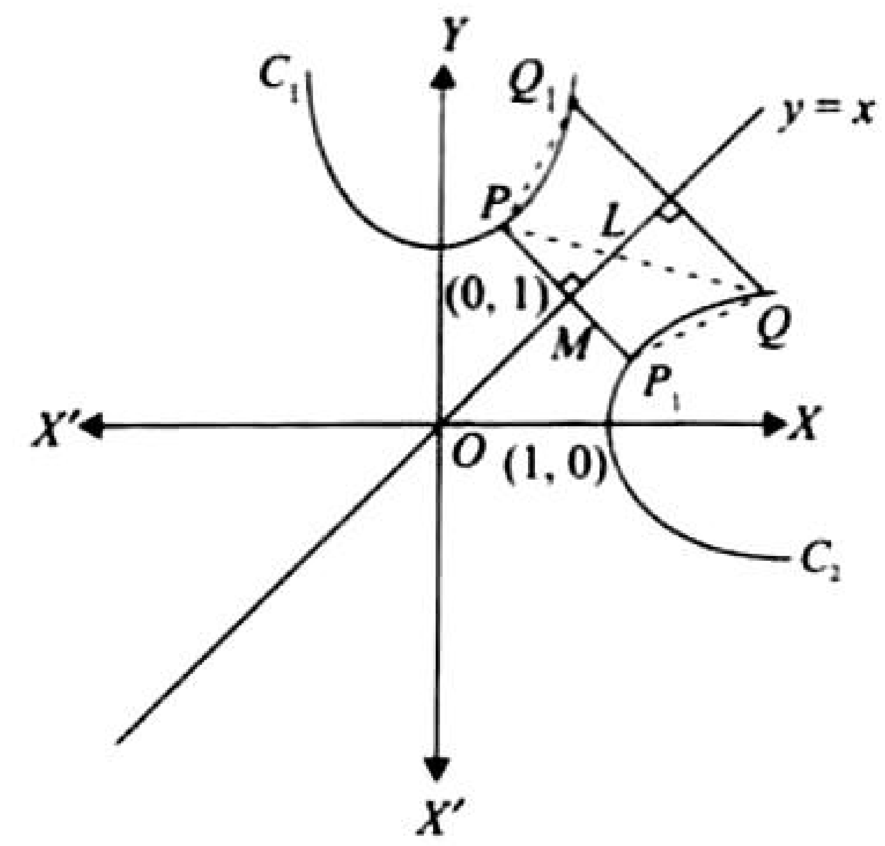
which is a parabola.

10. Given that $C_1: x^2 = y - 1$, $C_2: y^2 = x - 1$.

Here, C_1 and C_2 are symmetrical about the line $y = x$.

Let $P(x_1, x_1^2 + 1)$ be on C_1 and $Q(y_2^2 + 1, y_2)$ be on C_2.

Then the image of P in $y = x$ is $P_1(x_1^2 + 1, x_1)$ on C_2. The image of Q in $y = x$ is $Q_1(y_2, y_2^2 + 1)$ on C_1.



Now, PP_1 and QQ_1 both are perpendicular to the mirror line $y = x$.

Also, M is the midpoint of PP_1, since P_1 is the mirror image of P in $y = x$. So,

$PM = \frac{1}{2}PP_1$

In ΔPML,

$PL > PM$

or $PL > \frac{1}{2}PP_1$ (i)

Similarly, $LQ > \frac{1}{2}QQ_1$ (ii)

Adding (i) and (ii), we get

$PL + LQ > \frac{1}{2}(PP_1 + QQ_1)$

or $PQ > \frac{1}{2}(PP_1 + QQ_1)$

Therefore, PQ is more than the mean of PP_1 and QQ_1 or

$PQ \geq \min(PP_1, QQ_1)$

Let $\min(PP_1, QQ_1) = PP_1$. Then,

$PQ^2 \geq PP_1^2 = (x_1^2 + 1 - x_1)^2 + (x_1^2 + 1 - x_1)^2$

$= 2(x_1^2 + 1 - x_1)^2$

$= f(x_1)$

or $f'(x_1) = 4(x_1^2 + 1 - x_1)(2x_1 - 1)$

$= 4\left\{\left(x_1 - \frac{1}{2}\right)^2 + \frac{3}{4}\right\}(2x_1 - 1)$

Therefore, $f'(x_1) = 0$ when $x_1 = 1/2$.

Also, $f'(x_1) < 0$ if $x_1 < 1/2$.

and $f'(x_1) > 0$ if $x_1 > 1/2$.

So, $f(x_1)$ is minimum when $x_1 = 1/2$.

Thus, at $x_1 = 1/2$, point P is P_0 on C_1.

$P_0\left(\frac{1}{2}, \left(\frac{1}{2}\right)^2 + 1\right) \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$

Similarly, Q_0 on C_2 will be the image of P_0 with respect to $y = x$. So,

$Q_0 \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

11. The equation of normal to the parabola $y^2 = 4x$ having slope m is

$y = mx - 2m - m^3$

It passes through the point $P(h, k)$. So,

$mh - k - 2m - m^3 = 0$

or $m^3 + (2 - h)m + k = 0$ (i)

which is cubic in m and has three roots such that the product of roots

$m_1 m_2 m_3 = -k$ [From (i)]

But given that $m_1 m_2 = \alpha$. So,

$m_3 = -\frac{k}{\alpha}$

But m_3 must satisfy (i). So,

$\frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$

or $k^2 + 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$

So, the locus of $P(h, k)$ is $y^2 = \alpha^2 x + (\alpha^3 - 2\alpha^2)$.

But given that the locus of P is a part of the parabola $y^2 = 4x$.

Therefore, comparing the two, we get

$\alpha^2 = 4$ and $\alpha^3 - 2\alpha^2 = 0$

$\therefore \alpha = 2$

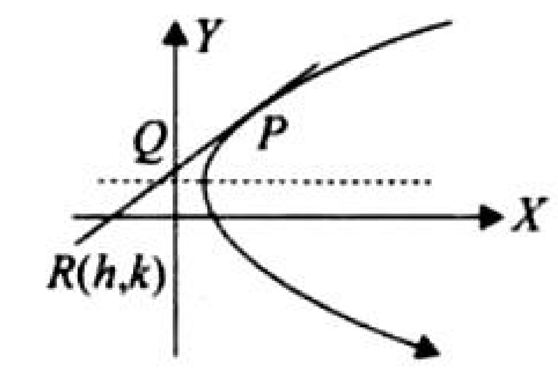
12. The parabola is $(y - 1)^2 = 4(x - 1)$ whose directrix is the y-axis or $x = 0$.

Any point on this parabola is $P(t^2 + 1, 2t + 1), t \in R$.

Therefore, the equation of tangent at $P(t^2 + 1, 2t + 1)$ is

$t(y - 1) = (x - 1) + t^2$

or $x - ty + (t^2 + t - 1) = 0$ (i)



Tangent meets the directrix at

$Q\left(0, \frac{t^2 + t - 1}{t}\right)$

$\frac{QR}{PR} = \frac{1/2}{1} = \frac{1}{2}$

$\therefore \frac{QR}{PQ} = \frac{1}{1}$

$\therefore Q$ is the midpoint of PR. Therefore,

$\frac{h + t^2 + 1}{2} = 0$

and $\frac{k + 2t + 1}{2} = \frac{t^2 + t - 1}{t}$

or $t^2 = -1 - h$

and $kt + 2t^2 + t = 2t^2 + 2t - 2$

or $t^2 = -1 - h$

and $2 = t(1 - k)$

Eliminating t, we have

$4 = (-1 - h)(1 - k)^2$

or $(x + 1)(y - 1)^2 + 4 = 0$