Chapter 9

Sequence and Series

Miscellaneous Exercise

Question 1:Show that the sum of $(m + n)^{th}$ and $(m - n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Answer 1:

Let a and d be the first term and the common difference of the A.P.

respectively. It is known that the kth term of an A. P. is given by

$a_k = a + (k - 1) d$

$\therefore a_{m+n} = a + (m + n - 1) d$

$a_{m-n} = a + (m - n - 1) d$

$a_m = a + (m - 1) d$

$\therefore a_{m+n} + a_{m-n} = a + (m + n - 1) d + a + (m - n - 1) d$

$= 2a + (m + n - 1 + m - n - 1) d$

$= 2a + (2m - 2) d$

$= 2a + 2 (m - 1) d$

$=2 [a + (m - 1) d]$

$= 2am$

Thus, the sum of $(m + n)^{th}$ and $(m - n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Question 2: If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Answer 2:

Let the three numbers in A.P. be a – d, a, and a + d.

According to the given information,

$(a - d) + (a) + (a + d) = 24$ ... (1)

$\Rightarrow 3a = 24$

$\therefore a = 8$

$(a - d)\, a\, (a + d) = 440$ ... (2)

$\Rightarrow (8 - d)(8)(8 + d) = 440$

$\Rightarrow (8 - d)(8 + d) = 55$

$\Rightarrow 64 - d^2 = 55$

$\Rightarrow d^2 = 64 - 55 = 9$

$\Rightarrow d = \pm 3$

Therefore, when d = 3, the numbers are 5, 8, and 11 and when d = –3, the numbers are 11, 8, and 5.

Thus, the three numbers are 5, 8, and 11.

Question 3: Let the sum of n, 2n, 3n terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$

Answer 3:

Let a and b be the first term and the common difference of the A.P. respectively. Therefore,

$S_1 = \frac{n}{2}[2a + (n - 1)d]$… (1)

$S_2 = \frac{2n}{2}[2a + (2n - 1)d] = n[2a + (2n - 1)d]$ … (2)

$S_3 = \frac{3n}{2}[2a + (3n - 1)d]$ … (3)

From (1) and (2), we obtain

$S_2 - S_1 = n\,[2a + + (2n - 1)\,d)] - \frac{n}{2}\,[2a + (n - 1)\,d)]$

$= n\left\{\frac{4a+4nd-2d-2a-nd+d}{2}\right\}$

$= n\left\{\frac{2a+3nd-d}{2}\right\}$

$= \frac{n}{2}\,[2a + 93n - 1)d]$

$\therefore 3\,(S_2 - S_1) = \frac{3n}{2}\,[2a + (3n - 1)d] = S_3$ [from (3)]

Hence, the given result is proved.

Question 4: Find the sum of all numbers between 200 and 400 which are divisible by 7.

Answer 4:

The numbers lying between 200 and 400, which are divisible by 7, are

203, 210, 217 ... 399

∴ First term, a = 203

Last term, l = 399

Common difference, d = 7

Let the number of terms of the A.P. be n.

$\therefore a_n = 399 = a + (n - 1)\,d$

⇒ 399 = 203 + (n –1) 7

⇒ 7 (n –1) = 196

⇒ n –1 = 28

⇒ n = 29

$S_{29} = \frac{29}{2}(203 + 399)$

$= \frac{29}{2}(602)$

= (29) (602)

= 8729

Thus, the required sum is 8729.

Question 5: Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Answer 5:

The integers from 1 to 100, which are divisible by 2, are 2, 4, 6... 100.

This forms an A.P. with both the first term and common difference equal to 2.

$\Rightarrow 100 = 2 + (n - 1)^{2}$

$\Rightarrow n = 50$

$\therefore 2 + 4 + 6 + \ldots + 100 = \frac{50}{2}[2(2) + (50 - 1)(2)]$

$= \frac{50}{2}[4 + 98]$

= (25) (102)

= 2550

The integers from 1 to 100, which are divisible by 5, are 5, 10... 100.

This forms an A.P. with both the first term and common difference equal to 5.

$\therefore 100 = 5 + (n - 1)^{5}$

$\Rightarrow 5n = 100$

$\Rightarrow n = 20$

∴ $5 + 10 + \ldots + 100 = \frac{20}{2}[2(5) + (20 - 1)5]$

$= 10 [10 + (19) 5]$

$= 10 [10 + 95] = 10 \times 105$

$= 1050$

The integers, which are divisible by both 2 and 5, are 10, 20, ... 100.

This also forms an A.P. with both the first term and common difference equal to 10.

∴ $100 = 10 + (n - 1)(10)$

$\Rightarrow 100 = 10n$

$\Rightarrow n = 10$

∴ $10 + 20 + \ldots + 100 = \frac{10}{2}[2(10) + (10 - 1)(10)]$

$= 5 [20 + 90] = 5 [110] = 550$

Required sum = 2550 + 1050 – 550 = 3050

Thus, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

Question 6: Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

Answer:

The two-digit numbers, which when divided by 4, yield 1 as remainder, are13, 17, ... 97.

This series forms an A.P. with first term 13 and common difference 4.

Let n be the number of terms of the A.P.

It is known that the nth term of an A.P. is given by, $a_n = a + (n - 1) d$

∴ 97 = 13 + (n –1) (4)

⇒ 4 (n –1) = 84

⇒ n – 1 = 21

⇒ n = 22

Sum of n terms of an A.P. is given by,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{22} = \frac{22}{2}$ [22 (13) + (22 – 1) (4)]

= 11 [26 + 84]

= 1210

Thus, the required sum is 1210.

Question 7: If f is a function satisfying f (x + y) = f(x). f(y) for all x, y ∈ N, such that f (1) = 3 and ∑ f(x) = 120 n1, find the value of n.

Answer 7:

It is given that,

f (x + y) = f (x) × f (y) for all x, y ∈ N .... (1)

f (1) = 3

Taking x = y = 1 in (1),

we obtain f (1 + 1) = f (2) = f (1) f (1) = 3 × 3 = 9

Similarly,

f (1 + 1 + 1) = f (3) = f (1 + 2) = f (1) f (2) = 3 × 9 = 27

f (4) = f (1 + 3) = f (1) f (3) = 3 × 27 = 81

∴ f (1), f (2), f (3), ..., that is 3, 9, 27, ..., forms a G.P. with both the first term and common ratio equal to 3.

It is known that, Sn = $\frac{a(r^n-1)}{r-1}$

It is given that, $\sum_{x=1}^{n} f(x)$ = 120

= 120 = $\frac{3(3^n-1)}{3-1}$

= 120 = $\frac{3}{2}(3^n - 1)$

= 3n – 1 = 80

= 3n = 81 = 34

= n = 4

Thus, the value of n is 4.

Question 8:The sum of some terms of G.P. is 315 whose first term and the common

ratio is 5 and 2, respectively. Find the last term and the number of terms.

Answer 8:

Let the sum of n terms of the G.P. be 315.

It is known that, $S_n = \frac{a(r^n-1)}{r-1}$

It is given that the first term a is 5 and common ratio r is 2.

∴ 315 = $\frac{5(2^n-1)}{2-1}$

= 2n – 1 = 63

= 2n = 64 = $(2)^6$

= n = 6

Last term of the G.P = 6^{th} term = ar^{6-1} = (5) $(2)^5$ = (5) (32) = 160

Thus, the last term of the G.P. is 160.

Question 9: The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Answer 9:

Let a and r be the first term and the common ratio of the G.P. respectively.

a = 1 $a_3 = ar^2 = r^2$ $a^5 = ar^4 = r^4$

$\therefore r^2 + r^4 = 90$

$\Rightarrow r^4 + r^2 - 90 = 0$

$= r^2 = \frac{-1 \pm \sqrt{1+360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = \frac{-1 \pm 19}{2}$ = - 10 or 9

= r ± 3 (taking real roots)

Thus, the common ratio of the G.P. is ± 3.

Question 10: The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Answer 10:

Let the three numbers in G.P. be a, ar, and ar^2

From the given condition,

$a + ar + ar^2 = 56$

$\Rightarrow a(1 + r + r^2) = 56$ .... (1)

a – 1, ar – 7, ar2 – 21 forms an A.P.

$\therefore (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$\Rightarrow ar - a - 6 = ar^2 - ar - 14$

$\Rightarrow ar^2 - 2ar + a = 8$

$\Rightarrow ar^2 - ar - ar + a = 8$

$\Rightarrow a\ (r^2 + 1 - 2r) = 8$

$\Rightarrow a\ (r - 1)^2 = 8$ .... (2)

From (1) and (2), we get

$\Rightarrow 7(r^2 - 2r + 1) = 1 + r + r^2$

$\Rightarrow 7r^2 - 14\ r + 7 - 1 - r - r^2 = 0$

$\Rightarrow 6r^2 - 15r + 6 = 0$

$\Rightarrow 6r^2 - 12r - 3r + 6 = 0$

$\Rightarrow 6r\ (r - 2) - 3\ (r - 2) = 0$

$\Rightarrow (6r - 3)\ (r - 2) = 0$

When r = 2, a = 8

When

Therefore, when r = 2, the three numbers in G.P. are 8, 16, and 32.

When, r=1/2, the three numbers in G.P. are 32, 16, and 8.

Thus, in either case, the three required numbers are 8, 16, and 32.

Question 11: A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Answer 11:

Let the G.P. be T_1, T_2, T_3, T_4 ... T_{2n}.

Number of terms = 2n

According to the given condition,

$T_1 + T_2 + T_3 + ... + T_{2n} = 5\ [T_1 + T_3 + ... + T_{2n-1}]$

$\Rightarrow T_1 + T_2 + T_3 + ... + T_{2n} - 5\ [T_1 + T_3 + ... + T_{2n-1}] = 0$

$\Rightarrow T_2 + T_4 + ... + T_{2n} = 4\ [T_1 + T_3 + ... + T_{2n-1}]$

Let the G.P. be a, ar, ar^2, ar^3 ...

$\therefore \frac{ar(r^n-1)}{r-1} = \frac{4\times a(r^n-1)}{r-1}$

= ar = 4a

= r = 4

Thus, the common ratio of the G.P. is 4.

Question 12: The sum of the first four terms of an A.P. is 56. The sum of the last four terms are 112. If its first term is 11, then find the number of terms.

Answer 12:

Let the A.P. be a, a + d, a + 2d, a + 3d ... a + (n – 2) d, a + (n – 1) d.

Sum of first four terms = a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d

Sum of last four terms

= [a + (n – 4) d] + [a + (n – 3) d] + [a + (n – 2) d] + [a + n – 1) d]

= 4a + (4n – 10) d

According to the given condition,

4a + 6d = 56

⇒ 4(11) + 6d = 56 [Since a = 11 (given)]

⇒ 6d = 12

⇒ d = 2

∴ 4a + (4n –10) d = 112

⇒ $4(11) + (4n - 10)^2 = 112$

⇒ $(4n - 10)^2 = 68$

⇒ 4n – 10 = 34

$\Rightarrow$ 4n = 44

$\Rightarrow$ n = 11

Thus, the number of terms of the A.P. is 11.

Question 13: If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$then show that a, b, c and d are in G.P.

Answer 13:

It is given that,

$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$

= (a + bx) (b – cx) = (b + cx) (a – bx)

= ab – acx + b^2x – bcx^2 = ab – b^2x + acx – bcx^2

= $2b^2x$ = 2acx

= b^2 = ac

= $\frac{b}{a} = \frac{c}{b}$ … (1)

Also, $\frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$

= (b + cx) (c – dx) = (b – cx) (c + dx)

= $bc - bdx$ + c^2x – cdx^2 = $bc + bdx$ – c^2x + cdx^2

= $2c^2x$ = 2bdx

= c^2 = bd

= $\frac{c}{d} = \frac{b}{c}$ … (2)

From (1) and (2), we obtain

= $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

Thus, a, b, c, and dare in G.P.

Question 14: The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively. Show that (q – r) a + (r – p) b + (p – q) c = 0

Answer 14:

Let, t and d be the first term and the common difference of the A.P. respectively.

The nth term of an A.P. is given by, an = t + (n – 1) d

Therefore,

= a_p = t + (p – 1) d = a … (1)

= a_q = t + (p – 1) d = b … (2)

= a_r = t + (r – 1) d = c … (3)

Subtracting eq. (2) from (1), we obtain

(p – 1 – q + 1) d = a – b

= (p – q) d = a – b

= d = $\frac{a-b}{p-q}$ …. (4)

Subtracting eq. (3) from (2), we obtain

(q – 1 – r + 1) d = b – c

= (q – r) d = b – c

= d = $\frac{b-c}{q-r}$ …. (5)

Equating both the values of d obtained in (4) and (5), we obtain

= $\frac{a-b}{p-q} = \frac{b-c}{q-r}$

= (a – b) (q – r) = (b – c) (p – q)

$= aq - ar - bq + br = bp - bq - cp + cq$

$= bp - cp + cq - aq + ar - br = 0$

= $(-aq + ar)$ + (bp – br) + (-cp + cq) = 0 [by rearranging terms]

= -a (q – r) – b (r – p) - c (p – q) = 0

= a (q – r) + b (r – p) + c (p – q) = 0

Thus, the given result is proved.

Question 15: If a $\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$are in A.P., prove that a, b, c are in A.P.

Answer 15:

It is given that a $\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

b $\left(\frac{1}{c}+\frac{1}{a}\right) - a\left(\frac{1}{b}+\frac{1}{c}\right) = c\left(\frac{1}{a}+\frac{1}{b}\right) - b\left(\frac{1}{c}+\frac{1}{a}\right)$

$= \frac{b(a+c)}{ac} - \frac{a(b+c)}{bc} = \frac{c(a+b)}{ab} - \frac{b(a+c)}{ac}$

$= \frac{b^2a+b^2c-a^2b-a^2c}{abc} = \frac{c^2a+c^2b-b^2a-b^2c}{abc}$

= $b^2a - a^2b + b^2c - a^2c = c^2a - b^2a + c^2b - b^2c$

= ab (b – a) + c ($b^2 - a^2$) = a ($c^2 - b^2$) + bc (c – b)

= ab (b – a) + c (b – a) (b + a) = a (c – b) (c + b) + bc (c – b)

= (b – a) (ab + cb + ca) = (c – b) (ac + ab + bc)

= b – a = c – b

Thus, a, b, and c, are in A.P.

Question 16: If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$are in G.P.

Answer 16:

It is given that a, b, c and d are in G.P.

$b^2 = ac$ … (1)

$c^2 = bd$ …. (2)

$ad = bc$ … (3)

It has to be proved that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P. i.e.,

$= (b^n + c^n)^2 = (an + b^n)(c^n + d^n)$

Consider L.H.S.

$= (b^n + c^n)^2 = b^2n + 2b^2cn + c^2n$

$= (ac)^n + 2b^nc^n + (bd)^n$ [using (1) and (2)]

$= a^n c^n + b^n c^n + b^n c^n + b^n d^n$

$= a^n c^n + b^n c^n + a^n d^n + b^n d^n$ [using (3)]

$= c^n (a^n + b^n) + d^n (a^n + b^n)$

$= (a^n + b^n)(c^n + d^n)$ R.H.S.

Thus, $(a^n + b^n), (b^n + c^n)$, and $(c^n + d^n)$ are in G.P.

Question 17: If a and b are the roots of are roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d, form a G.P.

Prove that $(q + p):(q - p) = 17:15$.

Answer 17:

It is given that a and b are the roots of $x^2 - 3x + p = 0$

$a + b = 3$ and $ab = p$ …. (1)

Also, c and d are the roots of $x^2 - 12x + q = 0$

$c + d = 12$ and $cd = q$ … (2)

It is given that a, b, c, d, are in G.P.

Let a = x, b = xr, c = xr^2, d = xr^3 from (1) and (2)

We obtain $x + xr = 3 = x(1 + r) = 3$

$= xr^2 + xr^3 = 12$

$= xr^2(1 + r) = 12$

On dividing, we obtain

$= \frac{xr^2(1+r)}{x(1+r)} = \frac{12}{3}$

$= r^2 = 4$

$= r = \pm 2$

When r = 2, x = $\frac{3}{1+2} = \frac{3}{3} = 1$

Where r = -2, x = $\frac{3}{1-2} = \frac{3}{-1} = -3$

Case I

When r = 2 and x = 1, ab = $x^2r = 2$, cd = $x^2r^5 = 32$

$= \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$

i.e., (q + p): (q – p) = 17:15

case II

When r = -2, x = -3, ad = $x^2r = -18$, cd = $x^2r^5 = -288$

$= \frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$

Thus, in both the cases, we obtain (q + p): (q – p) = 17: 15

Question 18: The ratio of the A.M and G.M. of two positive numbers a and b, is m: n. Show that

$$a : b = \left(m + \sqrt{m^2 + n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$$

Answer 18:

Let the two numbers be a and b.

A.M. = $\frac{a+b}{2}$ and G.M. = $\sqrt{ab}$

According to the given condition,

$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$

= $\frac{(a+b)^2}{4(ab)} = \frac{m^2}{n^2}$

= $(a+b)^2 = \frac{4abm^2}{n^2}$

= (a + b) = $\frac{2\sqrt{ab}m}{n}$ … (1)

= $(a - b)^2 = \frac{4abm^2}{n^2} - 4ab = \frac{4ab(m^2-n^2)}{n^2}$

= (a – b) = $\frac{2\sqrt{ab}\sqrt{m^2+n^2}}{n}$ ….(2)

Adding (1) and (2), we obtain

$2a = \frac{2\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$

= $a = \frac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$

Substituting the value of a in (1), we obtain

= $b = \frac{2\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$

= $\frac{\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\sqrt{m^2 - n^2}$

= $\frac{\sqrt{ab}}{n}\left(m - \sqrt{m^2 - n^2}\right)$

$= a : b = \frac{\frac{\sqrt{ab}}{n}(m+\sqrt{m^2+n^2})}{\frac{\sqrt{ab}}{n}(m-\sqrt{m^2-n^2})} = \frac{(m+\sqrt{m^2+n^2})}{(m-\sqrt{m^2-n^2})}$

Thus, a; b = (m + $\sqrt{m^2 + n^2}$) : (m - $\sqrt{m^2 - n^2}$)

Question 19: If a, b, c are in A.P; b, c, d are in G.P and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that

a, c, e is in G.P.

Answer 19:

It is given that a, b, c is in A.P.

b – a = c – b … (1)

It is given that b, c, d, are in G.P.

= $c^2 = bd$ … (2)

Also, $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P.

$= \frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d}$

$= \frac{2}{d} = \frac{1}{c} + \frac{1}{e}$ … (3)

It has to be proved that a, c, e, are in G.P. i.e., c2 = ae

From (1), we obtain

2b = a + c

= b = $\frac{a+c}{2}$

From (2), we obtain

= d = $\frac{c^2}{b}$

Subtracting this value in (3), we obtain

$= \frac{2b}{c^2} = \frac{1}{c} + \frac{1}{e}$

$= \frac{2(a+c)}{2c^2} = \frac{1}{c} + \frac{1}{e}$

$= \frac{a+c}{c^2} = \frac{e+c}{ec}$

$= \frac{a+c}{c} = \frac{e+c}{e}$

= (a + c) e = (e + c) c

= ae $+ ce = ec +$ c^2

= c^2 = ae

Thus, a, c, and e are in G.P.

Question 20: Find the sum of the following series up to n terms:

(i) 5 + 55 + 555 + ... (ii) .6 +.66 +. 666 +...

Answer 20:

(1) 5 + 55 + 555 + …

Let, Sn = 5 + 55 + 555 + … to n terms

$= \frac{5}{9}$ [9 + 99 + 999 + … to n terms]

$= \frac{5}{9}$ [(10 – 1) + (10^2 – 1) + (10^3 – 1) + … to n terms]

$= \frac{5}{9}$ [(10 + 10^2 + 10^3 + … n terms) – (1 + 1 + … n terms)]

$= \frac{5}{9}\left[\frac{10(10^n-1)}{10-1} - n\right]$

$= \frac{5}{9}\left[\frac{10(10^n-1)}{9}\right]$

$= \frac{50}{81}(10^n - 1) - \frac{5n}{9}$

(ii) .6 + .66 + .666 + …

Let, Sn = 0.6 + 0.66 + 0.666 + … to n terms

= 6 [0.1 + 0.11 + 0.111 + … to n terms]

$= \frac{6}{9}$ [0.9 + 0.99 + 0.999 + to n terms]

$= \frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\cdots to\ n\ terms\right]$

$= \frac{2}{3}\left[(1+1+\cdots n\ terms)-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\cdots nterms\right)\right]$

$= \frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$

$= \frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}(1-10^{-n})$

$= \frac{2}{3}n-\frac{2}{27}(1-10^{-n})$

Question 21: Find the 20^{th} term of the series 2 × 4 + 4 × 6 + 6 × 8 + ... + n terms.

Answer 21:

The given series is 2 × 4 + 4 × 6 + 6 × 8 + ... + n terms

n^{th} term = $a_n = 2n \times (2n + 2) = 4n^2 + 4n$

$a_{20} = 4(20)^2 + 4(20) = 4(400) + 80 = 1600 + 80 = 1680$

Thus, the 20^{th} term of the series is 1680.

Question 22:

Find the sum of the first n terms of the series: 3 + 7 + 13 + 21 + 31 + ...

Answer 22:

The given series is 3 + 7 + 13 + 21 + 31 + …

$S = 3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n$

$S = 3 + 7 + 13 + 21 + \ldots + a_{n-2} + a_{n-1} + a_n$

On subtracting both the eq., we obtain

S – S = [3 + (7 + 13 + 21 31 + … + a_{n-1} + a_n)] – [(3 + 7 + 13 + 21 + 31 + … + a_{n-1} + a_n)]

S – S = 3 + [(7 – 3) + (13 – 7) + (21 – 13) + … + (a_n – a_{n-1}] – a_n

0 = 3 + [4 + 6 + 8 + … (n – 1) terms] – a_n

a_n = 3 + [4 + 6 + 8 + … (n – 1) terms]

$a_n = 3 + \left(\frac{n-1}{2}\right)$ [2 × 4 + (n – 1 – 1) 2]

$= 3 + \left(\frac{n-1}{2}\right)$ [8 + (n – 2) 2]

$= 3 + \left(\frac{n-1}{2}\right)$ [2n + 4]

= 3 + (n – 1) (n – 2)

= 3 + (n^2 + n – 2)

= n^2 + n + 1

$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$

$= n\left[\frac{(n+1)(2n+1)+3(n+1)+6}{6}\right]$

$= n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right]$

$= n\left(\frac{2n^2+6n+10}{6}\right)$

$= \frac{n}{3}(n^2 + 3n + 5)$

Question 23: If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3 (1+ 8S_1)$

Answer 23:

From the given information

$S_1 = \frac{n(n+1)}{2}$

$S_2 = \frac{n^2(n+1)^2}{4}$

Here, $S_3 (1 + 8S_1) = \frac{n^2(n+1)^2}{4}\left[1 + \frac{8n(n+1)}{2}\right]$

$= \frac{n^2(n+1)^2}{4}[4n^2 + 4n + 1]$

$= \frac{n^2(n+1)^2}{4}(2n + 1)^2$

$= \frac{[n(n+1)(2n+1)]^2}{4}$ … (1)

Also, $9S_2^2 = 9\ \frac{[n(n+1)(2n+1)]^2}{6^2}$

$= \frac{9}{36}[n(n + 1)(2n + 1)]^2$

$= \frac{[n(n+1)(2n+1)]^2}{4}$ …. (2)

Thus, from (1) and (2), we obtain $9S_2^2 = S_3 (1 + 8S_1)$

Question 24: Find the sum of the following series up to n terms:

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$$

Answer 24:

The nth term of the given series is $\frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\cdots+(2n-1)}$

Here, 1, 3, 5, … (2n – 1) is an A.P. with first term a, last term (2n – 1) and number of terms as n

$= 1 + 3 + 5 + …. + (2n – 1) = \frac{n}{2}[2 \times 1 + (n – 1) 2] = n^2$

$a_n = \frac{n^2(n+1)^2}{4n^2} = \frac{(n+1)^2}{4} = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$

$S_n = \sum_{k=1}^{a} a_k = \sum_{k=1}^{n}\left(\frac{1}{4}k^2 + \frac{1}{2}k + \frac{1}{4}\right)$

$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{1}{4}n$

$= \frac{n[(n+1)(2n+1)+6(n+1)+6]}{24}$

$= \frac{n[2n^2+3n+1+6n+6+6]}{24}$

$= \frac{n(2n^2+9n+12)}{24}$

Question 25: Show that $\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)} = \frac{3n+5}{3n+1}$

Answer 25:

n^{th} term of the numerator = $n(n+1)^2 = n^3 + 2n^2 + n$

n^{th} term of the denominator = $n^2(n+1) = n^3 + n^2$

$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)} = \frac{\sum_{k=1}^{n} n_k}{\sum_{k=1}^{n} n_k} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)}$ …. (1)

Here, $\sum_{k=1}^{n}(k^3 + 2k^2 + k)$

$= \frac{n^n(n+1)^2}{6} = \frac{2n(n+1)(2n+1)}{4} = \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right]$

$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$

$= \frac{n(n+1)}{12}[3n^2 + 11n + 10]$

$= \frac{n(n+1)}{12}[3n^2 + 6n + 5n + 10]$

$= \frac{n(n+1)}{12}[3n(n+2)+5(n+2)]$

$= \frac{n(n+1)(n+2)(3n+5)}{12} \dots (2)$

Also, $\sum_{k=1}^{n}(k^2+k^2) = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right]$

$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$

$= \frac{n(n+1)}{12}[3n^2+7n+2]$

$= \frac{n(n+1)}{12}[3n^2+6n+n+2]$

$= \frac{n(n+1)}{12}[3n(n+2)+1(n+2)]$

$= \frac{n(n+1)(n+2)(3n+1)}{12} \dots (3)$

From (1), (2) and (3), we obtain

$= \frac{1\times2^2+2\times3^2+\cdots+n\times(n+1)^2}{1^2\times2+2^2\times3+\cdots+n^2\times(n+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$

$= \frac{3n+5}{3n+1}$

Hence, proved.

Question 26: A farmer buys a used tractor for ₹12000. He pays ₹6000 cash and agrees to pay the balance in annual installments of ₹500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

Answer 26:

It is given the farmer pays ₹6000 in cash.

Therefore, unpaid amount = ₹12000 - ₹6000 = ₹6000

According to the given conditions, the interest paid annually is

12% of 6000, 12% of 5500, 12% of 5000 … 12% of 500

Thus, total interest to be paid

= 12% of 6000 + 12% of 5500 + 12% of 5000 + … + 12% of 500

= 12% of (6000 + 5500 +5000 + … + 500)

= 12% of (500 + 1000 + 1500 + … + 6000)

Now, the series 500, 1000, 1500, 6000 is an A.P. with both the first term and common difference equal to 500

Let, the number of terms of the A.P. be n.

6000 = 500 + (n – 1) 500

= 1 + (n – 1) = 12

= n = 12

Sum of A.P.

= $\frac{12}{2}[2(500) + (12 - 1)(500)]$ = 6 [1000 + 5500] = 6 [6500] = 39000

Thus total interest to be paid

= 12% of (500 + 1000 + 1500 + … + 6000))

= 12% of 39000 = ₹4680

Thus, cost of tractor = (₹12000 + ₹4680) = ₹16680

Question 27: Sham shed Ali buys a scooter for ₹22000. He pays ₹4000 cash and agrees to pay the balance in annual installment of ₹1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Answer:

It is given that sham shed Ali buys a scooter for ₹22000 and pays ₹4000 in cash.

Unpaid amount = ₹22000 - ₹4000 = ₹18000

According to the given condition, the interest paid annually is 10% of 18000, 10% of 17000, 10% of 16000 …. 10% of 1000.

Thus, total interest to be paid

= 10% of 18000, 10% of 17000, 10% of 16000 …. 10% of 1000

= 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of (1000 + 200 + 3000 + … + 18000)

Here, 1000, 2000, 3000 … 18000 forms an A.P. with first term and common difference both equal to 1000.

Let, the numbers of terms be n.

= 18000 = 1000 + (n – 1) (1000)

= n = 18

1000 + 2000 + … + 18000 = $\frac{18}{2}$ [2 (1000) + (18 – 1) (1000)]

= 9 [2000 + 17000]

= 171000

Total interest paid = 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of ₹171000 = ₹17100

Cost of scooter = ₹22000 + ₹17100 = ₹39100

Question 28: A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paisa to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.

Answer:

The numbers of letters mailed forms a G.P. 4, 42, …48

First term = 4

Common ratio = 4

Numbers of item = 8

It is known that the sum of n terms of a G.P. is given by

$S_n = \frac{a(r^n - 1)}{1 - r}$

$S_8 = \frac{4(4^8 - 1)}{4 - 1} = \frac{4(65536 - 1)}{3} = \frac{4(65535)}{3}$ = 4 (21845) = 87380

It is given that the cost to mail one letter is 50 paisa.

Cost of mailing 87380 letters = ₹87380 $\times \frac{50}{100}$ = ₹43690

Thus, the amount spent when 8^{th} set of letter is mailed is ₹43690.

Question 29: A man deposited ₹10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

Answer:

It is given that the man deposited ₹10000 in a bank at the rate of 5%

Simple interest annually = $\frac{5}{100} \times$ ₹10000 = ₹ 500

Interest in first year = 10000 + 500 + 500 + … + 500

Amount in 15^{th} year = ₹

= ₹10000 + 14 × ₹500

= ₹10000 + ₹7000

= ₹17000

Amount after 20 years = ₹10000 + 500 + 500 + … + 500

= ₹10000 + 20 × ₹500

= ₹10000 +₹10000

= ₹20000

Question 30: A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Answer:

Cost of machine = ₹15625

Machine depreciates by 20% every year.

Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

Value at the end of 5 years = $15625 \times \frac{4}{5} \times \frac{4}{5} \ldots \times \frac{4}{5} = 5 \times 1024 = 5120$

Thus, the value of the machine at the end of 5 years is ₹= 5120.

Question 31: 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Answer:

Let, x be the numbers of days in which 150 workers finish the work.

According to the given information,

150x = 150 + 146 + 142 + … (x + 8) terms

The series 150 + 146 + 142 + … (x + 8) terms is an A.P. with first term 146, common differences – 4 and number of terms as (x + 8)

$= 150x = \frac{(x+8)}{2}$ [2 (150) + (x + 8 – 1) (-4)]

= 150x = (x + 8) [150 + (x + 7) (-2)]

= 150x = (x + 8) (150 – 2x – 14)

= 150x = (x + 8) (136 – 2x)

= 75x = 69x – x^2 + 544 – 8x

= x^2 + 15x -544 = 0

= x^2 + 32x – 17x – 544 = 0

= x (x + 32) – 17 (x + 32) = 0

= (x + 32) (x – 17) = 0

= x = 17 or x = - 32

However, x cannot be

= negative x = 17

Therefore, originally, the number of days in which the work was completed is 17. Thus, required number of days = (17 + 8) = 25