COMPLEX NUMBERS [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answers Type

1. If the cube roots of unity are 1, ω, ω^2, then the roots of the equation $(x-1)^3 + 8 = 0$ are

a. $-1, 1+2\omega, 1+2\omega^2$ **b.** $-1, 1-2\omega, 1-2\omega^2$
c. $-1, -1, -1$ **d.** none of these

(IIT-JEE 1979)

2. The smallest positive integer n for which $[(1+i)/(1-i)]^n = 1$ is

a. $n = 8$ **b.** $n = 16$
c. $n = 12$ **d.** none of these

(IIT-JEE 1980)

3. The complex numbers $z = x + iy$ which satisfy the equation $|(z-5i)/(z+5i)| = 1$ lie on

a. the x-axis
b. the straight line $y = 5$
c. a circle passing through the origin
d. none of these **(IIT-JEE 1981)**

4. If $z = [(\sqrt{3}/2) + i/2]^5 + [(\sqrt{3}/2) - i/2]^5$, then

a. $\text{Re}(z) = 0$ **b.** $\text{Im}(z) = 0$
c. $\text{Re}(z) > 0, \text{Im}(z) > 0$ **d.** $\text{Re}(z) > 0, \text{Im}(z) < 0$

(IIT-JEE 1982)

5. The inequality $|z-4| < |z-2|$ represents the region given by

a. $\text{Re}(z) \geq 0$ **b.** $\text{Re}(z) < 0$
c. $\text{Re}(z) > 0$ **d.** none of these

(IIT-JEE 1982)

6. If $z = x + iy$ and $\omega = (1 - iz)/(z - i)$, then $|\omega| = 1$ implies that in the complex plane

a. z lies on the imaginary axis
b. z lies on the real axis
c. z lies on the unit circle
d. none of these **(IIT-JEE 1983)**

7. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

a. $z_1 + z_4 = z_2 + z_3$ **b.** $z_1 + z_3 = z_2 + z_4$
c. $z_1 + z_2 = z_3 + z_4$ **d.** none of these

(IIT-JEE 1983)

8. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles

a. have the same area b. are similar
c. are congruent d. none of these

(IIT-JEE 1985)

9. If z_1 and z_2 are two nonzero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to

a. $-\pi$ b. $-\frac{\pi}{2}$ c. 0
d. $\frac{\pi}{2}$ e. π **(IIT-JEE 1987)**

10. The value of $\sum_{k=1}^{6}(\sin(2\pi k/7) - i\cos(2\pi k/7))$ is

a. -1 b. 0 c. $-i$
d. i e. none **(IIT-JEE 1987)**

11. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

a. $x = n\pi$ b. $x = 0$
c. $x = (n + 1/2)\pi$ d. no value of x

(IIT-JEE 1988)

12. If $\omega\ (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B, are respectively

a. 0, 1 b. 1, 1 c. 1, 0 d. -1, 1

(IIT-JEE 1995)

13. Let z and ω be two nonzero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals

a. ω b. $-\omega$ c. $\bar{\omega}$ d. $-\bar{\omega}$

(IIT-JEE 1995)

14. Let z and ω be two complex numbers such that $|z| \leq 1$, $|\omega| \leq 1$ and $|z - i\omega| = |z - i\bar{\omega}| = 2$, then z equals

a. 1 or i b. i or $-i$ c. 1 or -1 d. i or -1

(IIT-JEE 1995)

15. For positive integers n_1, n_2 the value of the expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$ is a real number if and only if

a. $n_1 = n_2 + 1$ b. $n_1 = n_2 - 1$
c. $n_1 = n_2$ d. $n_1 > 0, n_2 > 0$

(IIT-JEE 1996)

16. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals

a. 128ω b. -128ω c. $128\omega^2$ d. $-128\omega^2$

(IIT-JEE 1998)

17. The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

a. i b. $i - 1$ c. $-i$ d. 0

(IIT-JEE 1998)

18. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

a. $x = 3, y = 0$ b. $x = 1, y = 3$
c. $x = 0, y = 3$ d. $x = 0, y = 0$

(IIT-JEE 1998)

19. If $i = \sqrt{-1}$, then $4 + 5[(-1/2) + i\sqrt{3}/2]^{334} + 3[(-1/2) + (i\sqrt{3}/2)]^{365}$ is equal to

a. $1 - i\sqrt{3}$ b. $-1 + i\sqrt{3}$
c. $i\sqrt{3}$ d. $-i\sqrt{3}$ **(IIT-JEE 1999)**

20. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$

a. π b. $-\pi$
c. $-\frac{\pi}{2}$ d. $\frac{\pi}{2}$ **(IIT-JEE 2000)**

21. If z_1, z_2, and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|(1/z_1) + (1/z_2) + (1/z_3)\right| = 1$, then $|z_1 + z_2 + z_3|$ is

a. equal to 1 b. less than 1
c. greater than 3 d. equal to 3

(IIT-JEE 2000)

22. Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form

a. $4k + 1$ b. $4k + 2$
c. $4k + 3$ d. $4k$ **(IIT-JEE 2001)**

23. The complex numbers z_1, z_2, and z_3 satisfying $[(z_1 - z_3)/(z_2 - z_3)] = [(1 - i\sqrt{3})/2]$ are the vertices of a triangle which is

a. of area zero b. right-angled isosceles
c. equilateral d. obtuse-angled isosceles

(IIT-JEE 2001)

24. Let $\omega = (-1/2) + i(\sqrt{3}/2)$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

a. 3ω b. $3\omega(\omega - 1)$
c. $3\omega^2$ d. $3\omega(1 - \omega)$

(IIT-JEE 2002)

25. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

a. 0 b. 2
c. 7 d. 17 **(IIT-JEE 2002)**

26. If $|z| = 1$ and $\omega = (z - 1)/(z + 1)$ (where $z \neq -1$), then $\text{Re}(\omega)$ is

a. 0 b. $\frac{1}{|z+1|^2}$

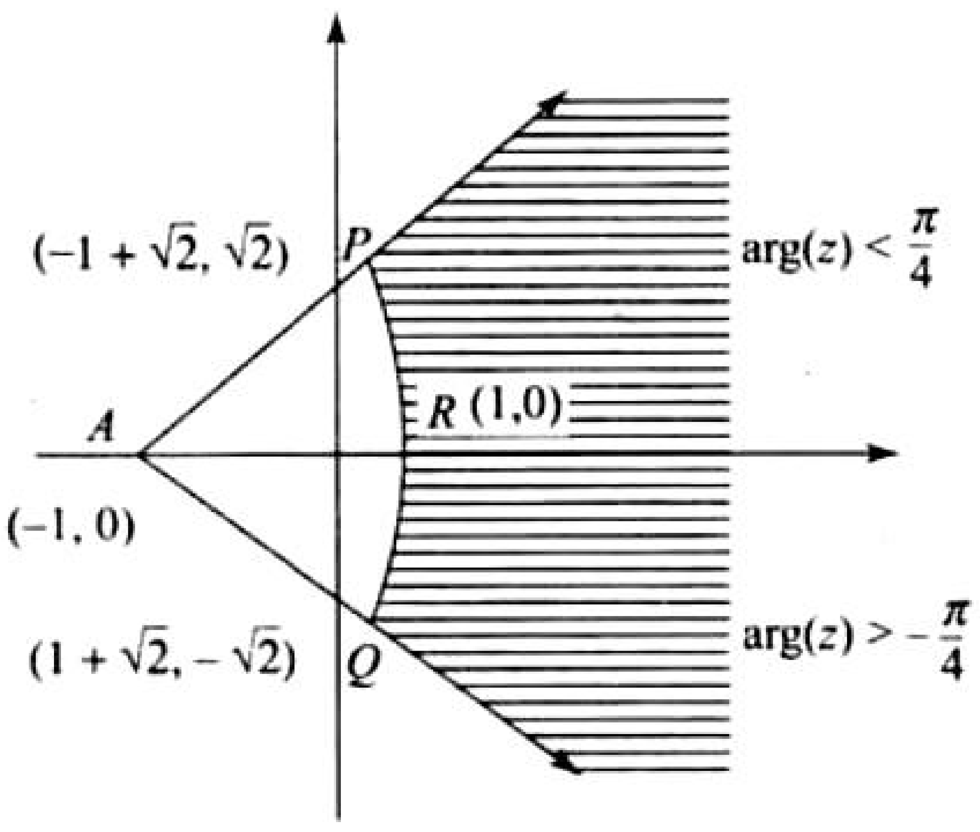
c. $\left|\frac{z}{z+1}\right|\frac{1}{|z+1|^2}$ d. $\frac{\sqrt{2}}{|z+1|^2}$

(IIT-JEE 2003)

27. If $\omega\ (\neq 1)$ is a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is

a. 2 b. 3
c. 5 d. 6 **(IIT-JEE 2004)**

28. The locus of z which lies in shaded region (excluding the boundaries) is best represented by



a. z: $|z+1| > 2$ and $|\arg(z+1)| < \pi/4$
b. z: $|z-1| > 2$ and $|\arg(z-1)| < \pi/4$
c. z: $|z+1| < 2$ and $|\arg(z+1)| < \pi/2$
d. z: $|z-1| < 2$ and $|\arg(z+1)| < \pi/2$ **(IIT-JEE 2005)**

29. a, b, c are integers, not all simultaneously equal, and ω is cube root of unity ($\omega \neq 1$), then minimum value of $|a + b\omega + c\omega^2|$ is

a. 0 **b.** 1 **c.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{1}{2}$

(IIT-JEE 2005)

30. If $(w - \bar{w}z)/(1 - z)$ is purely real where $w = \alpha + i\beta, \beta \neq 0$ and $z \neq 1$, then the set of the values of z is

a. $\{z : |z| = 1\}$ **b.** $\{z : z = \bar{z}\}$
c. $\{z : z \neq 1\}$ **d.** $\{z : |z| = 1, z \neq 1\}$

(IIT-JEE 2006)

31. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is

a. $3e^{i\pi/4} + 4i$ **b.** $(3 - 4i)e^{i\pi/4}$
c. $(4 + 3i)e^{i\pi/4}$ **d.** $(3 + 4i)e^{i\pi/4}$

(IIT-JEE 2007)

32. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $z/(1 - z^2)$ lie on

a. a line not passing through the origin
b. $|z| = \sqrt{2}$
c. the x-axis
d. the y-axis **(IIT-JEE 2007)**

33. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from the origin by 5 units and then vertically away from the origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\pi/2$ in anticlockwise direction on a circle with center at the origin to reach a point z_2. Then point z_2 is given by

a. $6 + 7i$ **b.** $-7 + 6i$ **c.** $7 + 6i$ **d.** $-6 + 7i$

(IIT-JEE 2008)

34. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

a. 48 **b.** 32 **c.** 40 **d.** 80

(IIT-JEE 2009)

35. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is

a. $\frac{1}{\sin 2°}$ **b.** $\frac{1}{3\sin 2°}$ **c.** $\frac{1}{2\sin 2°}$ **d.** $\frac{1}{4\sin 2°}$

(IIT-JEE 2009)

36. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a cannot take the value

a. -1 **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{3}{4}$

(IIT-JEE 2012)

37. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

a. $1/\sqrt{2}$ **b.** $1/2$ **c.** $1/\sqrt{7}$ **d.** $1/3$

(JEE Advanced 2013)

Multiple Correct Answers Type

1. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies

a. $|\omega_1| = 1$ **b.** $|\omega_2| = 1$
c. $\text{Re}(\omega_1\bar{\omega}_2) = 0$ **d.** $\omega_1\bar{\omega}_2 = 0$

(IIT-JEE 1985)

2. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $(z_1 + z_2)/(z_1 - z_2)$ may be

a. zero **b.** real and positive
c. real and negative **d.** purely imaginary

(IIT-JEE 1986)

3. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\arg(w)$ denotes the principal argument of a nonzero complex number w, then

a. $|z - z_1| + |z - z_2| = |z_1 - z_2|$

b. $(z - z_1) = (z - z_2)$

c. $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

d. $\arg(z - z_1) = \arg(z_2 - z_1)$ **(IIT-JEE 2010)**

4. Let $w = \dfrac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ..\}$. Further $H_1 = \left\{z \in C : \text{Re } z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re } z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$, and O represents the origin, then $\angle z_1 O z_2 =$

a. $\pi/2$ **b.** $\pi/6$
c. $2\pi/3$ **d.** $5\pi/6$

(JEE Advanced 2013)

Linked Comprehension Type

For Problems 1–3

Let A, B, C be three sets of complex numbers as defined below:

$A = \{z: \text{Im } z \geq 1\}$
$B = \{z: |z - 2 - i| = 3\}$
$C = \{z: \text{Re}((1 - i)z) = \sqrt{2}\}$ **(IIT-JEE 2008)**

1. The number of elements in the set $A \cap B \cap C$ is
a. 0 **b.** 1 **c.** 2 **d.** ∞

2. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between
a. 25 and 29 **b.** 30 and 34
c. 35 and 39 **d.** 40 and 44

3. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between
a. -6 and 3 **b.** -3 and 6
c. -6 and 6 **d.** -3 and 9

For Problems 4 and 5

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\dfrac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \text{Re } z > 0\}$

(JEE Advanced 2013)

4. Area of A =
a. $\dfrac{10\pi}{3}$ **b.** $\dfrac{20\pi}{3}$ **c.** $\dfrac{16\pi}{3}$ **d.** $\dfrac{32\pi}{3}$

5. $\min\limits_{z \in S} |1 - 3i - z| =$
a. $\dfrac{2-\sqrt{3}}{2}$ **b.** $\dfrac{2+\sqrt{3}}{2}$ **c.** $\dfrac{3-\sqrt{3}}{2}$ **d.** $\dfrac{3+\sqrt{3}}{2}$

Matching Column Type

1. Match the conics in Column I with the statements/expressions in Column II.

Column I	Column II
(a) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(b) Parabola	(q) Points z in the complex plane satisfying $\|z+2\| - \|z-2\| = \pm 3$
(c) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\dfrac{1-t^2}{1+t^2}\right), y = \dfrac{2t}{1+t^2}$
(d) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \leq x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \|z\|^2 + 1$

(IIT-JEE 2009)

2. Match the statements in Column I with those in Column II. [Note: Here z takes the values in the complex plane and Im(z) and Re(z) denote, respectively, the imaginary part and the real part of z]

Column I	Column II
(a) The set of points z satisfying $\|z - i\|z\|\| - \|z + i\|z\|\| = 0$ is contained in or equal to	(p) an ellipse with eccentricity 4/5
(b) The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	(q) the set of points z satisfying Im $z = 0$
(c) If $\|\omega\| = 2$, then the set of points $z = \omega - (1/\omega)$ is contained in or equal to	(r) the set of points z satisfying $\|\text{Im } z\| \leq 1$
(d) If $\|\omega\| = 1$, then the set of points $z = \omega + 1/\omega$ is contained in or equal to	(s) the set of points z satisfying $\|\text{Re } z\| \leq 1$
	(t) the set of points z satisfying $\|z\| \leq 3$

(IIT-JEE 2010)

3. Match the statements given in Column I with the values given in Column II.

Column I	Column II
(a) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\dfrac{\pi}{6}$
(b) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\dfrac{\pi}{6}\right)$ is	(q) $\dfrac{2\pi}{3}$
(c) The value of $\dfrac{\pi^2}{\log_e 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\dfrac{\pi}{3}$

(d) The maximum value of $\left\lvert \text{Arg}\left(\frac{1}{1-z}\right)\right\rvert$ for $\lvert z \rvert = 1, z \neq 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

(IIT-JEE 2011)

4. Match the statements given in Column I with the intervals/union of intervals given in Column II.

Column I	Column II
(a) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, \lvert z \rvert = 1, z \neq \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(b) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$
(c) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2, \infty)$
(d) If $f(x) = x^{3/2}(3x - 10), x \geq 0$, then $f(x)$ is increasing in	(s) $(-\infty, -1] \cup [1, \infty)$
	(t) $(-\infty, 0] \cup [2, \infty)$

(IIT-JEE 2011)

5. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) - i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots, 9$

Column I	Column II
(p) For each z_k there exists a z_j such $z_k \cdot z_j = 1$	(1) True
(q) There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	(2) False
(r) $\frac{\lvert 1-z_1 \rvert \lvert 1-z_2 \rvert \ldots \lvert 1-z_9 \rvert}{10}$ equals	(3) 1
(s) $1 - \sum_{k=0}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	(4) 2

Codes:

	(p)	(q)	(r)	(s)
a.	(4)	(3)	(2)	(1)
b.	(2)	(4)	(3)	(1)
c.	(4)	(3)	(1)	(2)
d.	(2)	(4)	(1)	(3)

(JEE Advanced 2014)

6. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(a) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(p) 1
(b) Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is(are)	(q) 2
(c) Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are)	(r) 3
(d) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $\lvert q - a \rvert$ is (are)	(s) 4
	(t) 5

(JEE Advanced 2015)

Integer Answer Type

1. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to.}$$ **(IIT-JEE 2010)**

2. If z is any complex number satisfying $\lvert z - 3 - 2i \rvert \leq 2$, then the minimum value of $\lvert 2z - 6 + 5i \rvert$ is. **(IIT-JEE 2011)**

3. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$
$a + b\omega^2 + c\omega = z$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is. **(IIT-JEE 2011)**

4. For any integer k, let $\alpha_k = \cos\dfrac{k\pi}{7} + i\sin\dfrac{k\pi}{7}$, where $i = \sqrt{-1}$. Value of the expression $\dfrac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$ is

(JEE Advanced 2015)

Fill in the Blanks Type

1. If the expression $\dfrac{\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) + i\tan(x)\right]}{\left[1 + 2i\sin\left(\frac{x}{2}\right)\right]}$ is real, then the set of all possible values of x is ______________.

(IIT-JEE 1987)

2. For any two complex numbers z_1, z_2 and any real numbers a and b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ______________.

(IIT-JEE 1988)

3. If a, b, c are the numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$, and $z_3 = 0$ form an equilateral triangle, then $a =$ ___________ and $b =$ ___________.

(IIT-JEE 1989)

4. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$, respectively, then A represents the complex number __________ or __________. **(IIT-JEE 1993)**

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_2 =$ _____________, $z_3 =$ _____________.

(IIT-JEE 1994)

6. The value of the expression $1 \times (2 - \omega) \times (2 - \omega^2) + 2 \times (3 - \omega) \times (3 - \omega^2) + \cdots + (n - 1) \times (n - \omega) \times (n - \omega^2)$, where ω is an imaginary cube root of unity, is ______________. **(IIT-JEE 1996)**

True/False Type

1. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $((1 - z)/(1 + z)) \cap 0$.

(IIT-JEE 1984)

2. If the complex numbers z_1, z_2, and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. **(IIT-JEE 1984)**

3. If three complex numbers are in A.P., then they lie on a circle in the complex plane. **(IIT-JEE 1985)**

4. The cube roots of unity when represented on an Argand diagram form the vertices of an equilateral triangle.

(IIT-JEE 1988)

Subjective Type

1. Express $1/(1 - \cos\theta + 2i\sin\theta)$ in the form $x + iy$.

(IIT-JEE 1978)

2. If $x = a + b$, $y = a\beta + b\gamma$, $z = a\gamma + b\beta$, where γ and β are complex cube roots of unity, show that $xyz = a^3 + b^3$.

(IIT-JEE 1978)

3. If $x + iy = \sqrt{(a + ib)/(c + id)}$, then prove that $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$. **(IIT-JEE 1979)**

4. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$. **(IIT-JEE 1980)**

5. Find the real values of x and y for which of the following equation is satisfied:

$$\frac{(1 + i)x - 2i}{3 + i} + \frac{(2 - 3i)y + i}{3 - i} = i$$

(IIT-JEE 1980)

6. Let the complex numbers z_1, z_2, and z_3, be the vertices of an equilateral triangle. Let z_0 be the circumcenter of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

(IIT-JEE 1981)

7. Prove that the complex numbers z_1, z_2, and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1z_2 = 0$.

(IIT-JEE 1983)

8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz, and $z + iz$ is $\frac{1}{2}|z|^2$.

(IIT-JEE 1986)

9. Complex numbers z_1, z_2, z_3 are the vertices A, B, C, respectively, of an isosceles right-angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

(IIT-JEE 1986)

10. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. **(IIT-JEE 1990)**

11. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

(IIT-JEE 1995)

12. If $|z| \le 1$, $|w| \le 1$, then show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$. **(IIT-JEE 1995)**

13. Find all nonzero complex numbers z satisfying $\bar{z} = iz^2$.

(IIT-JEE 1996)

14. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$. **(IIT-JEE 1997)**

15. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane, respectvely. If $\angle AOB = \theta \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2(\theta/2)$. **(IIT-JEE 1997)**

16. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$. **(IIT-JEE 1999)**

17. Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \cdots + \alpha^{q-1} = 0$, but not both together. **(IIT-JEE 2002)**

18. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $|(1 - z_1\bar{z}_2)/(z_1 - z_2)| < 1$. **(IIT-JEE 2003)**

19. Prove that there exists no complex number z such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$. **(IIT-JEE 2003)**

20. Find the center and radius of the circle given by $\left|(z-\alpha)/(z-\beta)\right| = k$, $k \neq 1$, where $z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. **(IIT-JEE 2004)**

21. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$, find the other vertices of the square. **(IIT-JEE 2005)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. b.	**2.** d.	**3.** a.	**4.** b.	**5.** d.
6. b.	**7.** b.	**8.** b.	**9.** c.	**10.** d.
11. d.	**12.** b.	**13.** d.	**14.** c.	**15.** d.
16. d.	**17.** b.	**18.** d.	**19.** c.	**20.** a.
21. a.	**22.** d.	**23.** c.	**24.** b.	**25.** b.
26. a.	**27.** b.	**28.** a.	**29.** b.	**30.** d.
31. d.	**32.** d.	**33.** d.	**34.** a.	**35.** d.
36. d.	**37.** c.			

Multiple Correct Answers Type

1. a., b., c. **2.** a., d. **3.** a., c., d. **4.** c., d.

Linked Comprehension type

1. b. **2.** c. **3.** d. **4.** b. **5.** c.

Matching Column Type

1. (d) – (q), (s); (b) – (s), (t)

2. (a) – (q), (r); b – (p); (c) – (p), (s), (t); (d) – (p), (q), (s), (t)

3. (d) – (t)

4. (a) – (s)

5. a.

6. (c) – (p), (q), (s), (t)

Integer Answer Type

1. (1) **2.** (5) **4.** (4)

Fill in the Blanks Type

1. $x = 2n\pi$ or $x = n\pi + \pi/4, n \in Z$

2. $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

3. $a = 2 - \sqrt{3}$, $b = 2 - \sqrt{3}$ **4.** $1 - \frac{3}{2}i$ or $3 - \frac{i}{2}$

5. $1 - i\sqrt{3}, -2$ **6.** $\frac{1}{4}n[n-1][n^2+3n+4]$

True/False Type

1. True **2.** True **3.** False **4.** True

Subjective Type

1. $\left(\frac{1}{5 + 3\cos\theta}\right) + \left(\frac{-2\cot\theta/2}{5 + 3\cos\theta}\right)$

5. $x = 3, y = -1$

20. Center $\equiv \frac{\alpha - k^2\beta}{1 - k^2}$ and radius $= \frac{k|\alpha - \beta|}{|1 - k^2|}$

21. $z_2 = (1 - \sqrt{3}) + i$, $z_3 = -i\sqrt{3}$, $z_4 = (\sqrt{3} + 1) - i$

Hints and Solutions

Single Correct Answer Type

1. b. $(x-1)^3 + 8 = 0 \quad \Rightarrow \quad \left(\dfrac{x-1}{-2}\right)^3 = 1$

$\Rightarrow \quad \dfrac{x-1}{-2} = 1, \omega, \omega^2$

$\Rightarrow \quad x = -1, 1 - 2\omega, 1 - 2\omega^2$

2. d. $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{1-1+2i}{2} = i$

Now $i^n = 1$. Hence, the smallest positive integral value of n should be 4.

3. a. We know that $|z - z_1| = |z - z_2|$. Then locus of z is the line, which is a perpendicular bisector of line segment joining z_1 and z_2. Hence,

$z = x + iy$

$\Rightarrow \quad |z - 5i| = |z + 5i|$

Therefore, z remains equidistant from $z_1 = 5i$ and $z_2 = -5i$. Hence, z lies on perpendicular bisector of line segment joining z_1 and z_2, which is clearly the real axis or $y = 0$.

Alternate solution:

$\left|\dfrac{z-5i}{z+5i}\right| = 1$

$\Rightarrow \quad |x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow \quad |x + (y-5)i| = |x + (y+5)i|$

$\Rightarrow \quad x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow \quad x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$

$\Rightarrow \quad 20y = 0$

$\Rightarrow \quad y = 0$

4. b. $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$

$= \left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)^5 + \left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)^5$

$= \left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) + \left(\cos\dfrac{5\pi}{6} - i\sin\dfrac{5\pi}{6}\right)$

$= 2\cos\dfrac{5\pi}{6}$

$= -\sqrt{3}$

$\Rightarrow \quad \text{Re}(z) < 0$ and $\text{Im}(z) = 0$

Alternate solution:

$z = z_1 + \bar{z}_1$

where $z_1 = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5$

$\Rightarrow \quad z$ is real

$\Rightarrow \quad \text{Im}(z) = 0$

5. d. $|z - 4| < |z - 2|$

$\Rightarrow \quad |(x-4) + iy| < |(x-2) + iy|$

or $\quad (x-4)^2 + y^2 < (x-2)^2 + y^2$

or $\quad -8x + 16 < -4x + 4$

or $\quad 4x - 12 > 0$

or $\quad x > 3$

$\Rightarrow \quad \text{Re}(z) > 3$

6. b. $|\omega| = 1$

$\Rightarrow \quad \left|\dfrac{1 - iz}{z - i}\right| = 1$

or $\quad |1 - iz| = |z - i|$

or $\quad |-i|\,|z + i| = |z - i|$

or $\quad |z + i| = |z - i|$

Hence, z is equidistant from $(0, -1)$ and $(0, 1)$. So, z lies on perpendicular bisector of $(0, -1)$ and $(0, 1)$, i.e., x-axis, and $y = 0$. Therefore, z lies on the real axis.

7. b. If vertices of a parallelogram are z_1, z_2, z_3, z_4, then as diagonals bisect each other comparing complex numbers of midpoint,

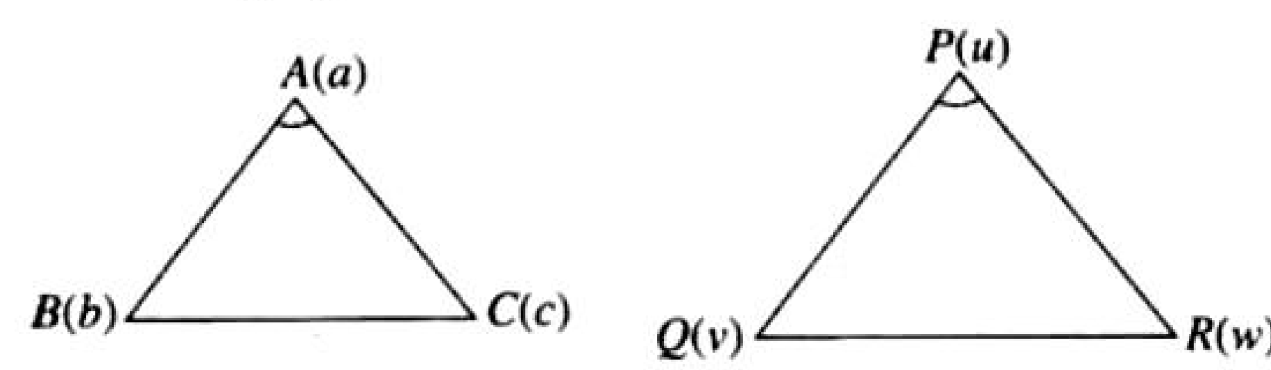
$\dfrac{z_1 + z_3}{2} = \dfrac{z_2 + z_4}{2}$

or $\quad z_1 + z_3 = z_2 + z_4$

8. b. We have $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$

$$\Rightarrow \quad r = \frac{c-a}{b-a} = \frac{w-u}{v-u} \qquad (1)$$

Consider triangles with vertices a, b, c and u, v, w as shown in the following figures.



From (1) $\left|\dfrac{a-c}{a-b}\right| = \left|\dfrac{u-w}{u-v}\right|$

$\Rightarrow \quad \dfrac{AC}{AB} = \dfrac{PR}{PQ}$ (2)

Also, $\arg\left(\dfrac{a-c}{a-b}\right) = \arg\left(\dfrac{u-w}{u-v}\right)$

$\Rightarrow \quad \angle BAC = \angle QPR$ (3)

From (2) and (3), using Side-Angle-Side criterion we can say that triangles ABC and PQR are similar.

9. c. Let $z_1 = |z_1| (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = |z_2| (\cos \theta_2 + i \sin \theta_2)$. Also,

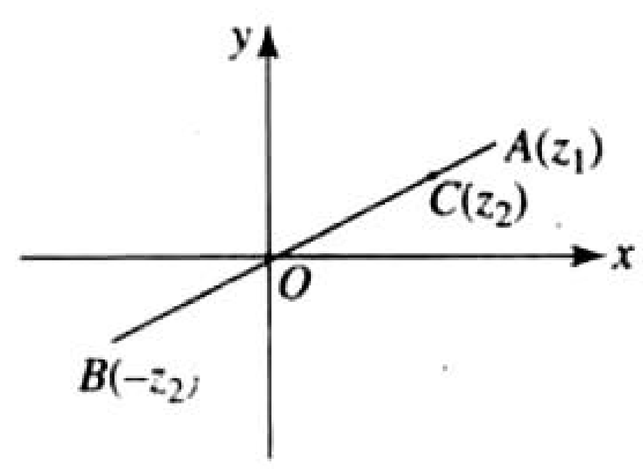
$$|z_1 + z_2| = |z_1| + |z_2|$$
$$\Rightarrow \quad |z_1 + z_2|^2 = (|z_1| + |z_2|)^2$$
$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$\Rightarrow \quad \text{Re}(z_1\bar{z}_2) = 2|z_1||z_2|$$
$$\Rightarrow \quad 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2|$$
$$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1$$
$$\Rightarrow \quad \theta_1 - \theta_2 = 0$$
$$\Rightarrow \quad \arg z_1 - \arg z_2 = 0$$

Alternative Method:

$$|z_1 + z_2| = |z_1| + |z_2|$$
$$\Rightarrow \quad |z_1 + (-z_2)| = |z_1| + |-z_2|$$

$\Rightarrow \quad AB = AO + OB$ for $A(z_1)$, $O(0)$ and $B(-z_2)$

Thus points $A(z_1)$, $O(0)$ and $B(-z_2)$ can be plotted as shown in the following figure.



So, $A(z_1)$, $O(0)$ and $C(z_2)$ are collinear as shown in figure.

$\Rightarrow \arg(z_1) = \arg(z_2)$

10. d. Let $z = \cos(2\pi/7) + i \sin(2\pi/7)$. Then by De Moivre's theorem, we have

$$z^k = \cos\frac{2\pi k}{7} + i \sin\frac{2\pi k}{7}$$

Now, $\displaystyle\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$$= \sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7}\right)$$
$$= (-i)\sum_{k=1}^{6} z^k$$
$$= -i\,\frac{z(1-z^6)}{1-z}$$
$$= -i\left(\frac{z-z^7}{1-z}\right)$$
$$= (-i)\left(\frac{z-1}{1-z}\right) \quad [\text{Using } z^7 = \cos 2k\pi + i \sin 2k\pi = 1]$$
$$= (i)\left(\frac{1-z}{1-z}\right)$$
$$= i$$

Alternative Method:

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$$
$$= \sum_{k=1}^{6}\sin\frac{2\pi k}{7} - i\sum_{k=1}^{6}\cos\frac{2\pi k}{7}$$
$$= \frac{\sin\dfrac{6\left(\frac{2\pi}{7}\right)}{2}}{\sin\dfrac{\left(\frac{2\pi}{7}\right)}{2}}\sin\left(\frac{\frac{2\pi}{7}+\frac{12\pi}{7}}{2}\right) - i\,\frac{\sin\dfrac{6\left(\frac{2\pi}{7}\right)}{2}}{\sin\dfrac{\left(\frac{2\pi}{7}\right)}{2}}\cos\left(\frac{\frac{2\pi}{7}+\frac{12\pi}{7}}{2}\right)$$
$$= \frac{\sin\frac{6\pi}{7}}{\sin\frac{\pi}{7}}\sin(\pi) - i\,\frac{\sin\frac{6\pi}{7}}{\sin\frac{\pi}{7}}\cos(\pi)$$
$$= 0 - i\,\frac{\sin\left(\pi - \frac{\pi}{7}\right)}{\sin\frac{\pi}{7}}(-1)$$
$$= i$$

11. d. Let $z_1 = \sin x + i \cos 2x$; $z_2 = \cos x - i \sin 2x$. Then

$$\bar{z}_1 = z_2$$
$$\Rightarrow \quad \sin x - i\cos 2x = \cos x - i \sin 2x$$
$$\sin x = \cos x \text{ and } \cos 2x = \sin 2x$$
$$\Rightarrow \quad \tan x = 1 \text{ and } \tan 2x = 1$$
$$\Rightarrow \quad x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8}$$

which is not possible. Hence, there is no value of x.

12. b. $(1 + \omega)^7 = A + B\omega$

$\Rightarrow \quad (-\omega^2)^7 = A + B\omega$ $\quad (\because 1 + \omega + \omega^2 = 0)$

$\Rightarrow \quad -\omega^{14} = A + B\omega$

$\Rightarrow \quad -\omega^2 = A + B\omega$ $\quad (\because \omega^3 = 1)$

$\Rightarrow \quad 1 + \omega = A + B\omega$

$\Rightarrow \quad A = 1, B = 1$

13. d. We have

$$|z| = |\omega| \text{ and } \arg z = \pi - \arg\omega$$

Let $\omega = re^{i\theta}$. Then

$$z = re^{i(\pi-\theta)}$$
$$\Rightarrow \quad z = re^{i\pi}e^{-i\theta} = (re^{-i\theta})(\cos\pi + i\sin\pi)$$
$$= \bar{\omega}(-1) = -\bar{\omega}$$

14. c. We have

$2 = |z - i\omega| \le |z| + |\omega|$ $\quad (\because |z_1 + z_2| \le |z_1| + |z_2|)$

$\therefore \quad |z| + |\omega| \ge 2$ (1)

But given that $|z| \le 1$ and $|\omega| \le 1$. Hence,

$|z| + |\omega| \le 2$ (2)

From (1) and (2),

$$|z| = |\omega| = 1$$

Also, $|z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow \quad |z - (-i\omega)| = |z - i\bar{\omega}|$

Hence, z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(i\bar{\omega})$, which is a real axis, as $(-i\omega)$ and $(i\bar{\omega})$ are conjugate to each other. For z, $\text{Im}(z) = 0$. If $z = x$, then

$$|z| \le 1 \Rightarrow x^2 \le 1$$

$\Rightarrow \quad -1 \le x \le 1$

15. d. $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$

$$= \left[(1+i)^{n_1} + (1-i)^{n_1}\right] + \left[(1+i)^{n_2} + (1-i)^{n_2}\right]$$

$$= \left[(1+i)^{n_1} + \overline{(1+i)^{n_1}}\right] + \left[(1+i)^{n_2} + \overline{(1+i)^{n_2}}\right]$$

$$= [\text{purely real number}] + [\text{purely real number}]$$

Hence, n_1 and n_2 are any integers.

16. d. We have,

$$(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7$$
$$= (-2)^7 (\omega^2)^7$$
$$= -128\omega^{14}$$
$$= -128\omega^2$$

17. b. $\sum_{i=1}^{13} (i^n + i^{n+1}) = \sum_{i=1}^{13} i^n (1 + i)$

$$= (1+i) \sum_{i=1}^{13} i^n$$

$$= i(1 + i) \frac{(1 - i^{13})}{1 - i}$$

$$= i - 1 \text{ as } i^{13} = i$$

18. d. Taking $-3i$ common from C_2, we get

$$-3i \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0 \qquad (\because C_2 \equiv C_3)$$

$\Rightarrow \quad x = 0, y = 0$

19. c. $E = 4 + 5(\omega)^{334} + 3(\omega)^{365}$

$$= 4 + 5\omega + 3\omega^2$$
$$= 1 + 2\omega + 3(1 + \omega + \omega^2)$$
$$= 1 + (-1 + i\sqrt{3})$$
$$= i\sqrt{3}$$

20. a. $\arg(-z) - \arg(z) = \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$

21. a. $|z_1| = |z_2| = |z_3| = 1$

Now, $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

Similarly,

$$z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1$$

Now, $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

or $\quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

or $\quad |\overline{z_1 + z_2 + z_3}| = 1$

or $\quad |z_1 + z_2 + z_3| = 1$

22. d. Let

$$z = (1)^{1/n} = (\cos 2k\pi + i \sin 2k\pi)^{1/n}$$
$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \ k = 0, 1, 2, \ldots, n - 1$$

Let $\quad z_1 = \cos\left(\frac{2k_1\pi}{n}\right) + i\sin\left(\frac{2k_1\pi}{n}\right)$

and $\quad z_2 = \cos\left(\frac{2k_2\pi}{n}\right) + i\sin\left(\frac{2k_2\pi}{n}\right)$

be the two values of z such that they subtend angle of 90° at origin. Then

$$\frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1 - k_2) = \pm n$$

As k_1 and k_2 are integers and $k_1 \ne k_2$, therefore $n = 4m$, $m \in Z$.

23. c. $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$

$\Rightarrow \quad \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{1 - i\sqrt{3}}{2}\right)$

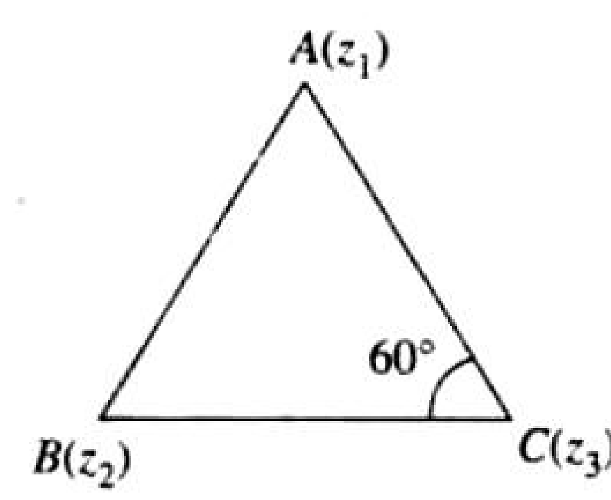
Hence, the angle between $z_1 - z_3$ and $z_2 - z_3$ is 60°. Also,

$$\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right|$$

$\Rightarrow \quad \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1$

or $\quad |z_1 - z_3| = |z_2 - z_3|$

or $\quad AC = BC$



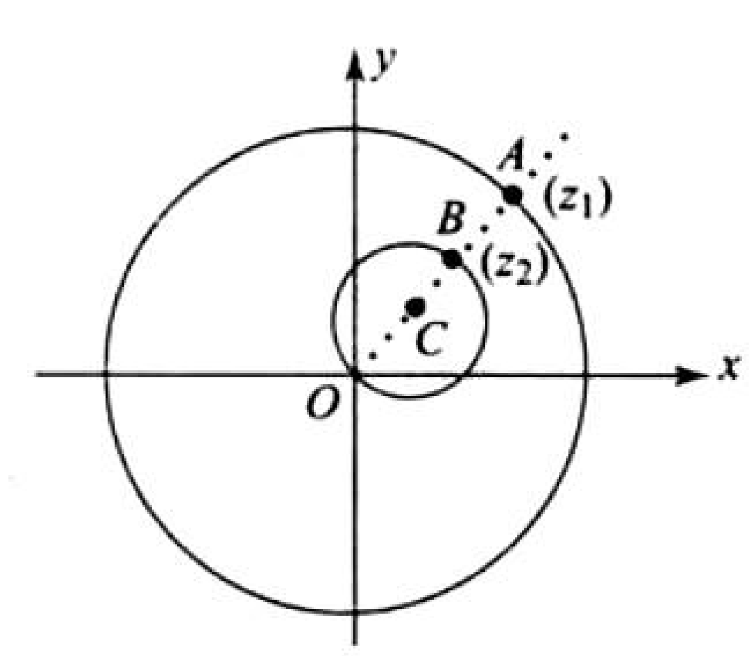
Hence, the triangle with vertices z_1, z_2, and z_3 is isosceles with vertical angle 60°. Hence, rest of the two angles should also be 60° each. Therefore, the required triangle is an equilateral triangle.

24. b. Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = 3[-\omega^4 - \omega^6 - \omega^4]$$

$$= 3(-1 - 2\omega)$$
$$= 3(\omega^2 - \omega)$$
$$= 3\omega(\omega - 1)$$

25. b. Given that $|z_1| = 12$. Therefore, z_1 lies on a circle with center (0, 0) and radius 12 units. As $|z_2 - 3 - 4i| = 5$, so z_2 lies on a circle with center (3, 4) and radius 5 units.



From the figure it is clear that $|z_1 - z_2|$, i.e., distance between z_1 and z_2 will be minimum when they lie at A and B, respectively. Then $|z_1 - z_2| = AB = OA - OB = 12 - 2(5) = 2$.

26. a. $\omega = \dfrac{z-1}{z+1}$

$\Rightarrow \quad z = \dfrac{1+\omega}{1-\omega}$

Now, $|z| = 1$

$\Rightarrow \quad \left|\dfrac{1+\omega}{1-\omega}\right| = 1$

$\Rightarrow \quad |\omega + 1| = |\omega - 1|$

Therefore, ω is equidistant from (1, 0) and (–1, 0) and hence must lie on perpendicular bisector of line segment joining (1, 0) and (–1, 0), i.e., imaginary axis. Hence, ω is purely imaginary, i.e., Re(ω) = 0.

27. b. $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow \quad (-\omega)^n = (1 + \omega)^n = (-\omega^2)^n$

$\Rightarrow \quad \omega^n = 1$

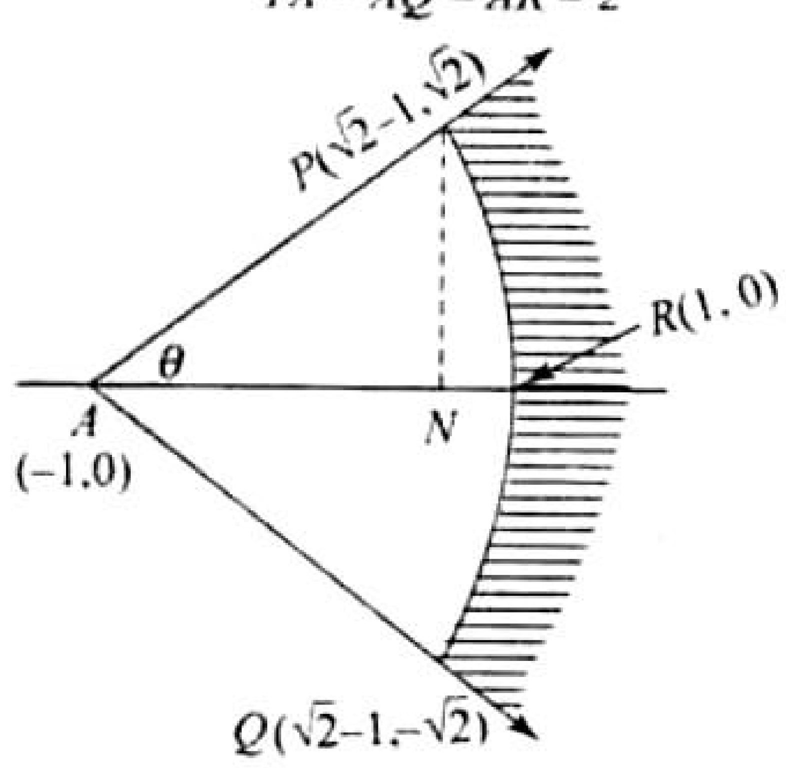
Hence, the least positive value of n is 3.

28. a. Here we observe that

$PA = AQ = AR = 2$

Therefore, PRQ is an arc of a circle with center at A and radius 2. Shaded region is outer (exterior) part of the sector $APRQA$. Hence, for any point x on arc PRQ, we should have

$PA = AQ = AR = 2$



$|z - (-1)| = 2$

and for shaded region,

$|z + 1| > 2 \qquad (1)$

Also, $\tan\theta = \dfrac{PN}{AN} = \dfrac{\sqrt{2}}{(\sqrt{2}-1)-(-1)} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

$\Rightarrow \quad \theta = \pi/4$

and by symmetry, arg(z + 1) varies from $-\pi/4$ to $\pi/4$ as it moves from Q to P on arc QRP. Hence, for shaded region, we also have

$-\pi/4 < \arg(z + 1) < \pi/4$

or $\quad |\arg(z + 1)| < \pi/4 \qquad (2)$

Combining (1) and (2), we find that (a) is the correct option.

29. b. Given that a, b, c are integers not all equal, ω is cube root of unity $\neq 1$. Then

$|a + b\omega + c\omega^2|$

$= \left|a + b\left(\dfrac{-1+i\sqrt{3}}{2}\right) + c\left(\dfrac{-1-i\sqrt{3}}{2}\right)\right|$

$= \left|\left(\dfrac{2a-b-c}{2}\right) + i\left(\dfrac{b\sqrt{3}-c\sqrt{3}}{2}\right)\right|$

$= \dfrac{1}{2}\sqrt{(2a-b-c)^2 + 3(b-c)^2}$

$= \dfrac{1}{2}\sqrt{4a^2 + b^2 + c^2 - 4ab + 2bc - 4ac + 3b^2 + 3c^2 - 6bc}$

$= \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}$

$= \sqrt{\dfrac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]}$

R.H.S. will be minmum when $a = b = c$, but we cannot take $a = b = c$ as per the question. Hence, the minimum value is obtained when any two are zero and third is a minimum magnitude integer, i.e., 1. Thus, $b = c = 0$, $a = 1$ gives us the minimum value of 1.

30. d. Since $(w - \bar{w}z)/(1 - z)$ is purely real, we have

$\overline{\left(\dfrac{w-\bar{w}z}{1-z}\right)} = \left(\dfrac{w-\bar{w}z}{1-z}\right)$

or $\quad \dfrac{\bar{w}-w\bar{z}}{1-\bar{z}} = \dfrac{w-\bar{w}z}{1-z}$

or $\quad \bar{w} - \bar{w}z - w\bar{z} + wz\bar{z} = w - w\bar{z} - \bar{w}z + \bar{w}z\bar{z}$

or $\quad w - \bar{w} = (w - \bar{w})|z|^2$

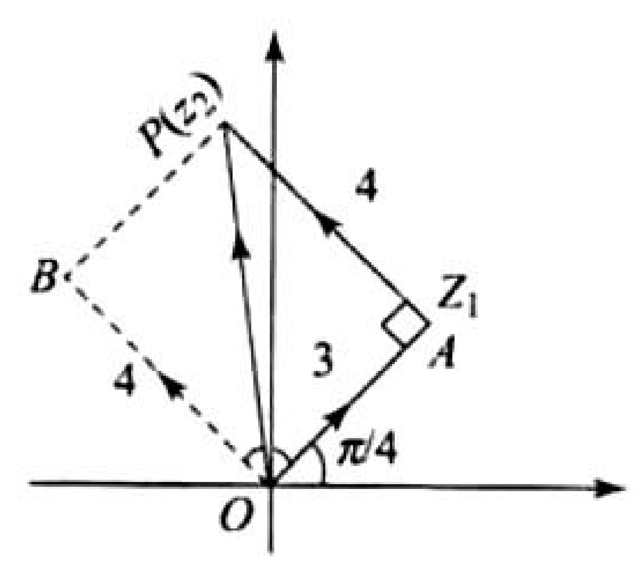
or $\quad |z|^2 = 1 \qquad (\because w = \alpha + i\beta \text{ and } \beta \neq 0)$

or $\quad |z| = 1$

Also given $z \neq 1$. Therefore, the required set is $\{z : |z| = 1, z \neq 1\}$.

31. d. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

Rotating OA by an angle 45° in anticlockwise direction to get OP, we have



$\dfrac{z_2 - 0}{z_1 - 0} = \dfrac{|z_2|}{|z_1|}e^{i\theta}$

(where $\tan\theta = 4/3$)

$\Rightarrow \quad \dfrac{z_2 - 0}{3e^{i\pi/4}} = \dfrac{5}{3}(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \dfrac{z_2 - 0}{e^{i\pi/4}} = 5\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\Rightarrow \quad z_2 = (3 + 4i)e^{i\pi/4}$

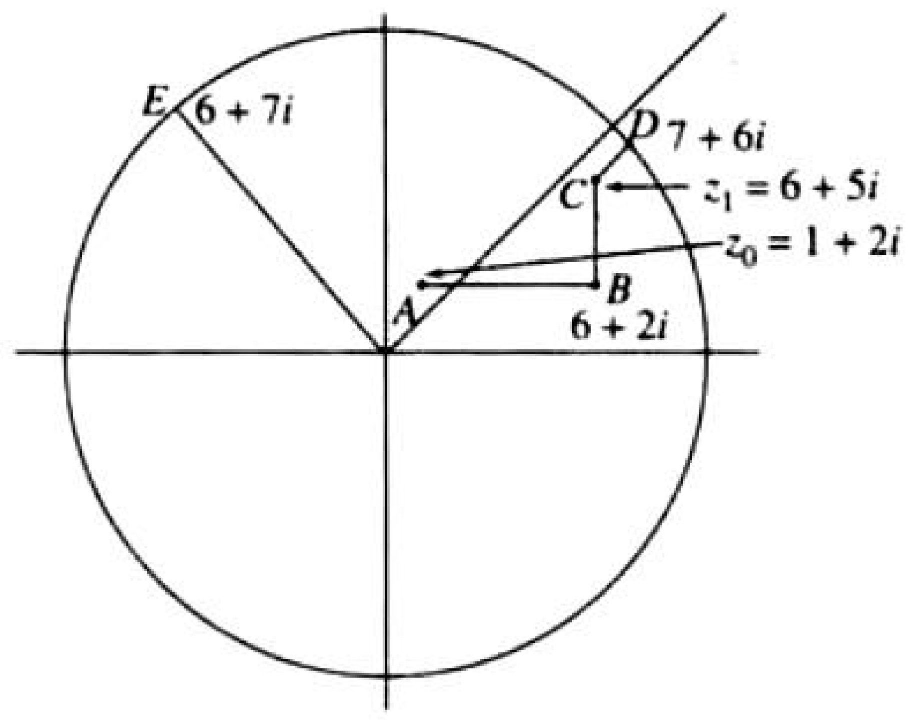
32. d. Given $|z| = 1$ and $z \neq \pm 1$. To find locus of $\omega = z/(1 - z^2)$. We have

$\omega = \dfrac{z}{1-z^2} = \dfrac{z}{z\bar{z} - z^2} \qquad (\because |z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1)$

$= \dfrac{1}{\bar{z} - z}$

which is a purely imaginary number. Therefore, ω must lie on the y-axis.

33. d.



We have $z_0 = 1 + 2i$

$z_1 = 6 + 5i$

Now z_1 moves $\sqrt{2}$ units in the direction of vector $\hat{i} + \hat{j}$ i.e., in the direction of line $y = x$.

Using parametric form of straight line, we get coordinates of point D as

$$\left(6 + \sqrt{2}\cos\frac{\pi}{4}, 5 + \sqrt{2}\sin\frac{\pi}{4}\right) \equiv (7, 6)$$

Now this point moves through an angle $\pi/2$ in anticlockwise direction on a circle with center at the origin to reach a point z_2.

$\therefore \quad z_2 = (7 + 6i)e^{i\pi/2} = -6 + 7i$

34. a. $z\bar{z}(\bar{z}^2 + z^2) = 350$

Puting $z = x + iy$, we have

$(x^2 + y^2)(x^2 - y^2) = 175$

$(x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$

$\therefore \quad x^2 + y^2 = 25$

and $\quad x^2 - y^2 = 7$

(as other combinations give non-integral values of x and y)

$\therefore \quad x = \pm 4, y = \pm 3$ $(x, y \in I)$

$\therefore$ Vertices of rectangle are $(\pm 4, \pm 3)$

Hence, area is $8 \times 6 = 48$ sq. units.

35. d. $S = \sum_{m=1}^{15} \text{Im}(z^{2m-1})$

$= \sin\theta + \sin 3\theta + \ldots + \sin 29\theta$

$\Rightarrow 2(\sin\theta)S = (1 - \cos 2\theta) + (\cos 2\theta - \cos 4\theta) + \ldots + (\cos 28\theta - \cos 30\theta)$

$$\Rightarrow \quad S = \frac{1 - \cos 30\theta}{2\sin\theta}$$

$$= \frac{1}{4\sin 2^\circ}$$

36. d. Given equation is $z^2 + z + 1 - a = 0$

Clearly this equation do not have real roots if

$D < 0$

$\Rightarrow \quad 1 - 4(1 - a) < 0$

$\Rightarrow \quad 4a < 3$

$\Rightarrow \quad a < \frac{3}{4}$

37. c. Given circles are $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$

or $\quad |z - z_0| = r \quad (1)$

and $\quad |z - z_0| = 2r \quad (2)$

where $z_0 = x_0 + iy_0$

Now α and $\frac{1}{\bar{\alpha}}$ lies on circle (1) and (2), respectively. Then

$$|\alpha - z_0| = r \text{ and } \left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$$

$$\Rightarrow \quad |\alpha - z_0| = r \text{ and } |1 - \bar{\alpha}z_0| = 2r|\bar{\alpha}|$$

$$\Rightarrow \quad |\alpha - z_0|^2 = r^2 \text{ and } |1 - \bar{\alpha}z_0|^2 = 4r^2|\alpha|^2$$

Subtracting, we get $|1 - \bar{\alpha}z_0|^2 - |\alpha - z_0|^2 = 4r^2|\alpha|^2 - r^2$

$$\Rightarrow \quad 1 + |\alpha z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0 - (|\alpha|^2 + |z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0) = 4r^2|\alpha|^2 - r^2$$

$$\Rightarrow \quad 1 + |\alpha|^2|z_0|^2 - |\alpha|^2 - |z_0|^2 = 4r^2|\alpha|^2 - r^2$$

$$\Rightarrow \quad (1 - |\alpha|^2)(1 - |z_0|^2) = 4r^2|\alpha|^2 - r^2$$

Given $2|z_0|^2 = r^2 + 2$

$$\Rightarrow \quad (1 - |\alpha|^2)\left(1 - \frac{r^2 + 2}{2}\right) = 4r^2|\alpha|^2 - r^2$$

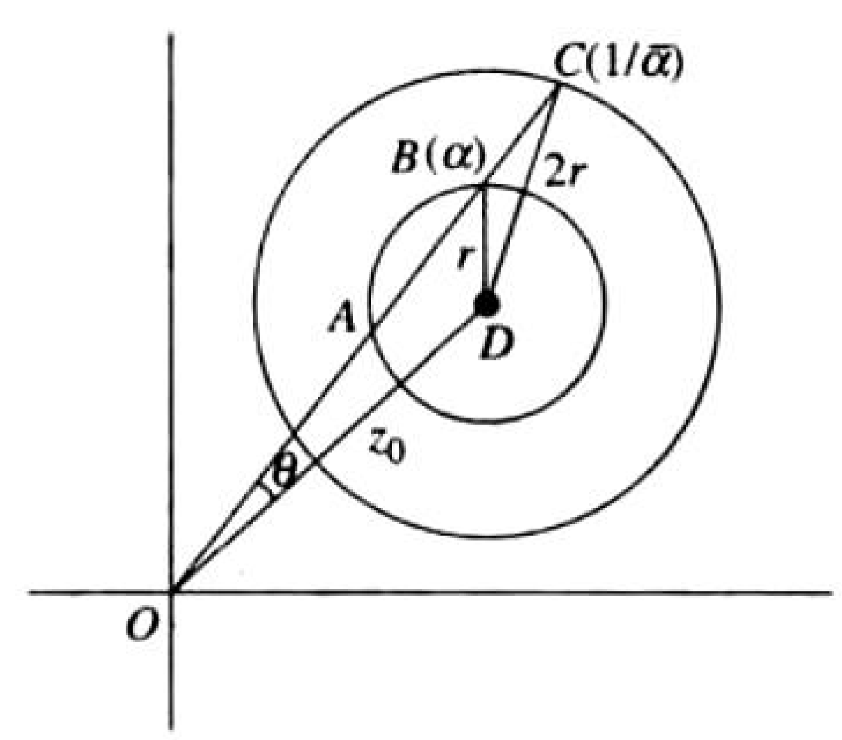
$$\Rightarrow \quad (1 - |\alpha|^2)\left(\frac{-r^2}{2}\right) = 4r^2|\alpha|^2 - r^2$$

$$\Rightarrow \quad |\alpha|^2 - 1 = 8|\alpha|^2 - 2 \Rightarrow |\alpha|^2 = \frac{1}{7} \quad \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$$

Alternative Method:

$$\arg\left(\frac{1}{\bar{\alpha}}\right) = -\arg(\bar{\alpha}) = \arg\alpha$$

Thus α and $\frac{1}{\bar{\alpha}}$ lies on the same ray as shown in the following figure.



$$\text{OB} = |\alpha|, \text{ OC} = \left|\frac{1}{\bar{\alpha}}\right| = \frac{1}{|\alpha|}$$

$$\text{In } \Delta\text{OBD}, \cos\theta = \frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|}$$

$$\text{In } \Delta\text{OCD}, \cos\theta = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$$\text{Thus, } \frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|} = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$\Rightarrow \quad |z_0|^2 + |\alpha|^2 - r^2 = |\alpha|^2 \left[|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2 \right]$

$\Rightarrow \quad |z_0|^2 + |\alpha|^2 - r^2 = |\alpha|^2 |z_0|^2 + 1 - 4r^2 |\alpha|^2$

Given $2|z_0|^2 = r^2 + 2$

$\Rightarrow \quad |z_0|^2 + |\alpha|^2 + 2 - 2|z_0|^2 = |\alpha|^2 |z_0|^2 + 1 + (8 - 8|z_0|^2)|\alpha|^2$

$\Rightarrow \quad 1 - |z_0|^2 = 7|\alpha|^2 - 7|z_0|^2 |\alpha|^2$

$\Rightarrow \quad 7|\alpha|^2 = 1$

$\Rightarrow \quad |\alpha| = \frac{1}{\sqrt{7}}$

Multiple Correct Answers Type

1. a., b., c.

We have,

$$|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1 \quad (1)$$

and

$$\text{Re}(z_1 \bar{z}_2) = 0 \Rightarrow \text{Re}\{(a + ib)(c - id)\} = 0 \Rightarrow ac + bd = 0 \quad (2)$$

Now from (1) and (2),

$$a^2 + b^2 = 1 \Rightarrow a^2 + \frac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2 \quad (3)$$

Also, $c^2 + d^2 = 1 \Rightarrow c^2 + \frac{a^2c^2}{b^2} = 1 \Rightarrow b^2 = c^2 \quad (4)$

$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$ [From (1) and (4)]

and $|\omega_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$ [From (1) and (4)]

Further,

$$\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a + ic)(b - id)\}$$
$$= ab + cd$$
$$= ab + \left(-\frac{ac^2}{b}\right) \quad \text{[From (2)]}$$
$$= \frac{ab^2 - ac^2}{b} = 0 \quad \text{[From (4)]}$$

Also, $\text{Im}(\omega_1\bar{\omega}_2)\ bc - ad$

$$= bc - a\left(-\frac{ac}{b}\right) = \frac{(a^2 + b^2)c}{b} = \frac{c}{b} = \pm 1 \neq 0$$

$\therefore \quad |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1\bar{\omega}_2) = 0$

2. a., d.

Let $z_1 = a + ib, a > 0$ and $b \in R$; $z_2 = c + id, d < 0, c \in R$. Given,

$|z_1| = |z_2|$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad a^2 - c^2 = d^2 - b^2 \quad (1)$

Now, $\frac{z_1 + z_2}{z_1 - z_2} = \frac{(a + c) + i(b + d)}{(a - c) + i(b - d)}$

$$= \frac{[(a^2 - c^2) + (b^2 - d^2)] + i[(a - c)(b + d) - (a + c)(b - d)]}{(a - c)^2 + (b - d)^2}$$

which is a purely imaginary number or zero in case $a + c = b + d = 0$.

3. a., c., d.

Given $z = (1 - t)z_1 + tz_2$

$\Rightarrow \quad z = \frac{(1 - t)z_1 + tz_2}{(1 - t) + t}$

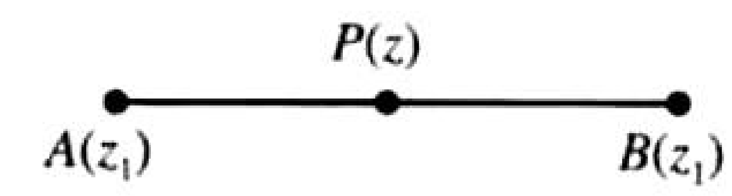
$\Rightarrow$ z divides the line segment joining z_1 and z_2 in ration $(1 - t) : t$ internally as $0 < t < 1$

$\Rightarrow$ z, z_1, and z_2 are collinear.

$\Rightarrow$ $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

$\Rightarrow \quad \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

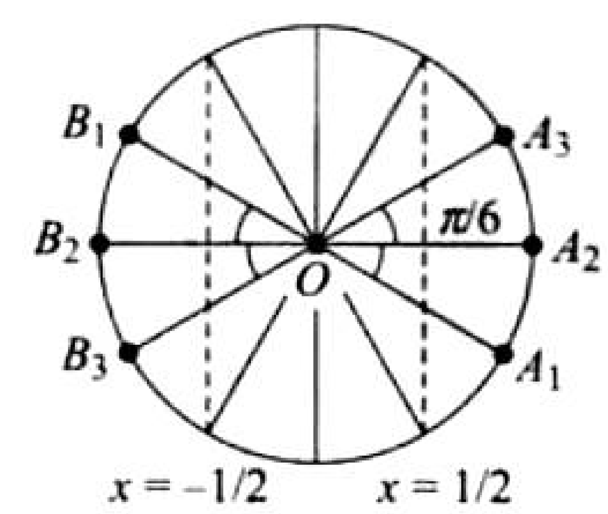


$AP + PB = AB$

$\Rightarrow \quad |z - z_1| + |z - z_2| = |z_1 - z_2|$

4. c., d.

$w = \frac{\sqrt{3} + i}{2} = e^{\frac{i\pi}{6}}$, so $w^n = e^{i\left(\frac{n\pi}{6}\right)}$, $n = 0, 1, 2, 3, \ldots, 12$

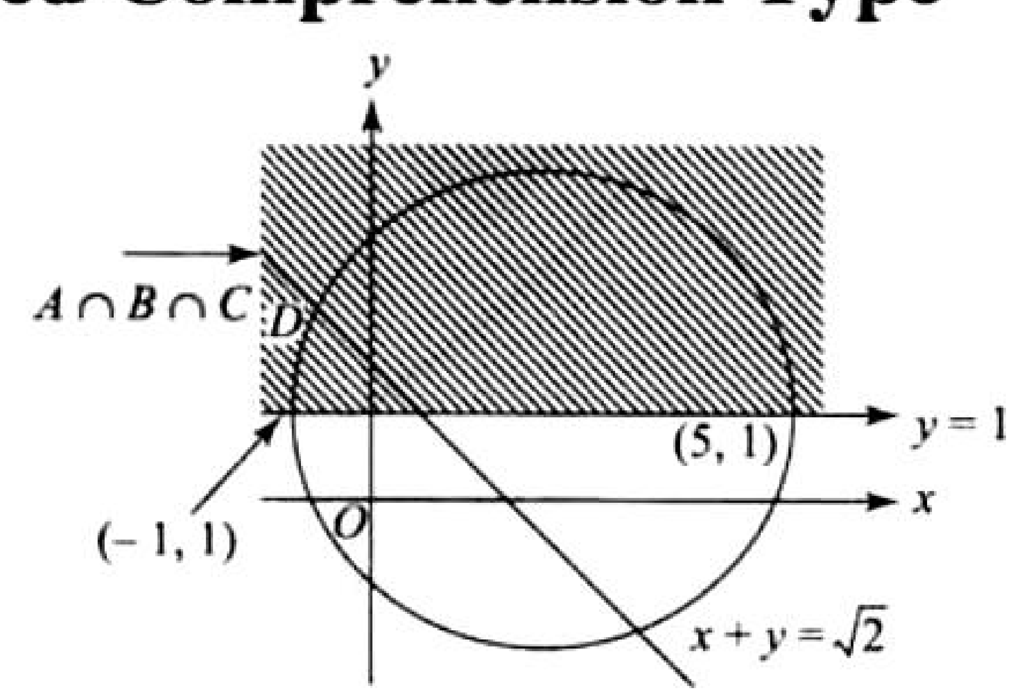


Now, for z_1, $\cos\frac{n\pi}{6} > \frac{1}{2}$ and for z_2, $\cos\frac{n\pi}{6} < -\frac{1}{2}$

Possible position of z_1 are A_1, A_2, A_3 whereas of z_2 are B_1, B_2, B_3 (as shown in the figure)

So possible value of $\angle z_1Oz_2$ according to the given options is $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

Linked Comprehension Type



1. b. A is the set of points on and above the line $y = 1$ in the Argand plane. B is the set of points on the circle $(x - 2)^2 + (y - 1)^2 = 9$ and

$$C = \text{Re}(1 - i)z = \text{Re}((1 - i)(x + iy)) = \sqrt{2}$$

$\Rightarrow \quad x + y = \sqrt{2}$

Hence, $A \cap B \cap C$ has only one point D of intersection.

2. c. The points (–1, 1) and (5, 1) are the extremities of a diameter of the given circle. Hence,

$$|z + 1 - i|^2 + |z - 5 - i|^2 = 36$$

3. d. $||z| - |w|| < |z - w|$ and $|z - w|$ is the distance between z and w. Here, z is fixed. Hence, distance between z and w would be maximum for diametrically opposite points. Therefore,

$$|z - w| < 6$$

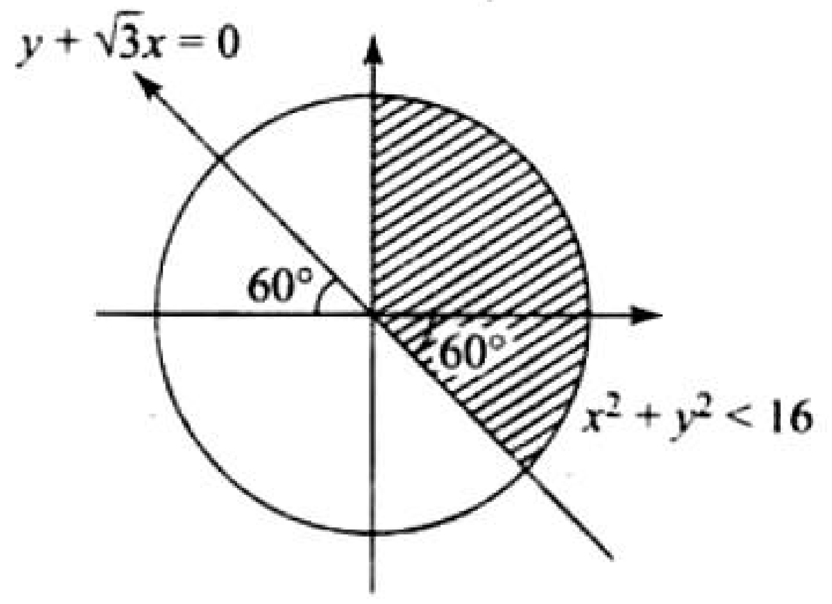
$\Rightarrow \quad -6 < |z| - |w| < 6$

$\Rightarrow \quad -3 < |z| - |w| + 3 < 9$

4. b. $S_1 : |z| < 4$, z lies inside the circle or radius 4

$S_2 : \sqrt{3}x + y > 0$, z lies above the line $\sqrt{3}x + y = 0$

S_3: Re(z) > 0, z lies to the right of imaginary axis.



Area of region $S_1 \cap S_2 \cap S_3$ = Shaded area

$$= \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6} = 4^2\pi\left\{\frac{1}{4} + \frac{1}{6}\right\} = \frac{20\pi}{3}$$

5. c. Distance of (1, – 3) from $y + \sqrt{3}\,x = 0$ is $\left|\frac{3 - \sqrt{3} \times 1}{2}\right| = \frac{3 - \sqrt{3}}{2}$

$$\Rightarrow \min_{z \in S} |1 - 3i - z| = \frac{3 - \sqrt{3}}{2}$$

Matching Column Type

1. (d) – (p), (s)

$|z + 2| - |z - 2| = \pm 3$

Now we know that if $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$ the locus is a hyperbola.

Also eccentricity of hyperbola is more than 1

(b) – (s), (t)

Let $z = x + iy$; $x, y \in R$

$\Rightarrow \quad (x + 1)^2 - y^2 = x^2 + y^2 + 1$

$\Rightarrow \quad y^2 = x$; which is a parabola.

Also eccentricity of parabola is 1

Note: Solutions of the remaining parts are given in their respective chapters.

2. (a) – (q), (r)

$$\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|, z \neq 0$$

$\frac{z}{|z|}$ is unimodular complex number

and lies on perpendicular bisector of i and $-i$

$$\Rightarrow \quad \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1|z|$$

$\Rightarrow \quad z$ is real number $\Rightarrow$ Im (z) = 0 .

(b) – (p)

$|z + 4| + |z - 4| = 10$

z lies on an ellipse whose foci are (4,0) and (–4,0) and length of major axis is 10

$\Rightarrow \quad 2ae = 8$ and $2a = 10 \Rightarrow e = 4/5$

$|\text{Re}(z)| \le 5$.

(c) – (p), (s), (t)

$|\omega| = 2 \Rightarrow w = 2(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad z = x + iy = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$$

$$= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$$

$$\Rightarrow \quad e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

(d) – (q), (r), (s), (t)

$|\omega| = 1 \Rightarrow x + iy = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$x + iy = 2\cos\theta$

$|\text{Re}(z)| \le 1$, Im(z) = 0.

3. (d) – (t) Let $u = \frac{1}{1 - z}$

$\Rightarrow \quad z = 1 - \frac{1}{u}$

Given $|z| = 1$

$\Rightarrow \quad \left|1 - \frac{1}{u}\right| = 1$

$\Rightarrow \quad |u - 1| = |u|$

Therefore, locus of u is perpendicular bisector of line segment joining 0 and 1.

$\Rightarrow \quad$ maximum arg u approaches $\frac{\pi}{2}$ but will not attain.

Note: Solutions of the remaining parts are given in their respective chapters.

4. (a) – (s)

$$\omega = \frac{2i(x + iy)}{1 - (x + iy)^2} = \frac{2i(x + iy)}{1 - (x^2 - y^2 + 2ixy)}$$

Using $1 - x^2 = y^2$

$$\omega = \frac{2ix - 2y}{2y^2 - 2ixy} = -\frac{1}{y}.$$

$\because \quad -1 \le y \le 1 \Rightarrow -\frac{1}{y} \le -1$ or $-\frac{1}{y} \ge 1$.

Note: Solutions of the remaining parts are given in their respective chapters.

5. a.

(p) z_k is 10th root of unity

So, $\quad \bar{z}_k$ will also be 10th root of unity.

Take z_j as $\bar{z}_k$.

(q) $z_1 \neq 0$ take $z = \frac{z_k}{z_1}$, we can always find z.

(r) $z^{10} - 1 = (z - 1)(z - z_1) \ldots (z - z_9)$

$\Rightarrow \quad (z - z_1)(z - z_2) \ldots\ldots (z - z_9)$

$\quad = 1 + z + z^2 + \ldots + z^9 \ \ \forall z \in$ complex number

Put $z = 1$

$\Rightarrow \quad (1 - z_1)(1 - z_2) \ldots\ldots (1 - z_9) = 10$

$$\Rightarrow \quad \frac{|1 - z_1||1 - z_2| \ldots |1 - z_9|}{10} = 1$$

(s) $1 + z_1 + z_2 + \ldots + z_9 = 0$

$\Rightarrow \quad \text{Re}(1) + \text{Re}(z_1) + \ldots + \text{Re}(z_9) = 0$

$\Rightarrow \quad \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = -1.$

6. (c) – (p), (q), (s), (t)

Let $\quad a = 3 - 3\omega + 2\omega^2$

$\Rightarrow \quad a\omega = 3\omega - 3\omega^2 + 2x$

$\Rightarrow \quad a\omega^2 = 3\omega^2 - 3 + 2\omega$

Now given $a^{4n+3}(1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$

Thus, n should not be a multiple of 3.

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (1) $\quad \omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying $(C_1 \to C_1 + C_2 + C_3)$

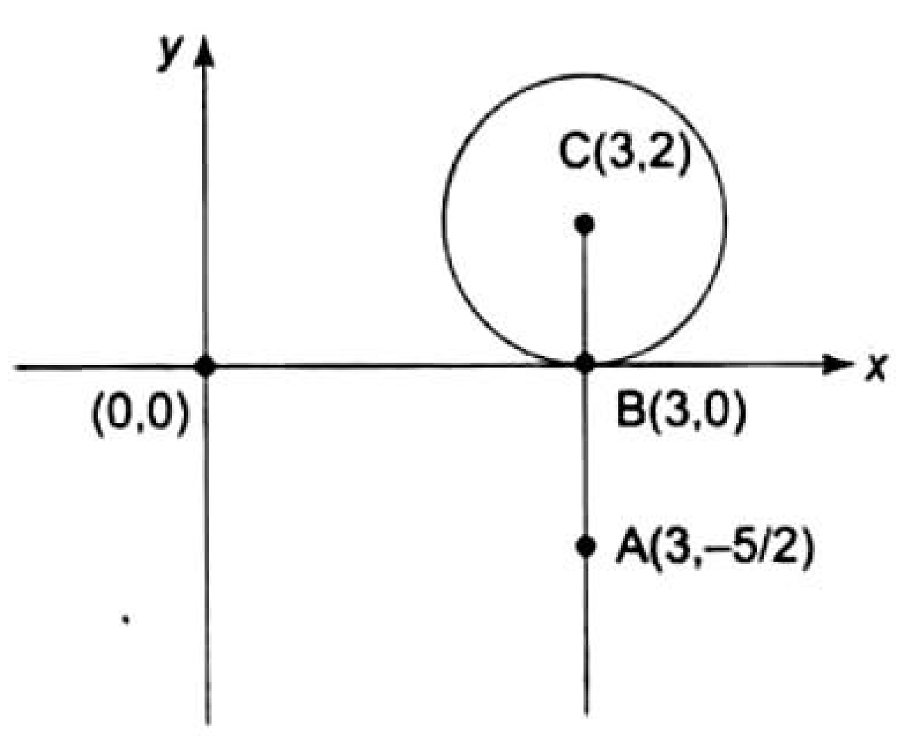
$$\Rightarrow \quad z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow \quad z^3 = 0$

$z = 0$ is only solution.

2.(5) $|z - 3 - 2i| \le 2$

$\Rightarrow z$ lies on or inside the circle radius 2 and center (3, 2)



$|2z - 6 + 5i|_{\text{min.}}$

$= 2|z - 3 + (5/2)i|_{\text{min.}}$

= 2(minimum distance of any point on the circle to the point (3, – 5/2)

$= 2(5/2) = 5$

3. The expression may not attain integral value for all a, b, c.

If we consider $a = b = c$, then

$y = a(1 + \omega + \omega^2) = a(1 + i\sqrt{3})$

$z = a(1 + \omega^2 + \omega) = a(1 + i\sqrt{3})$

$\therefore \quad |x|^2 + |y|^2 + |z|^2 = 9|a|^2 + 4|a|^2 + 4|a|^2 = 17|a|^2$

$$\therefore \quad \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{17}{3}$$

Note: However if $\omega = e^{i(2\pi/3)}$, then the value of the expression = 3.

4. (4) $\alpha_k = \cos\dfrac{k\pi}{7} + i\sin\dfrac{k\pi}{7} = e^{\frac{k\pi}{7}i}$

$$\therefore \quad \frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{\frac{(k+1)\pi}{7}i} - e^{\frac{k\pi}{7}i}\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-1)\pi}{7}i} - e^{\frac{(4k-2)\pi}{7}i}\right|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{\frac{k\pi}{7}i}\right|\left|e^{\frac{\pi i}{7}} - 1\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-2)\pi}{7}i}\right|\left|e^{\frac{\pi i}{7}} - 1\right|}$$

$$= \frac{\sum_{k=1}^{12} 1}{\sum_{k=1}^{3} 1} \qquad \left(\because \left|e^{\frac{k\pi}{7}i}\right| = \left|e^{\frac{(4k-2)\pi}{7}i}\right| = 1\right)$$

$$= \frac{12}{3} = 4$$

Fill in the Blanks Type

1. Let

$$z = \frac{\sin\frac{x}{2} + \cos\frac{x}{2} + i\tan x}{1 + 2i\sin\frac{x}{2}}$$

$$= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2} + i\tan x\right)\left(1 - 2i\sin\frac{x}{2}\right)}{\left(1 + 2i\sin\frac{x}{2}\right)\left(1 - 2i\sin\frac{x}{2}\right)}$$

$$= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2} + 2\sin\frac{x}{2}\tan x\right) + i\left(\tan x - 2\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{1 + 4\sin^2\frac{x}{2}}$$

Now, $\text{Im}(z) = 0$ (as z is real)

$$\Rightarrow \quad \tan x - 2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \frac{\sin x}{\cos x} - 2\sin^2(x/2) - 2\sin(x/2)\cos(x/2) = 0$$

$$\Rightarrow \quad \frac{\sin x}{\cos x} - (1 - \cos x) - \sin x = 0$$

$\Rightarrow \quad \sin x\left[\dfrac{1}{\cos x}-1\right]-[1-\cos x]=0$

$\Rightarrow \quad (1-\cos x)\left[\dfrac{\sin x}{\cos x}-1\right]=0$

$\Rightarrow \quad \cos x = 1 \Rightarrow x = 2n\pi$ or $\tan x = 1 \Rightarrow x = n\pi + \pi/4,\ n \in Z$

2. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\text{Re}(z_1\bar{z}_2)$

$\quad + b^2|z_1|^2 + a^2|z_2|^2 + 2ab \times \text{Re}(z_1\bar{z}_2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

3. As $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ from an equilateral triangle, therefore

$|z_1 - z_3| = |z_2 - z_3| = |z_1 - z_2|$

$\Rightarrow \quad |a + i| = |1 + bi| = |(a-1) + i(1-b)|$

$\Rightarrow \quad a^2 + 1 = 1 + b^2 = (a-1)^2 + (1-b)^2$

$\Rightarrow \quad a^2 = b^2 = a^2 + b^2 - 2a - 2b + 1$

$\Rightarrow \quad a = b \qquad (\because a, b > 0;\ \therefore a \neq -b) \qquad (1)$

and $\quad b^2 - 2a - 2b + 1 = 0 \qquad (2)$

Solving $a^2 - 2a - 2a + 1 = 0$, we get

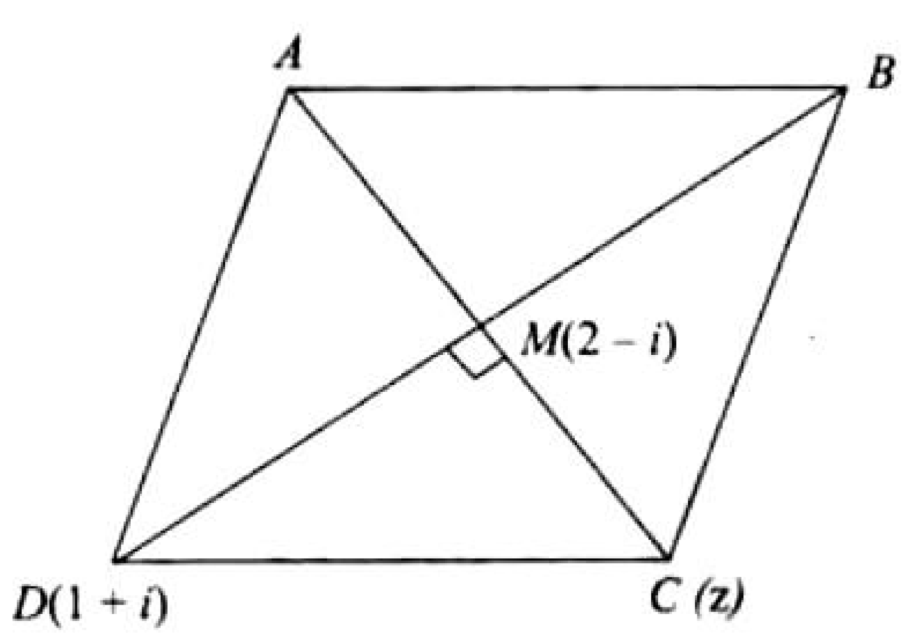
$a^2 - 4a + 1 = 0$

$\Rightarrow \quad a = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

But $0 < a, b < 1$.

$\therefore \quad a = 2 - \sqrt{3}$ and $b = 2 - \sqrt{3}$

4.



Rotating DM about M by an angle 90°, we have

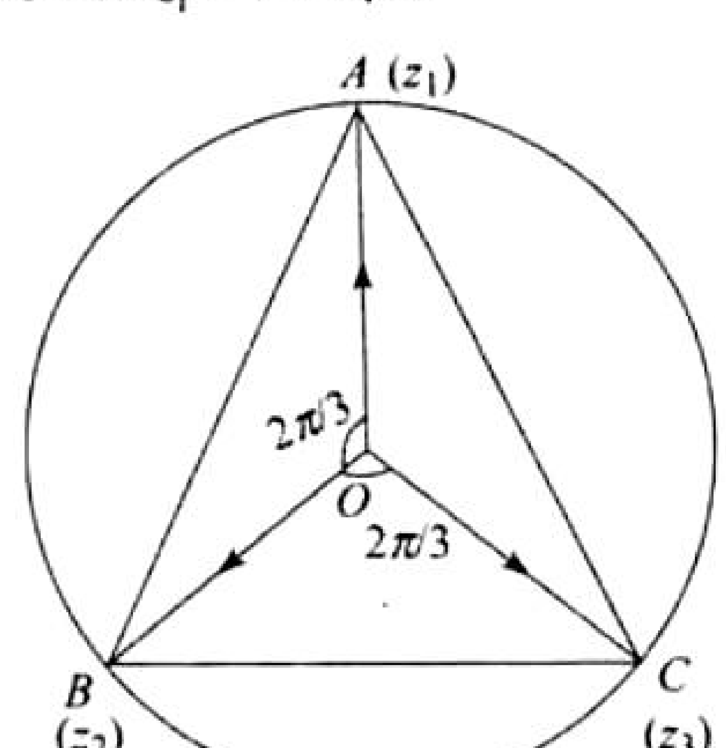
$$\frac{z-(2-i)}{(1+i)-(2-i)} = \frac{|z-(2-i)|}{|(1+i)-(2-1)|}e^{\pm i\frac{\pi}{2}}$$

$\Rightarrow \quad \dfrac{z-(2-i)}{-1+2i} = \pm\dfrac{i}{2}$

$\Rightarrow \quad 2z = (-i-2) + (4-2i)$ or $(i+2) + (4-2i)$

$\Rightarrow \quad z = 1 - \dfrac{3}{2}i$ or $3 - \dfrac{i}{2}$

5. Let z_1, z_2, z_3 be the vertices A, B, and C, respectively, of equilateral ΔABC, inscribed in a circle $|z| = 2$ with center (0, 0) and radius = 2. Given $z_1 = 1 + i\sqrt{3}$.



Rotating OA about O by an angle $2\pi/3$, we have

$$\frac{z-0}{1+i\sqrt{3}-0} = \frac{|z-0|}{|1+i\sqrt{3}-0|}e^{\pm i\frac{2\pi}{3}}$$

$\Rightarrow \quad z = (1+i\sqrt{3})\left(\cos\dfrac{2\pi}{3} \pm i\sin\dfrac{2\pi}{3}\right)$

$\Rightarrow \quad z = \left(1+i\sqrt{3}\right)\left(-\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad z = -\dfrac{(1+i\sqrt{3})^2}{2}$ or $-\dfrac{(1+i\sqrt{3})(1-i\sqrt{3})}{2}$

$\Rightarrow \quad z = 1 - i\sqrt{3}$ or -2

6. $S = 1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$

Here,

$T_n = (n-1)(n-\omega)(n-\omega^2)$

$= n^3 - 1$

$S = \displaystyle\sum_{n=2}^{n}\left(n^3 - 1\right)$

$= \displaystyle\sum_{n=1}^{n}\left(n^3 - 1\right)$

$= \left[\left(\dfrac{n(n+1)}{2}\right)^2 - n\right]$

$= \dfrac{n^2(n^2 + 2n + 1) - 4n}{4}$

$= \dfrac{1}{4}n(n^3 + 2n^2 + n - 4)$

$= \dfrac{1}{4}n[n-1]\left[n^2 + 3n + 4\right]$

True/False Type

1. True Let $z = x + iy$. Then

$1 \cap z \Rightarrow 1 \leq x$ and $0 \leq y$ (by definition)

$\dfrac{1-z}{1+z} = \dfrac{1-(x+iy)}{1+(x+iy)}$

$= \dfrac{(1-x)-iy}{(1+x)+iy} \times \dfrac{(1+x)-iy}{(1+x)-iy}$

$= \dfrac{1-x^2-y^2}{(1+x)^2+y^2} - \dfrac{iy(1-x+1+x)}{(1+x)^2+y^2}$

$= \dfrac{1-x^2-y^2}{(1+x)^2+y^2} - \dfrac{2iy}{(1+x)^2+y^2}$

Now, $\dfrac{1-z}{1+z} \cap 0 \Rightarrow \dfrac{1-x^2-y^2}{(1+x)^2+y^2} \leq 0$ and $\dfrac{-2y}{(1+x)^2+y^2} \leq 0$

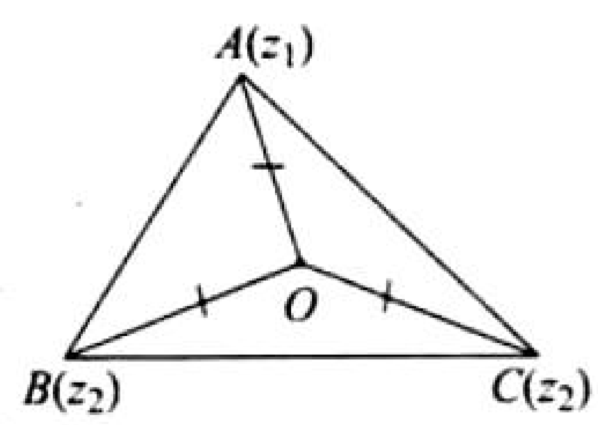
$\Rightarrow \quad 1 - x^2 - y^2 \leq 0$ and $-2y \leq 0$

$\Rightarrow \quad x^2 + y^2 \geq 1$ and $y \geq 0$

which is true as $x > 1$ and $y > 0$. Therefore, the given statement is true, $\forall z \in C$.

2. True As $|z_1| = |z_2| = |z_3|$, therefore, z_1, z_2, z_3 are equidistant from origin.

Hence, O is circumcenter of ΔABC. But according to the question, ΔABC is equilateral and we know that in an equilateral triangle circumcenter and centroid coincide. Hence, centroid of ΔABC is 0.



Hence, $\dfrac{z_1 + z_2 + z_3}{3} = 0$

or $z_1 + z_2 + z_3 = 0$

Therefore, the statement is true.

3. **False** If z_1, z_2, z_3 are in A.P., then $(z_1 + z_3)/2 = z_2$. So, z_2 is mid-point of line joining z_1 and z_3. Hence, z_1, z_2, z_3 lie on a straight line. Hence, given statement is false.

4. **True** Cube roots of unity are $z_1 = 1$, $z_2 = \dfrac{-1+i\sqrt{3}}{2}$ and $z_3 = \dfrac{-1-i\sqrt{3}}{2}$

$$|z_1 - z_2| = \left|\frac{3-i\sqrt{3}}{2}\right| = \sqrt{3}$$

$$|z_2 - z_3| = \left|\frac{i2\sqrt{3}}{2}\right| = \sqrt{3}$$

and $|z_1 - z_3| = \left|\dfrac{3+i\sqrt{3}}{2}\right| = \sqrt{3}$

Thus, cube roots of unity are vertices of equilateral triangle. Hence, the statement is true.

Subjective Type

1. $$\frac{1}{1-\cos\theta+2i\sin\theta} = \frac{1}{2\sin^2\theta/2+4i\sin\theta/2\ \cos\theta/2}$$

$$= \frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$= \frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$= \frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

$$= \frac{1}{\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$= \left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

2. As β and γ are the complex cube roots of unity, therefore let $\beta = \omega$ and $\gamma = \omega^2$ so that $\omega + \omega^2 + 1 = 0$ and $\omega^3 = 1$. Then,

$$\begin{aligned} xyz &= (a+b)(a\omega+b\omega^2)(a\omega^2+b\omega) \\ &= (a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3) \\ &= (a+b)(a^2+ab\omega+ab\omega^2+b^2) \quad (\text{Using } \omega^3 = 1) \\ &= (a+b)(a^2+ab(\omega+\omega^2)+b^2) \\ &= (a+b)(a^2-ab+b^2) \quad (\text{Using } \omega+\omega^2 = -1) \\ &= a^3+b^3 \end{aligned}$$

3. Given,

$$x+iy = \sqrt{\frac{a+ib}{c+id}}$$

or $\quad (x+iy)^2 = \dfrac{a+ib}{c+id}$ (1)

or $\quad \left|(x+iy)^2\right| = \left|\dfrac{a+ib}{c+id}\right|$

or $\quad |x+iy|^4 = \left|\dfrac{a+ib}{c+id}\right|^2$

or $\quad (x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$

4. Given that n is an odd integer >3 and n is not a multiple of 3. Let

$$p(x) = (x+1)^n - x^n - 1$$

and
$$\begin{aligned} q(x) &= x^3+x^2+x \\ &= x(x^2+x+1) \\ &= x(x-\omega)(x-\omega^2) \end{aligned}$$

where ω and ω^2 are cube roots of unity. Clearly, 0, ω, ω^2 are zeros of the polynomial $q(x)$. Now,

$$p(0) = 1^n - 0 - 1 = 0$$

Hence, 0 is a zero of $p(x)$.

$$\begin{aligned} p(\omega) &= (\omega+1)^n - \omega^n - 1 \\ &= (-\omega^2)^n - \omega^n - 1 \\ &= -(\omega^{2n}+\omega^n+1) \quad [\because n \text{ is odd}] \\ &= 0 \quad [\because \omega^n+\omega^{2n}+1 = 0 \text{ if } n \neq 3m] \end{aligned}$$

Therefore, ω is a zero of $p(x)$. Also,

$$\begin{aligned} p(\omega^2) &= (\omega^2+1)^n - (\omega^2)^n - 1 \\ &= (-\omega)^n - \omega^{2n} - 1 \\ &= -\omega^n - \omega^{2n} - 1 \\ &= -(1+\omega^n+\omega^{2n}) \\ &= 0 \quad [\text{for } n \neq 3m] \end{aligned}$$

Hence, ω^2 is a zero of $p(x)$.

Since 0, ω, ω^2 are zeros of $p(x)$, hence x, $x-\omega$, $x-\omega^2$ are factors of $p(x)$. Hence, $x(x-\omega)(x-\omega^2)$ is a factor of $p(x)$, i.e., x^3+x^2+x is a factor of $p(x)$.

5. $$\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$$

$\Rightarrow \quad (4+2i)x - 6i - 2 + (9-7i)y + 3i - 1 = 10i$

$\Rightarrow \quad (4x+9y-3) + (2x-7y-3)i = 10i$

$\Rightarrow \quad 4x+9y-3 = 0$ and $2x-7y-3 = 10$

On solving, we get $x = 3$, $y = -1$.

6. $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of an equilateral triangle. Hence,

$$z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$$

Now, $(z_1+z_2+z_3)^2 = z_1^2+z_2^2+z_3^2 + 2(z_1z_2+z_2z_3+z_3z_1)$

$$= 3(z_1^2+z_2^2+z_3^2)$$

We also have,

$$z_0 = \frac{z_1+z_2+z_3}{3}$$

(as centroid will coincide with circumcenter)

$\Rightarrow \quad z_1^2+z_2^2+z_3^2 = 3z_0^2$

7. We know that if z_1, z_2, z_3 form an equilateral triangle, then

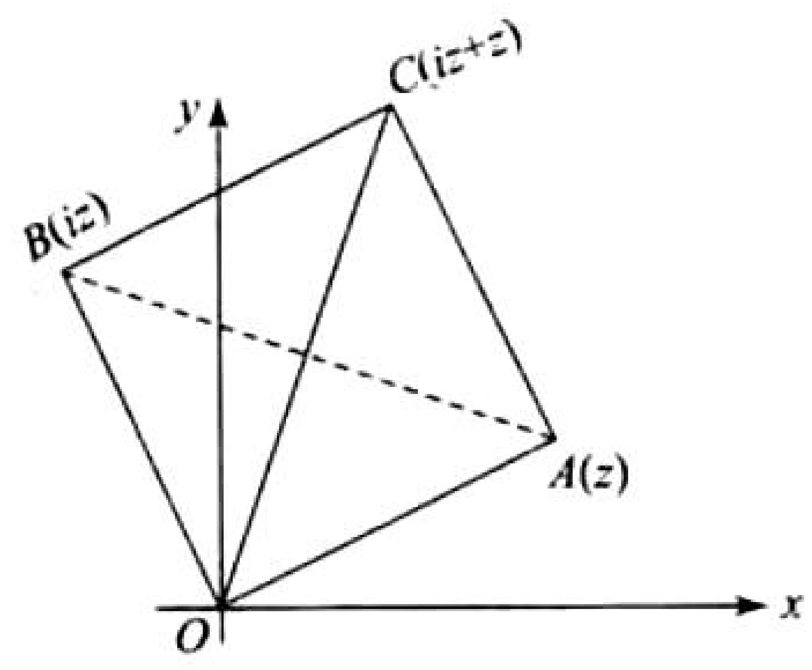
$$z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$$

Putting $z_3 = 0$, we get

$$z_1^2+z_2^2 = z_1z_2$$

$\Rightarrow \quad z_1^2+z_2^2-z_1z_2 = 0$

8. Let the vertices of the triangle be $A(z)$, $B(iz)$, $C(z + iz)$. We know that iz is obtained by rotating OA through an angle $90°$. Also point $z + iz$ can be obtained by completing the parallelogram two of whose adjacent sides are OA and OB. From Argand diagram, it is clear that



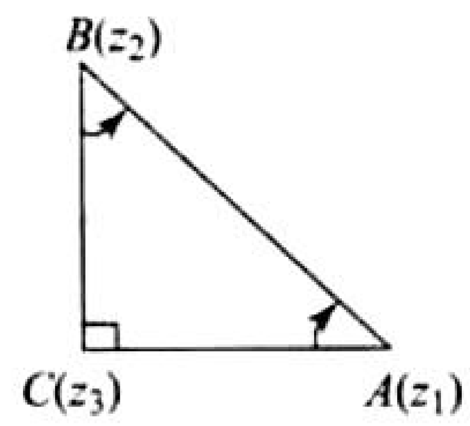
Area of ΔABC = Area of ΔOAB

$$= \frac{1}{2} \times OA \times OB \qquad [\because \text{ it is right angled at point } O]$$

$$= \frac{1}{2}|z| \times |iz|$$

$$= \frac{1}{2}|z|^2$$

9. Applying rotation about point C,



$$\frac{z_2 - z_3}{z_1 - z_3} = e^{i\pi/2} \qquad (1)$$

Applying rotation about point B,

$$\frac{z_1 - z_2}{z_3 - z_2} = \sqrt{2}e^{i\pi/4} \qquad (2)$$

Applying rotation about point A,

$$\frac{z_2 - z_1}{z_3 - z_1} = \sqrt{2}\, e^{-i\pi/4} \qquad (3)$$

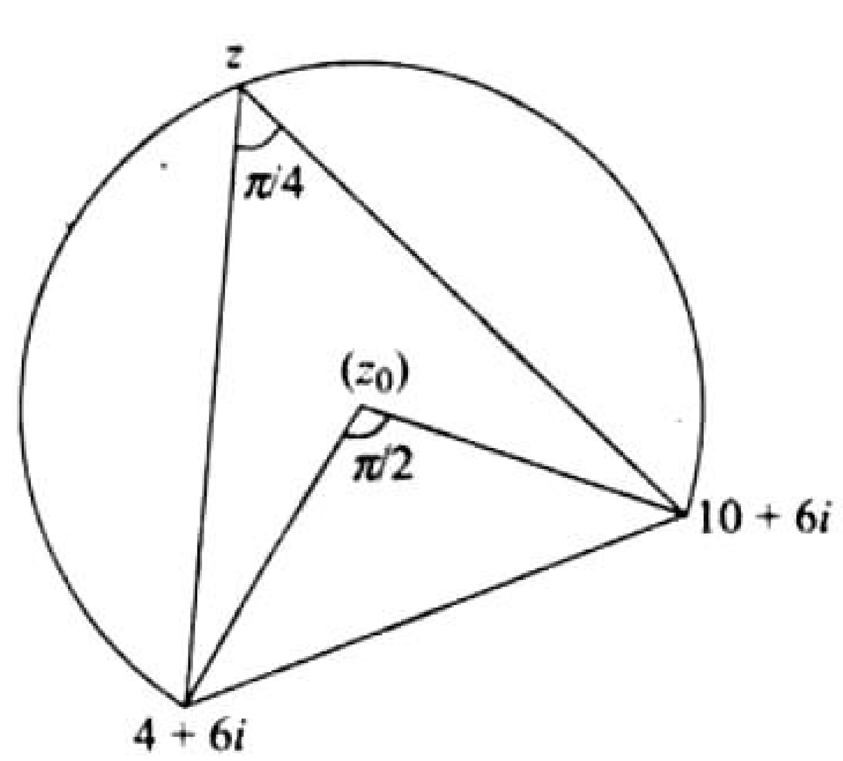
Multiplying (2) and (3), we get

$$\frac{(z_1 - z_2)(z_2 - z_1)}{(z_3 - z_2)(z_3 - z_1)} = 2$$

$$(z_1 - z_2)^2 = -2(z_3 - z_2)(z_3 - z_1)$$

$$= 2(z_1 - z_3)(z_3 - z_2)$$

10.



$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$$

Locus of z is the major arc whose center is at z_0. Applying rotation at z_0, we have

$$\frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} = \frac{|z_0 - (10 + 6i)|}{|z_0 - (4 + 6i)|} e^{i\frac{\pi}{2}}$$

or $$\frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} = i$$

or $z_0 - 10 - 6i = iz_0 - 4i + 6$

or $z_0 = 7 + 9i$

Thus, center is at $7 + 9i$ and z is any point on the arc.

Hence, $|z - (7 + 9i)| = |10 + 6i - (7 + 9i)| = 3\sqrt{2}$.

11. Dividing throughout by i, we get

$z^3 - iz^2 + iz + 1 = 0$

$\Rightarrow$ $z^2(z - i) + i(z - i) = 0$ as $1 = -i^2$

$\Rightarrow$ $(z - i)(z^2 + i) = 0$

$\Rightarrow$ $z = i$ or $z^2 = -i$

$\Rightarrow$ $|z| = |i| = 1$ or $|z^2| = |z|^2 = |-i| = 1$

$\Rightarrow$ $|z| = 1$

Hence, in either case $|z| = 1$.

12. Let $z = |z|e^{i\alpha}$ and $w = |w|e^{i\beta}$. Now,

$|z - w|^2 = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$

$= (|z| - |w|)^2 + 2|z||w| - |z||w|e^{i(\alpha-\beta)} - |z||w|e^{-i(\alpha-\beta)}$

$= (|z| - |w|)^2 + |z||w|(2 - 2\cos(\alpha - \beta))$

$\le (|z| - |w|)^2 + 4\sin^2\left(\frac{\alpha - \beta}{2}\right)^2 \qquad (\because |z| \le 1, |w| \le 1)$

$\le (|z| - |w|)^2 + 4\left(\frac{\alpha - \beta}{2}\right)^2 \qquad [\because \sin\theta < \theta \text{ for } \theta \in (0, \pi/2)]$

$= (|z| - |w|)^2 + 4(\alpha - \beta)^2$

$= (|z| - |w|)^2 + (\arg z - \arg w)^2$

13. Let $z = x + iy$. Then

$\bar{z} = iz^2$

$\Rightarrow$ $x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow$ $x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow$ $x(1 + 2y) = 0 \qquad (1)$

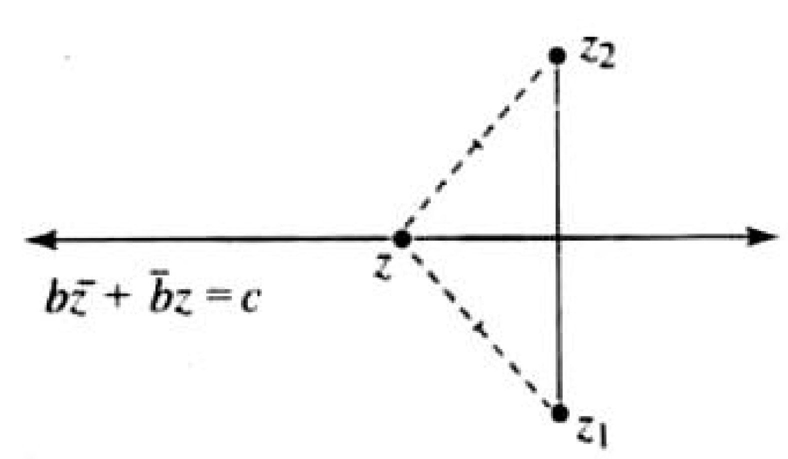
and $x^2 - y^2 + y = 0 \qquad (2)$

From (1), $x = 0$ or $y = -1/2$. From (2), when $x = 0$, $y = 0$, 1 and when $y = -1/2$, $x = \pm(\sqrt{3}/2)$. For nonzero complex number z,

$$z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

14. Given that z_1 is the reflection of z_2 through the line

$b\bar{z} + \bar{b}z = c \qquad (1)$



Therefore, for any arbitrary point z on the line, we must have

$|z - z_1| = |z - z_2|$

or $|z - z_1|^2 = |z - z_2|^2$

or $|z|^2 + |z_1|^2 - z\bar{z}_1 - \bar{z}z_1 = |z|^2 + |z_2|^2 - z\bar{z}_2 - \bar{z}z_2$

or $(\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} = |z_2|^2 - |z_1|^2 \qquad (1)$

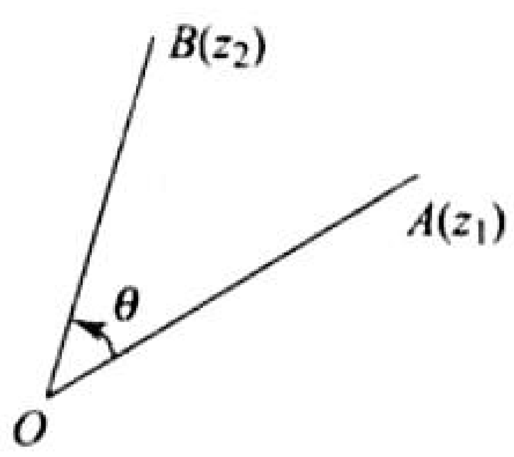
Comparing (1) with (2), we have

$b = z_2 - z_1$ and $c = |z_2|^2 - |z_1|^2$

$\Rightarrow$ $\bar{z}_1 b + z_2\bar{b} = \bar{z}_1(z_2 - z_1) + z_2(\bar{z}_2 - \bar{z}_1) = |z_2|^2 - |z_1|^2 = c$

15. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$. Then,

$$z_1 + z_2 = -p,\ z_1 z_2 = q$$



Also, $\frac{z_2}{z_1} = e^{i\theta}$ or $z_2 = z_1 e^{i\theta}$

$\Rightarrow \quad z_1(1 + e^{i\theta}) = -p,\ z_1^2 e^{i\theta} = q$

$$z_1^2 = q\, e^{-i\theta} = \frac{p^2}{(1 + e^{i\theta})^2}$$

$\Rightarrow \quad p^2 = qe^{-i\theta}(1 + e^{2i\theta} + 2e^{i\theta})$

$$= q(e^{-i\theta} + e^{i\theta} + 2)$$

$$= q(2\cos\theta + 2)$$

$$= 4q\cos^2\frac{\theta}{2}$$

16. Given that z and ω are two complex numbers. To prove

$$|z|^2 w - |w|^2 z = z - w \Leftrightarrow z = w \text{ or } z\bar{w} = 1$$

First let us consider

$$|z|^2 w - |w|^2 z = z - w \qquad (1)$$

$\Rightarrow \quad z(1 + |w|^2) = w(1 + |z|^2)$

$\Rightarrow \quad \frac{z}{w} = \frac{1 + |z|^2}{1 + |w|^2} = \text{a real number}$

$\Rightarrow \quad \overline{\left(\frac{z}{w}\right)} = \frac{z}{w} \Rightarrow \frac{\bar{z}}{\bar{w}} = \frac{z}{w}$

$\Rightarrow \quad \bar{z}w = z\bar{w} \qquad (2)$

Again from Eq. (1),

$$z\bar{z}\,w - w\bar{w}z = z - w$$

$$z(\bar{z}w - 1) - w(\bar{w}z - 1) = 0$$

$$z(z\bar{w} - 1) - w(z\bar{w} - 1) = 0 \qquad \text{[Using Eq. (2)]}$$

$\Rightarrow \quad (z\bar{w} - 1)(z - w) = 0$

$\Rightarrow \quad z\bar{w} = 1$ or $z = w$

Conversely if $z = w$, then L.H.S. of (1) is $|w|^2 w - |w|^2 w = 0$ and R.H.S. of (1) is $w - w = 0$. Therefore, Eq. (1) holds. Also, if $\bar{w}z = 1$, then $w\bar{z} = 1$. L.H.S. of (1) is $z\bar{z}w - w\bar{w}z = zz\bar{w} - w\bar{w}z$ = R.H.S. Hence proved.

17. $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$

$\Rightarrow \quad z = (1)^{1/p}$ or $(1)^{1/q} \qquad (1)$

where p and q are distinct prime numbers. Hence, both the equations will have distinct roots and as $z \neq 1$, both will be simultaneously zero for any value of z given by Eq. (1). Also,

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = \frac{1 - \alpha^p}{1 - \alpha} \qquad (\alpha \neq 1)$$

or $$1 + \alpha + \alpha^2 + \cdots + \alpha^q = \frac{1 - \alpha^q}{1 - \alpha} \qquad (\alpha \neq 1)$$

Because of (1), either $\alpha^p = 1$ or $\alpha^q = 1$ but not both simultaneously as p and q are distinct primes.

18. Given that $|z_1| < 1 < |z_2|$. Now,

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$$

or $\quad |1 - z_1\bar{z}_2| < |z_1 - z_2|$

or $\quad |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

or $\quad (1 - z_1\bar{z}_2)\overline{(1 - z_1\bar{z}_2)} < (z_1 - z_2)\overline{(z_1 - z_2)}$

or $\quad (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

or $\quad 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$

or $\quad 1 + |z_1|^2|z_2|^2 < |z_1|^2 + |z_2|^2$

or $\quad (1 - |z_1|^2)(1 - |z_2|^2) < 0$

which is obviously true as

$$|z_1| < 1 < |z_2|$$

$\Rightarrow \quad |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

19. $\sum_{r=1}^{n} a_r z^r = 1$ (where $|a_r| < 2$)

$\Rightarrow \quad a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n = 1$

$\Rightarrow \quad |a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n| = 1 \qquad (1)$

$\Rightarrow \quad 1 = |a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n|$

$$\leq |a_1 z| + |a_2 z^2| + \cdots + |a_n z^n|$$

$$= |a_1||z| + |a_2||z^2| + |a_3||z^3| + \cdots + |a_n||z^n|$$

$$< 2\,[|z| + |z|^2 + |z|^3 + \cdots + |z|^n]$$

$(\because |a_r| < 2,\ \forall\, r$ and $|z^n| = |z|^n)$

$$= 2\left[\frac{|z|(1 - |z|^n)}{1 - |z|}\right]$$

$$= 2\left[\frac{|z| - |z|^{n+1}}{1 - |z|}\right]$$

$\Rightarrow \quad 2[|z| - |z|^{n+1}] > 1 - |z| \qquad (\because 1 - |z| > 0$ as $|z| < 1/3)$

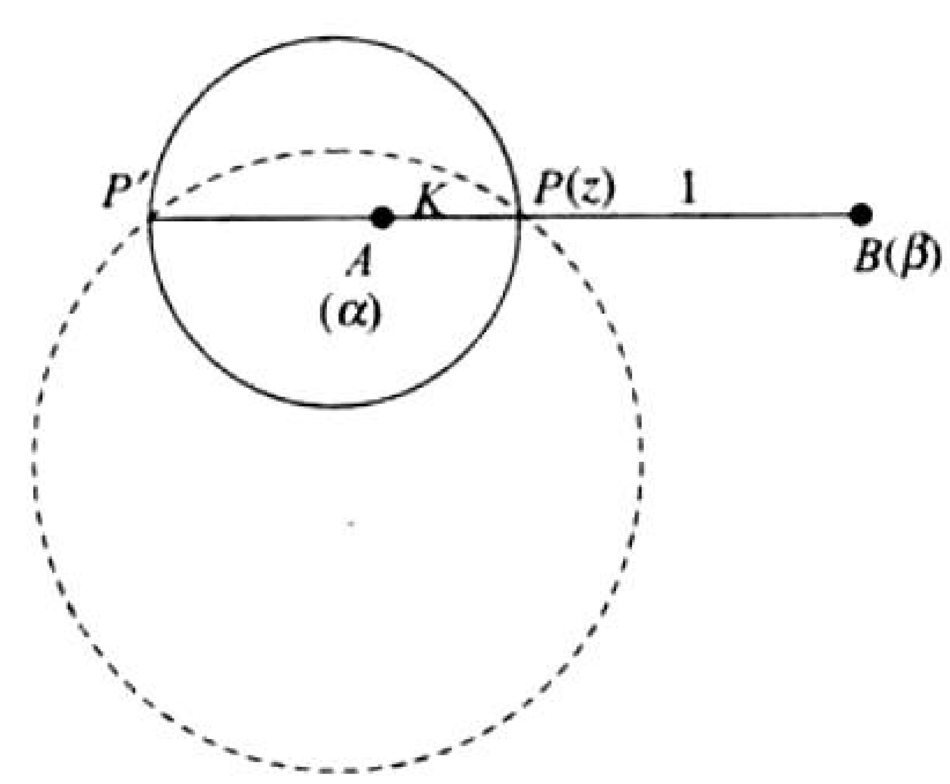
$\Rightarrow \quad \frac{3}{2}|z| > \frac{1}{2} + |z|^{n+1}$

$\Rightarrow \quad |z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1}$

$\Rightarrow \quad |z| > \frac{1}{3}$

which is a contradiction. Hence, there exists no such complex number.

20.



$$\left|\frac{z - \alpha}{z - \beta}\right| = k$$

$\Rightarrow \quad |z - \alpha| = k\,|z - \beta|$

Let points A, B, and P represent complex numbers α, β, and z, respectively. Then,

$$|z - \alpha| = k\,|z - \beta|$$

Therefore, z is the complex number whose distance from A is k times its distance from B, i.e.,

$$PA = k\,PB$$

Hence, P divides AB in the ratio k:1 internally or externally (at P'). Then

$$P \equiv \left(\frac{k\beta+\alpha}{k+1}\right) \text{ and } P' \equiv \left(\frac{k\beta-\alpha}{k-1}\right)$$

Now through PP' there can pass a number of circles, but with given data we can find radius and center of that circle for which PP' is diameter. Hence, the center is the midpoint of PP' and is given by

$$\frac{\left(\frac{k\beta+\alpha}{k+1}+\frac{k\beta-\alpha}{k-1}\right)}{2}$$

$$= \frac{k^2\beta+k\alpha-k\beta-\alpha+k^2\beta-k\alpha+k\beta-\alpha}{2(k^2-1)}$$

$$= \frac{k^2\beta-\alpha}{k^2-1}$$

$$= \frac{\alpha-k^2\beta}{1-k^2}$$

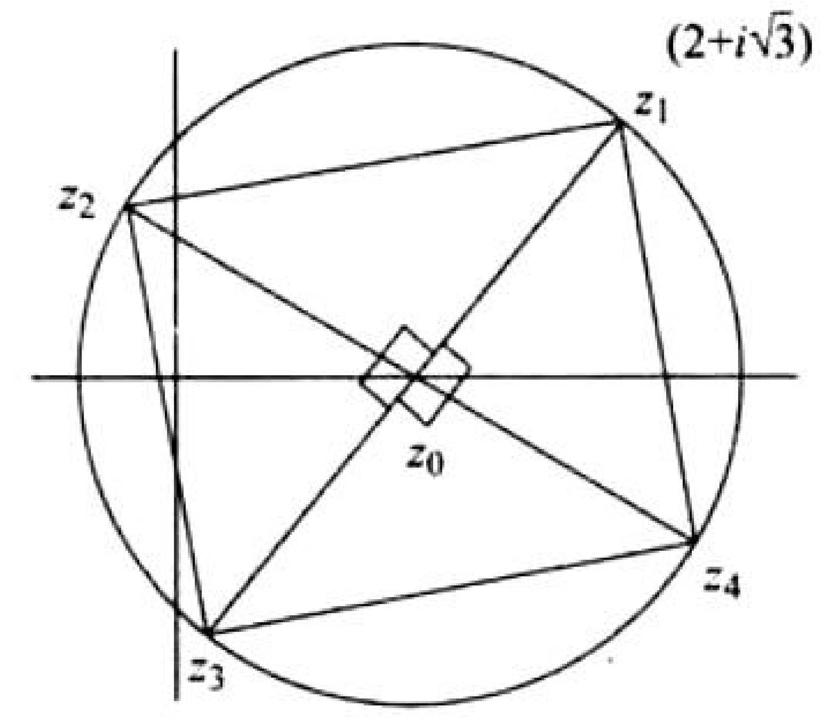
$$\text{Radius} = \frac{1}{2}|PP'|$$

$$= \frac{1}{2}\left|\frac{k\beta+\alpha}{k+1}-\frac{k\beta-\alpha}{k-1}\right|$$

$$= \frac{1}{2}\left|\frac{k^2\beta+k\alpha-k\beta-\alpha-k^2\beta+k\alpha-k\beta+\alpha}{k^2-1}\right|$$

$$= \frac{k|\alpha-\beta|}{|1-k^2|}$$

21. The given circle is $|z-1| = \sqrt{2}$, where $z_0 = 1$ is the center and $\sqrt{2}$ is radius of the circle. z_1 is one of the vertices of the square inscribed in the given circle.



Clearly, z_2 can be obtained by rotating z_1 by an angle of 90° in anticlockwise sense about center z_0. Thus,

$$z_2 - z_0 = (z_1 - z_0)\, e^{i\pi/2}$$

$$\Rightarrow \quad z_2 - 1 = (2 + i\sqrt{3} - 1)i$$

$$\Rightarrow \quad z_2 = i - \sqrt{3} + 1$$

$$\Rightarrow \quad z_2 = (1-\sqrt{3}) + i$$

Now z_0 is midpoint of z_1 and z_3 and z_2 and z_4

$$\therefore \quad \frac{z_1+z_3}{2} = z_0 \Rightarrow \frac{2+i\sqrt{3}+z_3}{2} = 1$$

$$\Rightarrow \quad z_3 = -i\sqrt{3}$$

and $$\frac{z_2+z_4}{2} = z_0 \Rightarrow \frac{(1-\sqrt{3})+i+z_4}{2} = 1$$

$$\Rightarrow \quad z_4 = (\sqrt{3}+1) - i$$