Inverse Trigonometric Functions

- If $\sin y = x$, then $y = \sin^{-1} x$ (We read it as sine inverse x)

  Here, $\sin^{-1} x$ is an inverse trigonometric function. Similarly, the other inverse trigonometric functions are as follows:

  - If $\cos y = x$, then $y = \cos^{-1} x$
  - If $\tan y = x$, then $y = \tan^{-1} x$
  - If $\cot y = x$, then $y = \cot^{-1} x$
  - If $\sec y = x$, then $y = \sec^{-1} x$
  - If $\text{cosec}\, y = x$, then $y = \text{cosec}^{-1} x$

- The domains and ranges (principle value branches) of inverse trigonometric functions can be shown in a table as follows:

Function	**Domain**	**Range (Principle value branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	**R**	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	**R**	$(0, \pi)$
$y = \sec^{-1} x$	**R** $- (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \text{cosec}^{-1} x$	**R** $- (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- Note that $y = \tan^{-1} x$ does not mean that $y = (\tan x)^{-1}$. This argument also holds true for the other inverse trigonometric functions.

- The principal value of an inverse trigonometric function can be defined as the value of inverse trigonometric functions, which lies in the range of principal branch.

  Example 1: What is the principal value of $\tan^{-1}\left(-\sqrt{3}\right) + \sin^{-1}(1)$?

  Solution:

  Let $\tan^{-1}\left(-\sqrt{3}\right) = y$ and $\sin^{-1}(1) = z$

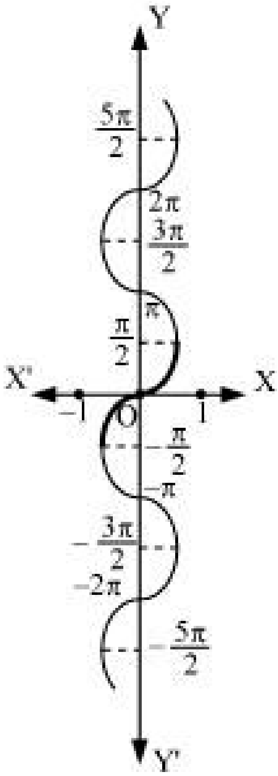
  $\Rightarrow \tan y = -\sqrt{3} = -\tan\left(\frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$ and $\sin z = 1 = \sin\frac{\pi}{2}$

  We know that the ranges of principal value branch of $\tan^{-1}$ and $\sin^{-1}$ are $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ respectively. Also, $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ $\sin\left(\frac{\pi}{2}\right) = 1$

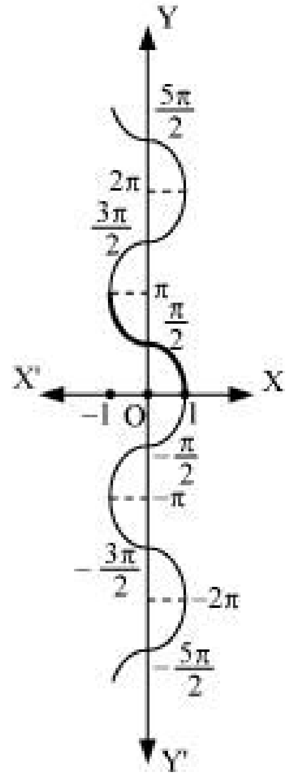
  Therefore, principal values of $\tan^{-1}\left(-\sqrt{3}\right) = \frac{-\pi}{3}$ and $\sin^{-1}(1) = \frac{\pi}{2}$

  $\therefore \tan^{-1}\left(-\sqrt{3}\right) + \sin^{-1} 1 = \frac{-\pi}{3} + \frac{\pi}{2} = \frac{\pi}{6}$

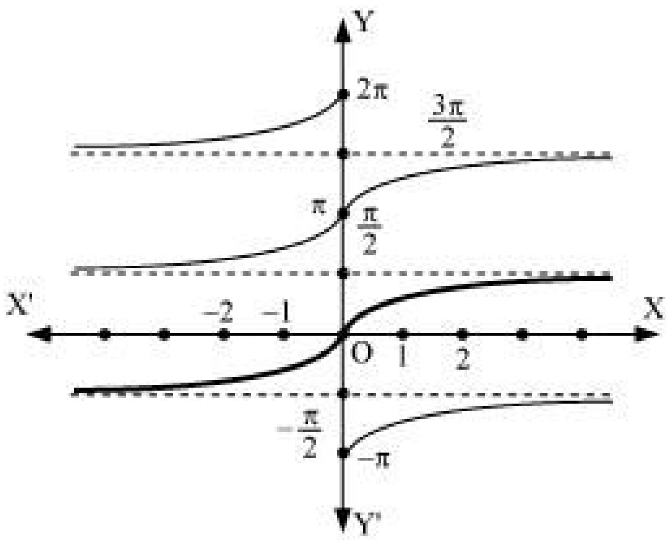
- Graphs of the six inverse trigonometric functions can be drawn as follows:



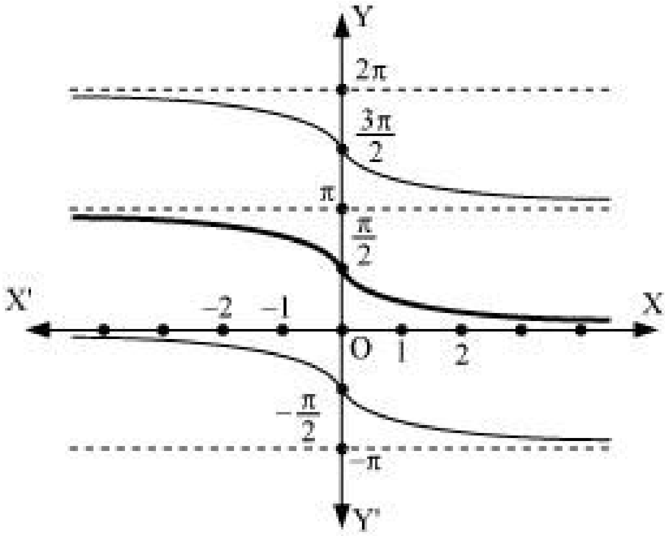
$y = \sin^{-1} x$



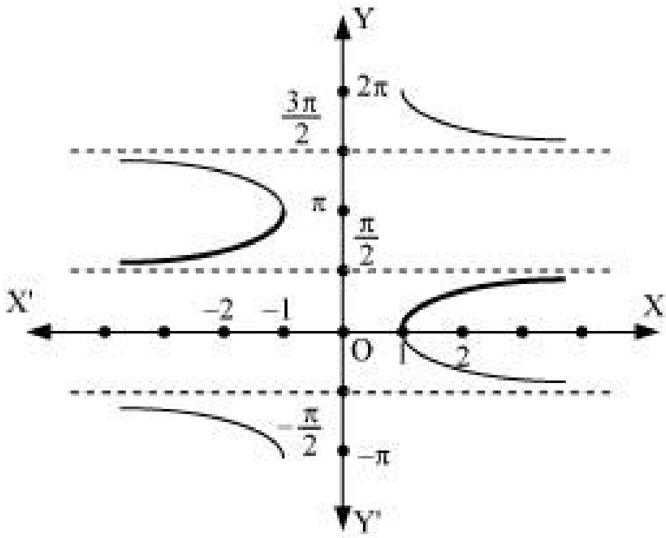
$y = \cos^{-1} x$



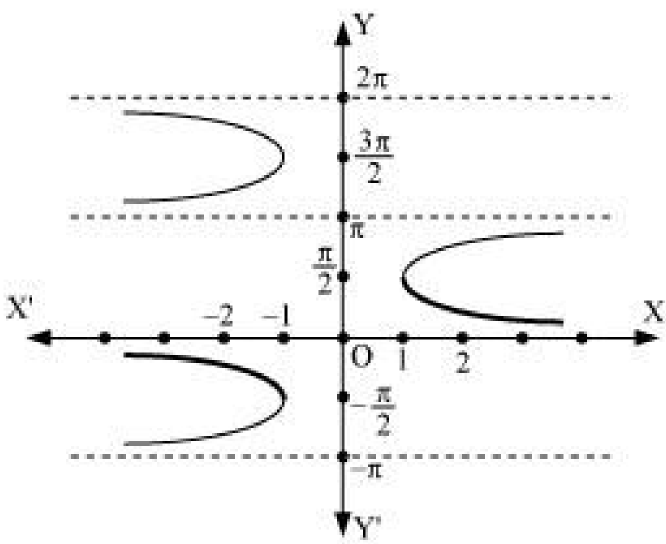
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \mathrm{cosec}^{-1} x$

- The relation $\sin y = x \Rightarrow y = \sin^{-1}x$ gives $\sin(\sin^{-1}x) = x$, where $x \in [-1, 1]$; and $\sin^{-1}(\sin x) = x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  This property can be similarly stated for the other inverse trigonometric functions as follows:

  - $\cos(\cos^{-1}x) = x, x \in [-1, 1]$ and $\cos^{-1}(\cos x) = x, x \in [0, \pi]$
  - $\tan(\tan^{-1}x) = x, x \in \mathbf{R}$ and $\tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x, x \in \mathbf{R} - (-1, 1)$ and $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
  - $\sec(\sec^{-1}x) = x, x \in \mathbf{R} - (-1, 1)$ and $\sec^{-1}(\sec x) = x, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
  - $\cot(\cot^{-1}x) = x, x \in \mathbf{R}$ and $\cot^{-1}(\cot x) = x, x \in (0, \pi)$

- For suitable values of domains;
  - $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x, x \in \mathbf{R} - (-1, 1)$
  - $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, x \in \mathbf{R} - (-1, 1)$
  - $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & x > 0 \\ \cot^{-1}\pi, & x - 0x < \end{cases}$
  - $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x, x \in [-1, 1]$
  - $\sec^{-1}\left(\frac{1}{x}\right) = \cos x, x \in [-1, 1]$
  - $\cot^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1}x, & x > 0 \\ \pi + \tan^{-1}x, & x < 0 \end{cases}$

  Note: While solving problems, we generally use the formulas $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$ and $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$ when the conditions for x (i.e., $x > 0$ or $x < 0$) are not given

- For suitable values of domains;
  - $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$
  - $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$
  - $\tan^{-1}(-x) = -\tan^{-1}x, \; x \in \mathbf{R}$

- **$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \geq 1$**
- $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in \mathbf{R}$

- For suitable values of domains;
  - $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$
  - $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbf{R}$
  - $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$, $|x| \geq 1$

- For suitable values of domains;
  - $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy}, & xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy}, & xy > 1 \end{cases}$
  - $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

Note: While solving problems, we generally use the formula $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$ when the condition for xy is not given.

- For $x \in [-1, 1]$, $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$
- For $x \in (-1, 1)$, $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$
- For x ³ 0, $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$

Example: 2

For $x, y \in [-1, 1]$, show that: $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

Solution:

We know that $\sin^{-1}x$ and $\sin^{-1}y$ can be defined only for $x, y \in [-1, 1]$

Let $\sin^{-1}x = a$ and $\sin^{-1}y = b$

$\Rightarrow x = \sin a$ and $y = \sin b$

Also, $\cos a = \sqrt{1-x^2}$ and $\cos b = \sqrt{1-y^2}$

We know that, $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\Rightarrow a + b = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Example: 3

If $\tan^{-1}\left(\frac{5}{6}\right) + \tan^{-1}\left(\frac{1}{11}\right) = x$, then find $\sec x$.

Solution:

We have $x = \tan^{-1}\left(\frac{5}{6}\right) + \tan^{-1}\left(\frac{1}{11}\right) = \tan^{-1}\left[\frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{6}\times\frac{1}{11}}\right]$

$\left[\text{Using the identity } \tan^{-1}x + \tan^{-1}y \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ where } x = \frac{5}{6} \text{ and } y = \frac{1}{11}\right]$

$\therefore x = \tan^{-1}\left[\frac{\frac{55+6}{66}}{\frac{66-5}{66}}\right]$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

$\sec x = \sec\frac{\pi}{4} = \sqrt{2}$

Example: 4

Show that: $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ where $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Solution:

We know that,

$$3\tan^{-1}x = \tan^{-1}x + 2\tan^{-1}x$$

$$= \tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2}$$

$$= \tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\times\frac{2x}{1-x^2}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{3x-x^3}{1-x^2}}{\frac{1-3x^2}{1-x^2}}\right]$$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$