Chapter 1: Differentiation

EXERCISE 1.1 [PAGES 11 - 13]

Exercise 1.1 | Q 1.1 | Page 11

Differentiate the following w.r.t.x: $(x^3 - 2x - 1)^5$

SOLUTION

Method 1 :

Let $y = (x^3 - 2x - 1)^5$

Put $u = x^3 - 2x - 1$. Then $y = u^5$

$\therefore \dfrac{dy}{du} = \dfrac{d}{du}(u^5) = 5u^4$

= 5(x3 – 2x– 1)4

and

$\dfrac{du}{dx} = \dfrac{d}{dx}(x^3 - 2x - 1)$

= 3x2 – 2 x 1 – 0

= 3x2 – 2

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 5(x^3 - 2x - 1)^4(3x^2 - 2)$

$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$

Method 2 :

Let $y = (x^3 - 2x - 1)^5$

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3 - 2x - 1)^5$

$= 5\left(x^3 - 2x - 1\right)4 \times \frac{\mathrm{d}}{\mathrm{dx}}\left(x^3 - 2x - 1\right)$

$= 5(x^3 - 2x - 1)^4 \times (3x^2 - 2 \times 1 - 0)$

$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$

Note : Out of the two methods given above, we will use Method 2 for solving the remaining problems.

Exercise 1.1 | Q 1.2 | Page 11

Differentiate the following w.r.t.x: $\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$

SOLUTION

Let y = $\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$

Differentiating w.r.t. x,we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$$

$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}-1} \times \frac{\mathrm{d}}{\mathrm{dx}}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)$$

$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}} \times \left(2 \times \frac{3}{2}x^{\frac{3}{2}-1} - 3 \times \frac{4}{3}x^{\frac{4}{3}-1} - 0\right)$$

$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}}\left(3x^{\frac{1}{2}} - 4x^{\frac{1}{3}}\right)$$

$$= \frac{5}{2}\left(3\sqrt{x} - 4\sqrt[3]{x}\right)\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}}.$$

Exercise 1.1 | Q 1.3 | Page 11

Differentiate the following w.r.t.x: $\sqrt{x^2 + 4x - 7}$

$y = \sqrt{x^2 + 4x - 7}\left[\sqrt{x} = \frac{1}{2\sqrt{x}}\right]$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{2\sqrt{x^2 + 4x - 7}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(x^2 + 4x - 7)$

$= \frac{1}{2\sqrt{x^2 + 4x - 7}}\left(\frac{\mathrm{d}}{\mathrm{dx}}x^2 + \frac{\mathrm{d}}{\mathrm{dx}}4x - \frac{\mathrm{d}}{\mathrm{dx}}7\right)$

= $\frac{1}{2\sqrt{x^2 + 4x - 7}}.(2x + 4 - 0)$

= $\frac{2(x + 2)}{2\sqrt{x^2 + 4x - 7}}$

= $\frac{(x + 2)}{\sqrt{x^2 + 4x - 7}}$.

Exercise 1.1 | Q 1.4 | Page 11

Differentiate the following w.r.t.x: $\sqrt{x^2 + \sqrt{x^2 + 1}}$

SOLUTION

Let y = $\sqrt{x^2 + \sqrt{x^2 + 1}}$

Differentiating w.r.t. x,we get,

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2 + \sqrt{x^2 + 1}\right)^{\frac{1}{2}}$

= $\frac{1}{2}\left(x^2 + \sqrt{x^2 + 1}\right)^{-\frac{1}{2}} . \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2 + \sqrt{x^2 + 1}\right)$

= $\frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} . \left[\frac{\mathrm{d}}{\mathrm{dx}}(x^2) + \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x^2 + 1}\right)\right]$

= $\frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} . \left[2x + \frac{1}{2\sqrt{x^2 + 1}} . \frac{\mathrm{d}}{\mathrm{dx}}(x^2 + 1)\right]$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[2x + \frac{1}{2\sqrt{x^2+1}}(2x+0)\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[2x + \frac{x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[\frac{2x\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{x\left(2\sqrt{x^2+1}+1\right)}{2\sqrt{x^2+1}.\sqrt{x^2+\sqrt{x^2+1}}}.$$

Exercise 1.1 | Q 1.5 | Page 11

Differentiate the following w.r.t.x: $\frac{3}{5\sqrt[3]{(2x^2-7x-5)^5}}$

SOLUTION

Let y = $\frac{3}{5\sqrt[3]{(2x^2-7x-5)^5}}$

Differentiating w.r.t. x,we get,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{3d}{5dx}(2x^2-7x-5)^{-\frac{5}{3}}$$

$$= \frac{3}{5} \times \left(-\frac{5}{3}\right)(2x^2-7x-5)^{-\frac{5}{3}-1} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(2x^2-7x-5)$$

$$= -(2x^2-7x-5)^{-\frac{8}{3}} \cdot (2 \times 2x - 7 \times 1 - 0)$$

$$= -\frac{4x-7}{(2x^2-7x-5)^{\frac{8}{3}}}.$$

Exercise 1.1 | Q 1.6 | Page 11

Differentiate the following w.r.t.x: $\left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}}\right)^5$

SOLUTION

Let y = $\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$

Differentiating w.r.t. x,we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.\left[\frac{d}{dx}(3x-5)^{\frac{1}{2}}-\frac{d}{dx}(3x-5)^{-\frac{1}{2}}\right)\Bigg]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times\left[\frac{1}{2}(3x-5)^{-\frac{1}{2}}.\frac{d}{dx}(3x-5)-\left(-\frac{1}{2}\right)(3x-5)^{-\frac{3}{2}}.\frac{d}{dx}(3x-5)\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times\left[\frac{1}{2\sqrt{3x-5}}.(3\times1-0)+\frac{1}{2(3x-5)^{\frac{3}{2}}}.(3\times1-0)\right]$$

$$=\frac{15}{2}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.\left[\frac{3}{2\sqrt{3x-5}}+\frac{3}{2(3x-5)^{\frac{3}{2}}}\right]$$

$$=\frac{15}{2}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.\left[\frac{3x-5+1}{(3x-5)^{\frac{3}{2}}}\right]$$

$$=\frac{15(3x-4)}{2(3x-5)^{\frac{3}{2}}}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.$$

Exercise 1.1 | Q 2.01 | Page 12

Differentiate the following w.r.t.x: cos(x^2 + a^2)

SOLUTION

Let y = cos(x^2 + a^2)

Differentiating w.r.t. x,we get,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\cos\left(x^2 + a^2\right)\right]$$

$$= -\sin\left(x^2 + a^2\right).\ \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2 + a^2\right)$$

= −sin(x^2 + a^2).(2x + 0)

= −2x sin(x^2 + a^2).

Exercise 1.1 | Q 2.02 | Page 12

Differentiate the following w.r.t.x: $\sqrt{e^{(3x+2)+\,5}}$

SOLUTION

Let y = $\sqrt{e^{(3x+2)+\,5}}$

Differentiating w.r.t. x,we get,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[e^{(3x+2)} + 5\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[e^{(3x+2)} + 5\right]^{-\frac{1}{2}}.\ \frac{\mathrm{d}}{\mathrm{dx}}\left[e^{(3x+2)} + 5\right]$$

$$= \frac{1}{2\sqrt{e^{(3x+2)} + 5}}.\ \left[e^{(3x+2)}.\ \frac{\mathrm{d}}{\mathrm{dx}}(3x + 2)\ +\ 0\right]$$

$$= \frac{1}{2\sqrt{e^{(3x+2)} + 5}}.\ \left[e^{(3x+2)}.\ (3 \times 1 + 0)\right]$$

$$= \frac{3e^{(3x+2)}}{2\sqrt{e^{(3x+2)+5}}}.$$

Exercise 1.1 | Q 2.03 | Page 12

Differentiate the following w.r.t.x: $\log\left[\tan\left(\frac{x}{2}\right)\right]$

SOLUTION

Let y = $\log\left[\tan\left(\frac{x}{2}\right)\right]$

Differentiating w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}\log\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)} \cdot \frac{d}{dx}\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)} \cdot \sec^2\left(\frac{x}{2}\right) \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{1}{\cos^2\left(\frac{x}{2}\right)} \cdot \frac{1}{2} \times 1$$

$$= \frac{1}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$= \frac{1}{\sin x}$$

= cosec x.

Exercise 1.1 | Q 2.04 | Page 12

Differentiate the following w.r.t.x: $\sqrt{\tan\sqrt{x}}$

SOLUTION

Let y = $\sqrt{\tan\sqrt{x}}$

Differentiating w.r.t. x,we get,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{\tan\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(\tan\sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x}. \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

Exercise 1.1 | Q 2.05 | Page 12

Differentiate the following w.r.t.x: $\cot^3[\log(x^3)]$

SOLUTION

Let y = $\cot^3[\log(x^3)]$

Differentiating w.r.t. x,we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot\left(\log x^3\right)\right]^3$$

$$= 3\left[\cot\left(\log x^3\right)\right]^2 . \frac{d}{dx}\left[\cot\left(\log x^3\right)\right]$$

$$= 3\cot^2\left[\log\left(x^3\right)\right] . \left[-\mathrm{cosec}^2\left(\log x^3\right)\right] . \frac{d}{dx}\left(\log x^3\right)$$

$$= -3\cot^2\left[\log\left(x^3\right)\right] . \mathrm{cosec}^2\left[\log\left(x^3\right)\right] .3 \frac{d}{dx}(\log x)$$

$$= -3\cot^2\left[\log\left(x^3\right)\right] . \mathrm{cosec}^2\left[\log\left(x^3\right)\right] .3 \times \frac{1}{x}$$

$$= \frac{-9\mathrm{cosec}^2\left[\log\left(x^3\right)\right] . \cot^2\left[\log\left(x^3\right)\right)}{x}$$

Exercise 1.1 | Q 2.06 | Page 12

Differentiate the following w.r.t.x: $5^{\sin^3 x+3}$

SOLUTION

Let y = $5^{\sin^3 x+3}$

Differentiating w.r.t. x,we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(5^{\sin^3 x+3}\right)$$

$$= 5^{\sin^3 x+3} . \log 5 . \frac{d}{dx}\left(\sin^3 x + 3\right)$$

$$= 5^{\sin^3 x+3} . \log 5 . \left[3\sin^2 x . \frac{d}{dx}(\sin x) + 0\right]$$

$$= 5^{\sin^3 x+3} . \log 5 . \left[3\sin^2 x \cos x\right]$$

$$= 3\sin^2 x \cos x . 5^{\sin^3 x+3} . \log 5.$$

Exercise 1.1 | Q 2.07 | Page 12

Differentiate the following w.r.t.x: $\operatorname{cosec}\left(\sqrt{\cos x}\right)$

SOLUTION

Let y = $\operatorname{cosec}\left(\sqrt{\cos x}\right)$

Differentiating w.r.t. x,we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\operatorname{cosec}\left(\sqrt{\cos x}\right)\right]$$

$$= -\operatorname{cosec}\left(\sqrt{\cos x}\right).\cot\left(\sqrt{\cos x}\right).\frac{d}{dx}\sqrt{\cos x}$$

$$= -\operatorname{cosec}\left(\sqrt{\cos x}\right).\cot\left(\sqrt{\cos x}\right).\frac{1}{2\sqrt{\cos x}}.\frac{d}{dx}(\cos x)$$

$$= -\operatorname{cosec}\left(\sqrt{\cos x}\right).\cot\left(\sqrt{\cos x}\right).\frac{1}{2\sqrt{\cos x}}.(-\sin x)$$

$$= \frac{\sin x.\operatorname{cosec}\left(\sqrt{\cos x}\right).\cot\left(\sqrt{\cos x}\right)}{2\sqrt{\cos x}}.$$

Exercise 1.1 | Q 2.08 | Page 12

Differentiate the following w.r.t.x: $\log[\cos(x^3 - 5)]$

SOLUTION

Let y = $\log[\cos(x^3 - 5)]$

Differentiating w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\log\left[\cos\left(x^3-5\right)\right]\right\}$$

$$= \frac{1}{\cos(x^3-5)}.\frac{d}{dx}\left[\cos\left(x^3-5\right)\right]$$

$$= \frac{1}{\cos(x^3-5)}.\left[-\sin\left(x^3-5\right)\right].\frac{d}{dx}\left(x^3-5\right)$$

$= -\tan(x^3 - 5) \times (3x^2 - 0)$

$= -3x^2\tan(x^3 - 5).$

Exercise 1.1 | Q 2.09 | Page 12

Differentiate the following w.r.t.x: $e^{3\sin^2 x - 2\cos^2 x}$

SOLUTION

Let y = $e^{3\sin^2 x - 2\cos^2 x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{3\sin^2 x - 2\cos^2 x}\right]$$

$$= e^{3\sin^2 x - 2\cos^2 x}.\ \frac{d}{dx}(3\sin^2 x - 2\cos^2 x)$$

$$= e^{3\sin^2 x - 2\cos^2 x}.\ \left[3\frac{d}{dx}(\sin x)^2 - 2\frac{d}{dx}(\cos^2 x)\right]$$

$$= e^{3\sin^2 x - 2\cos^2 x}.\ \left[3 \times 2\sin x.\ \frac{d}{dx}(\sin x) - 2 \times 2\cos x.\ \frac{d}{dx}(\cos x)\right]$$

$$= e^{3\sin^2 x - 2\cos^2 x}.\ [6\sin x\cos x - 4\cos x(-\sin x)]$$

$$= e^{3\sin^2 x - 2\cos^2 x}.\ (10\sin x\cos x)$$

$$= 5(2\sin x\cos x).\ e^{3\sin^2 x - 2\cos^2 x}$$

$$= 5\sin 2x.\ e^{3\sin^2 x - 2\cos^2 x}.$$

Exercise 1.1 | Q 2.1 | Page 12

Differentiate the following w.r.t.x: $\cos^2[\log(x^2 + 7)]$

SOLUTION

Let y = $\cos^2[\log(x^2 + 7)]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos[\log(x^2+7)]\}^2$$

$$= 2\cos[\log(x^2+7)].\frac{d}{dx}\{\cos[\log(x^2+7)]\}$$

$$= 2\cos[\log(x^2+7)].\{-\sin[\log(x^2+7)]\}.\frac{d}{dx}[\log(x^2+7)]$$

$$= -2\sin[\log(x^2+7)].\cos[\log(x^2+7)] \times \frac{1}{x^2+7}.\frac{d}{dx}(x^2+7)$$

$$= -\sin[2\log(x^2+7)] \times \frac{1}{x^2+7}.(2x+0) \quad ...[\because 2\sin x \cdot \cos x = \sin 2x]$$

$$= \frac{-2x.\sin[2\log(x^2+7)]}{x^2+7}.$$

Exercise 1.1 | Q 2.11 | Page 12

Differentiate the following w.r.t.x: tan[cos (sinx)]

SOLUTION

Let y = tan[cos (sinx)]

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\tan[\cos(\sin x)]\}$$

$$= \sec^2[\cos(\sin x)].\frac{d}{dx}[\cos(\sin x)]$$

$$= \sec^2[\cos(\sin x)].[-\sin(\sin x)].\frac{d}{dx}(\sin x)$$

$$= -\sec^2[\cos(\sin x)] \cdot \sin(\sin x) \cdot \cos x.$$

Exercise 1.1 | Q 2.12 | Page 12

Differentiate the following w.r.t.x: sec[tan $(x^4 + 4)$]

SOLUTION

Let y = sec[tan $(x^4 + 4)$]

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left\{\sec\left[\tan\left(x^4+4\right)\right]\right\}$$

$$= \sec\left[\tan\left(x^4+4\right)\right].\tan\left[\tan\left(x^4+4\right)\right].\frac{\mathrm{d}}{\mathrm{dx}}\left[\tan\left(x^4+4\right)\right]$$

$$= \sec\left[\tan\left(x^4+4\right)\right].\tan\left[\tan\left(x^4+4\right)\right].\sec^2\left(x^4+4\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(x^4+4\right)$$

$= \sec[\tan(x^4 + 4)]\cdot\tan[\tan(x^4 + 4)]\cdot\sec^2(x^4 + 4).(4x^3 + 0)$

$= 4x^3\cdot\sec^2(x^4 + 4)\cdot\sec[\tan(x^4 + 4)]\cdot\tan[\tan(x^4 + 4)]$.

Exercise 1.1 | Q 2.13 | Page 12

Differentiate the following w.r.t.x: $e^{\log\left[(\log x)^2 - \log x^2\right]}$

SOLUTION

Let y = $e^{\log\left[(\log x)^2 - \log x^2\right]}$

$= (\log x)^2 - \log x^2$ ...[$\because e^{\log x} = x$]

$= (\log x)^2 - 2\log x$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\log x)^2 - 2\log x\right]$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}(\log x)^2 - 2\frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$= 2\log x.\frac{\mathrm{d}}{\mathrm{dx}}(\log x) - 2\times\frac{1}{x}$$

$= 2\log x \times \frac{1}{x} - \frac{2}{x}$

$= \frac{2\log x}{x} - \frac{2}{x}.$

Exercise 1.1 | Q 2.14 | Page 12

Differentiate the following w.r.t.x: $\sin\sqrt{\sin\sqrt{x}}$

SOLUTION

Let y = $\sin\sqrt{\sin\sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\sin\sqrt{\sin\sqrt{x}}\right)$$

$$= \cos\sqrt{\sin\sqrt{x}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{\sin\sqrt{x}}\right)$$

$$= \cos\sqrt{\sin\sqrt{x}} \times \frac{1}{2\sqrt{\sin\sqrt{x}}}.\frac{\mathrm{d}}{\mathrm{dx}}(\sin\sqrt{x})$$

$$= \frac{\cos\sqrt{\sin\sqrt{x}}}{2\sqrt{\sin\sqrt{x}}} \times \cos\sqrt{x}.\frac{\mathrm{d}}{\mathrm{dx}}(\sqrt{x})$$

$$= \frac{\cos\sqrt{\sin\sqrt{x}}.\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos\sqrt{\sin\sqrt{x}}.\cos\sqrt{x}}{4\sqrt{x}.\sqrt{\sin\sqrt{x}}}.$$

Exercise 1.1 | Q 2.15 | Page 12

Differentiate the following w.r.t.x: $\log\left[\sec\left(e^{x^2}\right)\right]$

SOLUTION

Let y = $\log\left[\sec\left(e^{x^2}\right)\right]$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\log\left[\sec\left(e^{x^2}\right)\right]$$

$$= \frac{1}{\sec\left(e^{x^2}\right)}\cdot\frac{\mathrm{d}}{\mathrm{dx}}\left[\sec\left(e^{x^2}\right)\right]$$

$$= \frac{1}{\sec\left(e^{x^2}\right)}.\sec\left(e^{x^2}\right)\tan\left(e^{x^2}\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{x^2}\right)$$

$$= \tan\left(e^{x^2}\right).e^{x^2}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$$

$$= \tan\left(e^{x^2}\right).e^{x^2}.2x$$

$$= 2x.e^{x^2}\tan\left(e^{x^2}\right).$$

Exercise 1.1 | Q 2.16 | Page 12

Differentiate the following w.r.t.x: $\log_{e^2}(\log x)$

SOLUTION

Let y = $\log_{e^2}(\log x)$

$$= \frac{\log(\log\mathrm{x})}{\log e^2}$$

$$= \frac{\log(\log\mathrm{x})}{2\log e}$$

$$= \frac{\log(\log\mathrm{x})}{2} \qquad \ldots[\because \log e = 1]$$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{2}\frac{\mathrm{d}}{2dx}[\log(\log x)]$$

$$= \frac{1}{2}\times\frac{1}{\log\mathrm{x}}.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$= \dfrac{1}{2\log x} \times \dfrac{1}{x}$

$= \dfrac{1}{2x\log x}$.

Exercise 1.1 | Q 2.17 | Page 12

Differentiate the following w.r.t.x: [log {log(logx)}]^2

SOLUTION

Let y = [log {log(logx)}]^2

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\log\{\log(\log x)\}]^2$$

$$= 2.\log\{\log(\log x)\} \times \frac{\mathrm{d}}{\mathrm{dx}}[\log\{\log(\log x)\}]$$

$$= 2.\log\{\log(\log x)\} \times \frac{1}{\log(\log x)}.\frac{\mathrm{d}}{\mathrm{dx}}[\log(\log x)]$$

$$= 2.\log\{\log(\log x)\} \times \frac{1}{\log(\log x)}.\frac{1}{\log x} \times \frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$= 2.\log\{\log(\log x)\} \times \frac{1}{\log(\log x)}.\frac{1}{\log x} \times \frac{1}{x}$$

$$= 2.\left[\frac{\log\{\log(\log x)\}}{x.\log x.\log(\log x)}\right].$$

Exercise 1.1 | Q 2.18 | Page 12

Differentiate the following w.r.t.x: $\sin^2 x^2 - \cos^2 x^2$

SOLUTION

Let y = $\sin^2 x^2 - \cos^2 x^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^2 x^2 - \cos^2 x^2\right]$$

$$= \frac{d}{dx}\left(\sin x^2\right)^2 - \frac{d}{dx}\left(\cos x^2\right)^2$$

$$= 2\sin x^2 . \frac{d}{dx}\left(\sin x^2\right) - 2\cos x^2 . \frac{d}{dx}\left(\cos x^2\right)$$

$$= 2\sin x^2 . \cos x^2 . \frac{d}{dx}\left(x^2\right) - 2\cos x2 . (-\sin x2) . \frac{d}{dx}\left(x^2\right)$$

= $2\sin x^2 . \cos x^2 \times 2x + 2\sin x^2 . \cos x^2 \times 2x$

= $4x\,(2\sin x^2 . \cos x^2)$

= $4x \sin(2x^2)$.

Exercise 1.1 | Q 3.01 | Page 12

Differentiate the following w.r.t.x: $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

SOLUTION

Let y = $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(x^2 + 4x + 1\right)^3 + \left(x^3 - 5x - 2\right)^4\right]$$

$$= \frac{d}{dx}\left(x^2 + 4x + 1\right)^3 + \frac{d}{dx}\left(x^3 - 5x - 2\right)^4$$

$$= 3\left(x^2 + 4x + 1\right)^2 . \frac{d}{dx}\left(x^2 + 4x + 1\right) + 4\left(x^3 - 5x - 2\right)^3 . \frac{d}{dx}\left(x^3 - 5x - 2\right)$$

= $3(x^2 + 4x + 1)^3 . (2x + 4 \times 1 + 0) + 4(x^3 - 5x - 2)^3 . (3x^2 - 5 \times 1 - 0)$

= $6(x + 2)(x^2 + 4x + 1)^2 + 4(3x^2 - 5)(x^3 - 5x - 2)^3$.

Exercise 1.1 | Q 3.02 | Page 12

Differentiate the following w.r.t.x: $(1 + 4x)^5 (3 + x - x2)^8$

SOLUTION

Let y = $(1 + 4x)^5 (3 + x - x2)^8$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(1+4x)^5\left(3+x-x^2\right)^8$$

$$= (1+4x)^5 . \frac{d}{dx}\left(3+x-x^2\right)^8 + \left(3+x-x^2\right)^8 . \frac{d}{dx}(1+4x)^5$$

$$= (1+4x)^5 \times 8\left(3+x-x^2\right)^7 . \frac{d}{dx}\left(3+x-x^2\right) + \left(3+x-x^2\right)^8 \times 5(1+4x)^4 . \frac{d}{dx}(1+4x)$$

$= 8(1 + 4x)^5 (3 + x - x^2)^7 . (0 + 1 - 2x) + 5(1 + 4x)^4 (3 + x - x2)^8 . (0 + 4 \text{ x } 1)$

$= 8(1 - 2x)(1 + 4x)^5(3 + x - x^2)^7 + 20(1 + 4x)^4(3 + x - x^2)^8.$

Exercise 1.1 | Q 3.03 | Page 12

Differentiate the following w.r.t.x: $\frac{x}{\sqrt{7-3x}}$

SOLUTION

Let y = $\dfrac{x}{\sqrt{7-3x}}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x}{\sqrt{7-3x}}\right)$$

$$= \frac{\sqrt{7-3x}.\frac{\mathrm{d}}{\mathrm{dx}}(x) - x\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{7-3x}\right)}{\left(\sqrt{7-3x}\right)^2}$$

$$= \frac{\sqrt{7-3x}\times 1 - x\times\frac{1}{2\sqrt{7-3x}}.\frac{\mathrm{d}}{\mathrm{dx}}(7-3x)}{7-3x}$$

$$= \frac{\sqrt{7-3x} - \frac{x}{2\sqrt{7-3x}}(0-3\times 1)}{7-3x}$$

$$= \frac{2(7-3x)+3x}{2(7-3x)^{\frac{3}{2}}}$$

$$= \frac{14-6x+3x}{2(7-3x)^{\frac{3}{2}}}$$

$$= \frac{14-3x}{2(7-3x)^{\frac{3}{2}}}.$$

Exercise 1.1 | Q 3.04 | Page 12

Differentiate the following w.r.t.x: $\dfrac{\left(x^3-5\right)^5}{\left(x^3+3\right)^3}$

SOLUTION

Let y = $\dfrac{\left(x^3-5\right)^5}{\left(x^3+3\right)^3}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{\left(x^3-5\right)^5}{\left(x^3+3\right)^3}\right]$$

$$=\frac{\left(x^3+3\right)^3.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^3-5\right)^5-\left(x^3-5\right)^5.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^3+3\right)^3}{\left[\left(x^3+3\right)^3\right]^2}$$

$$=\frac{\left(x^3+3\right)^3\times 5\left(x^3-5\right)^4.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^3-5\right)-\left(x^3-5\right)^5\times 3\left(x^3+3\right)^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^3+3\right)}{\left(x^3+3\right)^6}$$

$$=\frac{5\left(x^3+3\right)^3\left(x^3-5\right)^4.\left(3x^3-0\right)-3\left(x^3-5\right)^5\left(x^3+3\right)^2.\left(3x^2+0\right)}{\left(x^3+3\right)^6}$$

$$=\frac{3x^2\left(x^3+3\right)^2\left(x^3-5\right)^4\left[5\left(x^3+3\right)-3\left(x^3-5\right)\right]}{\left(x^3+3\right)^6}$$

$$=\frac{3x^2\left(x^3-5\right)^4\left(5x^3+15-3x^3+15\right)}{\left(x^3+3\right)^4}$$

$$=\frac{3x^2\left(x^3-5\right)^4\left(2x^3+30\right)}{\left(x^3+3\right)^4}$$

$$=\frac{6x^2\left(x^2+15\right)\left(x^3-5\right)^4}{\left(x^3+3\right)^4}.$$

Exercise 1.1 | Q 3.05 | Page 12

Differentiate the following w.r.t.x: $(1 + \sin^2 x)^2 (1 + \cos^2 x)^3$

SOLUTION

Let y = $(1 + \sin^2 x)^2 (1 + \cos^2 x)^3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(1+\sin^2 x\right)^2\left(1+\cos^2 x\right)^3\right]$$

$$= \left(1+\sin^2 x\right)^2 . \frac{d}{dx}\left(1+\cos^2 x\right)^3 + \left(1+\cos^2 x\right)^3 . \frac{d}{dx}\left(1+\sin^2 x\right)^2$$

$$= \left(1+\sin^2 x\right)^2 \times 3\left(1+\cos^2 x\right)^2 . \frac{d}{dx}\left(1+\cos^2 x\right) + \left(1+\cos^2 x\right)^3 \times 2\left(1+\sin^2 x\right) . \frac{d}{dx}\left(1+\sin^2 x\right)$$

$$= 3\left(1+\sin^2 x\right)^2\left(1+\cos^2 x\right)^2 . \left[0+2\cos x . \frac{d}{dx}(\cos x)\right] + 2\left(1+\sin^2 x\right)\left(1+\cos^2 x\right)^3 . \left[0+2\sin x . \frac{d}{dx}(\sin x)\right.$$

$= 3(1 + \sin^2 x)^2(1 + \cos^2 x)^2.[2\cos x(- \sin x)] + 2(1 + \sin^2 x)(1 + \cos^2 x)^3[2\sin x .\cos x]$
$= 3(1 + \sin^2 x)^2(1 + \cos^2 x)^2(- \sin 2x) + 2(1 + \sin^2 x)(1 + \cos^2 x)^3(\sin 2x)$

$= \sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2 [- 3(1 + \sin^2 x) + 2(1 + \cos^2 x)]$
$= \sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2(- 3 - 3\sin^2 x + 2 + 2\cos^2 x)$

$= \sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2[- 1 - 3\sin^2 x + 2(1 - \sin^2 x)]$
$= \sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2(-1 - 3\sin^2 x + 2 - 2\sin^2 x)$
$= \sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2(1 - 5\sin^2 x)$.

Exercise 1.1 | Q 3.06 | Page 12

Differentiate the following w.r.t.x: $\sqrt{\cos x} + \sqrt{\cos\sqrt{x}}$

SOLUTION

Let y = $\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}\right]$$

$$= \frac{d}{dx}(\cos x)^{\frac{1}{2}} + \frac{d}{dx}\left(\cos \sqrt{x}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}(\cos x)^{-\frac{1}{2}} . \frac{d}{dx}(\cos x) + \frac{1}{2}\left(\cos \sqrt{x}\right)^{-\frac{1}{2}} . \frac{d}{dx}\left(\cos \sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{\cos x}} . (-\sin x) + \frac{1}{2\sqrt{\cos \sqrt{x}}} \times \left(-\sin \sqrt{x}\right) . \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin \sqrt{x}}{2\sqrt{\cos \sqrt{x}}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin \sqrt{x}}{4\sqrt{x}\sqrt{\cos \sqrt{x}}}.$$

Exercise 1.1 | Q 3.07 | Page 12

Differentiate the following w.r.t.x: log (sec 3x+ tan 3x)

SOLUTION

Let y = log (sec 3x+ tan 3x)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\sec 3x + \tan 3x)]$$

$$= \frac{1}{\sec 3x + \tan 3x} \cdot \frac{d}{dx}(\sec 3x + \tan 3x)$$

$$= \frac{1}{\sec 3x + \tan 3x} \times \left[\frac{d}{dx}(\sec 3x) + \frac{d}{dx}(\tan 3x)\right]$$

$$= \frac{1}{\sec 3x + \tan 3x} \times \left[\sec 3x \tan 3x . \frac{d}{dx}(3x) + \sec^2 3x . \frac{d}{dx}(3x)\right]$$

$$= \frac{1}{\sec 3x + \tan 3x} \times [\sec 3x \tan 3x \times 3 + \sec^2 3x \times 3]$$

$$= \frac{3 \sec 3x(\tan 3x + \sec 3x)}{\sec 3x + \tan 3x}$$

= 3sec3x.

Exercise 1.1 | Q 3.08 | Page 12

Differentiate the following w.r.t.x: $\frac{1 + \sin x^\circ}{1 - \sin x^\circ}$

SOLUTION

Let y = $\dfrac{1+\sin x^\circ}{1-\sin x^\circ}$

= $\dfrac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)}$ $\quad \ldots\left[\because x^\circ = \left(\dfrac{\pi x}{180}\right)^\circ\right]$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)}\right]$$

$$= \frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right].\frac{\mathrm{d}}{\mathrm{dx}}\left[1+\sin\left(\frac{\pi x}{180}\right)\right]-\left[1+\sin\left(\frac{\pi x}{180}\right)\right].\frac{\mathrm{d}}{\mathrm{dx}}\left[1-\sin\left(\frac{\pi x}{180}\right)\right]}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$= \frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right].\left[0+\cos\left(\frac{\pi x}{180}\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi x}{180}\right)-\left[1+\sin\left(\frac{\pi x}{180}\right)\right].\left[0-\cos\left(\frac{\pi x}{180}\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi x}{180}\right)\right]\right)}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$= \frac{(1-\sin x^\circ)\left[(\cos x^\circ)\times\frac{\pi}{180}\times 1\right]-(1+\sin x^\circ)\left[(-\cos x^\circ)\times\frac{\pi}{180}\times 1\right]}{(1-\sin x^\circ)^2}$$

$$= \frac{\frac{\pi}{180}\cos x^\circ(1-\sin x^\circ+1+\sin x^\circ)}{(1-\sin x^\circ)^2}$$

$$= \frac{\pi\cos x^\circ}{90(1-\sin x^\circ)^2}.$$

Exercise 1.1 | Q 3.09 | Page 12

Differentiate the following w.r.t.x: $\cot\left(\dfrac{\log x}{2}\right)-\log\left(\dfrac{\cot x}{2}\right)$

SOLUTION

Let y = $\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)\right]$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left[\cot\left(\frac{\log x}{2}\right)\right] - \frac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(\frac{\cot x}{2}\right)\right]$$

$$= -\mathrm{cosec}^2\left(\frac{\log x}{2}\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\log x}{2}\right) - \frac{1}{\left(\frac{\cot x}{2}\right)}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\cot x}{2}\right)$$

$$= -\mathrm{cosec}^2\left(\frac{\log x}{2}\right) \times \frac{1}{2} \times \frac{1}{x} - \frac{2}{\cot x} \times \frac{1}{2} \times \left(-\mathrm{cosec}^2 x\right)$$

$$= -\frac{\mathrm{cosec}^2\left(\frac{\log x}{2}\right)}{2x} + \tan x.\,\mathrm{cosec}^2 x.$$

Exercise 1.1 | Q 3.1 | Page 12

Differentiate the following w.r.t.x: $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

SOLUTION

Let y = $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

= $\dfrac{e^{2x}-\frac{1}{e^{2x}}}{e^{2x}+\frac{1}{e^{2x}}}$

= $\dfrac{e^{4x}-1}{e^{4x}+1}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{e^{4x}-1}{e^{4x}+1}\right)$$

= $\dfrac{\left(e^{4x}+1\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{4x}-1\right)-\left(e^{4x}-1\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{4x}+1\right)}{\left(e^{4x}+1\right)^2}$

= $\dfrac{\left(e^{4x}+1\right)\left[e^{4x}.\frac{\mathrm{d}}{\mathrm{dx}}(4x)-0\right]-\left(e^{4x}-1\right)\left[e^{4x}.\frac{\mathrm{d}}{\mathrm{dx}}(4x)+0\right]}{\left(e^{4x}+1\right)^2}$

= $\dfrac{\left(e^{4x}+1\right).e^{4x}\times 4-\left(e^{4x}-1\right).e^{4x}\times 4}{\left(e^{4x}+1\right)^2}$

= $\dfrac{4e^{4x}\left(e^{4x}+1-e^{4x}+1\right)}{\left(e^{4x}+1\right)^2}$

= $\dfrac{8e^{4x}}{\left(e^{4x}+1\right)^2}$.

Exercise 1.1 | Q 3.11 | Page 12

Differentiate the following w.r.t.x: $\dfrac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

SOLUTION

Let y = $\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}\right)$$

$$=\frac{\left(e^{\sqrt{x}}-1\right)\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{\sqrt{x}}+1\right)-\left(e^{\sqrt{x}}+1\right)\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{\sqrt{x}}-1\right)}{\left(e^{\sqrt{x}}-1\right)^{2}}$$

$$=\frac{\left(e^{\sqrt{x}}-1\right)\left[e^{\sqrt{x}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}\right)+0\right]-\left(e^{\sqrt{x}}+1\right)\left[e^{\sqrt{x}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}\right)-0\right]}{\left(e^{\sqrt{x}}-1\right)^{2}}$$

$$=\frac{\left(e^{\sqrt{x}}-1\right)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]-\left(e^{\sqrt{x}}+1\right)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]}{\left(e^{\sqrt{x}}-1\right)^{2}}$$

$$=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\left(e^{\sqrt{x}}-1-e^{\sqrt{x}}-1\right)}{\left(e^{\sqrt{x}}-1\right)^{2}}$$

$$=\frac{-e^{\sqrt{x}}}{\sqrt{x}\left(e^{\sqrt{x}}-1\right)^{2}}.$$

Exercise 1.1 | Q 3.12 | Page 12

Differentiate the following w.r.t.x: $\log[\tan^3x.\sin^4x.(x^2+7)^7]$

SOLUTION

Let y = $\log[\tan^3 x.\sin^4 x.(x^2 + 7)^7]$

= $\log \tan^3 x + \log \sin^4 x + \log(x^2 + 7)^7$

= $3\log \tan x + 4\log \sin x + 7\log(x^2 + 7)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[3\log\tan x + 4\log\sin x + 7\log\left(x^2+7\right)\right]$$

$$= 3\frac{\mathrm{d}}{\mathrm{dx}}(\log\tan x) + 4\frac{\mathrm{d}}{\mathrm{dx}}(\log\sin x) + 7\frac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(x^2+7\right)\right]$$

$$= 3\times\frac{1}{\tan x}.\frac{\mathrm{d}}{\mathrm{dx}}(\tan x) + 4\times\frac{1}{\sin x}.\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) + 7\times\frac{1}{x^2+7}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+7\right)$$

$$= 3\times\frac{1}{\tan x}.\sec^2 x + 4\times\frac{1}{\sin x}.\cos x + 7\times\frac{1}{x^2+7}.(2x+0)$$

$$= 3\times\frac{\mathrm{cosx}}{\mathrm{sinx}}\times\frac{1}{\cos^2 x} + 4\cot x + \frac{14x}{x^2+7}$$

$$= \frac{6}{2\sin x\cos x} + 4\mathrm{cot} + \frac{14x}{x^2+7}$$

$$= \frac{6}{\sin 2x} + 4\cot x + \frac{14x}{x^2+7}$$

$$= 6\mathrm{cosec}2x + 4\cot x + \frac{14x}{x^2+7}.$$

Exercise 1.1 | Q 3.13 | Page 12

Differentiate the following w.r.t.x: $\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$

SOLUTION

Let y = $\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$

= $\log\left(\sqrt{\frac{2\sin^3\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}}\right)$

= $\log\tan\left(\frac{3x}{2}\right)$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\log\tan\left(\frac{3x}{2}\right)\right]$

= $\frac{1}{\tan\left(\frac{3x}{2}\right)} \times \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan\left(\frac{3x}{2}\right)\right]$

= $\frac{1}{\tan\left(\frac{3x}{2}\right)} \times \sec^2\left(\frac{3x}{2}\right) . \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{3x}{2}\right)$

= $\frac{\cos\left(\frac{3x}{2}\right)}{\sin\left(\frac{3x}{2}\right)} \times \frac{1}{\cos^2\left(\frac{3x}{2}\right)} \times \frac{3}{2} \times 1$

= $3 \times \frac{1}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}$

= $3 \times \frac{1}{\sin 3x}$

= 3 cosec3x.

Exercise 1.1 | Q 3.14 | Page 12

Differentiate the following w.r.t.x: $\left(\sqrt{\frac{1+\cos\left(\frac{5x}{2}\right)}{1-\cos\left(\frac{5x}{2}\right)}}\right)$

SOLUTION

Using

$\log\left(\frac{a}{b}\right)$ = log a - log b

log a^b = b log a

y = $\log\left(\sqrt{1+\cos\ \frac{5x}{2}}\right) - \log\left(\sqrt{1-\cos\left(\frac{5x}{2}\right)}\right)$

y = $\log\left(1+\cos\left(\frac{x}{2}\right)^{\frac{1}{2}} - \log\left(1-\cos\left(\frac{5x}{2}\right)\right)\right)^{\frac{1}{2}}$

y = $\frac{1}{2}\log\left[1+\cos\left(\frac{5x}{2}\right)\right] - \frac{1}{2}\log\left[\left(1-\cos\left(\frac{5x}{2}\right)\right]\right.$

Differentiating w.r.t. x

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{2}\frac{1}{1+\cos\left(\frac{5x}{2}\right)}\frac{\mathrm{d}}{\mathrm{dx}}\left(1+\cos\ \frac{5x}{2}\right) - \frac{1}{2}\times\frac{1}{1-\cos\left(\frac{5x}{2}\right)}\frac{\mathrm{d}}{\mathrm{dx}}\left(1-\cos\ \frac{5x}{2}\right)$$

$$= \frac{1}{2\left(1+\cos\left(\frac{5x}{2}\right)\right)}\left(-\sin\left(\frac{5x}{2}\right).\frac{5}{2} - \frac{1}{2\left(1-\cos\left(\frac{5x}{2}\right)\right)}\left(\sin\left(\frac{5x}{2}\right).\frac{5}{2}\right.\right.$$

$$= \frac{-5\sin\left(\frac{5x}{2}\right)}{4\left(1+\cos\left(\frac{5x}{2}\right)\right)} - \frac{5\sin\left(\frac{5x}{2}\right)}{4\left(1-\cos\left(\frac{5x}{2}\right)\right)}$$

$$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right)\left[\frac{1}{1+\cos\left(\frac{5x}{2}\right)} + \frac{1}{1-\cos\left(\frac{5x}{2}\right)}\right]$$

$$= \frac{\frac{-5}{2}\sin\left(\frac{5x}{2}\right)\left[1-\cos\left(\frac{5x}{2}\right)+1+\cos\ \frac{5x}{2}\right]}{\left[1-\cos^2\left(\frac{5x}{2}\right)\right]}$$

$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right)\times\frac{2}{\sin^2\left(\frac{5x}{2}\right)}$ ...[$\because$ 1 − $\cos^2$x = $\sin^2$x]

$$= \frac{-5}{4}\frac{1}{\sin\left(\frac{5x}{2}\right)}$$

$$-\frac{5}{2}\times \mathrm{cosecx}$$

$$-\frac{5}{2}\mathrm{cosec}\left(\frac{5x}{2}\right).$$

Exercise 1.1 | Q 3.15 | Page 12

Differentiate the following w.r.t.x: $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

SOLUTION

Let y = $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

= $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}}\right)$

= $\log\left(\sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}}\right)$

= $\log\left(\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}\right)$

= $\log\left(\frac{1-\sin x}{\cos x}\right)$

= $\log\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)$

= log(sec x – tan x)

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}[\log(\sec x-\tan x)]$$

$= \dfrac{1}{\sec x - \tan x} . \dfrac{\mathrm{d}}{\mathrm{dx}}(\sec x - \tan x)$

$= \dfrac{1}{\sec x - \tan x} \times \left(\sec x \tan x - \sec^2 x\right)$

$= \dfrac{- \sec x(\sec x - \tan x)}{\sec x - \tan x}$

= −sec x.

Exercise 1.1 | Q 3.16 | Page 12

Differentiate the following w.r.t.x: $\log\left[4^{2x}\left(\dfrac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$

SOLUTION

Let y = $\log\left[4^{2x}\left(\dfrac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$

$= \log 4^{2x} + \log\left(\dfrac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}$

$= 2x \log 4 + \dfrac{3}{2}\log\left(\dfrac{x^2+5}{\sqrt{2x^3-4}}\right)$

$= 2x \log 4 + \dfrac{3}{2}\left[\log\left(x^2+5\right) - \log\left(2x^3-4\right)^{\frac{1}{2}}\right]$

$= 2x \log 4 + \dfrac{3}{2}\left[\log\left(x^2+5\right) - \log\left(2x^3-4\right)\right.$

Differentiating w.r.t. x, we get

$\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[2x \log 4 + \dfrac{3}{2}\log\left(x^2+5\right) - \dfrac{3}{4}\log\left(2x^3-4\right)\right]$

$= (2 \log 4)\dfrac{\mathrm{d}}{\mathrm{dx}}(x) + \dfrac{3}{2}\dfrac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(x^2+5\right)\right] - \dfrac{3}{4}\dfrac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(2x^3-4\right)\right]$

$$= (2\log 4) \times 1 + \frac{3}{2} \times \frac{1}{x^2+5} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+5\right) - \frac{3}{4} \times \frac{1}{2x^3-4} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(2x^3-4\right)$$

$$= 2\log 4 + \frac{3}{2(x^2+5)} \times (2x+0) - \frac{3}{4(2x^3-4)} \times \left(2 \times 3x^2 - 0\right)$$

$$= 2\log 4 + \frac{3x}{x^2+5} - \frac{9x^2}{2(2x^3-4)}.$$

Exercise 1.1 | Q 3.17 | Page 12

Differentiate the following w.r.t.x: $\log\left[\frac{ex^2(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

SOLUTION

Let y = $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

Using

log(A.B) = logA + logB

y = $\log e^{x^2} + \log\left(\frac{(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right)$

= $\log e^{x^2} + \log(5-4x)^{\frac{3}{2}} - \log\left(\sqrt[3]{7-6x}\right)$

= $x^2 \log e + \frac{3}{2}\log(5-4x) - \log(7-6x)^{\frac{1}{3}}$

= $x^2 + \frac{3}{2}\log(5-4x) - \frac{1}{3}\log(7-6x)$

Now,

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}x^2 + \frac{3}{2}\frac{\mathrm{d}}{\mathrm{dx}}\log(5-4x) - \frac{1}{3}\frac{\mathrm{d}}{\mathrm{dx}}\log(7-6x)$$

$$= 2x + \frac{3}{2}\frac{1}{5-4x}(-4) - \frac{1}{3}\frac{1}{(7-6x)}x(-6)$$

= $2x - \frac{6}{(5-4x)} + \frac{2}{(7-6x)}$

$$2x - \frac{6}{5-4x} + \frac{2}{7-6x}.$$

Exercise 1.1 | Q 3.18 | Page 12

Differentiate the following w.r.t.x:

$$\log\left[\frac{a^{\cos x}}{\left(x^2-3\right)^3 \log x}\right]$$

SOLUTION

Let y = $\log\left[\frac{a^{\cos x}}{\left(x^2-3\right)^3 \log x}\right]$

= $\log a^{\cos x} - \log(x^2 - 3)^3 - \log(\log x)$

= $(\cos x)(\log a) - 3\log(x^2 - 3) - \log(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[(\cos x)(\log a) - 3\log\left(x^2-3\right) - \log(\log x)\right]$$

$$= (\log a).\frac{d}{dx}(\cos x) - 3\frac{d}{dx}\left[\log\left(x^2-3\right)\right] - \frac{d}{dx}[\log(\log x)]$$

$$= (\log a)(-\sin x) - 3\times\frac{1}{x^2-3}.\frac{d}{dx}\left(x^2-3\right) - \frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$= -(\sin x)(\log a) - \frac{3}{x^2-3}\times(2x-0) - \frac{1}{\log x}\times\frac{1}{x}$$

$$= -(\sin x)(\log a) - \frac{6x}{x^2-3} - \frac{1}{x\log x}.$$

Exercise 1.1 | Q 3.19 | Page 12

Differentiate the following w.r.t.x: y = $(25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

SOLUTION

y = $(25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

= $5^{2\log_5(\sec x)} - 4^{2\log_4(\tan x)}$

= $5^{2\log_5(\sec^2 x)} - 4^{2\log_4(\tan^2 x)}$

= $\sec^2 x - \tan^2 x$ ...[$\because a^{\log_a x} = x$]

$\therefore$ y = 1

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(1)$ = 0.

Exercise 1.1 | Q 3.2 | Page 12

Differentiate the following w.r.t.x: $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

SOLUTION

Let y = $\dfrac{\left(x^2+2\right)^4}{\sqrt{x^2+5}}$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{\left(x^2+2\right)^4}{\sqrt{x^2+5}}\right]$$

$$= \frac{\sqrt{x^2+5}.\ \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+2\right)^4 - \left(x^2+2\right)^4.\ \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x^2+5}\right)}{\left(\sqrt{x^2+5}\right)^2}$$

$$= \frac{\sqrt{x^2+5}\times 4\left(x^2+2\right)^3.\ \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+2\right) - \left(x^2+2\right)^4\times\frac{1}{2\left(\sqrt{x^2+5}\right)}.\ \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+5\right)}{x^2+5}$$

$$= \frac{\sqrt{x^2+5}\times 4\left(x^2+2\right)^3.\left(2x+0\right) - \frac{\left(x^2+2\right)^4}{2\sqrt{x^2+5}}\times\left(2x+0\right)}{x^2+5}$$

$$= \frac{8x\left(x^2+5\right)\left(x^2+2\right)^3 - x\left(x^2+2\right)^4}{\left(x^2+5\right)^{\frac{3}{2}}}$$

$$= \frac{x\left(x^2+2\right)^3\left[8\left(x^2+5\right)-\left(x^2+2\right)\right]}{\left(x^2+5\right)^{\frac{3}{2}}}$$

$$= \frac{x\left(x^2+2\right)^3\left(8x^2+40-x^2-2\right)}{\left(x^2+5\right)^{\frac{3}{2}}}$$

$$= \frac{x\left(x^2+2\right)^3\left(7x^2+38\right)}{\left(x^2+5\right)^{\frac{3}{2}}}.$$

Exercise 1.1 | Q 4.1 | Page 12

A table of values of f, g, f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)

2	1	6	–3	4
4	3	4	5	-6
6	5	2	–4	7

If r(x) =f [g(x)] find r' (2).

SOLUTION

r(x) =f[g(x)]

$\therefore$ r'(x) $= \frac{\mathrm{d}}{\mathrm{dx}} f[\mathrm{g}(x)]$

$= f\prime[\mathrm{g}(x)].\frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{g}(x)]$

$= f\prime[\mathrm{g}(x)].[\mathrm{g}\prime(x)]$

$\therefore$ r'(2) $= f\prime[\mathrm{g}(2)].\mathrm{g}\prime(2)$

$= f\prime(6).\mathrm{g}\prime(2)$ ...[$\because$ g(x) = 6, when x = 2]

= –4 x 4 ...[From the table]

= –16.

Exercise 1.1 | Q 4.2 | Page 12

A table of values of f, g, f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
2	1	6	–3	4
4	3	4	5	-6
6	5	2	–4	7

If R(x) =g[3 + f(x)] find R'(4).

SOLUTION

R(x) =g[3 + f(x)]

$\therefore$ R'(x) = $\frac{\mathrm{d}}{\mathrm{dx}}\{g[3+f(x)]\}$

= $\mathrm{g}\prime[3+f(x)].\frac{\mathrm{d}}{\mathrm{dx}}[3+f(x)]$

= $\mathrm{g}\prime[3+f(x)].[0+f\prime(x)]$

= $\mathrm{g}\prime[3+f(x)].f\prime(x)$

$\therefore$ R'(4) = $g\prime[3+f(4)].f\prime(4)$

= $\mathrm{g}\prime[3+3].f\prime(4)$ ...[$\because$ f(x) = 3, when x = 4]

= $\mathrm{g}\prime(6).f\prime(4)$

= 7 x 5 ...[Frrom the table]

= 35.

Exercise 1.1 | Q 4.3 | Page 12

A table of values of f, g, f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
2	1	6	–3	4
4	3	4	5	-6
6	5	2	–4	7

If s(x) =f[9 − f (x)] find s'(4).

SOLUTION

$s(x) = f[9 - f(x)]$

$\therefore s'(x) = \frac{d}{dx}\{f[9 - f(x)]\}$

$= f'[9 - f(x)].\frac{d}{dx}[9 - f(x)]$

$= f'[9 - f(x)].[0 - f'(x)]$

$= - f'[9 - f(x)].f'(x)$

$\therefore s'(4) = - f'[9 - f(4)].f'(4)$

$= - f'[9 - 3].f'(4)$ ...$[\because f(x) = 3, \text{ when } x = 4]$

$= - f'(6).f'(4)$

$= - (- 4)(5)$ ...[From the table]

$= 20.$

Exercise 1.1 | Q 4.4 | Page 12

A table of values of f, g, f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
2	1	6	–3	4
4	3	4	5	-6
6	5	2	–4	7

If $S(x) = g[g(x)]$ find $S'(6)$.

SOLUTION

S(x) =g [g(x)]

∴ S'(x) = $\dfrac{\mathrm{d}}{\mathrm{dx}}\mathrm{g}[\mathrm{g}(x)]$

= $\mathrm{g}\prime[\mathrm{g}(x)].\dfrac{\mathrm{d}}{\mathrm{dx}}[\mathrm{g}(x)]$

= $\mathrm{g}\prime[\mathrm{g}(x)].\mathrm{g}\prime(x)$

∴ S'(6) = $\mathrm{g}\prime[\mathrm{g}(6).\mathrm{g}\prime(6)$

= $\mathrm{g}\prime(2).\mathrm{g}\prime(6)$ ...[∵ g(x) = 2, when x = 6]

= 4 x 7 ...[From the table]

= 28.

Exercise 1.1 | Q 5 | Page 12

Assume that $f\prime(3)=-1,\mathrm{g}\prime(2)=5,\mathrm{g}(2)=3\text{ and }y=f[\mathrm{g}(x)],\mathrm{then}\left[\dfrac{\mathrm{dy}}{\mathrm{dx}}\right]_{x=2}=?$

y = $f[\mathrm{g}(x)]$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}}=\dfrac{\mathrm{d}}{\mathrm{dx}}\{f[\mathrm{g}(x)]\}$

= $f\prime[\mathrm{g}(x)].\dfrac{\mathrm{d}}{\mathrm{dx}}[\mathrm{g}(x)]$

= $f\prime[\mathrm{g}(x).\mathrm{g}\prime(x)$

$\therefore \left[\dfrac{\mathrm{dy}}{\mathrm{dx}}\right]_{x=2}$

= $f\prime[\mathrm{g}(2)].\mathrm{g}\prime(2)$

= $f\prime(3).\mathrm{g}\prime(2)$ ...[∵ g(2) = 3]

= −1 x 5 ...(Given)

= −5.

Exercise 1.1 | Q 6 | Page 12

If h(x) = $\sqrt{4f(x)+3\mathrm{g}(x)}, f(1)=4, \mathrm{g}(1)=3, f\prime(1)=3, \mathrm{g}\prime(1)=4, \text{find } \mathrm{h}\prime(1).$

SOLUTION

Given : $f(1)=4, \mathrm{g}(1)=3, f\prime(1)=3, \mathrm{g}\prime(1)=4$ ...(1)

Now, h(x) = $\sqrt{4f(x)+3\mathrm{g}(x)}$

∴ h'(x) = $\frac{\mathrm{d}}{\mathrm{dx}}\left[\sqrt{4f(x)+3\mathrm{g}(x)}\right]$

= $\frac{1}{2\sqrt{4f(x)+3\mathrm{g}(x)}}\cdot\frac{\mathrm{d}}{\mathrm{dx}}[4f(x)+3\mathrm{g}(x)]$

= $\frac{1}{2\sqrt{4f(x)+3\mathrm{g}(x)}}.[4f\prime(x)+3\mathrm{g}\prime(x)]$

∴ h'(1) = $\frac{1}{2\sqrt{4f(1)+3\mathrm{g}(1)}}.[4f\prime(1)+3\mathrm{g}\prime(1)]$

= $\frac{1}{2\sqrt{4\times 4+3\times 3}}\times[4\times 3+3\times 4]$ ...[By (1)]

= $\frac{1}{2\sqrt{25}}\times 24$

= $\frac{1}{2\times 5}\times 24$

= $\frac{12}{5}$.

Exercise 1.1 | Q 7 | Page 12

Find the x co-ordinates of all the points on the curve y = sin 2x − 2 sin x, 0 ≤ x < 2π, where dy/dx = 0.

SOLUTION

$y = \sin 2x - 2 \sin x, 0 \le x < 2\pi$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(\sin 2x - 2\sin x)$

$= \frac{\mathrm{d}}{\mathrm{dx}}(\sin 2x) - 2\frac{\mathrm{d}}{\mathrm{dx}}(\sin x)$

$= \cos 2x . \frac{\mathrm{d}}{\mathrm{dx}}(2x) - 2\cos x$

$= \cos^2 x \text{ x } 2 - 2 \cos x$

$= 2 (2 \cos^2 x - 1) -2 \cos x$

$= 4 \cos^2 x - 2 - 2 \cos x$

$= 4 \cos^2 x - 2 \cos x - 2$

If $\frac{\mathrm{dy}}{\mathrm{dx}} = 0$, then $4 \cos^2 x - 2 \cos x - 2 = 0$

$\therefore 4 \cos^2 x - 4 \cos x \;\; 2 \cos x - 2 = 0$

$\therefore 4 \cos x (\cos x - 1) + 2 (\cos x - 1) = 0$

$\therefore (\cos x - 1)(4 \cos x + 2) =$

$\therefore \cos x - 1 = 0$ or $4 \cos x + 2 = 0$

$\therefore \cos x = 1$ or $\cos x = -\frac{1}{2}$

$\therefore \cos x = \cos 0$

or

$\cos x = -\cos \frac{\pi}{3}$

$= \cos\left(\pi - \frac{\pi}{3}\right)$

$= \frac{\cos 2\pi}{3}$

or

$\cos x = -\cos \frac{\pi}{3}$

$= \cos\left(\pi - \frac{\pi}{3}\right)$

$= \frac{\cos 4\pi}{3}$ ...[$\because 0 \le x < 2\pi$]

$\therefore x = 0$ or $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$.

$\therefore x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Exercise 1.1 | Q 8 | Page 13

Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]:

"Let f (x) $= x^2 + 5$ and g (x) $= e^x + 3$ then
f[g(x)] = ... and g[f(x)] =...........

Now f'(x) = ... and g'(x) = ...
The derivative of f[g(x)] w. r. t. x in terms of f and g is ....

Therefore $\frac{d}{dx}[f[g(x)]]$ = ... and

$\left[\frac{d}{dx}[f[g(x)]]\right]_{x=0}$ = ...

The derivative of g[f(x)] w. r. t. x in terms of f and g is

Therefore $\frac{d}{dx}[g[f(x)]]$ = ... and

$\left[\frac{d}{dx}[g[f(x)]]\right]_{x=-1}$ = ..."

Hint basket : $\left\{f'[g(x)] \cdot g'(x), 2e^{2x} + 6e^x, 8, g'[f(x)] \cdot f'(x), 2xe^{x^2+5}, \ -2e^6, e^{2x} + 6e^x + 14, e^{x^2+5} + 3, 2x, e^x\right\}$

SOLUTION

f[g(x)] = $e^{2x} + 6e^{x} + 14$

g[f(x)] = $e^{x^2+5} + 3$

f'(x) = 2x, g'f(x) = e^{x}

The derivative of f[g(x)] w. r. t. x in terms of f and g is $f\prime[\mathrm{g}(x)].\,\mathrm{g}\prime(x)$.

$\therefore \dfrac{\mathrm{d}}{\mathrm{dx}}\{f[\mathrm{g}(x)]\} = 2e^{2x} + 6e^{x}$ and $\dfrac{\mathrm{d}}{\mathrm{dx}}\{f[\mathrm{g}(x)]\}_{x=0}$ = 8

The derivative of g[f(x)] w. r. t. x in terms of f and g is $\mathrm{g}\prime[f(x)].\,f\prime(x)$.

$\therefore \dfrac{\mathrm{d}}{\mathrm{dx}}\{\mathrm{g}[f(x)]\} = 2e^{x^2+5}$ and

$\dfrac{\mathrm{d}}{\mathrm{dx}}\{\mathrm{g}[f(x)]\}_{x=-1}$ = $-2e^{6}$.

EXERCISE 1.2 [PAGES 29 - 30]

Exercise 1.2 | Q 1.1 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = $f^{-1}(y)$ in the following: y = √x

SOLUTION

y = $\sqrt{y}$

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$y^2 = x$

$\therefore x = y^2$

$\therefore x = f^{-1}(y) = y^2$

$\therefore \frac{dx}{dy} = \frac{d}{dy}\left(y^2\right) = 2y$

$= 2\sqrt{x}$ ...[By (1)]

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

$= \frac{1}{2\sqrt{x}}$.

Exercise 1.2 | Q 1.2 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = $f^{-1}(y)$ in the following: y = $\sqrt{2 - \sqrt{x}}$

SOLUTION

y = $\sqrt{2-\sqrt{x}}$ ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$y^2 = 2-\sqrt{x} \quad \therefore \sqrt{x} = 2-y^2$

$\therefore x = (2-y^2)^2$

$\therefore x = f^{-1}(y) = (2-y^2)^2$

$\therefore \dfrac{dx}{dy} = \dfrac{d}{dy}(2-y^2)^2$

$= 2(2-y^2).\dfrac{d}{dy}(2-y^2)$

$= 2(2-y^2).(0-2y)$

$= -4y(2-y^2)$

$= -4\sqrt{2-\sqrt{x}}(2-2+\sqrt{x})$ ...[By (1)]

$= -4\sqrt{x}\sqrt{2-\sqrt{x}}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\left(\frac{dx}{dy}\right)}$

$= \dfrac{1}{4\sqrt{x}\sqrt{2-\sqrt{x}}}$.

Exercise 1.2 | Q 1.3 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = f^{-1}(y) in the following: y = $\sqrt[3]{x-2}$

SOLUTION

y = $\sqrt[3]{x-2}$ ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$y^3 = x - 2$

$\therefore x = y^3 + 2$

$\therefore x = f^{-1}(y) = y^3 + 2$

$\therefore \dfrac{dx}{dy} = \dfrac{d}{dy}\left(y^3 + 2\right)$

$= 3y^2 + 0 = 3y^2$

$= 3\left(\sqrt[3]{(x-2)}\right)^2$ ...[By (1)]

$= 3(x-2)^{\frac{2}{3}}$

$= 3.\left(\sqrt[3]{(x-2)^2}\right)$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\left(\frac{dx}{dy}\right)}$

$= \dfrac{1}{3\sqrt[3]{(x-2)^2}}, x > 2.$

Exercise 1.2 | Q 1.4 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = f^{-1}(y) in the following: y = log(2x – 1)

SOLUTION

y = log(2x – 1) ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$2x - 1 = e^y$

$\therefore 2x = e^y + 1$

$\therefore x = f^{-1}(y)$

$= \frac{1}{2}(e^y + 1)$

$\therefore \frac{dx}{dy} = \frac{1}{2}\frac{d}{dy}(e^y + 1)$

$= \frac{1}{2}(e^y + 0)$

$= \frac{1}{2}e^y$

$= \frac{1}{2}e^{\log(2x-1)}$ ...[By (1)]

$= \frac{1}{2}(2x - 1)$ ...[$\because e^{\log x} = x$]

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

$= \frac{2}{2x - 1}$.

Exercise 1.2 | Q 1.5 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function $x = f^{-1}(y)$ in the following: y = 2x + 3

SOLUTION

y = 2x + 3 ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

2x = y – 3

$\therefore$ x = $\frac{y-3}{2}$

$\therefore$ x = $f^{-1}(y)$

= $\frac{y-3}{2}$

$$\therefore \frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dy}}(y-3)$$

$$= \frac{1}{2}(1-0)$$

$$= \frac{1}{2}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{\left(\frac{\mathrm{dx}}{\mathrm{dy}}\right)}$$

$$= \frac{1}{\left(\frac{1}{2}\right)}$$

= 2.

Exercise 1.2 | Q 1.6 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = $f^{-1}(y)$ in the following: y = e^x – 3

SOLUTION

$y = e^x - 3$ ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$e^x = y + 3$

$\therefore x = \log(y + 3)$

$\therefore x = f^{-1}(y) = \log(y + 3)$

$\therefore \frac{dy}{dx} = \frac{d}{dy}[\log(y + 3)$

$= \frac{1}{y+3} \cdot \frac{d}{dy}(y+3)$

$= \frac{1}{y+3} \cdot (1+0)$

$= \frac{1}{y+3}$

$= \frac{1}{e^x - 3 + 3}$ ...[By (1)]

$= \frac{1}{e^x}$

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

$= \frac{1}{\left(\frac{1}{e^x}\right)}$

$= e^x$.

Exercise 1.2 | Q 1.7 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = e^{2x-3}$

SOLUTION

$y = e^{2x-3}$ ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$2x - 3 = \log y$

$\therefore 2x = \log y + 3$

$\therefore x = f^{-1}(y)$

$= \frac{1}{2}(\log y + 3)$

$\therefore \frac{dx}{dy} = \frac{1}{2}\frac{d}{dy}(\log y + 3)$

$= \frac{1}{2}\left(\frac{1}{y} + 0\right)$

$= \frac{1}{2y}$

$= \frac{1}{2e^{2x-3}}$ ...[By (1)]

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

$= \frac{1}{\left(\frac{1}{2e^{2x-3}}\right)}$

$= 2e^{2x-3}$.

Exercise 1.2 | Q 1.8 | Page 29

Find the derivative of the function y = f(x) using the derivative of the inverse function x = $f^{-1}(y)$ in the following: y = $\log_2\left(\frac{x}{2}\right)$

SOLUTION

y = $\log_2\left(\frac{x}{2}\right)$ ...(1)

We have to find the inverse function of y = f(x), i.e x in terms of y.

From (1),

$\frac{x}{2} = 2^y$

$\therefore x = 2.2^y = 2^{y+1}$

$\therefore x = f^{-1}(y) = 2^{y+1}$

$\therefore \frac{\text{dx}}{\text{dy}} = \frac{\text{d}}{dy}\left(2^{y+1}\right)$

$= 2^{y+1}. \log 2. \frac{\text{d}}{\text{dy}}(y+1)$

$= 2^{y+1}. \log 2. (1+0)$

$= 2^{y+1}. \log 2$

$= 2^{\log_2\left(\frac{x}{2}\right)+1}. \log 2$ ...[By (1)]

$= 2^{\log_2\left(\frac{x}{2}\right)+\log_2 2}. \log 2$

$= 2^{\log_2\left(\frac{x}{2}\times 2\right).\log 2}$

$= 2^{\log_2 x}.\log 2$

$= x \log 2$ ...[$\because a^{\log_a x} = x$]

$\therefore \frac{\text{dy}}{\text{dx}} = \frac{1}{\left(\frac{\text{dx}}{\text{dy}}\right)}$

$= \frac{1}{x \log 2}$.

Exercise 1.2 | Q 2.1 | Page 29

Find the derivative of the inverse function of the following : $y = x^2 \cdot e^x$

SOLUTION

$y = x^2 \cdot e^x$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2 . e^x\right)$$

$$= x^2 \frac{\mathrm{d}}{\mathrm{dx}}(e^x) + e^x \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$$

$= x^2 \cdot e^x + e^x \times 2x$

$= xe^x (x + 2)$

The derivative of inverse function of y = f(x) is given by

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}$$

$$= \frac{1}{xe^x(x+2)}.$$

Exercise 1.2 | Q 2.2 | Page 29

Find the derivative of the inverse function of the following : $y = x \cos x$

SOLUTION

$y = x \cos x$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x \cos x)$$

$$= x \frac{\mathrm{d}}{\mathrm{dx}}(\cos x) + \cos x \frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$= x(-\sin x) + \cos x \times 1$

$= \cos x - x \sin x$

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{\cos x - x\sin x}.$$

Exercise 1.2 | Q 2.3 | Page 29

Find the derivative of the inverse function of the following : $y = x \cdot 7^x$

SOLUTION

$y = x \cdot 7^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x.7^x)$$

$$= x\frac{d}{dx}(7^x) + 7^x\frac{d}{dx}(x)$$

$= x \cdot 7^x \log 7 + 7^x \times 1$

$= 7x\ (x \log 7 + 1)$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{7^x(x\log 7 + 1)}.$$

Exercise 1.2 | Q 2.4 | Page 29

Find the derivative of the inverse function of the following : $y = x^2 + \log x$

SOLUTION

y = x^2 + log x

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2 + \log x\right)$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right) + \frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$= 2x + \frac{1}{x}$$

$$= \frac{2x^2 + 1}{x}$$

The derivative of inverse function of y = f(x) is given by

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}$$

$$= \frac{1}{\left(\frac{2x^2+1}{x}\right)}$$

$$= \frac{x}{2x^2 + 1}.$$

Exercise 1.2 | Q 2.5 | Page 29

Find the derivative of the inverse function of the following : y = x log x

SOLUTION

y = x log x

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x \log x)$$

$$= x\frac{d}{dx}(\log x) + (\log x).\frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

= 1 + logx

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{1 + \log x}.$$

Exercise 1.2 | Q 3.1 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. $y = x^5 + 2x^3 + 3x$, *at* $x = 1$

SOLUTION

y = $x^5 + 2x^3 + 3x$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^5 + 2x^3 + 3x\right)$$

= $5x^4 + 2 \times 3x^2 + 3 \times 1$

= $5x^4 + 6x^2 + 3$

The derivative of inverse function of y = f(x) is given by

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}$$

= $\frac{1}{5x^4 + 6x^2 + 3}$

At x = 1, $\frac{\mathrm{dx}}{\mathrm{dy}}$

= $\frac{1}{\left(5x^4 + 6x^2 + 3\right)_{at\ x=1}}$

= $\frac{1}{5(1)^4 + 6(1)^2 + 3}$

= $\frac{1}{5 + 6 + 3}$

= $\frac{1}{14}$.

Exercise 1.2 | Q 3.2 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. y = $e^x + 3x + 2$

SOLUTION

y = e^x + 3x + 2

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}(e^x + 3x + 2)$$

= e^x + 3 x 1 + 0

= e^x + 3

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{e^x + 3}$$

At x = 0, $\frac{dx}{dy}$

$$= \frac{1}{(e^x + 3)_{at x=0}}$$

$$= \frac{1}{e^0 + 3}$$

$$= \frac{1}{1 + 3}$$

$$= \frac{1}{4}.$$

Exercise 1.2 | Q 3.3 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. y = $3x^2 + 2\log x^3$

SOLUTION

y = $3x^2 + 2\log x^3$

= $3x^2 + 6\log x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^2 + 6\log x\right)$$

= $3 \times 2x + 6 \times \frac{1}{x}$

= $6x + \frac{6}{x}$

= $\frac{6x^2 + 6}{x}$

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

= $\frac{1}{\left(\frac{6x^2+6}{x}\right)}$

= $\frac{x}{6x^2 + 6}$

At x = $1, \frac{dx}{dy}$

= $\left(\frac{x}{6x^2 + 6}\right)_{at\ x=1}$

= $\frac{1}{6(1)^2 + 6}$

= $\frac{1}{12}$.

Exercise 1.2 | Q 3.4 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. y = sin(x – 2) + x^2

SOLUTION

y = sin(x – 2) + x^2

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(x-2) + x^2\right]$$

$$= \frac{d}{dx}[\sin(x-2)] + \frac{d}{dx}(x^2)$$

$$= \cos(x-2).\frac{d}{dx}(x-2) + 2x$$

= cos(x – 2).(1 – 0) + 2x

= cos(x – 2) + 2x

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{\cos(x-2) + 2x}$$

At x = 2, $\frac{dx}{dy}$

$$= \left(\frac{1}{[\cos(x-2) + 2x]}\right)_{\text{at } x=2}$$

$$= \frac{1}{1+4}$$

$$= \frac{1}{5}.$$

Exercise 1.2 | Q 4 | Page 29

If f(x) = x^3 + x – 2, find $(f^{-1})'(-2)$.

SOLUTION

$f(x) = x^3 + x - 2$ ...(1)

Differentiating w.r.t. x, we get

$f'(x) = \dfrac{d}{dx}\left(x^3 + x - 2\right)$

$= 3x^2 + 1 = 0$

$= 3x^2 + 1$

We know that

$(f^{-1})'(y) = \dfrac{1}{f\prime(x)}$ ...(2)

From (1), $y = f(x) = -2$, when $x = 0$

$\therefore$ from (2), $(f^{-1})'(-2)$

$= \dfrac{1}{f\prime(0)}$

$= \dfrac{1}{(3x^2 + 1)_{\text{at}x=0}}$

$= \dfrac{1}{3(0) + 1}$

$= 1.$

Exercise 1.2 | Q 5.1 | Page 29

Using derivative, prove that: $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

SOLUTION

Let f(x) = $\tan^{-1}x + \cot^{-1}x$ ...(1)

Differentiating w.r.t. x, we get

f'(x) = $\frac{d}{dx}\left(\tan^{-1}x + \cot^{-1}x\right)$

= $\frac{d}{dx}\left(\tan^{-1}x\right) + \frac{d}{dx}\left(\cot^{-1}x\right)$

= $\frac{1}{1+x^2} - \frac{1}{1+x^2}$

= 0

Since f'(x) = 0, f(x) is a constant function.

Let f(x) = k.

For any value of x, f(x) = k

Let x = 0.

Then f(0) = k ...(2)

From (1), f(0) = $\tan^{-1}(0) + \cot^{-1}(0)$

= $0 + \frac{\pi}{2} = \frac{\pi}{2}$

∴ k = $\frac{\pi}{2}$ ...[By (2)]

∴ f(x) = k = $\frac{\pi}{2}$

Hence, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$. ...[By (1)]

Exercise 1.2 | Q 5.2 | Page 29

Using derivative, prove that: $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ ...[for |x| ≥ 1]

SOLUTION

Let f(x) = $\sec^{-1}x + \text{cosec}^{-1}x$ for $|x| \geq 1$ ...(1)

Differentiating w.r.t. x, we get

f'(x) = $\frac{d}{dx}\left(\sec^{-1}x + \text{cosec}^{-1}x\right)$

= $\frac{d}{dx}\left(\sec^{-1}x\right) + \frac{d}{dx}\left(\text{cosec}^{-1}x\right)$

= $\frac{1}{x\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}}$

= 0

Since, f'(x) = 0, f(x) is a constant function.

Let f(x) = k.

For any value of x,f(x) = k, where |x| > 1

Let x = 2.

Then, f(2) = k ...(2)

From (1),f(2) = $\sec^{-1}(2) + \text{cosec}^{-1}(2)$

= $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

∴ k = $\frac{\pi}{2}$ ...[By (2)]

∴ f(x) = k = $\frac{\pi}{2}$

Hence, $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$. ...[By (1)]

Exercise 1.2 | Q 6.01 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}(\log x)$

SOLUTION

Let y = $\tan^{-1}(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(\log x)\right]$$

$$= \frac{1}{1+(\log x)^2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{1+(\log x)^2} \times \frac{1}{x}$$

$$= \frac{1}{x\left[1+(\log x)^2\right]}.$$

Exercise 1.2 | Q 6.02 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}(e^{-x})$

SOLUTION

Let y = $\operatorname{cosec}^{-1}(e^{-x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\operatorname{cosec}^{-1}(e^{-x})\right]$$

$$= \frac{-1}{e^{-x}\sqrt{(e^{-x})^2 - 1}} \cdot \frac{d}{dx}(e^{-x})$$

$$= \frac{1}{e^{-x}\sqrt{e^{-2}x - 1}} \times e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{-1}{\sqrt{e^{-2}x - 1}} \times -1$$

$$= \frac{1}{\sqrt{\frac{1}{e^{2x}} - 1}}$$

$$= \frac{e^x}{\sqrt{1 - e^2x}}.$$

Exercise 1.2 | Q 6.03 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(x^3)$

SOLUTION

Let y = $\cot^{-1}(x^3)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}(x^3)\right]$$

$$= \frac{-1}{1 + (x^3)^2} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{-1}{1 + x^6} \times 3x^2$$

$$= \frac{-3x^2}{1 + x^6}.$$

Exercise 1.2 | Q 6.04 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(4^x)$

SOLUTION

Let y = $\cot^{-1}(4^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}(4^x)\right]$$

$$= \frac{-1}{1 + (4^x)^2} \cdot \frac{d}{dx}(4^x)$$

$$= \frac{-1}{1 + 4^2x} \times 4^x \log 4$$

$$= \frac{4^x \log 4}{1 + 4^2x}.$$

Exercise 1.2 | Q 6.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}\left(\sqrt{x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\sqrt{x}\right)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\sqrt{x}\right)\right]$$

$$= \frac{1}{1+\left(\sqrt{x}\right)^2} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}\right)$$

$$= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1+x)}.$$

Exercise 1.2 | Q 6.06 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x^2}{2}}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{\left(1-\frac{1+x^2}{2}\right)}} \times \frac{1}{\sqrt{2}}\frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{2}-1-x^2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx}(1+x^2)$$

$$= \frac{1}{\sqrt{1-x^2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot (0+2x)$$

$$= \frac{x}{\sqrt{(1-x^2)(1+x^2)}}$$

$$= \frac{x}{\sqrt{1-x^4}}.$$

Exercise 1.2 | Q 6.07 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}(1-x^2)$

SOLUTION

Let y = $\cos^{-1}(1 - x^2)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}\left[\cos^{-1}\left(1-x^2\right)\right]$$

$$= \frac{-1}{\sqrt{1-(1-x^2)^2}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(1-x^2\right)$$

$$= \frac{-1}{\sqrt{1-(1-2x^2+x^4)}} \cdot (0-2x)$$

$$= \frac{2x}{\sqrt{2x^2-x^4}}$$

$$= \frac{2x}{x\sqrt{2-x^2}}$$

$$= \frac{2}{\sqrt{2-x^2}}.$$

Exercise 1.2 | Q 6.08 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\sin^{-1}\left(x^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2}x^{\frac{3}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

Exercise 1.2 | Q 6.09 | Page 29

Differentiate the following w.r.t. x : $\cos^3[\cos^{-1}(x^3)]$

SOLUTION

Let y = $\cos^3[\cos^{-1}(x^3)]$

= $[\cos(\cos^{-1}x^3)]^3$

= $(x^3)^3$

= x^9

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{dx}\left(x^9\right)$$

= $9x^8$.

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

SOLUTION

Let y = $\sin^4\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

= $\left\{\sin\left[\sin^{-1}\left(\sqrt{x}\right)\right]\right\}^4$

= $\left(\sqrt{x}\right)^4$

= x^2

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$$

= 2x.

Exercise 1.2 | Q 6.02 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}(e^{-x})$

SOLUTION

Let y = $\operatorname{cosec}^{-1}(e^{-x})$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\operatorname{cosec}^{-1}\left(e^{-x}\right)\right]$$

$$= \frac{-1}{e^{-x}\sqrt{\left(e^{-x}\right)^2 - 1}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{-x}\right)$$

$$= \frac{1}{e^{-x}\sqrt{e^{-2}x - 1}} \times e^{-x}. \frac{\mathrm{d}}{\mathrm{dx}}(-x)$$

$$= \frac{-1}{\sqrt{e^{-2}x - 1}} \times -1$$

$$= \frac{1}{\sqrt{\frac{1}{e^{2x}} - 1}}$$

$$= \frac{e^x}{\sqrt{1 - e^2x}}.$$

Exercise 1.2 | Q 6.03 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(x^3)$

SOLUTION

Let y = $\cot^{-1}(x^3)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}\left(x^3\right)\right]$$

$$= \frac{-1}{1+\left(x^3\right)^2} \cdot \frac{d}{dx}\left(x^3\right)$$

$$= \frac{-1}{1+x^6} \times 3x^2$$

$$= \frac{-3x^2}{1+x^6}.$$

Exercise 1.2 | Q 6.04 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(4^x)$

SOLUTION

Let y = $\cot^{-1}(4^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}(4^x)\right]$$

$$= \frac{-1}{1+(4^x)^2} \cdot \frac{d}{dx}(4^x)$$

$$= \frac{-1}{1+4^2x} \times 4^x \log 4$$

$$= \frac{4^x \log 4}{1+4^2x}.$$

Exercise 1.2 | Q 6.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}(\sqrt{x})$

SOLUTION

Let y = $\tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(\sqrt{x})\right]$$

$$= \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1+x)}.$$

Exercise 1.2 | Q 6.06 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x^2}{2}}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{\left(1 - \frac{1+x^2}{2}\right)}} \times \frac{1}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{1+x^2}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{2} - 1 - x^2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(1+x^2\right)$$

$$= \frac{1}{\sqrt{1-x^2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot (0+2x)$$

$$= \frac{x}{\sqrt{(1-x^2)(1+x^2)}}$$

Exercise 1.2 | Q 6.07 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}(1-x^2)$

SOLUTION

Let y = $\cos^{-1}(1-x^2)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}\left[\cos^{-1}\left(1-x^2\right)\right]$$

$$= \frac{-1}{\sqrt{1-(1-x^2)^2}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(1-x^2\right)$$

$$= \frac{-1}{\sqrt{1-(1-2x^2+x^4)}} \cdot (0-2x)$$

$$= \frac{2x}{\sqrt{2x^2-x^4}}$$

$$= \frac{2x}{x\sqrt{2-x^2}}$$

$$= \frac{2}{\sqrt{2-x^2}}.$$

Exercise 1.2 | Q 6.08 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\sin^{-1}\left(x^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2}x^{\frac{3}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

Exercise 1.2 | Q 6.09 | Page 29

Differentiate the following w.r.t. x : $\cos^3[\cos^{-1}(x^3)]$

SOLUTION

Let y = $\cos^3[\cos^{-1}(x^3)]$

= $[\cos(\cos^{-1}x^3)]^3$

= $(x^3)^3$

= x^9

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{dx}\left(x^9\right)$$

= $9x^8$.

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

SOLUTION

Let y = $\sin^4\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

= $\left\{\sin\left[\sin^{-1}\left(\sqrt{x}\right)\right]\right\}^4$

= $\left(\sqrt{x}\right)^4$

= x^2

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$

= 2x.

Exercise 1.2 | Q 7.01 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}\left[\cot\left(e^{x^2}\right)\right]$

SOLUTION

Let y = $\cot^{-1}\left[\cot\left(e^{x^2}\right)\right] = e^{x^2}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{x^2}\right)$

= $e^{x^2} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$

= $e^{x^2} \times 2x$

= $2xe^2$.

Exercise 1.2 | Q 7.02 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}\left[\frac{1}{\cos(5^x)}\right]$

SOLUTION

Let y = $\text{cosec}^{-1}\left[\dfrac{1}{\cos(5^x)}\right]$

= $\text{cosec}^{-1}[\sec(5^x)]$

= $\text{cosec}^{-1}\left[\text{cosec}\left(\dfrac{\pi}{2} - 5^x\right)\right]$

= $\dfrac{\pi}{2} - 5^x$

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 5^x\right)$

= $\dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) - \dfrac{d}{dx}(5^x)$

= $0 - 5^x.\log 5$

= $-5^x.\log 5$.

Exercise 1.2 | Q 7.03 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}\left(\sqrt{\dfrac{1+\cos x}{2}}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$

= $\cos^{-1}\left(\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2}}\right)$

= $\cos^{-1}\left[\cos\left(\frac{x}{2}\right)\right]$

= $\frac{x}{2}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x}{2}\right)$

= $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $\frac{1}{2} \times 1$

= $\frac{1}{2}$.

Exercise 1.2 | Q 7.04 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{\sqrt{1-\cos(x^2)}}{2}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\frac{\sqrt{1-\cos(x^2)}}{2}\right)$

= $\cos^{-1}\left(\sqrt{\frac{2\sin^2\left(\frac{x^2}{2}\right)}{2}}\right)$

= $\cos^{-1}\left[\sin\left(\frac{x^2}{2}\right)\right]$

= $\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\frac{x^2}{2}\right)\right]$

= $\frac{\pi}{2}-\frac{x^2}{2}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}-\frac{x^2}{2}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x^2)$

= $0-\frac{1}{2}\times 2x$

= − x.

Exercise 1.2 | Q 7.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}\left[\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right]$

SOLUTION

Let y = $\tan^{-1}\left[\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right]$

= $\tan^{-1}\left[\frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{\pi}{4}\right).\tan\left(\frac{x}{2}\right)}\right]$ $\ldots\left[\because \tan\frac{\pi}{4}=1\right]$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$

= $\frac{\pi}{4}-\frac{x}{2}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{4}-\frac{x}{2}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{4}\right)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $0-\frac{1}{2}\times 1$

= $-\frac{1}{2}$.

Exercise 1.2 | Q 7.06 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}\left(\frac{1}{4\cos^3 2x-3\cos 2x}\right)$

Let y = $\operatorname{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$

= $\operatorname{cosec}^{-1}\left(\frac{1}{\cos 6x}\right)$ ...[$\because$ cos3x = 4cos^{3}x − 3cosx]

= cosec^{-1}(sec6x)

= $\operatorname{cosec}^{-1}\left[\operatorname{cosec}\left(\frac{\pi}{2} - 6x\right)\right]$

= $\frac{\pi}{2} - 6x$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2} - 6x\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right) - 6\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= 0 − 6 x 1

= −6.

Exercise 1.2 | Q 7.07 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}\left[\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right]$

SOLUTION

Let y = $\tan^{-1}\left[\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right]$

= $\tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{6}\right)}{2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)}\right]$

= $\tan^{-1}\left[\cot\left(\frac{x}{6}\right)\right]$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{6}\right)\right]$

= $\frac{\pi}{2}-\frac{x}{6}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}-\frac{x}{6}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right)-\frac{1}{6}\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $0-\frac{1}{6}\times 1$

= $-\frac{1}{6}$.

Exercise 1.2 | Q 7.08 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

SOLUTION

Let y = $\cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

= $\cot^{-1}\left[\frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}\right]$

= $\cot^{-1}\left[\tan\left(\frac{3x}{2}\right)\right]$

= $\cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{3x}{2}\right)\right]$

= $\frac{\pi}{2}-\frac{3x}{2}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}-\frac{3x}{2}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right)-\frac{3}{2}\frac{\mathrm{dx}}{\mathrm{dx}}$

= $0-\frac{3}{2}\times 1$

= $-\frac{3}{2}$.

Exercise 1.2 | Q 7.09 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$

= $\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-7x\right)}{1+\cos\left(\frac{\pi}{2}-7x\right)}\right]$

= $\tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{7x}{2}\right).\cos\left(\frac{\pi}{4}-\frac{7x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{7x}{2}\right)}\right]$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{7x}{2}\right)\right]$

= $\frac{\pi}{4}-\frac{7x}{2}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}-\frac{7x}{2}\right)$

= $\frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{7}{2}\frac{d}{dx}(x)$

= $0-\frac{7}{2}\times 1$

= $-\frac{7}{2}$.

Exercise 1.2 | Q 7.1 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

= $\tan^{-1}\left[\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}}\right]$

= $\tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$

= $\frac{\pi}{2}-\frac{\pi}{2}$

Differentiate the following w.r.t. x :

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}-\frac{x}{2}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $0-\frac{1}{2}\times 1$

= $-\frac{1}{2}$.

Exercise 1.2 | Q 7.11 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}(\operatorname{cosec} x+\cot x)$

SOLUTION

Let y = $\tan^{-1}$ (cosec x + cot x)

$$= \tan^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{.}2\right).\cos\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{2}\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$= 0 - \frac{1}{2} \times 1$$

$$= -\frac{1}{2}.$$

Exercise 1.2 | Q 7.12 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left[\frac{\sqrt{1+\sin\ \left(\frac{4}{3}\right)} + \sqrt{1-\sin\ \left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\ \left(\frac{4x}{3}\right)} - \sqrt{1-\sin\ \left(\frac{4x}{3}\right)}}\right]$

SOLUTION

Let y = $\cot^{-1}\left[\frac{\sqrt{1+\sin\ \left(\frac{4}{3}\right)}+\sqrt{1-\sin\ \left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\ \left(\frac{4x}{3}\right)}-\sqrt{1-\sin\ \left(\frac{4x}{3}\right)}}\right]$

= $1+\sin\left(\frac{4x}{3}\right)$

= $1+\cos\left(\frac{\pi}{2}-\frac{4x}{3}\right)$

= $2\cos^2\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

$\therefore\ \sqrt{1+\sin\left(\frac{4x}{3}\right)}=\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

Also, $1-\sin\left(\frac{4x}{3}\right)$

= $1-\cos\left(\frac{\pi}{2}-\frac{4x}{3}\right)$

= $2\sin^2\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

= $2\sin^2\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

$\therefore\ \sqrt{1-\sin\left(\frac{4x}{3}\right)}=\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

. $\frac{\sqrt{1+\sin\ \left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{}$

..

$$\overline{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}$$

$$= \frac{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$= \frac{1+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots\left[\text{Dividing by}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)\right.$$

$$= \frac{\tan\ \frac{\pi}{4}+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\ \frac{\pi}{4}.\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots\left[\because \tan\ \frac{\pi}{4}=1\right]$$

$$= \tan\left[\frac{\pi}{4}+\frac{\pi}{4}-\frac{2x}{3}\right]$$

$$= \tan\left(\frac{\pi}{2}-\frac{2x}{3}\right)$$

$$= \cot\left(\frac{2x}{3}\right)$$

$$\therefore y = \cot^{-1}\left[\cot\left(\frac{2x}{3}\right)\right]=\frac{2x}{3}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{3}\right)$$

$$= \frac{2}{3}\frac{d}{dx}(x)$$

$$= \frac{2}{3}\times 1$$

$$= \frac{2}{3}.$$

Exercise 1.2 | Q 8.1 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{4\sin x + 5\cos x}{\sqrt{41}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\frac{4\sin x + 5\cos x}{\sqrt{41}}\right)$

= $\sin^{-1}\left[(\sin x)\left(\frac{4}{\sqrt{41}}\right) + (\cos x)\left(\frac{5}{\sqrt{41}}\right)\right]$

Since, $\left(\frac{4}{\sqrt{41}}\right)^2 + \left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41} + \frac{25}{41} = 1$,

we can write, $\frac{4}{\sqrt{41}} = \cos\infty$ and $\frac{5}{\sqrt{41}} = \sin\infty$.

∴ y = $\sin^{-1}$ (sin x cos ∞ + cos x sin ∞)

= $\sin^{-1}$[sin(x + ∞)]

= x + ∞, where ∞ is a constant

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(x + \infty)$

= $\frac{d}{dx}(x) + \frac{d}{dx}(\infty)$

= 1 + 0

= 1.

Exercise 1.2 | Q 8.2 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{\sqrt{3}\cos x - \sin x}{2}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\frac{\sqrt{3}\cos x - \sin x}{2}\right)$

= $\cos^{-1}\left[(\cos x)\left(\frac{\sqrt{3}}{2}\right) - (\sin x)\left(\frac{1}{2}\right)\right]$

= $\cos^{-1}\left(\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6}\right)$ ...$\left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \sin \frac{\pi}{6} = \frac{1}{2}\right]$

= $\cos^{-1}\left[\cos\left(x + \frac{\pi}{6}\right)\right]$

= $x + \frac{\pi}{6}$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(x + \frac{\pi}{6}\right)$

= $\frac{\mathrm{d}}{\mathrm{dx}}(x) + \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{6}\right)$

= 1 + 0

= 1.

Exercise 1.2 | Q 8.3 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{\cos\sqrt{x} + \sin\sqrt{x}}{\sqrt{2}}\right)$

SOLUTION

y = $\sin^{-1}\left(\frac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

= $\sin^{-1}\left(\frac{1}{\sqrt{2}}\cos\sqrt{x}+\frac{1}{\sqrt{2}}\sin\sqrt{x}\right)$

Put,

$\frac{1}{\sqrt{2}}$ = sinx

$\frac{1}{\sqrt{2}}$ = cosα

Also,

sin^{2}α + cos^{2}α = $\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2$ = 1

And,

tanα = 1

∴ α = tan^{-1}1

y = $\sin^{-1}(\sin\alpha.\cos\sqrt{x}+\cos\alpha.\sin(x)$

= $\sin^{-1}(\sin(\alpha+\sqrt{x}))$

y = $\alpha+\sqrt{x}$

y = $\tan^{-1}(1)+\sqrt{x}$

y = $\alpha+\sqrt{x}$

y = $\tan^{-1}(1)+\sqrt{x}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}+\sqrt{x})$

= $0+\frac{1}{2\sqrt{x}}$

= $\frac{1}{2\sqrt{x}}$.

Exercise 1.2 | Q 8.4 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{3\cos 3x - 4\sin 3x}{5}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\frac{3\cos 3x - 4\sin 3x}{5}\right)$

= $\cos^{-1}\left[(\cos 3x)\left(\frac{3}{5}\right) - (\sin 3x)\left(\frac{4}{5}\right)\right]$

Since, $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

= $\frac{9}{25} + \frac{16}{25}$ = 1

we can write, $\frac{3}{5} = \cos\infty$ and $\frac{4}{5} = \sin\infty$.

∴ y = $\cos^{-1}$(cos3x cos∞ – sin3x sin∞)

= $\cos^{-1}$[cos(3x + ∞)

= 3x + ∞, where ∞ is a constant

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(3x + \infty)$

= $3\frac{\mathrm{d}}{\mathrm{dx}}(x) + \frac{\mathrm{d}}{\mathrm{dx}}(\infty)$

= 3 x 1 + 0

= 3.

Exercise 1.2 | Q 8.5 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left[\frac{3\cos(e^x) + 2\sin(e^x)}{\sqrt{13}}\right]$

SOLUTION

y = $\cos^{-1}\left(\frac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$

= $\cos^{-1}\left(\cos(e^x).\frac{3}{\sqrt{13}}+\sin(e^x)\frac{2}{\sqrt{13}}\right)$

Put,

$\frac{3}{\sqrt{13}}$ = cosx

$\frac{2}{\sqrt{3}}$ = sinx

Also,

$\sin^2\alpha + \cos^2\alpha = \frac{9}{13}+\frac{4}{13} = 1$

And,

$\tan\alpha = \frac{\sin x}{\cos \alpha} = \frac{2}{3}$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$

$y = \cos^{-1}(\cos e^x . \cos\alpha + \sin e^x . \cos\alpha)$

$y = \cos^{-1}(\cos e^x - \alpha))$ $\quad \because \cos^{-1}x . (\cos x) = x$

$y = e^x - \alpha$

$= e^x = \tan^{-1}\left(\frac{2}{3}\right)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= e^x - 0$

$= e^x$.

Exercise 1.2 | Q 8.5 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left[\dfrac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right]$

SOLUTION

$$y = \cos^{-1}\left(\frac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$$

$$= \cos^{-1}\left(\cos(e^x).\ \frac{3}{\sqrt{13}}+\sin(e^x)\frac{2}{\sqrt{13}}\right)$$

Put,

$$\frac{3}{\sqrt{13}} = \text{cosx}$$

$$\frac{2}{\sqrt{3}} = \text{sinx}$$

Also,

$$\sin^2\alpha + \cos^2\alpha = \frac{9}{13}+\frac{4}{13} = 1$$

And,

$$\tan\alpha = \frac{\sin x}{\cos\alpha} = \frac{2}{3}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$$

$$y = \cos^{-1}(\cos e^x\ .\ \cos\alpha + \sin e^x\ .\ \cos\alpha)$$

$$y = \cos^{-1}(\cos e^x - \alpha)) \qquad \because \cos^{-1}x\ .\ (\cos x) = x$$

$$y = e^x - \alpha$$

$$= e^x = \tan^{-1}\left(\frac{2}{3}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

= $e^x - 0$

= e^x.

Exercise 1.2 | Q 8.6 | Page 30

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}\left[\frac{10}{6\sin(2^x) - 8\cos(2^x)}\right]$

SOLUTION

Let y = $\operatorname{cosec}^{-1}\left[\frac{10}{6\sin(2^x) - 8\cos(2^x)}\right]$

= $\sin^{-1}\left[\frac{6\sin(2^x) - 8\cos(2^x)}{10}\right]$ $\ldots\left[\because \operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)\right]$

= $\sin^{-1}\left[\{\sin(2^x)\}\left(\frac{6}{10}\right) - \{\cos(2^x)\}\left(\frac{8}{10}\right)\right]$

Since, $\left(\frac{6}{10}\right)^2 + \left(\frac{8}{0}\right)^2 = \frac{36}{100} + \frac{64}{100}$ = 1,

we can write, $\frac{6}{10} = \cos\infty$ and $\frac{8}{10} = \sin\infty$.

$\therefore$ y = $\sin^{-1}[\sin(2^x).\cos\infty - \cos(2^x).\sin\infty]$

= $\sin^{-1}[\sin(2^x - \infty)]$

= $2^x - \infty$, where ∞ is a constant

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(2^x - \infty)$

= $\frac{d}{dx}(2^x) - \frac{d}{dx}(\infty)$

= $2^x.\log 2 - 0$

= $2^x.\log 2$.

Exercise 1.2 | Q 9.01 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put x = tanθ.

Then θ = $\tan^{-1}x$

∴ y = $\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\cos^{-1}(\cos 2\theta)$

= 2θ

= $2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\tan^{-1}x\right)$$

= $2\frac{d}{dx}\left(\tan^{-1}x\right)$

= $2 \times \frac{1}{1+x^2}$

= $\frac{2}{1+x^2}$.

Exercise 1.2 | Q 9.02 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put x = tanθ.

Then θ = $\tan^{-1}x$

∴ y = $\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

= $\tan^{-1}(\tan 2\theta)$

= 2θ

= $2\tan^{-1}x$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(2\tan^{-1}x\right)$

= $2\frac{d}{dx}\left(\tan^{-1}x\right)$

= $2 \times \frac{1}{1+x^2}$

= $\frac{2}{1+x^2}$.

Exercise 1.2 | Q 9.03 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put x = tanθ.

Then θ = $\tan^{-1}x$

∴ y = $\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\sin^{-1}(\cos 2\theta)$

= $\sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]$

= $\frac{\pi}{2}-2\theta$

= $\frac{\pi}{2}-2\tan^{-1}x$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-2\tan^{-1}x\right)$

= $\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}\left(\tan^{-1}x\right)$

= $0-2\times\frac{1}{1+x^2}$

= $\frac{-2}{1+x^2}$.

Exercise 1.2 | Q 9.04 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

SOLUTION

Let y = $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put x = sinθ.

Then θ = $\sin^{-1}$x

∴ y = $\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

= $\sin^{-1}$(2sinθ cosθ)

= $\sin^{-1}$(sin2θ)

= 2θ

= $2\sin^{-1}$x

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(2\sin^{-1}x\right)$$

= $2\frac{\mathrm{d}}{\mathrm{dx}}\left(\sin^{-1}x\right)$

= $2 \times \frac{1}{\sqrt{1-x^2}}$

= $\frac{2}{\sqrt{1-x^2}}$

We can also put x = cosθ.

Then θ = $\cos^{-1}$x

∴ y = $\sin^{-1}\left(2\cos\theta\sqrt{1-\cos^2\theta}\right)$

= $\sin^{-1}$(2cosθ sinθ)

= $\sin^{-1}$(sin2θ)

= 2θ

= $2\cos^{-1}$x

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\cos^{-1}x\right)$$

$$= 2\frac{d}{dx}\left(\cos^{-1}x\right)$$

$$= 2 \times \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-2}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx} = \pm\frac{2}{\sqrt{1-x^2}}$.

Exercise 1.2 | Q 9.05 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}(3x - 4x^3)$

SOLUTION

Let $y = \cos^{-1}(3x - 4x^3)$

Put $x = \sin\theta$.

Then $\theta = \sin^{-1}x$

$\therefore$ $y = \cos^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \cos^{-1}(\sin 3\theta)$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 3\theta\right)\right]$$

$$= \frac{\pi}{2} - 3\theta$$

$$= \frac{\pi}{2} - 3\sin^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 3\sin^{-1}x\right)$$

$$= \frac{dy}{dx}\left(\frac{\pi}{2}\right) - 3\frac{d}{dx}\left(\sin^{-1}x\right)$$

$= 0 - 3 \times \dfrac{1}{\sqrt{1 - x^2}}$

$= \dfrac{-3}{\sqrt{1 - x^2}}.$

Exercise 1.2 | Q 9.06 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$= \cos^{-1}\left[\dfrac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}\right]$

$= \cos^{-1}\left(\dfrac{e^{2x} - 1}{e^{2x} + 1}\right)$

Put $e^x = \tan\theta$.

Then $\theta = \tan^{-1}(e^x)$

$\therefore$ y = $\cos^{-1}\left(\dfrac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$

$= \cos^{-1}\left[-\left(\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)\right]$

$= \cos^{-1}(-\cos 2\theta)$

$= \cos^{-1}[\cos(\pi - 2\theta)]$

$= \pi - 2\theta$

$= \pi - 2\tan^{-1}(e^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\pi - 2\tan^{-1}(e^x)\right]$$

$$= \frac{d}{dx}(\pi) - 2\frac{d}{dx}\left[\tan^{-1}(e^x)\right]$$

$$= 0 - 2 \times \frac{1}{1+(e^x)^2} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{-2}{1+e^{2x}} \times e^x$$

$$= \frac{2e^x}{1+e^{2x}}.$$

Exercise 1.2 | Q 9.07 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\frac{(1-9^x)}{(1+9^x)}$

SOLUTION

Let y = $\cos^{-1}\frac{(1-9^x)}{(1+9^x)}$

$= \cos^{-1}\left[\frac{1-(3^x)^2}{1+(3^x)^2}\right]$

Put 3x = tanθ.

Then θ = $\tan^{-1}(3^x)$

$\therefore$ y = $\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\cos^{-1}(\cos 2\theta)$

= 2θ

= $2\tan^{-1}(3^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[2\tan^{-1}(3^x)\right]$$

$$= 2\frac{d}{dx}\left[\tan^{-1}(3^x)\right]$$

$$= 2 \times \frac{1}{1+(3^x)^2} \cdot \frac{d}{dx}(3^x)$$

$$= \frac{2}{1+3^{2x}} \times 3^x \log 3$$

$$= \frac{2.3^x \log 3}{1+3^{2x}}.$$

Exercise 1.2 | Q 9.08 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1-2^{4x}}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1-2^{4x}}\right)$

$$= \sin^{-1}\left[\frac{4^x.4^{\frac{1}{2}}}{1+(2^2)^{2c}}\right]$$

$$= \sin^{-1}\left(\frac{2.4^x}{1+4^{2x}}\right)$$

Put $4^x = \tan\theta$,

Then $\theta = \tan^{-1}(4^x)$

$\therefore$ y = $\sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}(4^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[2\tan^{-1}(4^x)\right]$$

$$= 2\frac{d}{dx}\left[\tan^{-1}(4^x)\right]$$

$$= 2 \times \frac{1}{1+(4^x)^2} \cdot \frac{d}{dx}(4^x)$$

$$= \frac{2}{1+4^{2x}} \times 4^x \log 4$$

$$= \frac{2.4^x \log 4}{1+4^{2x}}$$

Note: The answer can also be written as :

$$\frac{dy}{dx} = \frac{4^{\frac{1}{2}}.4^x \log 4}{1+4^{2x}}$$

$$= \frac{4^{x+\frac{1}{2}}.\log 4}{1+4^{2x}}.$$

Exercise 1.2 | Q 9.09 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right)$

= $\sin^{-1}\left[\frac{1-(5x)^2}{1+(5x)^2}\right]$

Put 5x = tanθ.

Then θ = $\tan^{-1}(5x)$

$\therefore$ y = $\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\sin^{-1}(\cos 2\theta)$

= $\sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]$

= $\frac{\pi}{2}-2\theta$

= $\frac{\pi}{2}-2\tan^{-1}(5x)$

Differentiating w.r.t. x, we get

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\frac{\pi}{2}-2\tan^{-1}(5x)\right]$

= $\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}\left[\tan^{-1}(5x)\right]$

= $0-2\times\frac{1}{1+(5)^2}\cdot\frac{d}{dx}(5x)$

= $\frac{-2}{1+25x^2}\times 5$

= $\frac{-10}{1+25x^2}$.

Exercise 1.2 | Q 9.1 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$

SOLUTION

Let y = $\sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$

= $\sin^{-1}\left[\frac{1-\left(\frac{x^3}{2}\right)^2}{1+\left(\frac{x^3}{2}\right)^2}\right]$

Put $x^{\frac{3}{2}} = \tan$ θ.

Then θ = $\tan^{-1}\left(x^{\frac{3}{2}}\right)$

∴ y = $\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\sin^{-1}(\cos 2\theta)$

= $\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]$

= $\frac{\pi}{2}-2\theta$

= $\frac{\pi}{2}-2\tan^{-1}\left(x^{\frac{3}{2}}\right)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{2}-2\tan^{-1}\left(x^{\frac{3}{2}}\right)\right]$

= $\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{3}{2}}\right)\right]$

= $0-2\times\frac{1}{1+\left(x^{\frac{3}{2}}\right)^2}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$

$= \frac{2}{1+x^3} \times \frac{3}{2} x^{\frac{1}{2}}$

$= -\frac{3\sqrt{x}}{1+x^3}.$

Exercise 1.2 | Q 9.11 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$

Put $x^{\frac{5}{2}}$ = tanθ.

Then θ = $\tan^{-1}\left(x^{\frac{5}{2}}\right)$

∴ y = $\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

= $\tan^{-1}$(tan2θ))

= 2θ

= $2\tan^{-1}\left(x^{\frac{5}{2}}\right)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left[2\tan^{-1}\left(x^{\frac{5}{2}}\right)\right]$

$= 2\frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{5}{2}}\right)\right]$

$= 2 \times \frac{1}{1+\left(x^{\frac{5}{2}}\right)^2} \cdot \frac{d}{dx}\left(x^{\frac{5}{2}}\right)$

$= \frac{2}{1+x^5} \times \frac{5}{2} x^{\frac{3}{2}}$

$= \dfrac{5x\sqrt{x}}{1+x^5}.$

Exercise 1.2 | Q 9.12 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

SOLUTION

Let y = $\cot^{-1}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

$= \tan^{-1}\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)$ ...$\left[\because \cot^{-1}x = \tan^{-1}\left(\dfrac{1}{x}\right)\right]$

$= \tan^{-1}\left(\dfrac{1+\sqrt{x}}{1-1\times\sqrt{x}}\right)$

$= \tan^{-1}(1) + \tan^{-1}(\sqrt{x})$ ...$\left[\because \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y\right]$

$= \dfrac{\pi}{4} + \tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\dfrac{\pi}{4} + \tan^{-1}(\sqrt{x})\right]$

$= \dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{\pi}{4}\right) + \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(\sqrt{x})\right]$

$= 0 + \dfrac{1}{1+(\sqrt{x})^2}\cdot\dfrac{\mathrm{d}}{\mathrm{dx}}(\sqrt{x})$

$= \dfrac{1}{1+x}\times\dfrac{1}{2\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}(1+x)}.$

Exercise 1.2 | Q 10.1 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

= $\tan^{-1}\left[\frac{5x+3x}{1-(5x)(3x)}\right]$

= $\tan^{-1}(5x) + \tan^{-1}(3x)$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(5x) + \tan^{-1}(3x)\right]$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(5x)\right] + \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(3x)\right]$

= $\frac{1}{1+(5x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(5x) + \frac{1}{1+(3x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(3x)$

= $\frac{1}{1+25x^2}\times 5 + \frac{1}{1+9x^2}\times 3$

= $\frac{5}{1+25x^2} + \frac{3}{1+9x^2}.$

Exercise 1.2 | Q 10.2 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

SOLUTION

Let y = $\cot^{-1}\left(\dfrac{1+35x^2}{2x}\right)$

= $\tan^{-1}\left(\dfrac{2x}{1+35x^2}\right)$ ...$\left[\because \cot^{-1}x = \tan^{-1}\left(\dfrac{1}{x}\right)\right]$

= $\tan^{1}\left[\dfrac{7x-5x}{1+(7x)(5x)}\right]$

= $\tan^{-1}(7x) - \tan^{-1}(5x)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(7x) - \tan^{-1}(5x)\right]$$

= $\dfrac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(x)\right] - \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(5x)\right]$

= $\dfrac{1}{1+(7)^2}\cdot\dfrac{\mathrm{d}}{\mathrm{dx}}(7x) - \dfrac{1}{1+(5x)^2}\cdot\dfrac{\mathrm{d}}{\mathrm{dx}}(5x)$

= $\dfrac{1}{1+49x^2}\times 7 - \dfrac{1}{1+25x^2}\times 5$

= $\dfrac{7}{1+49x^2} - \dfrac{5}{1+25x^2}$.

Exercise 1.2 | Q 10.3 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\dfrac{2\sqrt{x}}{1+3x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\dfrac{2\sqrt{x}}{1+3x}\right)$

= $\tan^{-1}\left[\dfrac{3\sqrt{x}-\sqrt{x}}{1+\left(3\sqrt{x}\right)\left(\sqrt{x}\right)}\right]$

$= \tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(\sqrt{x}) - \tan^{-1}(\sqrt{x})\right]\Bigg]$$

$$= \frac{d}{dx}\left[\tan^{-(3\sqrt{x}]} - \frac{d}{dx}\left[\tan^{-1}(\sqrt{x})\right]\right.$$

$$= \frac{1}{1+(3\sqrt{x})^2} \cdot \frac{d}{dx}(3\sqrt{x}) - \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+9x} \times 3 \times \frac{1}{2\sqrt{x}} - \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}\left[\frac{3}{1+9x} - \frac{1}{1+x}\right].$$

Exercise 1.2 | Q 10.4 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left[\frac{2^x+2}{1-3(4^x)}\right]$

SOLUTION

Let y = $\tan^{-1}\left[\frac{2^x+2}{1-3(4^x)}\right]$

$= \tan^{-1}\left[\frac{2^2.2^x}{1-3(4^x)}\right]$

$= \tan^{-1}\left[\frac{4.2^x}{1-3(4^x)}\right]$

$= \tan^{-1}\left[\frac{3.2^x+2^x}{1-(3.2^x \times 2^x)}\right]$

$= \tan^{-1}(3.2^x) + \tan^{-1}(2^x)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(3.2^x) + \tan^{-1}(2^x)\right]$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(3.3x^x)\right] + \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(2^x)\right]$$

$$= \frac{1}{1+(3.2^x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(3.2^x) + \frac{1}{1+(2^x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(2^x)$$

$$= \frac{1}{1+9(2^{2x})} \times 3 \times 2^x \log 2 + \frac{1}{1+2^{2x}} \times 2^x \log 2$$

$$= 2^x \log 2\left[\frac{3}{1+9(2^{2x})} + \frac{1}{1+2^{2x}}\right].$$

Exercise 1.2 | Q 10.5 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

$$= \tan^{-1}\left[\frac{2.2^x - 2^x}{1+(2.2^x)(2^x)}\right]$$

= $\tan^{-1}(2.2^x) - \tan^{-1}(2^x)$

Differentiating w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(2.2^x) - \tan^{-1}(2^x)\right]$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(2.2^x)\right] - \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}(2^x)\right]$$

$$= \frac{1}{1+(2.2^x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(2.2^x) - \frac{1}{1+(2^x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(2^x)$$

= $\frac{1}{1+4\left(2^{2x}\right)} \times 2 \times 2^x \log 2-\frac{1}{1+2^{2x}} \times 2^x \log 2$

= $2^x \log 2\left[\frac{2}{1+4\left(2^{2x}\right)}-\frac{1}{1+2^{2x}}\right]$.

Exercise 1.2 | Q 10.6 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

SOLUTION

Let y = $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

= $ta^{-1}\left(\frac{5ax}{a^2-6x^2}\right) \ldots\left[\because \cot^{-1} x=\tan^{-1}\left(\frac{1}{x}\right)\right]$

= $\tan^{-1}\left[\frac{5\left(\frac{x}{a}\right)}{1-6\left(\frac{x}{a}\right)^2}\right]$ ...[Dividing by a^2]

= $\tan\left[\frac{3\left(\frac{x}{a}\right)+2\left(\frac{x}{a}\right)}{1-3\left(\frac{x}{a}\right) \times 2\left(\frac{x}{a}\right)}\right]$

= $\tan^{-1}\left(\frac{3x}{a}\right)+\tan^{-1}\left(\frac{2x}{a}\right)$

Differentiating w.r.t. x, we get

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\frac{3x}{a}\right)+\tan^{-1}\left(\frac{x}{a}\right)\right]$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\frac{3x}{a}\right)\right]+\frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\frac{2x}{a}\right)\right]$

= $\frac{1}{1+\left(\frac{9x^2}{a^2}\right)} \times \frac{3}{a} \times 1+\frac{1}{1+\left(\frac{4x^2}{a^2}\right)} \times \frac{2}{a} \times 1$

$= \dfrac{a^2}{a^2+9x^2} \times \dfrac{3}{a} + \dfrac{a^2}{a^2+4x^2} \times \dfrac{2}{a}$

$= \dfrac{3a}{a^2+9x^2} + \dfrac{2a}{a^2+4x^2}.$

Exercise 1.2 | Q 10.7 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\dfrac{a+b\tan x}{b-a\tan x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\dfrac{a+b\tan x}{b-a\tan x}\right)$

$= \tan^{-1}\left[\dfrac{\frac{a}{b}+\tan x}{1-\frac{a}{b}.\tan x}\right]$

$= \tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}(\tan x)$

$= \tan^{-1}\left(\dfrac{a}{b}\right) + x$

Differentiating w.r.t. x, we get

$\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\dfrac{a}{b}\right)+x\right]$

$= \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\dfrac{a}{b}\right)\right] + \dfrac{\mathrm{d}}{\mathrm{dx}}(x)$

= 0 + 1

= 1.

Exercise 1.2 | Q 10.8 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$

= $\tan^{-1}\left[\dfrac{5-x}{1+(6x^2-5x-4)}\right]$

= $\tan^{-1}\left[\dfrac{(2x-1)-(3x-4)}{1+(2x+1)(3x-4)}\right]$

= $\tan^{-1}(2x + 1) - \tan^{-1}(3x - 4)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}(2x+1)\right]-\frac{d}{dx}\left[\tan^{-1}(3x-4)\right]$$

$$=\frac{1}{1+(2x+1)^2}\cdot\frac{d}{dx}(2x+1)-\frac{1}{1+(3x-4)^2}\cdot\frac{d}{dx}(3x-4)$$

$$=\frac{1}{1+(2x+1)^2}\cdot(2\times1+0)-\frac{1}{1+(3x-4)^2}\cdot(3\times1-0)$$

$$=\frac{2}{1+(2x+1)^2}-\frac{3}{1+(3x-4)^2}.$$

Exercise 1.2 | Q 10.9 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\dfrac{4-x-2x^2}{3x+2}\right)$

SOLUTION

Let y = $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$

= $\tan^{-1}\left(\frac{3x+2}{4-x-2x^2}\right)$ ... $\left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$

= $\tan^{-1}\left[\frac{3x+2}{1-(2x^2+x-3)}\right]$

= $\tan^{-1}\left[\frac{(2x+3)+(x-1)}{1-(2x+3)(x-1)}\right]$

= $\tan^{-1}(2x + 3) + \tan^{-1}(x - 1)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(2x+3)+\tan^{-1}(x-1)\right]$

= $\frac{d}{dx}\left[\tan^{-1}(2x+3)\right] + \frac{d}{dx}\left[\tan^{-1}(x-1)\right]$

= $\frac{1}{1+(2x+3)^2}\cdot\frac{d}{dx}(2x+3) + \frac{1}{1+(x-1)^2}\cdot\frac{d}{dx}(x-1)$

= $\frac{1}{1+(2x+3)^2}\cdot(2\times 1+0) + \frac{1}{1+(x-1)^2}\cdot(1-0)$

= $\frac{2}{1+(2x+3)^2} + \frac{1}{1+(x-1)^2}$.

EXERCISE 1.3 [PAGES 39 - 40]

Exercise 1.3 | Q 1.1 | Page 39

Differentiate the following w.r.t. x : $\frac{(x+\ 1)^2}{(x+2)^3(x+3)^4}$

SOLUTION

Let y = $\dfrac{(x+\ 1)^2}{(x+2)^3(x+3)^4}$

Then, log y = $\log\left[\dfrac{(x+1)^2}{(x+2)^3(x+3)^4}\right]$

= $\log(x + 1)^2 - \log(x + 2)^3 - \log(x + 3)^4$

= $2\log(x + 1) - 3\log(x + 2) - 4\log(x + 3)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} = 2\frac{\mathrm{d}}{\mathrm{dx}}[\log(x+1)] - 3\frac{\mathrm{d}}{\mathrm{dx}}[\log(x+2)] - 4\frac{\mathrm{d}}{\mathrm{dx}}[\log(x+3)]$$

$$= 2 \times \frac{1}{x+1}.\frac{\mathrm{d}}{\mathrm{dx}}(x+1) - 3 \times \frac{1}{x+2}.\frac{\mathrm{d}}{\mathrm{dx}}(x+2) - 4 \times \frac{1}{x+3}.\frac{\mathrm{d}}{\mathrm{dx}}(x+3)$$

$$= \frac{2}{x+1}.(1+0) - \frac{3}{x+2}.(1+0) - \frac{4}{x+3}.(1+0)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = y\left[\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}\right]$$

$$= \frac{(x+1)^2}{(x+2)^2(x+3)^4}.\left[\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}\right].$$

Exercise 1.3 | Q 1.2 | Page 39

Differentiate the following w.r.t. x : $\sqrt[3]{\dfrac{4x-1}{(2x+3)(5-2x)^2}}$

SOLUTION

Let y = $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

Then log y = $\log\left[\frac{4x-1}{(2x+3)(5-2x)^2}\right]^{\frac{1}{3}}$

= $\frac{1}{3}\log\left[\frac{4x-1}{(2x+3)(5-2x)^2}\right]$

= $\frac{1}{3}\left[\log(4x-1)-\log(2x+3)(5-2x)^2\right]$

= $\frac{1}{3}\log(4x-1)-\frac{1}{3}\log(2x+3)-\frac{2}{3}\log(5-2x)$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y}.\frac{\text{dy}}{\text{dx}}=\frac{1}{3}\frac{\text{d}}{\text{dx}}[\log(4x-1)]-\frac{1}{3}\frac{\text{d}}{\text{dx}}[\log(2x+3)]-\frac{2}{3}\frac{\text{d}}{\text{dx}}[\log(5-2x)]$

= $\frac{1}{3}\times\frac{1}{4x-1}.\frac{\text{d}}{\text{dx}}(4x-1)-\frac{1}{3}\times\frac{1}{2x+3}.\frac{\text{d}}{\text{dx}}(2x+3)-\frac{2}{3}\times\frac{1}{5-2x}.\frac{\text{d}}{\text{dx}}(5-2x)$

= $\frac{1}{3(4x-1)}.(4\times1-0)-\frac{1}{3(2x+3)}.(2\times1+0)-\frac{2}{3(5-2x)}.(0-2\times1)$

$\therefore\frac{\text{dy}}{\text{dx}}=y\left[\frac{4}{3(4x-1)}-\frac{2}{3(2x+3)}+\frac{4}{3(5-2x)}\right]$

= $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}\left[\frac{4}{3(4x-1)}-\frac{2}{3(2x+3)}+\frac{4}{3(5-2x)}\right].$

Exercise 1.3 | Q 1.3 | Page 39

Differentiate the following w.r.t. x : $(x^2+3)^{\frac{3}{2}}.\sin^3 2x.2^{x^2}$

SOLUTION

Let y = $\left(x^2+3\right)^{\frac{3}{2}}.\sin^3 2x.2^{x^2}$

Then log y = $\log\left[x^2+3\right)^{\frac{3}{2}}.\sin^3 2x.2^{x^2}$

= $\log\left(x^2+3\right)^{\frac{3}{2}}+\log\sin^3 2x+\log 2^{x^2}$

= $\frac{3}{2}\log\left(x^2+3\right)+3\log(\sin 2x)+x^2.\log 2$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{3}{2}\frac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(x^2+3\right)\right]+3\frac{\mathrm{d}}{\mathrm{dx}}[\log(\sin 2x)]+\log 2.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right)$

= $\frac{3}{2}\times\frac{1}{x^2+3}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+3\right)+3\times\frac{1}{\sin 2x}.\frac{\mathrm{d}}{\mathrm{dx}}(\sin 2x)+\log 2\times 2x$

= $\frac{3}{2(x^2+3)}.(2x+0)+\frac{3}{\sin 2x}\times\cos 2x.\frac{\mathrm{d}}{\mathrm{dx}}(2x)+2x\log 2$

= $\frac{6x}{2(x^2+3)}+3\cot 2x\times 2+2x\log 2$

$\therefore\frac{\mathrm{dy}}{\mathrm{dx}}=y\left[\frac{3x}{x^2+3}+6\cot 2x+2x\log 2\right]$

= $\left(x^2+3\right)^{\frac{3}{2}}.\sin^3 2x.2^{x^2}\left[\frac{3x}{x^2+3}+6\cot 2x+2x\log 2\right].$

Exercise 1.3 | Q 1.4 | Page 39

Differentiate the following w.r.t. x : $\frac{\left(x^2+2x+2\right)^{\frac{3}{2}}}{\left(\sqrt{x}+3\right)^3(\cos x)^x}$

SOLUTION

Let y = $\dfrac{\left(x^2+2x+2\right)^{\frac{3}{2}}}{\left(\sqrt{x}+3\right)^3(\cos x)^x}$

Then log y = $\log\left[\dfrac{\left(x^3+2x+2\right)^{\frac{3}{2}}}{\left(\sqrt{x}+3\right)^3(\cos x)^x}\right]$

= $\log\left(x^2+2x+2\right)^{\frac{3}{2}}-\log\left(\sqrt{x}+3\right)^3(\cos x)^x$

= $\dfrac{3}{2}\log\left(x^2+2x+2\right)-3\log\left(\sqrt{x}+3\right)-x\log(\cos x)$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{y}.\dfrac{\mathrm{dy}}{\mathrm{dx}}=\dfrac{3}{2}\dfrac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(x^2+2x+2\right)\right]-3\dfrac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(\sqrt{x}+3\right)\right]-\dfrac{\mathrm{d}}{\mathrm{dx}}[x\log(\cos x)]$

=

$\dfrac{3}{2}\times\dfrac{1}{x^2+2x+2}.\dfrac{\mathrm{d}}{\mathrm{dx}}\left(x^2+2x+2\right)-3\times\dfrac{1}{\sqrt{x}+3}.\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}+3\right)-\left\{x\dfrac{\mathrm{d}}{\mathrm{dx}}[\log(\cos x)]+\log(\cos x).\dfrac{\mathrm{d}}{\mathrm{dx}}(x)\right.$

=

$\dfrac{3}{2\left(x^2+2x+2\right)}\times(2x+2\times1+0)-\dfrac{3}{\sqrt{x}+3}\times\left(\dfrac{1}{2\sqrt{x}+0}\right)-x\times\dfrac{1}{\cos x}.\dfrac{\mathrm{d}}{\mathrm{dx}}(\cos x)+\log(\cos x)\times1\Big\}$

= $\dfrac{3(2x+2)}{2\left(x^2+2x+2\right)}-\dfrac{3}{2\sqrt{x}\left(\sqrt{x}+3\right)}-\left\{x\times\dfrac{1}{\cos x}.(-\sin x)+\log(\cos x)\right\}$

$\therefore\dfrac{\mathrm{dy}}{\mathrm{dx}}=y\left[\dfrac{3(x+1)}{x^3+2x+2}-\dfrac{3}{2\sqrt{x}\left(\sqrt{x}+3\right)}+x\tan x-\log(\cos x)\right]$

= $\dfrac{\left(x^2+2x+2\right)^{\frac{3}{2}}}{\left(\sqrt{x}+3\right)^3(\cos x)^x}\left[\dfrac{3(x+1)}{x^2+2x+2}-\dfrac{3}{2\sqrt{x}\left(\sqrt{x}+3\right)}+x\tan x-\log(\cos x)\right].$

Exercise 1.3 | Q 1.5 | Page 39

Differentiate the following w.r.t. x : $\dfrac{x^5.\tan^3 4x}{\sin^2 3x}$

SOLUTION

Let y = $\frac{x^5 . \tan^3 4x}{\sin^2 3x}$

Then log y = $\log\left[\frac{x^5 . \tan^3 4x}{\sin^{23} x}\right]$

= $\log x^5 + \log \tan^3 ax - \log \sin^2 3x$

= 5logx + 3log (tan4x) – 2log (sin3x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} . \frac{\mathrm{dy}}{\mathrm{dx}} = 5\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + 3\frac{\mathrm{d}}{\mathrm{dx}}[\log(\tan 4x)] - 2\frac{\mathrm{d}}{\mathrm{dx}}[\log(\sin 3x)]$$

$$= 5 \times \frac{1}{x} + 3 \times \frac{1}{\tan 4x} . \frac{\mathrm{d}}{\mathrm{dx}}(\tan 4x) - 2 \times \frac{1}{\sin 3x} . \frac{\mathrm{d}}{\mathrm{dx}}(\sin 3x)$$

$$= \frac{5}{x} + 3 \times \frac{1}{\tan 4x} \times \sec^{24} x . \frac{\mathrm{d}}{\mathrm{dx}}(4x) - 2 \times \frac{1}{\sin 3x} \times \cos 3x . \frac{\mathrm{d}}{\mathrm{dx}}(3x)$$

$$= \frac{5}{x} + 3 . \frac{\cos 4x}{\sin 4x} \times \frac{1}{\cos^{24} x} \times 4 - 2 \cot 3x \times 3$$

$$= \frac{5}{x} + \frac{24}{2 \sin 4x . \cos 4x} - 6 \cot 3x$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = y\left[\frac{5}{x} + \frac{24}{\sin 8x} - 6 \cot 3x\right]$$

$$= \frac{x^5 . \tan^3 4x}{\sin^2 3x}\left[\frac{5}{x} + 24\mathrm{cosec} 8x - 6 \cot 3x\right].$$

Exercise 1.3 | Q 1.6 | Page 39

Differentiate the following w.r.t. x : $x^{\tan^{-1} x}$

SOLUTION

Let y = $x^{\tan^{-1} x}$

Then log y = $\log\left(x^{\tan^{-1} x}\right) = \left(\tan^{-1} x\right)(\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\left(\tan^{-1} x\right)(\log x)\right]$$

$$= \left(\tan^{-1} x\right).\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x).\frac{\mathrm{d}}{\mathrm{dx}}\left(\tan^{-1} x\right)$$

$$= \left(\tan^{-1} x\right) \times \frac{1}{x} + (\log x) \times \frac{1}{1+x^2}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = y\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right]$$

$$= x^{\tan^{-1} x}\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right].$$

Exercise 1.3 | Q 1.7 | Page 39

Differentiate the following w.r.t. x : $(\sin x)^x$

SOLUTION

Let y = $(\sin x)^x$

Then log y = $\log(\sin x)^x$ = x.log(sin x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[x.\log(\sin x)]$$

$$= x.\frac{\mathrm{d}}{\mathrm{dx}}[\log(\sin x)] + \log(\sin x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$= x \times \frac{1}{\sin x}.\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) + \log(\sin x) \times 1$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = y\left[x \times \frac{1}{\sin x}.\cos x + \log(\sin x)\right]$$

= $(\sin x)^x[x \cot x + \log(\sin x)]$.

Exercise 1.3 | Q 1.8 | Page 39

Differentiate the following w.r.t. x : $(\sin x^x)$

SOLUTION

Let $y = (\sin x^x)$

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[(\sin x^x)]$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \cos(x^x).\frac{\mathrm{d}}{\mathrm{dx}}(x^x)$ ...(1)

Let $u = x^x$

Then $\log u = \log x^x = x.\log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}.\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x.\log x)$$

$$= x.\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dx}} = u(1 + \log x)$$

$\therefore \frac{\mathrm{d}}{\mathrm{dx}}(x^x) = x^x(1 + \log x)$ ...(2)

From (1) and (2), we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \cos(x^x).x^x(1 + \log x).$$

Exercise 1.3 | Q 2.1 | Page 40

Differentiate the following w.r.t. x : $x^e + x^x + e^x + e^e$

SOLUTION

Let $y = x^e + x^x + e^x + e^e$

Let $u = x^x$

Then $\log u = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}.\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x \log x)$$

$$= x.\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$= x \times \frac{1}{x} + (\log x)(1)$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dx}} = u(1 + \log x) = x^x(1 + \log x) \quad ...(1)$$

Now, $y = x^e + u + e^x + e^e$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x^e) + \frac{\mathrm{du}}{\mathrm{dx}} + \frac{\mathrm{d}}{\mathrm{dx}}(e^x) + \frac{\mathrm{d}}{\mathrm{dx}}(e^x)$$

$= ex^{e-1} + x^x(1 + \log x) + e^x + 0$ ...[By (1)]

$= ex^{e-1} + x^x(1 + \log x) + e^x$

$= ex^{e-1} + e^x + x^x(1 + \log x)$.

Exercise 1.3 | Q 2.2 | Page 40

Differentiate the following w.r.t. x : $x^{x^x} + e^{x^x}$

SOLUTION

Let y = $x^{x^x} + e^{x^x}$

Put u = x^{x^x} and $v = e^{x^x}$

Then y = u + v

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{du}}{\mathrm{dx}} + \frac{\mathrm{dv}}{\mathrm{dx}}$ ...(1)

Take u = x^{x^x}

$\therefore$ log u = $\log x^{x^x} = x^x . \log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{u} . \frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x^2 . \log x)$

= $x^x . \frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x) . \frac{\mathrm{d}}{\mathrm{dx}}(x^x)$

= $x^x \times \frac{1}{x} + (\log x) . \frac{\mathrm{d}}{\mathrm{dx}}(x^3)$ ...(2)

To find $\frac{\mathrm{d}}{\mathrm{dx}}(x^3)$

Let ω = x^x.

Then log ω = xlogx

Differentiating both sides w.r.t. x, we get

$\frac{1}{\omega} . \frac{\mathrm{d\omega}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x \log x)$

= $x . \frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x) . \frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $x \times \frac{1}{x} + (\log x) \times 1$

$\therefore \frac{\mathrm{d\omega}}{\mathrm{dx}} = \omega(1 + \log x)$

$\therefore \frac{\mathrm{d}}{\mathrm{dx}}(x^x) = x^x(1 + \log x)$ ...(3)

$\therefore$ from (2),

$$\frac{1}{u}.\frac{\mathrm{du}}{\mathrm{dx}} = x^x \times \frac{1}{x} + (\log x).\, x^x(1+\log x)$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dx}} = y\left[x^x \times \frac{1}{x} + (\log x).\, x^x(1+\log x)\right]$$

$$= x^{x^x}.\, x^x\left[\frac{1}{x} + (\log x).\,(1+\log x)\right]$$

$$= x^{x^x}.\, x^x.\log x\left[1+\log x + \frac{1}{x\log x}\right] \quad ...(4)$$

Also, v = e^{x^x}

$$\therefore \frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{x^x}\right)$$

$$= e^{x^x}.\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{x^x}\right)$$

$$= e^{x^x}.\, x^x(1+\log x) \quad ...(5)$$

From (1),(4) and (5), we get ...[By (3)]

$$\frac{\mathrm{dy}}{\mathrm{dx}} = x^{x^x}.\, x^x.\log x\left[1+\log x+\frac{1}{x\log x}\right] + e^{x^x}.\, x^x(1+\log x).$$

Exercise 1.3 | Q 2.3 | Page 40

Differentiate the following w.r.t. x : $(\log x)^x - (\cos x)^{\cot x}$

SOLUTION

Let y = $(\log x)^x - (\cos x)^{\cot x}$

Put u = $(\log x)^x$ and v = $(\cos x)^{\cot x}$

Then y = u – v

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{du}}{\mathrm{dx}} - \frac{\mathrm{dv}}{\mathrm{dx}} \quad ...(1)$$

Take u = $(\log x)^x$

∴ log u = $\log(\log x)^x$ = x.log(log x)

Differentiating both sides w.r.t. x, we get

$\frac{1}{u}.\frac{\mathrm{du}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}[x.\log(\log x)]$

$= x\frac{\mathrm{d}}{\mathrm{dx}}[\log(\log x)]+\log(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$

$= x\times\frac{1}{\log x}.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+\log(\log x)\times 1$

$= x\times\frac{1}{\log x}\times\frac{1}{x}+\log(\log x)$

$\therefore \frac{\mathrm{du}}{,}dx = u\left[\frac{1}{\log x}+\log(\log x)\right]$

$= (\log x)x\left[\frac{1}{\log x}+\log(\log x)\right]$ ...(2)

Also v = $(\cos x)^{\cot x}$

$\therefore$ log c = $\log(\cos x)^{\cot x}$ = (cot x).(log cos x)

Differentiating both sides w.r.t. x, we get

$\frac{1}{v}=\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{\mathrm{d}}{dx}[(\cot x).\log(\cos x)]$

$= (\cot x).\frac{\mathrm{d}}{\mathrm{dx}}(\log\cos x)+(\log\cos x).\frac{\mathrm{d}}{\mathrm{dx}}(\cot x)$

$= \cot x\times\frac{1}{\cos x}.\frac{\mathrm{d}}{\mathrm{dx}}(\cos x)+(\log\cos x)\left(-\mathrm{cosec}^2x\right)$

$= \cot x\times\frac{1}{\cos x}\times(-\sin x)-\left(\mathrm{cosec}^2x\right)(\log\cos x)$

$\therefore \frac{\mathrm{dv}}{\mathrm{dx}}=v\left[\frac{1}{\tan x}\times(-\tan x)-\left(\mathrm{cosec}^2x\right)(\log\cos x)\right]$

= $-(\cos x)^{\cot x}[1 + (\mathrm{cosec}^2x)(\log\cos x)]$

From (1), (2) and (3), we get

$\therefore \frac{\mathrm{dv}}{\mathrm{dx}}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]+(\cos x)^{\cot x}\left[1+\left(\mathrm{cosec}^2x\right)(\log\cos x)\right].$

Exercise 1.3 | Q 2.4 | Page 40

Differentiate the following w.r.t. x : $x^{x^x} + (\log x)^{\sin x}$

SOLUTION

Let y = $x^{x^x} + (\log x)^{\sin x}$

Put u = x^{e^x} $\text{ and } v = (\log x)^{\sin x}$

Then y = u + v

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{du}}{\mathrm{dx}} + \dfrac{\mathrm{dv}}{\mathrm{dx}}$ ...(1)

Take u = x^{e^x}

∴ log u = $\log x^{e^x} = e^x . \log x$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{u} . \dfrac{\mathrm{du}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}(e^x \log x)$

= $e^x \dfrac{\mathrm{d}}{\mathrm{dx}}(\log x) + \log x \dfrac{\mathrm{d}}{\mathrm{dx}}(e^x)$

= $e^x . \dfrac{1}{x} + (\log x)(e^x)$

$\therefore \dfrac{\mathrm{du}}{\mathrm{dx}} = y\left[\dfrac{e^x}{x} + e^x . \log x\right]$

= $e^x . x^{e^x}\left[\dfrac{1}{x} + \log x\right]$ ...(2)

Also, v = $(\log x)^{\sin x}$

∴ log v = $\log(\log x)^{\sin x}$ = (sin x).(log log x)

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{v} . \dfrac{\mathrm{dv}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}[(\sin x) . (\log \log x)]$

= $(\sin x) . \dfrac{\mathrm{d}}{\mathrm{dx}}\left[(\log \log x) + (\log \log x) . \dfrac{\mathrm{d}}{\mathrm{dx}}(\sin x)\right]$

$= \sin x \times \dfrac{1}{\log x} . \dfrac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log\log x).(\cos x)$

$\therefore \dfrac{\mathrm{dv}}{\mathrm{dx}} = v\left[\dfrac{\sin x}{\log x} \times \dfrac{1}{x} + (\cos x)(\log\log x)\right]$

$= (\log x)^{\sin x}\left[\dfrac{\sin x}{x}\log x + (\cos x)(\log\log x)\right]$ ...(2)

From (1), (2) and (3), we get

$\dfrac{\mathrm{dy}}{\mathrm{dx}} - e^x . x^{e^x}\left[\dfrac{1}{x} + \log x\right] + (\log x)^{\sin x}\left[\dfrac{\sin x}{x\log x} + (\cos x)(\log\log x)\right].$

Exercise 1.3 | Q 2.5 | Page 40

Differentiate the following w.r.t. x : $e^{\tan x} + (\log x)^{\tan x}$

SOLUTION

Let $y = e^{\tan x} + (\log x)^{\tan x}$

Put $u = (\log x)^{\tan x}$

$\therefore \log u = \log (\log x)^{\tan x} = (\tan x).(\log\log x)$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{u} . \dfrac{\mathrm{du}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}[(\tan x).(\log\log x)]$

$= (\tan x).\dfrac{\mathrm{d}}{\mathrm{dx}}(\log\log x) + (\log\log x).\dfrac{\mathrm{d}}{\mathrm{dx}}(\tan x)$

$= \tan x \times \dfrac{1}{\log x} . \dfrac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log\log x)\left(\sec^2 x\right)$

$= \tan x \times \dfrac{1}{\log x} \times \dfrac{1}{x} + (\log\log x)\left(\sec^2 x\right)$

$\therefore \mathrm{du}'/dx = u\left[\dfrac{\tan x}{x\log x} + (\log\log x)\left(\sec^2 x\right)\right]$

$= (\log x)^{\tan x}\left[\tan x(x\log x) + (\log\log x)\left(\sec^2 x\right)\right]$

Now, y = $e^{\tan x}$ + u

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{\tan x}\right) + \frac{\mathrm{du}}{\mathrm{dx}}$

$= e^{\tan x}.\frac{\mathrm{d}}{\mathrm{dx}}(\tan x) + \frac{\mathrm{du}}{\mathrm{dx}}$

$= e^{\tan x}.\sec^2 x + (\log x)^{\tan x}\left[\frac{\tan x}{x\log x} + (\log\log x)\left(\sec^2 x\right)\right].$

Exercise 1.3 | Q 2.6 | Page 40

Differentiate the following w.r.t. x : $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

SOLUTION

Let y = $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

Put u = $(\sin x)^{\tan x}$ and v = $(\cos x)^{\cot x}$

Then y = u + v

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{\mathrm{dv}}{\mathrm{dx}}$ ...(1)

Take u = $(\sin x)^{\tan x}$

$\therefore$ log u = $\log(\sin x)^{\tan x}$ = (tan x).(log sinx)

Differentiating both sides w.r.t. x, we get

$\frac{1}{u}.\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[(\tan x)(\log\sin x)]$

$= (\tan x).\frac{\mathrm{d}}{\mathrm{dx}}(\log\sin x) + (\log\sin x).\frac{\mathrm{d}}{\mathrm{dx}}(\tan x)$

$= \frac{\tan x}{\sin x}.\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) + (\log\sin x)\left(\sec^2 x\right)$

$= \dfrac{\frac{\sin x}{\cos x}}{\sin x} . \cos x + (\sec^2 x)(\log \sin x)$

= 1 + (sec^{2}x)(log sinx)

$\therefore \dfrac{\mathrm{du}}{\mathrm{dx}} = y[1 + (\sec^2 x)(\log \sin x)]$

= (sin x)$^{\tan x}$[1 + (sec^{2}x)(log sinx)] ...(2)

Also, v = (cos x)$^{\cot x}$

∴ log v = log(cos x)$^{\cot x}$ = (cot x).(log cosx)

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{v} . \dfrac{\mathrm{dv}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}[(\cot x) . (\log \cos x)]$

$= (\cot x) . \dfrac{\mathrm{d}}{\mathrm{dx}}(\log \cos x) + (\log \cos x) . \dfrac{\mathrm{d}}{\mathrm{dx}}(\cot x)$

$= \cot x \times \dfrac{1}{\cos x} . \dfrac{\mathrm{d}}{\mathrm{dx}}(\cos x) + (\log \cos x) . (-\mathrm{cosec}^2 x)$

$= \cot x \times \dfrac{1}{\cos x} \times (-\sin x) - (\mathrm{cosec}^2 x)(\log \cos x)$

$\therefore \dfrac{\mathrm{dv}}{\mathrm{dx}} = v\left[\dfrac{1}{\tan x} \times (-\tan x) - (\mathrm{cosec}^2 x)(\log \cos x)\right]$

= −(cos x)$^{\cot x}$ [1 + (cosec^{2}x)(log cosx)] ...(3)

From (1), (2) and (3), we get

$\dfrac{\mathrm{dy}}{\mathrm{dx}} = (\sin x)^{\tan x}[1 + (\sec^2 x)(\log \sin x)] - (\cos x)^{\cot x}[1 + (\mathrm{cosec}^2 x)(\log \cos x)]$.

Exercise 1.3 | Q 2.7 | Page 40

Differentiate the following w.r.t. x : $10^{x^x} + x^{x(10)} + x^{10x}$

SOLUTION

Let y = $10^{x^x}+x^{x(10)}+x^{10x}$

Put u = $10^{x^x}, v=x^{x^{10}}$ and $\omega=x^{10^x}$

Then y = u + v + ω

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{u}}{\mathrm{dx}}+\frac{\mathrm{dv}}{\mathrm{dx}}+\frac{\mathrm{d\omega}}{\mathrm{dx}}$ ...(1)

Take, u = 10^{x^x}

$\therefore \frac{\mathrm{du}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(10^{x^x}\right.$

= $10^{x^x}.\log 10.\frac{\mathrm{d}}{\mathrm{dx}}(x^x)$

To find $\frac{\mathrm{d}}{\mathrm{dx}}(x^x)$

Let z = x^x

∴ logz = $\log x^x$ = xlogx

Differentiating both sides w.r.t. x, we get

$\frac{1}{z}.\frac{\mathrm{dz}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}(x\log x)$

= $x.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$

= $x\times\frac{1}{x}+(\log x)(1)$

$\therefore \frac{\mathrm{dz}}{\mathrm{dx}}=z(1+\log x)$

$\therefore \frac{\mathrm{d}}{\mathrm{dx}}(x^x)=x^x(1+\log x)$

$\therefore \frac{\mathrm{du}}{\mathrm{dx}}=10^{x^x}.\log 10.x^x(1+\log x)$ ...(2)

Take, v = $x^{x^{10}}$

∴ log v = $\log x^{x^{10}}=x^{10}.\log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{v}.\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}\left(x^{10}\log x\right)$

= $x^{10}.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+(\log x).\frac{\mathrm{d}}{\mathrm{dx}}\left(x^{10}\right)$

= $x^{10}\times\frac{1}{x}+(\log x)\left(10x^{9}\right)$

$\therefore \frac{\mathrm{dv}}{\mathrm{dx}}=v\left[x^{9}+10x^{9}\log x\right]$

$\therefore \frac{\mathrm{dv}}{\mathrm{dx}}=x^{x^{10}}.x^{9}(1+10\log x)$ ...(3)

Also, ω = x^{10x}

∴ log ω = $\log x^{10x}=10^{x}.\log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{\omega}.\frac{\mathrm{d\omega}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}(10^{x}.\log x)$

= $10^{x}.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(10^{x})$

= $10^{x}\times\frac{1}{x}+(\log x)(10^{x}.\log 10)$

$\therefore \frac{\mathrm{d\omega}}{\mathrm{dx}}=\omega\left[\frac{10^{x}}{x}+10^{x}.(\log x)(\log 10)\right]$

$\therefore \frac{\mathrm{d\omega}}{\mathrm{dx}}=x^{10x}.10^{x}\left[\frac{1}{x}+(\log x)(\log 10)\right]$ ...(4)

From (1),(2),(3) and (4), we get

$\frac{\mathrm{dy}}{\mathrm{dx}}=10^{x\cdot x}.\log 10.\,x^{x}(1+\log x)+x^{x^{10}}.x^{9}(1+10\log x)+x^{10x}.10^{x}\left[\frac{1}{x}+(\log x)(\log 10)\right].$

Exercise 1.3 | Q 2.8 | Page 40

Differentiate the following w.r.t. x : $\left[(\tan x)^{\tan x}\right]^{\tan x}$ at $x=\frac{\pi}{4}$

SOLUTION

Let y = $\left[(\tan x)^{\tan x}\right]^{\tan x}$

$\therefore$ log y = $\log\left[(\tan x)^{\tan x}\right] \tan x$

= tanx. log(tanx)$^{\text{tanx}}$

= tanx. tanx log(tan x)

= (tanx)^2. log(tan x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\tan x)^2.\log(\tan x)\Big]$$

= $(\tan x)^2.\frac{\mathrm{d}}{\mathrm{dx}}(\log\tan x) + (\log\tan x).\frac{\mathrm{d}}{\mathrm{dx}}(\tan x)^2$

= $(\tan x)^2.\times\frac{1}{\tan x}.\frac{\mathrm{d}}{\mathrm{dx}}(\tan x) + (\log\tan x)\times 2\tan x.\frac{\mathrm{d}}{\mathrm{dx}}(\tan x)$

= $(\tan x)^2\times\frac{1}{\tan x}.\sec^2 x + (\log\tan x)\times 2\tan x\sec^2 x$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = y\left[(\tan x)\left(\sec^2 x\right) + (\log\tan x)\left(2\tan x\sec^2 x\right)\right]$

= [(tanx)$^{\text{tanx}}$]$^{\text{tanx}}$.(tanxsec^{2}x)[1 + 2logtanx]

If x = $\frac{\pi}{4}$, then

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \left[\left(\frac{\tan\pi}{4}\right)^{\tan\frac{\pi}{4}}\right]^{\tan\frac{\pi}{4}}\left(\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}\right)\left[1+2\log\tan\frac{\pi}{4}\right]$$

= $\left[(1)^1\right]^1.\left[1\left(\sqrt{2}\right)^2\right][1+2\log 1]$

= 1 x 2 x 1 ...[$\because$ log 1 = 0]

= 2.

Exercise 1.3 | Q 3.01 | Page 40

Find $\dfrac{\mathrm{dy}}{\mathrm{dx}}$ if $\sqrt{x}+\sqrt{y}=\sqrt{a}$

SOLUTION

$\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\cdot\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = 0

$\therefore \dfrac{1}{2\sqrt{y}}\dfrac{\mathrm{dy}}{\mathrm{dx}}=-\dfrac{1}{2\sqrt{x}}$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}}=-\sqrt{\dfrac{y}{x}}.$

Exercise 1.3 | Q 3.02 | Page 40

Find $\dfrac{\mathrm{dy}}{\mathrm{dx}}$ if $x\sqrt{x}+y\sqrt{y}=a\sqrt{a}$

SOLUTION

$x\sqrt{x}+y\sqrt{y}=a\sqrt{a}$

$\therefore x^{\frac{3}{2}}+y^{\frac{3}{2}}=a^{\frac{3}{2}}$

Differentiating both sides w.r.t. x, we get

$\dfrac{3}{2}\cdot x^{\frac{1}{2}}+\dfrac{3}{2}\cdot y^{\frac{1}{2}}\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = 0

$\therefore \dfrac{3}{2}\cdot y^{\frac{1}{2}}\dfrac{\mathrm{dy}}{\mathrm{dx}}=-\dfrac{3}{2}x^{\frac{1}{2}}$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}}=\dfrac{-x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$

= $-\sqrt{\dfrac{x}{y}}.$

Exercise 1.3 | Q 3.03 | Page 40

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if $x + \sqrt{xy} + y = 1$

SOLUTION

$x + \sqrt{xy} + y = 1$

Differentiating both sides w.r.t. x, we get

$$1 + \frac{1}{2\sqrt{xy}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(xy) + \frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore 1 + \frac{1}{2\sqrt{x}} \cdot \left[x\frac{\mathrm{dy}}{\mathrm{dx}} + y \times 1\right] + \frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore 1 + \frac{1}{2}\sqrt{\frac{x}{y}}\frac{\mathrm{dy}}{\mathrm{dx}} + \frac{1}{2}\sqrt{\frac{y}{x}} + \frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore \left(\frac{1}{2}\sqrt{\frac{x}{y}} + 1\right)\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{1}{2}\sqrt{\frac{y}{x}} - 1$$

$$\therefore \left(\frac{\sqrt{x} + 2\sqrt{y}}{2\sqrt{y}}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-\sqrt{y} - 2\sqrt{y}}{2\sqrt{x}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-\sqrt{y}(2\sqrt{x} + \sqrt{y})}{\sqrt{x}(\sqrt{x} + 2\sqrt{y})}.$$

Exercise 1.3 | Q 3.04 | Page 40

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ If $x^3 + x^2y + xy^2 + y^3 = 81$

SOLUTION

$x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. x, we get

$$3x^2 + x^2\frac{\mathrm{dy}}{\mathrm{dx}} + y\frac{\mathrm{d}}{\mathrm{dx}}(x^2) + x\frac{\mathrm{d}}{\mathrm{dx}}(y^2) + y^2\frac{\mathrm{d}}{\mathrm{dx}}(x) + 3y^2\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore 3x^2 + x^2\frac{\mathrm{dy}}{\mathrm{dx}} + y \times 2x + x \times 2y\frac{\mathrm{dy}}{\mathrm{dx}} + y^2 \times 1 + 3y^2\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore 3x^2 + x^2\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy + 2xy\frac{\mathrm{dy}}{\mathrm{dx}} + y^2 + 3y^2\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore \left(x^2 + 2xy + 3y^2\right)\frac{\mathrm{dy}}{\mathrm{dx}} = -3x^2\ 2xy - y^2$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(-3x^2 + 2xy + y^2\right)}{x^2 + 2xy + 3y^2}.$$

Exercise 1.3 | Q 3.05 | Page 40

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if $x^2y^2 - \tan^{-1}\left(\sqrt{x^2 + y^2}\right) = \cot^{-1}\left(\sqrt{x^2 + y^2}\right)$

SOLUTION

$$x^2y^2 - \tan^{-1}\left(\sqrt{x^2 + y^2}\right) = \cot^{-1}\left(\sqrt{x^2 + y^2}\right)$$

$$\therefore x^2y^2 - \tan^{-1}\left(\sqrt{x^2 + y^2}\right) + \cot^{-1}\left(\sqrt{x^2 + y^2}\right)$$

$$\therefore x^2y^2 = \frac{\pi}{2} \quad \ldots\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

Differentiating both sides w.r.t. x, we get

$$x^2.\frac{\mathrm{d}}{\mathrm{dx}}(y^2) + y^2.\frac{\mathrm{d}}{\mathrm{dx}}(x^2) = 0$$

$$\therefore x^2 \times 2y\frac{\mathrm{dy}}{\mathrm{dx}} + y^2 \times 2x = 0$$

$\therefore 2x^2y\frac{\text{dy}}{\text{dx}}$ = − 2xy^2

$\therefore x\frac{\text{dy}}{\text{dx}}$ = − y

$$\therefore \frac{\text{dy}}{\text{dx}} = -\frac{y}{x}.$$

Exercise 1.3 | Q 3.06 | Page 40

Find $\frac{\text{dy}}{\text{dx}}$ if xe^y + ye^x = 1

SOLUTION

xe^y + ye^x = 1

Differentiating both sides w.r.t. x, we get

$\frac{\text{d}}{\text{dx}}(xe^y) + \frac{\text{d}}{\text{dx}}(ye^x)$ = 0

$\therefore x.\frac{\text{d}}{\text{dx}}(e^y) + e^y.\frac{\text{d}}{\text{dx}}(x) + y.\frac{\text{d}}{\text{dx}}(e^x) + e^x.\frac{\text{dy}}{\text{dx}}$ = 0

$\therefore x.e^y\frac{\text{dy}}{\text{dx}} + e^y \times 1 + y \times e^x + e^x\frac{\text{dy}}{\text{dx}}$ = 0

$\therefore (e^x + xe^y)\frac{\text{dy}}{\text{dx}}$ = − e^y − ye^x

$$\therefore \frac{\text{dy}}{\text{dx}} = -\left(\frac{e^y + ye^x}{e^x + xe^y}\right).$$

Exercise 1.3 | Q 3.07 | Page 40

Find $\frac{\text{dy}}{\text{dx}}$ if e^{x+y} = cos(x − y)

SOLUTION

$e^{x+y} = \cos(x - y)$

Differentiating both sides w.r.t. x, we get

$$e^{x+y}.\ \frac{\mathrm{d}}{\mathrm{dx}}(x+y) = -\sin(x-y).\ \frac{\mathrm{d}}{\mathrm{dx}}(x-y)$$

$$\therefore\ e^{x+y}\left(1+\frac{\mathrm{dy}}{\mathrm{dx}}\right) = -\sin(x-y).\ \frac{\mathrm{dy}}{\mathrm{dx}}(x-y)$$

$$\therefore\ e^{x+y} + e^{x+y}.\ \frac{\mathrm{dy}}{\mathrm{dx}} = -\sin(x-y)\left(1-\frac{\mathrm{dy}}{\mathrm{dx}}\right)$$

$$\therefore\ \left[e^{x+y} - \sin(x-y)\right]\frac{\mathrm{dy}}{\mathrm{dx}} = -\sin(x-y) - e^{x+y}$$

$$\therefore\ \frac{\mathrm{dy}}{\mathrm{dx}} = -\left[\frac{\sin(x-y)+e^{x+y}}{e^{x+y}-\sin(x-y)}\right] = \frac{\sin(x-y)+e^{x+y}}{\sin(x-y)-e^{x+y}}.$$

Exercise 1.3 | Q 3.08 | Page 40

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if cos (xy) = x + y

SOLUTION

cos (xy) = x + y

Differentiating both sides w.r.t. x, we get

$$-\sin(xy).\ \frac{\mathrm{d}}{\mathrm{dx}}(xy) = 1 + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore\ -\sin(xy)\left[x\frac{\mathrm{dy}}{\mathrm{dx}} + y\frac{\mathrm{d}}{\mathrm{dx}}(x)\right] = 1 + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore\ -\sin(xy)\left[x\frac{\mathrm{dy}}{\mathrm{dx}} + y \times 1\right] = 1 + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore\ -x\sin(xy)\frac{\mathrm{dy}}{\mathrm{dx}} - y\sin(xy) = 1 + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore -\frac{\mathrm{dy}}{\mathrm{dx}} - \sin(xy)\frac{\mathrm{dy}}{\mathrm{dx}} = 1 + \ y\sin(xy)$$

$$\therefore -[1 + x\sin(xy)]\frac{\mathrm{dy}}{\mathrm{dx}} = 1 + \ y\sin(xy)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-[1 + y\sin(xy)]}{1 + x\sin(xy)}.$$

Exercise 1.3 | Q 3.09 | Page 40

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if $e^{e^{x-y}} = \frac{x}{y}$

SOLUTION

$$e^{e^{x-y}} = \frac{x}{y}$$

$$\therefore e^{x-y} = \log\left(\frac{x}{y}\right)$$ ...[∵ e^x = y ⇒ x = log y]

∴ e^{x-y} = log x – log y

Differentiating both sides w.r.t. x, we get

$$e^{x-y}.\frac{\mathrm{d}}{\mathrm{dx}}(x - y) = \frac{1}{x} - \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore e^{x-y}\left(1 - \frac{\mathrm{dy}}{\mathrm{dx}}\right) = \frac{1}{x} - \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore e^{x-y} - e^{x-y}\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x} - \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \left(\frac{1}{y} - e^{x-y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x} - \ e^{x-y}$$

$$\left(\frac{1 - ye^{x-y}}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1 - xe^{x-y}}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y(1 - xe^{x-y})}{(x(1 - ye^{x-y})}.$$

Exercise 1.3 | Q 4.1 | Page 40

Show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$ in the following, where a and p are constants : $x^7.y^5 = (x + y)^{12}$

SOLUTION

$x^7.y^5 = (x + y)^{12}$

$\therefore (\log x^7.y^5) = \log(x + y)12$

$\therefore \log x^7 + \log y^5 = \log(x + y)^{12}$

$\therefore 7\log x + 5\log y = 12\log(x + y)$

Differentiating both sides w.r.t. x, we get

$$7 \times \frac{1}{x} + 5 \times \frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = 12 \times \frac{1}{x+y}.\frac{\mathrm{d}}{\mathrm{dx}}(x+y)$$

$$\therefore \frac{7}{x} + \frac{5}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{12}{x+y}.\left(1 + \frac{\mathrm{dy}}{\mathrm{dx}}\right)$$

$$\therefore \frac{7}{x} + \frac{5}{y}.\frac{\mathrm{y}}{\mathrm{dx}} = \frac{12}{x+y} + \frac{12}{x+y}.\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \left(\frac{5}{y} - \frac{12}{x+y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{12}{x+y} - \frac{7}{x}$$

$$\therefore \left[\frac{5x + 5y - 12y}{y(x+y)}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{12x - 7x - 7y}{x(x+y)}$$

$$\therefore \left[\frac{5x - 7y}{y(x+y)}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{5x - 7y}{x(x-y)}$$

$$\therefore \frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.2 | Page 40

Show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$ in the following, where a and p are constants : $x^p y^4 = (x + y)^{p+4}$, $p \in N$

SOLUTION

$x^p y^4 = (x + y)^{p+4}$

Taking log

$\log(x^p y^4) = \log(x + y)^{p+4}$

$\log x^p + \log y^4 = (p + 4) \log(x + y)$

$p \log x + 4 \log y = (p + 4) \log(x + y)$

Differentiating both sides w.r.t. x, we get

$$p. \frac{\mathrm{d}}{\mathrm{dx}} \log x + 4. \frac{\mathrm{d}}{\mathrm{dx}} \log y = (p+4) \frac{\mathrm{d}}{\mathrm{dx}} \log(x+y)$$

$$\frac{p}{x} + 4 \frac{1}{y} \frac{\mathrm{dy}}{\mathrm{dx}} = (p+4) \frac{1}{x+y} \left(1 + \frac{\mathrm{dy}}{\mathrm{dx}}\right)$$

$$\frac{\mathrm{p}}{4} + \frac{4}{y} \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{(p+4)}{(x+y)} + \frac{p+4}{(x+y)} \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} \left[\frac{4}{y} - \frac{(p+4)}{(x+y)}\right] = \frac{p+4}{x+y} - \frac{p}{x}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} \left[\frac{4(x+y) - y(p+4)}{y(x+y)}\right] = \frac{x(p+4) - p(x+y)}{x(x+y)}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} \left[\frac{4x + 4y - py - 4y}{y(x+y)}\right] = \frac{px + 4x - px - py}{x(x+y)}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} \left[\frac{4x - py}{y}\right] = \frac{4x - py}{x}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.3 | Page 40

Show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$ in the following, where a and p are constants : $\sec\left(\frac{x^5 + y^5}{x^5 - y^5}\right) = a^2$

SOLUTION

$\sec\left(\frac{x^5+y^5}{x^5-y^5}\right)$ = a^2

$\therefore \frac{x^5+y^5}{x^5-y^5} = \sec^{-1}\left(a^2\right)$ = k

∴ $x^5 + y^5 = kx^5 - ky^5$

∴ $(1 + k)y^5 = (k - 1)x5$

$\therefore \frac{y^5}{x^5} = \frac{k-1}{k+1}$

$\therefore \frac{y}{x} = \left(\frac{k-1}{k+1}\right)^{\frac{1}{5}}$, a constant

Differentiating both sides w.r.t. x, we get

$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x}\right)$ = 0

$\therefore \frac{x.\frac{\mathrm{dy}}{\mathrm{dx}} - y.\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$ = 0

$\therefore x\frac{\mathrm{dy}}{\mathrm{dx}} - y \times 1$ = 0

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$

Alternative Method :

$\sec\left(\frac{x^5+y^5}{x^5-y^5}\right)$ = a^2

$\therefore \frac{x^5+y^5}{x^5-y^5}$ = $\sec^{-1}a^2$ = k ...(Say)

∴ $x^5 + y^5 = kx^5 - ky^5$

∴ $(1 + k)y^5 = (k - 1)x^5$

$\therefore \dfrac{y^5}{x^5} = \dfrac{k-1}{k+1}$ ...(1)

$\therefore$ y^5 = $k'x^5$, where k' = $\dfrac{k-1}{k+1}$

Differentiating both sides w.r.t. x, we get

$5y^4\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = k' x 5x4

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = k\prime.\dfrac{x^4}{y^4}$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \left(\dfrac{k-1}{k+1}\right).\dfrac{x^4}{y^4}$

= $\dfrac{y^5}{x^5} \times \dfrac{x^4}{y^4}$ ...[By (1)]

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}.$

Exercise 1.3 | Q 4.4 | Page 40

Show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$ in the following, where a and p are constants : $\tan^{-1}\left(\dfrac{3x^2-4y^2}{3x^2+4y^2}\right)$ = a^2

SOLUTION

$\tan^{-1}\left(\dfrac{3x^2-4y^2}{3x^2+4y^2}\right)$ = a^2

$\therefore \dfrac{3x^2-4y^2}{3x^2+4y^2}$ = $\tan a^2$ = k ...(Say)

$\therefore$ $3x^2 - 4y^2 = 3kx^2 + 4ky^2$

$\therefore$ $(4k + 4)y^2 = (3 - 3k)x^2$

$\therefore \dfrac{y^2}{x^2} = \dfrac{3-3k}{4k+4}$

$\therefore \dfrac{y}{x} = \sqrt{\dfrac{3-3k}{4k+4}}$, a constant

Differentiating both sides w.r.t. x, we get

$\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{y}{x}\right)$ = 0

$\therefore \dfrac{x\frac{\mathrm{dy}}{\mathrm{dx}} - y.\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$ = 0

$\therefore x\dfrac{\mathrm{dy}}{\mathrm{dx}} - y \times 1$ = 0

$\therefore x.\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = y

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$.

Exercise 1.3 | Q 4.5 | Page 40

Show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$ in the following, where a and p are constants : $\cos^{-1}\left(\dfrac{7x^4+5y^4}{7x^4-5y^4}\right) = \tan^{-1} a$

SOLUTION

$\cos^{-1}\left(\dfrac{7x^4+5y^4}{7x^4-5y^4}\right) = \tan^{-1} a$

$\dfrac{7x^4+5y^4}{7x^4-5y^4}$ = cos($\tan^{-1}$ a) = b

$\dfrac{7x^4+5y^4}{7x^4-5y^4}$ = b

$7x^4 + 5y^4 = b(7x^4 – 5y^4)$

$7x^4 + 5y^4 = 7bx^4 – 5by^4$

$5y^4 + 5by^4 = 7bx^4 – 7x^4$

$5y^4(1 + b) = 7x^4(b – 1)$

$$\frac{5y^4}{7x^4} = \frac{b-1}{1+b}$$

$$\frac{y^4}{x^4} = \frac{7(b-1)}{5(1+b)} = x$$

$$\frac{y^4}{x^4} = c....(1)$$

$y^4 = cx^4$

Differentiating both sides w.r.t. x, we get

$$4.\,y^3\frac{\mathrm{dy}}{\mathrm{dx}} = c.4x^3$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{c.4x^3}{4y^3}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{c.\,x^3}{y^3}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y^4}{x^4}.\,\frac{x^3}{y^3} \quad \text{...from..(1)}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.6 | Page 40

Show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$ in the following, where a and p are constants : $\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right)$ = 20

SOLUTION

$\log\left(\dfrac{x^{20}-y^{20}}{x^{20}+y^{20}}\right)$ = 20

$\therefore \dfrac{x^{20}-y^{20}}{x^{20}+y^{20}}$ = e^{20} = k ...(Say)

$\therefore$ $x^{20} - y^{20}$ = kx^{20} + ky^{20}

$\therefore$ (1 + k)y^{20} = kx^{20} + ky^{20}

$\therefore \dfrac{y^{20}}{x^{20}} = \dfrac{1-k}{1+k}$

$\therefore \dfrac{y}{x} = \left(\dfrac{1-k}{1+k}\right)^{\frac{1}{20}}$, a constant

Differentiating both sides w.r.t. x, we get

$\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{y}{x}\right)$ = 0

$\therefore \dfrac{x\frac{\mathrm{dy}}{\mathrm{dx}} - y.\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$ = 0

$\therefore x\dfrac{\mathrm{dy}}{\mathrm{dx}} - y \times 1$ = 0

$\therefore x\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = y

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$.

Exercise 1.3 | Q 4.7 | Page 40

Show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$ in the following, where a and p are constants : $e^{\frac{x^7-y^7}{x^7+y^7}}$ = a

SOLUTION

$e^{\frac{x^7-y^7}{x^7+y^7}}$ = a

$\therefore \dfrac{x^7-y^7}{x^7+y^7}$ = log a = k ...(Say)

$\therefore$ $x^7 - y^7 = kx^7 + ky^7$

$\therefore$ $(1 + k)y^7 = (1 - k)x^7$

$$\therefore \frac{y^7}{x^7} = \frac{1-k}{1+k}$$

$\therefore \dfrac{y}{x} = \left(\dfrac{1-k}{1+k}\right)^{\frac{1}{7}}$, a constant

Differentiating both sides w.r.t. x, we get

$\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{y}{x}\right)$ = 0

$\therefore \dfrac{x\frac{\mathrm{dy}}{\mathrm{dx}} - y.\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$ = 0

$\therefore x\dfrac{\mathrm{dy}}{\mathrm{dx}} - y \times 1$ = 0

$\therefore x\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = y

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.8 | Page 40

Show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$ in the following, where a and p are constants : $\sin\left(\dfrac{x^3-y^3}{x^3+y^3}\right)$ = a^3

SOLUTION

$\sin\left(\frac{x^3-y^3}{x^3+y^3}\right)$ = a^3

$\frac{x^3-y^3}{x^3+y^3}$ = $\sin a^3$ = b

$\frac{x^3-y^3}{x^3+y^3}$ = b

$x^3 - y^3 = b(x^3 + y^3)$

$x^3 - y^3 = bx^3 + by^3$

$x^3 - bx^3 = by^3 + y^3$

$x^3(1 - b) = y^3(b + 1)$

$\frac{y^3}{x^3} = \frac{1-b}{1+b}$ = e

$\frac{y^3}{x^3}$ = c .....(1)

$y^3 = cx^3$

Differentiating both sides w.r.t. x, we get

$3y^2\frac{\mathrm{dy}}{\mathrm{dx}}$ = $c.3x^2$

$\frac{y^2 dy}{dx}$ = cx^2

$\frac{\mathrm{dy}}{\mathrm{dx}}c\frac{x^2}{y^2}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y^3}{x^3}\cdot\frac{x^2}{y^2}$ ....from(1)

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}.$

Exercise 1.3 | Q 5.01 | Page 40

If log (x + y) = log(xy) + p, where p is a constant, then prove that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-y^2}{x^2}$.

SOLUTION

log (x + y) = log(xy) + p

∴ log(x + y) = logx + logy + p

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x+y} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(x+y) = \frac{1}{x} + \frac{1}{y} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} + 0$$

$$\therefore \frac{1}{x+y}\left(1 + \frac{\mathrm{dy}}{\mathrm{dx}}\right) = \frac{1}{x} + \frac{1}{y} \cdot \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \frac{1}{x+y} + \frac{1}{x+y} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x} + \frac{1}{y} \cdot \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \left(\frac{1}{x+y} - \frac{1}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x} - \frac{1}{x+y}$$

$$\therefore \left[\frac{y-x-y}{y(x+y)}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x+y-x}{x(x+y)}$$

$$\therefore \left[\frac{-x}{y(x+y)}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x(x+y)}$$

$$\therefore \left(-\frac{x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{y^2}{x^2}.$$

Exercise 1.3 | Q 5.02 | Page 40

If $\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2, \text{show that} \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{99x^2}{101y^2}$

SOLUTION

$\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right)$ = 2

$\therefore \frac{x^3 - y^3}{x^3 + y^3}$ = 102 = 100

∴ $x^3 - y^3 = 110x^3 + 100y^3$

∴ $101y^3 = -99x^3$

∴ $y^3 = \frac{-99}{101}x^3$

Differentiating both sides w.r.t. x, we get

$$3y^2\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-99}{101} \times 3x^2$$

$$\therefore \mathrm{dy?dx} = -\frac{99x^2}{101y^2}.$$

Exercise 1.3 | Q 5.03 | Page 40

If $\log_5\left(\frac{x^4 + y^4}{x^4 - y^4}\right) = 2, \text{show that} \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{12x^3}{13y^3}.$

SOLUTION

$\log_5\left(\frac{x^4 + y^4}{x^4 - y^4}\right)$ = 2

$\frac{x^4 + y^4}{x^4 - y^4} = 5^2$

$\frac{x^4 + y^4}{x^4 - y^4}$ = 25

$x^4 + y^4 = 25x^4 - 25y^4$

$26y^4 = 24x^4$

Differentiating both sides w.r.t. x, we get

$$26\frac{\mathrm{d}}{\mathrm{dx}}y^4 = 24\frac{\mathrm{d}}{\mathrm{dx}}x^4$$

$$26.4y^3.\frac{\mathrm{dy}}{\mathrm{dx}} = 24.4.\,x^3$$

$$26y^3.\frac{\mathrm{dy}}{\mathrm{dx}} = 24.\,x^3$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{24x^3}{26y^3}$$

$$\frac{\mathrm{dx}}{\mathrm{dx}} = \frac{12x^3}{13y^3}$$

Exercise 1.3 | Q 5.04 | Page 40

If $e^x + e^y = e^{x+y}$, then show that $\frac{\mathrm{dy}}{\mathrm{dx}} = -e^{y-x}$.

SOLUTION

$e^x + e^y = e^{x+y}$ ...(1)

Differentiating both sides w.r.t. x, we get

$$e^x + e^y.\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x+y}.\frac{\mathrm{d}}{\mathrm{dx}}(x+y)$$

$$\therefore e^x + e^y.\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x+y}.\left(1 + \frac{\mathrm{dy}}{\mathrm{dx}}\right)$$

$$\therefore e^x + e^y\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x+y} + e^{x+y}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \left(e^y - e^{x+y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x+y} - e^x$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

$= \dfrac{e^x + e^y - e^x}{e^y - e^x - e^y}$ ...[By (1)]

$= \dfrac{e^y}{-e^x}$

$= - e^{y-x}$.

Exercise 1.3 | Q 5.05 | Page 40

If $\sin^{-1}\left(\dfrac{x^5 - y^5}{x^5 + y^5}\right) = \dfrac{\pi}{6}$, show that $\dfrac{\text{dy}}{\text{dx}} = \dfrac{x^4}{3y^4}$

SOLUTION

$\sin^{-1}\left(\dfrac{x^5 - y^5}{x^5 + y^5}\right) = \dfrac{\pi}{6}$

$\therefore \dfrac{x^5 - y^5}{x^5 + y^5} = \sin\ \dfrac{\pi}{6} = \dfrac{1}{2}$

$\therefore 2x^5 - 2y^5 = x^5 + y^5$

$\therefore 3y^5 = x^5$

Differentiating both sides w.r.t. x, we get

$3 \times 5y^4 \dfrac{\text{dy}}{\text{dx}} = 5x^4$

$\therefore \dfrac{\text{dy}}{\text{dx}} = \dfrac{x^4}{3y^4}$.

Exercise 1.3 | Q 5.06 | Page 40

If $x^y = e^{x-y}$, then show that $\dfrac{\text{dy}}{\text{dx}} = \dfrac{\log x}{(1 + \log x)^2}$.

SOLUTION

$x^y = e^{x-y}$

$\therefore$ log xy = log e^{x-y}

$\therefore$ y logx = (x - y)log e

$\therefore$ y logx = x - y ...[$\because$ log e = 1]

$\therefore$ y + ylogx = x $\therefore$ y(1 + logx) = x

$\therefore$ y = $\dfrac{x}{1+\log x}$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x}{1+\log x}\right)$$

$$= \frac{(1+\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x) - x\frac{\mathrm{d}}{\mathrm{dx}}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x).1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}.$$

Exercise 1.3 | Q 5.07 | Page 40

If y = $\sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$, then show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\sin x}{1-2y}$.

SOLUTION

y = $\sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$

$\therefore$ y^2 = $\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}$

$\therefore$ y^2 = cos x + y

Differentiating both sides w.r.t. x, we get

$$2y\frac{\mathrm{dy}}{\mathrm{dx}} = -\sin x + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$\therefore (1-2y)\frac{\mathrm{dy}}{\mathrm{dx}}$ = sinx

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\sin x}{1-2y}.$$

Exercise 1.3 | Q 5.08 | Page 40

If y = $\sqrt{\log x + \log x + \sqrt{\log x + ...\infty}}$, show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x(2y-1)}$.

SOLUTION

y = $\sqrt{\log x + \log x + \sqrt{\log x + ...\infty}}$

$\therefore$ y^2 = $\log x + \sqrt{\log x + \sqrt{\log x + ...\infty}}$

$\therefore$ y^2 = log x + y

Differentiating both sides w.r.t. x, we get

$$2y.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x} + \frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore (2y-1)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x(2y-1)}.$$

Exercise 1.3 | Q 5.09 | Page 40

If y = $x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$, show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y^2}{x(1-\log y).}$.

SOLUTION

y = $x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$

$\therefore$ log y = $\log\left(x^{x^{x^{\cdot^{\cdot^{\infty}}}}}\right)$

= $\log x^{x^{x^{\cdot^{\cdot^{\infty}}}}} . \log x$

$\therefore$ log y = y log x ...(1)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} = y.\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x)\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \frac{1}{t}\frac{\mathrm{dy}}{\mathrm{dx}} = y \times \frac{1}{x} + (\log x)\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore \left(\frac{1}{y} - \log x\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$$

$$\therefore \left(\frac{1 - y\log x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y^2}{x(1 - y\log x)}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y^2}{x(1 - y\log)}. \qquad \text{...[By (1)]}$$

Exercise 1.3 | Q 5.1 | Page 40

If $e^y = y^x$, then show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{(\log y)^2}{\log y - 1}$.

SOLUTION

$e^y = y^x$

$\therefore \log e^y = \log y^x$

$\therefore y \log e = x \log y$

$\therefore y = x \log y$ ...[$\because \log e = 1$] ...(1)

Differentiating both sides w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = x\frac{\mathrm{d}}{\mathrm{dx}}(\log y) + (\log y).\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = x \times \frac{1}{y}.\frac{\mathrm{dy}}{\mathrm{dx}} + (\log y) \times 1$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x}{y}\frac{\mathrm{dy}}{\mathrm{dx}} + \log y$$

$$\therefore \left(1 - \frac{x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \log y$$

$$\therefore \left(\frac{y - x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \log y$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y \log y}{y - x}$$

$$= \frac{y \log y}{y - \left(\frac{y}{\log y}\right)} \quad \text{...[By (1)]}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{(\log y)^2}{\log y - 1}.$$

Alternative Method :

ey = yx

$\therefore$ log ey = log yx

$\therefore y \log e = x \log y$

$\therefore y = x \log y$ ...[$\because \log e = 1$]

$$\therefore x = \frac{y}{\log y}$$

Differentiating both sides w.r.t. x, we get

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{\mathrm{d}}{\mathrm{dy}}\left(\frac{y}{\log y}\right)$$

$$= \frac{(\log y).\frac{\mathrm{d}}{\mathrm{dy}}(y) - y.\frac{\mathrm{d}}{\mathrm{dy}}(\log y)}{(\log y)^2}$$

$$= \frac{(\log y) \times 1 - y \times \frac{1}{y}}{(\log y)^2}$$

$$= \frac{\log y - 1}{(\log y)^2}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{(\log y)^2}{\log y - 1}.$$

EXERCISE 1.4 [PAGES 48 - 49]

Exercise 1.4 | Q 1.1 | Page 48

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if x = at^2, y = 2at

SOLUTION

x = at^2, y = 2at

Differentiating x and y w.r.t. x, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{at}^2\right) = a\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{t}^2\right)$$

= a x 2t = 2at

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(2\mathrm{at}) = 2a\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{t})$$

= 2a x 1 = 2a

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

Exercise 1.4 | Q 1.2 | Page 48

Find $\frac{dy}{dx}$ if x = a cot θ, y = b cosec θ

SOLUTION

a cot θ, y = b cosec θ

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cot\theta) = a(-\operatorname{cosec}^2\theta)$$

$$= -\operatorname{cosec}^2\theta$$

and

$$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\operatorname{cosec}\theta) = b(-\operatorname{cosec}\theta\cot\theta)$$

$$= -b\operatorname{cosec}\theta\cot\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\operatorname{cosec}\theta}{-a\operatorname{cosec}^2\theta}$$

$$= \frac{b}{a}\cdot\frac{\cot\theta}{\operatorname{cosec}\theta}$$

$$= \frac{b}{a}\times\frac{\cos\theta}{\sin\theta}\times\sin\theta$$

$$= \left(\frac{b}{a}\right)\cos\theta.$$

Exercise 1.4 | Q 1.3 | Page 48

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$, if : x = $\sqrt{a^2+m^2}, y=\log\left(a^2+m^2\right)$

SOLUTION

x = $\sqrt{a^2+m^2}, y=\log\left(a^2+m^2\right)$

Differentiating x and y w.r.t. x, we get

$\frac{\mathrm{dx}}{\mathrm{dm}}=\frac{\mathrm{d}}{\mathrm{dm}}\left(\sqrt{a^2+m^2}\right)$

= $\frac{1}{2\sqrt{a^2+m^2}}\cdot\frac{\mathrm{d}}{\mathrm{dm}}\left(a^2+m^2\right)$

= $\frac{1}{2\sqrt{a^2+m^2}}\times(0+2m)=\frac{m}{\sqrt{a^2+m^2}}$

and

$\frac{\mathrm{dy}}{\mathrm{dm}}=\frac{\mathrm{d}}{\mathrm{dm}}\left[\log\left(a^2+m^2\right)\right]$

= $\frac{1}{a^2+m^2}\cdot\frac{\mathrm{d}}{\mathrm{dm}}\left(a^2+m^2\right)$

= $\frac{1}{a^2+m^2}\times(0+2m)=\frac{2m}{a^2+m^2}$

$\therefore\ \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\left(\frac{\mathrm{dy}}{\mathrm{dm}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dm}}\right)}$

= $\frac{\left(\frac{2m}{a^2+m^2}\right)}{\left(\frac{m}{\sqrt{a^2+m^2}}\right)}$

= $\frac{2}{\sqrt{a^2+m^2}}$.

Exercise 1.4 | Q 1.4 | Page 48

Find $\dfrac{dy}{dx}$, if : x = sinθ, y = tanθ

SOLUTION

x = sinθ, y = tanθ

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \sec^2\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{\sec^2\theta}{\cos\theta}$$

$= \sec^3\theta$.

Exercise 1.4 | Q 1.5 | Page 48

Find $\dfrac{dy}{dx}$, if : x = a(1 − cosθ), y = b(θ − sinθ)

SOLUTION

x = a(1 – cosθ), y = b(θ – sinθ)

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(1 - \cos\theta)$$

= a[0 – (– sin θ)] = a sin θ

and

$$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\theta - \sin\theta)$$

= b(1 – cos θ)

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b(1 - \cos\theta)}{a\sin\theta}$$

$$= \frac{b \times \sin^2\left(\frac{\theta}{2}\right)}{a \times 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \left(\frac{b}{a}\right)\tan\left(\frac{\theta}{2}\right).$$

Exercise 1.4 | Q 1.6 | Page 48

Find $\frac{dy}{dx}$, if : x = $\left(t + \frac{1}{t}\right), y = a\left(t + \frac{1}{t}\right)$, where a > 0, a ≠ 1, t ≠ 0.

SOLUTION

x = $\left(t+\dfrac{1}{t}\right), y = a\left(t+\dfrac{1}{t}\right)$ ...(1)

Differentiating x and y w.r.t. x, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(t+\frac{1}{t}\right)^{a}$$

$$= a\left(t+\frac{1}{t}\right)^{a-1}.\frac{\mathrm{d}}{\mathrm{dt}}\left(t+\frac{1}{t}\right)$$

$$= a\left(t+\frac{1}{t}\right)^{a-1}.\left(1-\frac{1}{t^2}\right)$$

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left[a^{\left(t+\frac{1}{t}\right)}\right]$$

$$= a^{\left(t+\frac{1}{t}\right)}.\log a.\frac{\mathrm{d}}{\mathrm{dt}}\left(t+\frac{1}{t}\right)$$

$$= a^{\left(t+\frac{1}{t}\right)}.\log a.\left(1-\frac{1}{t^2}\right)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{a^{\left(t+\frac{1}{t}\right)}.\log a.\left(1-\frac{1}{t^2}\right)}{a\left(t+\frac{1}{t}\right)^{a-1}.\left(1-\frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}}.\log a.\left(t+\frac{1}{t}\right)}{a.\left(t+\frac{1}{t}\right)^{a}}$$

$= \dfrac{y\log a.\left(\frac{t^2+1}{t}\right)}{\mathrm{ax}}$ ...[By (1)]

$= \dfrac{y(t^2+1)\log a}{\mathrm{axt}}.$

Exercise 1.4 | Q 1.7 | Page 48

Find $\dfrac{\mathrm{dy}}{\mathrm{dx}}$, if : $x = \cos^{-1}\left(\dfrac{2t}{1+t^2}\right), y = \sec^{-1}\left(\sqrt{1+t^2}\right)$

SOLUTION

$x = \cos^{-1}\left(\dfrac{2t}{1+t^2}\right), y = \sec^{-1}\left(\sqrt{1+t^2}\right)$

Put t = tanθ.

Then θ = $\tan^{-1}$t.

$\therefore$ x = $\cos^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right), y = \sec^{-1}\left(\sqrt{1+\tan^2\theta}\right)$

$\therefore$ x = $\cos^{-1}(\sin 2\theta), y = \sec^{-1}\left(\sqrt{\sec^2\theta}\right)$

$\therefore$ x = $\cos^{-1}\left[\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right], y = \sec^{-1}(\sec\theta)$

$\therefore$ x = $\dfrac{\pi}{2} - 2\theta, y = \theta$

$\therefore$ x = $\dfrac{\pi}{2} - 2\tan^{-1}t, y = \tan^{-1}t$

Differentiating x and y w.r.t. x, we get

$\dfrac{\mathrm{dx}}{\mathrm{dt}} = \dfrac{\mathrm{d}}{\mathrm{dt}}\left(\dfrac{\pi}{2}\right) - 2\dfrac{\mathrm{d}}{\mathrm{dt}}(\tan^{-1}t)$

$= 0 - 2 \times \dfrac{1}{1+t^2}$

$= \dfrac{-2}{1+t^2}$

and

$$\frac{dy}{dt} = \frac{d}{dt}\left(\tan^{-1} t\right)$$

$$= \frac{1}{1+t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\left(\frac{1}{1+t^2}\right)}{\left(\frac{-2}{1+t^2}\right)}$$

$$= -\frac{1}{2}.$$

Exercise 1.4 | Q 1.8 | Page 48

Find $\frac{dy}{dx}$, if : $x = \cos^{-1}\left(4t^3 - 3t\right), y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$.

SOLUTION

$$x = \cos^{-1}\left(4t^3 - 3t\right), y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$$

Put $t = \cos\theta$.

Then $\theta = \cos^{-1}t$.

$\therefore x = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$,

$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$

$\therefore x = \cos^{-1}(\cos 3\theta), y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$

$\therefore x = 3\theta$ and $y = \theta$

$\therefore$ x = 3$\cos^{-1}$t and y = $\cos^{-1}$t

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{dt} = 3\frac{d}{dt}(\cos^{-1})$$

= $3 \times \frac{-1}{\sqrt{1-t^2}}$

= $\frac{-3}{\sqrt{1-t^2}}$

and

$$\frac{dy}{dt} = \frac{d}{dt}(\cos^{-1} t)$$

= $\frac{-1}{\sqrt{1-t^2}}$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

= $\frac{\left(\frac{-1}{\sqrt{1-t^2}}\right)}{\left(\frac{-3}{\sqrt{1-t^2}}\right)}$

= $\frac{1}{3}$.

Alternative Method :

x = $\cos^{-1}$ (4t^3 – 3t), t = $\tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Put t = cosθ.

Then x = $\cos^{-1}$(4cos^{3}θ – 3cosθ), y = $\tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$

∴ x = $\cos^{-1}$ (cos3θ), y = $\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$

∴ x = 3θ, y = θ

∴ x = 3y

∴ y = $\frac{1}{3}x$

∴ $\frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}(x)$

= $\frac{1}{3}\times 1$

= $\frac{1}{3}$.

Exercise 1.4 | Q 2.1 | Page 48

Find $\frac{dy}{dx}$ if : x = cosec²θ, y = cot³θ at θ= $\frac{\pi}{6}$

SOLUTION

x = $\text{cosec}^2\theta$, y = $\cot^3\theta$

Differentiating x and y w.r.t. θ, we get

$\frac{dx}{d\theta} = \frac{d}{d\theta}(\text{cosec}\theta)^2 = 2\text{cosec}\theta.\ \frac{d}{d\theta}(\text{cosec}\theta)$

= 2cosecθ(– cosecθ cotθ)

= – 2cosec²θ cotθ

and

$\frac{dy}{d\theta} = \frac{d}{d\theta}(\cot\theta)^3 = 3\cot^2\theta.\ \frac{d}{d\theta}(\cot\theta)$

= 3cot²θ.(–cosec²θ)

= –3cot²θ.cosec²θ

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-3\cot^2\theta.\,\text{cosc}^2\theta}{-2\text{cosec}^2\theta.\cot\theta}$$

$= \frac{3}{2}\cot\theta$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{6}}$

$= \frac{3}{2}\cot\ \frac{\pi}{6}$

$= \frac{3\sqrt{3}}{2}$.

Exercise 1.4 | Q 2.2 | Page 48

Find $\frac{dy}{dx}$ if : x = a $\cos^3\theta$, y = a $\sin^3\theta$ at θ = $\frac{\pi}{3}$

SOLUTION

a $\cos^3\theta$, y = a $\sin^3\theta$

Differentiating x and y w.r.t. θ, we get

$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta)^3$

$= a \times 3\cos^2\theta.\ \frac{d}{d\theta}(\cos\theta)$

= 3a $\cos^2\theta(-\sin\theta)$

= − 3a $\cos^2\theta\sin\theta$

and

$\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta)^3$

$= a \times 3\sin^2\theta . \frac{d}{d\theta}(\sin\theta)$

$= 3a \sin^2\theta \cos\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$= \frac{3a\sin^2\cos\theta}{-3a\cos^2\theta\sin\theta}$

$= -\tan\theta$

$\therefore \left(\frac{dx}{dy}\right)_{\text{at } \theta - \frac{\pi}{3}}$

$= -\tan\ \frac{\pi}{3}$

$= -\sqrt{3}.$

Exercise 1.4 | Q 2.3 | Page 48

Find $\frac{dy}{dx}$ if : $x = t^2 + t + 1$, $y = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$ at $t = 1$

SOLUTION

x = $t^2 + t + 1$, y = $\sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$

Differentiating x and y w.r.t. t, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(t^2 + t + 1)$$

= 2t + 1 + 0 = 2t + 1

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left[\sin\left(\pi\frac{t}{2}\right)\right] + \frac{\mathrm{d}}{\mathrm{dt}}\left[\cos\left(\frac{\pi}{t}\right)\right]$$

$$= \cos\left(\frac{\pi t}{2}\right).\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\pi t}{2}\right) + \left[-\sin\left(\frac{\pi t}{2}\right)\right].\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\pi t}{2}\right)$$

$$= \cos\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1 - \sin\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1$$

$$= \frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right)\right]$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{\frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right)\right]}{2t + 1}$$

$$\therefore \left(\frac{dx}{dy}\right)_{\text{at } t=1}$$

$$= \frac{\frac{\pi}{2}\left[\cos\ \frac{\pi}{2} - \sin\ \frac{\pi}{2}\right]}{2(1) + 1}$$

$$= \frac{\frac{\pi}{2}(0 - 1)}{3}$$

$$= -\frac{\pi}{6}.$$

Exercise 1.4 | Q 2.4 | Page 48

Find $\frac{\text{dy}}{\text{dx}}$ if : x = 2 cot t + cos 2t, y = 2 sin t – sin 2t at t = $\frac{\pi}{4}$

SOLUTION

2 cot t + cos 2t, y = 2 sin t – sin 2t

Differentiating x and y w.r.t. t, we get

$$\frac{\text{dx}}{\text{dt}} = \frac{\text{d}}{\text{dt}}(2\cos t + \cos 2t)$$

$$= 2\frac{\text{d}}{\text{dt}}(\cos t) + \frac{\text{d}}{\text{dt}}(\cos 2t)$$

$$= 2(-\sin t) + (-\sin 2t).\frac{\text{d}}{\text{dt}}(2t)$$

= 2 sin t – sin 2t x 2 x 1

= – 2 sin t – 2 sin 2t

and

$$\frac{\text{dy}}{\text{dx}} = \frac{\text{d}}{\text{dt}}(2\sin t - \sin 2t)$$

$$= 2\frac{\text{d}}{\text{dt}}(\sin t) - \frac{\text{d}}{\text{dt}}(\sin 2t)$$

$$= 2\cos t - \cos 2t.\frac{\text{d}}{\text{dt}}(2t)$$

= 2 cos t – cos 2t x 2 x 1

= 2 cos t – 2 cos 2t

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{2\cos t - 2\cos 2t}{-2\sin t - 2\sin 2t}$$

$$= \frac{\cos t - \cos 2t}{-\sin t - \sin 2t}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}}$$

$$= \frac{\cos\ \frac{\pi}{4} - \cos\ \frac{\pi}{2}}{-\sin\ \frac{\pi}{4} - \sin\ \frac{\pi}{2}}$$

$$= \frac{\frac{1}{\sqrt{2}} - 0}{-\frac{1}{\sqrt{2}} - 1}$$

$$= \frac{-1}{1+\sqrt{2}}$$

$$= \frac{-1}{1+\sqrt{2}} + \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$-\left(1-\sqrt{2}\right)$$

$$= 1 - \sqrt{2}.$$

Exercise 1.4 | Q 2.5 | Page 48

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if : x = t + 2sin (πt), y = 3t − cos (πt) at t = $\frac{1}{2}$

SOLUTION

x = t + 2sin (πt), y = 3t − cos (πt)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}[t + 2\sin(\pi t)]$$

$$= \frac{d}{dt}(t) + 3.\frac{d}{dt}[\sin(\pi t)]$$

$$= 1 + 2 \times \cos(\pi t).\frac{d}{dx}(\pi t)$$

= 1 + 2cos(πt) x π x 1

= 1 + 2π cos (πt)

and

$$\frac{dy}{dt} = \frac{d}{dt}[3t - \cos(\pi t)]$$

$$= 3 \times 1 - [-\sin(\pi t)].\frac{d}{dt}(\pi t)$$

= 3 + sin (πt) x π x 1

= 3 + π sin (πt)

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{3 + \pi\sin(\pi t)}{1 + 2\pi\cos(\pi t)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{1}{2}}$$

$$= \frac{3 + \sin\left(\frac{\pi}{2}\right)}{1 + 2\pi\cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{3 + \pi \times 1}{1 + 2\pi(0)}$$

= 3 + π.

Exercise 1.4 | Q 3.1 | Page 48

If x = $a\sqrt{\sec\theta - \tan\theta}, y = a\sqrt{\sec\theta + \tan\theta}$, show that $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$.

SOLUTION

x = $a\sqrt{\sec\theta - \tan\theta}, y = a\sqrt{\sec\theta + \tan\theta}$

$\therefore \frac{x}{a} = \sqrt{\sec\theta - \tan\theta}, y = a\sqrt{\sec\theta + \tan\theta}$

$\therefore \sec\theta - \tan\theta = \frac{x^2}{a^2}$ ...(1)

$\sec\theta + \tan\theta = \frac{y^2}{a^2}$ ...(2)

Adding (1) and (2), we get

2secθ = $\frac{x^2}{a^2} + \frac{y^2}{a^2}$

= $\frac{x^2 + y^2}{a^2}$

∴ secθ = $\frac{x^2 + y^2}{2a^2}$

Subtracting (1) from (2), we get

2tanθ = $\frac{y^2}{a^2} - \frac{x^2}{a^2}$

= $\frac{y^2 - x^2}{a^2}$

$\therefore$ tanθ $= \dfrac{y^2 - x^2}{2a^2}$

$\therefore$ sec^{2}θ - tan^{2}θ = 1 gives,

$\left(\dfrac{x^2+y^2}{a^2}\right)^2 - \left(\dfrac{y^2-x^2}{2a^2}\right)^2$ = 1

$\therefore$ $(x^2 + y^2)^2 - (y^2 - x^2)^2 = 4a^2$

$\therefore$ $(4 + 2x^2y^2 + y^4) - (y^4 - 2x^2y^2 + x^4) = 4a^4$

$\therefore$ $4x^2y^2 = 4a^4$

$\therefore$ $x^2y^2 = a^4$

Differentiating both sides w.r.t. x, we get

$x^2 . \dfrac{\mathrm{d}}{\mathrm{dx}}(y^2) + y^2 \dfrac{\mathrm{d}}{\mathrm{dx}}(x^3)$ = 0

$\therefore$ $x^2 \times 2y\dfrac{\mathrm{dy}}{\mathrm{dx}} + y62 \times 2x$ = 0

$\therefore$ $2x^2y\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = $-2xy^2$

$\therefore$ $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y}{x}$.

Exercise 1.4 | Q 3.2 | Page 48

If $x = e^{\sin 3t}$, $y = e^{\cos 3t}$, then show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = -\dfrac{y \log x}{x \log y}$.

SOLUTION

$x = e^{\sin 3t}$, $y = e^{\cos 3t}$

$\therefore$ $\log x = \log e^{\sin 3t}$, $\log y = \log e^{\cos 3t}$

$\therefore$ log x = (sin 3t)(log e), log y = (cos 3t)(log e)

$\therefore$ log x = sin 3t, log y = cos 3t ...(1) ... [$\because$ log e = 1]

Differentiating both sides w.r.t. t, we get

$$\frac{1}{x} . \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(\sin 3t) = \cos 3t . \frac{\mathrm{d}}{\mathrm{dt}}(3t)$$

= cos 3t x 3 = 3 cos 3t

and

$$\frac{1}{y} . \frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(\cos 3t) = -\sin 3t . \frac{\mathrm{d}}{\mathrm{dx}}(3t)$$

= – sin 3t x 3 = 3 cos 3t

$$\therefore \frac{\mathrm{dx}}{\mathrm{dt}} = 3x \cos 3t \text{ and } \frac{\mathrm{dy}}{\mathrm{dt}} = -3y \sin 3t$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{-3y \sin 3t}{3x \cos 3t}$$

$$= \frac{-ysi3t}{x \cos 3t}$$

$$= \frac{-y \log x}{x \log y}. \qquad \text{...[By (1)]}$$

Exercise 1.4 | Q 3.3 | Page 48

If x = $\frac{t+1}{t-1}, y = \frac{1-t}{t+1}$, then show that $y^2 - \frac{\mathrm{dy}}{\mathrm{dx}}$ = 0.

SOLUTION

x = $\frac{t+1}{t-1}, y = \frac{1-t}{t+1}$

∴ y = $\frac{1}{\left(\frac{t+1}{1-t}\right)} = \frac{-1}{\left(\frac{t+1}{t-1}\right)}$

∴ y = $-\frac{1}{x}$

∴ xy = –1 ...(1)

Differentiating both sides w.r.t. t, we get

$x\frac{\mathrm{dy}}{\mathrm{dx}} + . \frac{\mathrm{d}}{\mathrm{dx}}(x)$ = 0

$\therefore x\frac{\mathrm{dy}}{\mathrm{dx}} + y \times 1$ = 0

$\therefore -\frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} + y$ = 0 ...[By (1)]

$\therefore -\frac{\mathrm{dy}}{\mathrm{dx}} + y^2$ = 0

$\therefore y^2 - \frac{\mathrm{dy}}{\mathrm{dx}}$ = 0.

Exercise 1.4 | Q 3.4 | Page 48

If x = a $\cos^3$t, y = a $\sin^3$t, show that $\frac{\mathrm{dy}}{\mathrm{dx}} = -\left(\frac{y}{x}\right)^3$.

SOLUTION

x = a $\cos^3$t, y = a $\sin^3$t

Differentiating x and y w.r.t. t, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = a\frac{\mathrm{d}}{\mathrm{dt}}(\cos t)^3 = a.3(\cos t)^2\frac{\mathrm{d}}{\mathrm{dt}}(\cos t)$$

= 3a$\cos^2$t(– sint) = –3a $\cos^2$t sint

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = a\frac{\mathrm{d}}{\mathrm{dt}}(\sin t)^3$$

= $a.3(\sin t)^2\frac{\mathrm{d}}{\mathrm{dt}}(\sin t)$

= 3a $\sin^2$t. cost

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{\mathrm{dt}}\right)}$$

= $\dfrac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}$

= $-\dfrac{\text{sint}}{\text{cost}}$ ...(1)

Now, x = a $\cos^3$t

∴ $\cos^3$t = $\dfrac{x}{a}$

∴ cos t = $\left(\dfrac{x}{a}\right)^{\frac{1}{3}}$

y = a$\sin^3$t

∴ $\sin^3$t = $\dfrac{y}{a}$

∴ $\cos^3$t = $\left(\dfrac{y}{a}\right)^{\frac{1}{3}}$

∴ from (1), $\dfrac{\text{dy}}{\text{dx}} = -\dfrac{\frac{y^{\frac{1}{3}}}{a^{\frac{1}{3}}}}{\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}}$

= $-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$.

Alternative Method :

x = a $\cos^3$t, y = a $\sin^3$t

$\therefore \cos^3 t = \dfrac{x}{a}, \sin^3 t = \dfrac{y}{a}$

$\therefore \cos t = \left(\frac{x}{a}\right)^{\frac{1}{3}}, \sin t = \left(\frac{y}{a}\right)^{\frac{1}{3}}$

$\therefore$ $\cos^2 t + \sin^2 t = 1$ gives

$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}}$ = 1

$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides w.r.t. t, we get

$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}, \frac{\mathrm{dy}}{\mathrm{dx}}$ = 0

$\therefore \frac{2}{3}y^{\frac{-1}{3}}\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{2}{3}x^{\frac{-1}{3}}$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = -\left(\frac{x}{y}\right)^{-\frac{1}{2}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}.$

Exercise 1.4 | Q 3.5 | Page 48

If x = $2\cos^4$(t + 3), y = $3\sin^4$(t + 3), show that $\frac{\mathrm{dy}}{\mathrm{dx}} = -\sqrt{\frac{3y}{2x}}$.

SOLUTION

x = $2\cos^4$(t + 3), y = $3\sin^4$(t + 3)

$\therefore \cos^4(t+3) = \frac{x}{2}, \sin^4(t+3) = \frac{y}{3}$

$\therefore \cos^2(t+3) = \sqrt{\frac{x}{2}}, \sin^2(t+3) = \sqrt{\frac{y}{3}}$

$\because$ $\cos^2$(t + 3) + $\sin^2$(t + 3) = 1

$\therefore \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}}$ = 1

Differentiating x and y w.r.t. t, we get

$$\frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x}\right)+\frac{1}{\sqrt{3}}\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{y}\right) = 0$$

$$\therefore \frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{3}}\times\frac{1}{2\sqrt{y}}.\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore \frac{1}{2\sqrt{3}.\sqrt{y}}.\frac{\mathrm{dy}}{\mathrm{dx}}=-\frac{1}{2\sqrt{2}.\sqrt{x}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=-\frac{\sqrt{3}.\sqrt{y}}{\sqrt{2}.\sqrt{x}}$$

$$= -\sqrt{\frac{3y}{2x}}.$$

Exercise 1.4 | Q 3.6 | Page 48

If x = log(1 + t^2), y = t – $\tan^{-1}$t,show that $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sqrt{e^x-1}}{2}$.

SOLUTION

x = log(1 + t^2), y = t – $\tan^{-1}$t

Differentiating x and y w.r.t. t, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left[\log\left(1+t^2\right)\right]$$

$$= \frac{1}{1+t^2}.\frac{\mathrm{d}}{\mathrm{dt}}\left(1-t^2\right)$$

$$= \frac{1}{1+t^2}\times(0+2t)$$

$$= \frac{2t}{1+t^2}$$

and

$$\frac{\mathrm{dy}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}(t)-\frac{\mathrm{d}}{\mathrm{dt}}\left(\tan^{-1}t\right)$$

$= 1 - \frac{1}{1+t^2}$

$= \frac{1+t^2-1}{1+t^2}$

$= \frac{t^2}{1+t2}$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$

$= \frac{\left(\frac{t2}{1+t^2}\right)}{\left(\frac{2t}{1+t^2}\right)}$

$= \frac{t}{2}$

Now, x = log (1 + t^2)

$\therefore$ 1 + t^2 = e^x

$\therefore$ t^2 = e^x - 1

$\therefore$ t = $\sqrt{e^x - 1}$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\sqrt{e^x-1}}{2}.$

Exercise 1.4 | Q 3.7 | Page 48

If x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}, \text{show that} \sin x + \frac{\mathrm{dy}}{dx}$ = 0

SOLUTION

x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left[\sin^{-1}\left(e^{1}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(e^{t}\right)^{2}}} \cdot \frac{d}{dt}\left(e^{t}\right)$$

$$= \frac{1}{\sqrt{1-e^{2t}}} \times e^{2} = \frac{e^{2}}{\sqrt{1-e^{2t}}}$$

and

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{1-e^{2}}\right)$$

$$= \frac{1}{2\sqrt{1-e^{2t}}} \cdot \frac{d}{dt}\left(1-e^{2t}\right)$$

$$= \frac{1}{2\sqrt{1-e^{2t}}} \cdot \times \left(0-e^{2t} \times 2\right)$$

$$= \frac{-e^{2t}}{\sqrt{1-e^{2t}}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\left(\frac{-e^{2t}}{\sqrt{1-e^{2t}}}\right)}{\left(\frac{e^{t}}{\sqrt{1-e^{2t}}}\right)}$$

$= -e^t$

$= -\sin x$ ...[$\because x = \sin^{-1}(e^t)$]

$\therefore \sin x + \frac{dy}{dx} = 0.$

Exercise 1.4 | Q 3.8 | Page 48

If x = $\dfrac{2bt}{1+t^2}, y = a\left(\dfrac{1-t^2}{1+t^2}\right)$, show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = -\dfrac{b^2 y}{a^2 x}$.

SOLUTION

x = $\dfrac{2bt}{1+t^2}, y = a\left(\dfrac{1-t^2}{1+t^2}\right)$

Put t = tanθ.

Then x = $b\left(\dfrac{2\tan\theta}{1+\tan\theta}\right), y = a\left(\dfrac{1-t^2}{1+\tan^2\theta}\right)$

∴ x = b sin 2θ, y = a cos 2θ

∴ $\dfrac{x}{b} = \sin 2\theta, \dfrac{y}{a} = \cos 2\theta$

∴ $\left(\dfrac{x}{b}\right)^2 + \left(\dfrac{y}{a}\right)^2$ = $\sin^2 2\theta + \cos^2 2\theta$

∴ $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2}$ = 1

Differentiating x and y w.r.t. y, we get

$\dfrac{1}{b^2} \times 2x\dfrac{\mathrm{dx}}{\mathrm{dy}} + \dfrac{1}{a^2} \times 2y$ = 0

∴ $\dfrac{2x dx}{b^2 dy} = \dfrac{-2y}{a^2}$

∴ $\dfrac{\mathrm{dx}}{\mathrm{dy}} = -\dfrac{b^2 y}{a^2 x}$.

Exercise 1.4 | Q 4.1 | Page 49

DIfferentiate x sin x w.r.t. tan x.

SOLUTION

Let u = x sin x and v = tan x.

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$

Differentiating u and v w.r.t. x, we get

$$\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x \sin x)$$

$$= x\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) + (\sin x).\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

= x cos x + (sin x) x 1

= x cos x + sin x

and

$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(\tan x)$ = $\sec^2 x$

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)}$$

$$= \frac{x\cos x + \sin x}{\sec^2 x}.$$

Exercise 1.4 | Q 4.2 | Page 49

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right) w.r.t. \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

SOLUTION

Let u = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and

v = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$.

Put x = tanθ.

Then $\theta = \tan^{-1}x$.

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right)$$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$$\therefore \frac{du}{dx} = 2\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$= 2 \times \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

$$\text{Also, } v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$$\therefore \frac{dv}{dx} = 2\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$= 2 \times \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{2}{1+x^2}\right)}$$

= 1.

Alternative Method :

Let u = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then we want to find $\frac{du}{dv}$

Put x = tanθ.

Then u = $\sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right)$

= $\sin^{-1}$ (sin2θ)

= 2θ

and

v = $\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\cos^{-1}$ (cos2θ)

= 2θ

∴ u = v

Differentiating both sides w.r.t. v, we get

$\frac{du}{dv}$ = 1.

Exercise 1.4 | Q 4.3 | Page 49

Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) w.r.t. \sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

SOLUTION

Let u = $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and

v = $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

Then we want to find $\frac{du}{dv}$.

Put x = cosθ.

Thenθ = $\cos^{-1}$x.

$\therefore$ u = $\tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)$

= $\tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

= $\tan^{-1}(\cot\theta)$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$

= $\frac{\pi}{2}-\theta$

= $\frac{\pi}{2}-\cos^{-1}x$

$\therefore \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(\cos^{-1}x)$

= $0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{-x^2}}$

v = $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

= $\cos^{-1}(2x^2 - 1)$

= $\cos^{-1}(2\cos^2\theta - 1)$

= $\cos^{-1}(\cos 2\theta)$

= 2θ

= 2 $\cos^{-1}$x

$\therefore \frac{\mathrm{dv}}{\mathrm{dx}} = 2. \frac{\mathrm{d}}{\mathrm{dx}}(\cos^{-1} x)$

= $\frac{-2}{\sqrt{1-x^2}}$

$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)}$

= $\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-2}$

= $-\frac{1}{2}$.

Exercise 1.4 | Q 4.4 | Page 49

Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) w.r.t. \tan^{-1} x.$

SOLUTION

Let u = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $v = \tan^{-1} x.$

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$.

Put x = tanθ.

Then = $\tan^{-1}$ x.

∴ u = $\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

= $\cos^{-1}$(cos2θ)

= 2θ

$\therefore$ u = 2$\tan^{-1}$x

$$\therefore \frac{du}{dx} = 2. \frac{d}{dx}(\tan^{-1} x)$$

= $2 \times \dfrac{1}{1+x^2}$

= $\dfrac{2}{1+x^2}$

Also, v =$\tan^{-1}$x

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

= $\dfrac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)}$

= 2.

Exercise 1.4 | Q 4.5 | Page 49

Differentiate 3^x w.r.t. $\log_x 3$.

SOLUTION

Let u = 3^x and v = $\log_x 3$.

Then we want to find $\frac{du}{dv}$.

Differentiating u and v w.r.t. x, we get

$$\frac{du}{dv} = \frac{d}{dx}(3^x)$$

= $3^x.\log 3$

and

$$\frac{dv}{dx} = \frac{d}{dx}(\log_x 3)$$

$$= \frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$$

$$= \log 3.\frac{d}{dx}(\log x)^{-1}$$

$$= (\log 3)(-1)(\log x)^{-2}.\frac{d}{dx}(\log x)$$

$$= \frac{-\log 3}{(\log x)^2} \times \frac{1}{x}$$

$$= \frac{-\log 3}{x(\log x)^2}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{3^x.\log 3}{\left[\frac{-\log 3}{x(\log x)^2}\right]}$$

= $-x(\log x)^2.3^x$.

Exercise 1.4 | Q 4.6 | Page 49

DIfferentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) w.r.t. \sec^{-1} x.$

SOLUTION

Let u = $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ and $v = \sec^{-1} x.$

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$.

Differentiate u and v w.r.t. x, we get

$$\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\left(\frac{\cos}{1+\sin x}\right)\right]$$

$$\frac{\cos x}{1+\sin x} = \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right).\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\right]$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\pi}{4}\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}(x)$$

$$= 0 - \frac{1}{2}\times 1$$

$$= -\frac{1}{2}$$

and

$$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\sec^{-1} x\right)$$

$= \dfrac{1}{x\sqrt{x^2-1}}$

$\therefore \dfrac{\mathrm{du}}{\mathrm{dx}} = \dfrac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)}$

$= \dfrac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{x\sqrt{x^2-1}}\right)}$

$= -\dfrac{x\sqrt{x^2-1}}{2}.$

Exercise 1.4 | Q 4.7 | Page 49

Differentiate x^x w.r.t. x^{six}.

SOLUTION

Let u = x^x and v = x^{sinx}

Then we want to find $\dfrac{\mathrm{du}}{\mathrm{dx}}$.

Take, u = x^x

∴ log u logxx = x log x

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{u}.\dfrac{\mathrm{du}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}(x\log x)$

$= x\dfrac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x).\dfrac{\mathrm{d}}{\mathrm{dx}}(x)$

$= x \times \dfrac{1}{x} + (\log x) \times 1$

$\therefore \dfrac{\mathrm{du}}{\mathrm{dx}} = u(1+\log x)$

= $x^x(1 + \log x)$

Also, v = x^{sinx}

$\therefore$ log v = log$x^{\sin x}$ = (sin x)(log x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}.\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}[(\sin x)(\log x)]$$

$$= (\sin x).\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(\sin x)$$

$$= (\sin x)\times\frac{1}{x}+(\log x)(\cos x)$$

$$\therefore \frac{\mathrm{dv}}{\mathrm{dx}}=v\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}}=\frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)}$$

$$= \frac{x^x(1+\log x)}{x^{\sin x}\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]}$$

$$= \frac{x(1+\log x)\times x}{x^{\sin x}[\sin x+x\cos x.\log x]}$$

$$= \frac{(1+\log x).x^{x+1-\sin x}}{\sin x+x\cos x.\log x}.$$

Exercise 1.4 | Q 4.8 | Page 49

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) w.r.t\ \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

SOLUTION

Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

and

v = $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

Then we want to find $\frac{du}{dv}$

u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put x = tanθ.

Thenθ = $\tan^{-1}$ x

and

$$\frac{\sqrt{1++x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

= $\frac{\sec\theta-1}{\tan\theta}$

$$= \frac{\frac{1}{\cos\theta} - 1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$= \frac{1 - \cos\theta}{\sin\theta}$$

$$= \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

$$\therefore u = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1}x)$$

$$= \frac{1}{2} \times \frac{1}{1 + x^2}$$

$$= \frac{1}{2(1 + x^2)}$$

$$v = \tan^{-1}\left(\frac{2x\sqrt{1 - x^2}}{1 - 2x^2}\right)$$

Put $x = \sin\theta$.

Then $\theta = \sin^{-1}x$

and

$$\frac{2x\sqrt{1 - x^2}}{1 - 2x^2}$$

$$= \frac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

= tan2θ

∴ v = $\tan^{-1}$(tan2θ)

= 2θ

= $2\sin^{-1}x$

$$\therefore \frac{dv}{dx} = 2\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left[\frac{1}{2(1+x^2)}\right]}{\left(\frac{2}{\sqrt{1-x^2}}\right)}$$

$$= \frac{1}{2(1+x^2)} \times \frac{\sqrt{1-x^2}}{2}$$

$$= \frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

EXERCISE 1.5 [PAGE 60]

Exercise 1.5 | Q 1.1 | Page 60

Find the second order derivatives of the following : $2x^5 - 4x^3 - \frac{2}{x^2} - 9$

SOLUTION

Let y = $2x^5 - 4x^3 - \frac{2}{x^2} - 9$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 4x^3 - \frac{2}{x^2} - 9\right)$

= $2\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^{-2}) - \frac{d}{dx}(9)$

= 2 x $5x^4$ – 4 x $3x^2$ – $2(-2)x^{-3}$ – 0

= $10x^4 - 12x^2 + 4x^{-3}$

and

$\frac{d^2y}{dx^2} = \frac{d}{dx}(10x^4 - 12x^2 + 4x^{-3})$

= $10\frac{d}{dx}(x^4) - 12\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-3})$

= 10 x $4x^3$ – 12 x 2x + $4(-3)x^{-4}$

= $40x^3 - 24x - \frac{12}{x^4}$.

Exercise 1.5 | Q 1.2 | Page 60

Find the second order derivatives of the following : e^{2x} . tan x

SOLUTION

Let y = e^{2x} . tan x

Then $\frac{dy}{dx} = \frac{d}{dx}\left(e^{2x}.\tan x\right)$

= $e^{2x}.\frac{d}{dx}(\tan x) + \tan x.\frac{d}{dx}\left(e^{2x}\right)$

= $e^{2x} \times \sec^2 x + \tan x \times e^{2x}.\frac{d}{dx}(2x)$

= e^{2x} . $\sec^2 x$ + e^{2x} . tanx x 2

= e^{2x} ($\sec^2 x$ + 2tan x)

and

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{2x}\left(\sec^2 x + 2\tan x\right)\right]$

= $e^{2x}.\frac{d}{dx}\left(\sec^2 x + 2\tan x\right) + \left(\sec^2 x + 2\tan x\right)\frac{d}{dx}\left(e^{2x}\right)$

= $e^{2x}\left[\frac{d}{dx}(\sec x)^2 + 2\frac{d}{dx}(\tan x)\right] + \left(\sec^2 x + 2\tan x\right) \times e^{2x}.\frac{d}{dx}(2x)$

= $e^{2x}\left[2\sec x.\frac{d}{dx}(\sec x) + 2\sec^2 x\right] + \left(\sec^2 x + 2\tan x\right)e^{2x} \times 2$

= e^{2x}(2sex . sec x tanx + 2$\sec^2$x) + 2e^{2x}($\sec^2$x + 2tanx)

= 2e^{2x}($\sec^2$x tanx + $\sec^2$x + $\sec^2$x + 2tanx)

= 2e^{2x}[$\sec^2$x(tanx + 1) + 1 + $\tan^2$x + 2tanx]

= 2e^{2x}[$\sec^2$x(1 + tanx) + (1 + tanx)^2]

= 2e^{2x}[(1 + tanx) ($\sec^2$x + 1 + tanx)]

= 2e^{2x}[(1 + tanx) (1 + $\tan^2$x + 1 + tanx)]

= 2e^{2x}(1 + tanx) (2 + tanx + $\tan^2$x).

Exercise 1.5 | Q 1.3 | Page 60

Find the second order derivatives of the following : e^{4x}. cos 5x

SOLUTION

Let y = e^{4x}. cos 5x

Then $\frac{dy}{dx} = \frac{d}{dx}\left(e^{4x}.\cos 5x\right)$

$= e^{4x}.\frac{d}{dx}(\cos 5x) + \cos 5x.\frac{d}{dx}\left(e^{4x}\right)$

$= e^{4}x.(-\sin 5x).\frac{d}{dx}(5x) + \cos 5x \times e^{4x}.\frac{d}{dx}(4x)$

= – e^{4x} . sin 5x x 5 + e^{4x} cos 5x x 4

= e^{4x} (4 cos 5x – 5 sin 5x)

and

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{4x}(4\cos 5x - 5\sin 5x)\right]$

$= e^{4x}\frac{d}{dx}(4\cos 5x - 5\sin 5x) + (4\cos 5x - 5\sin 5x).\frac{d}{dx}(4x)$

$= e^{4x}\left[4 - \sin 5x).\frac{d}{dx}(5x) - 5\cos 5x.\frac{d}{dx}(5x)\right] + (4\cos 5x - 5\sin 5x) \times e^{4x}.\frac{d}{dx}(4x)$

= e^{4x} [– 4 sin5x x 5 – 5 cos5x x 5] + (4 cos 5x – 5 sin 5x)e^{4x} x 4

= e^{4x}(– 20 sin 5x –25 cos 5x + 16 cos 5x – sin 5x)

= e^{4x} (– 9 cos 5x – 40 sin 5x)

= –e^{4x} (9 cos 5x + 40 sin 5x).

Exercise 1.5 | Q 1.4 | Page 60

Find the second order derivatives of the following : x^3.logx

SOLUTION

Let y = x^3.logx

Then, $\frac{dy}{dx} = \frac{d}{dx}(x^3 . \log x)$

= $x^3 \frac{d}{dx}(\log x) + (\log x). \frac{d}{dx}(x^3)$

= $x^3 \times \frac{1}{x} + (\log x) \times 3x^2$

= $x^2 + 3x^2 \log x$

= $x^2(1 + 3 \log x)$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[x^2(1 + 3\log x)]$$

= $x^2. \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x) \times 2x$

= $x^2\left(0 + 3 \times \frac{1}{x}\right) + (1 + 3\log x) \times 2x$

= 3x + 2x + 6x log x

= 5x + 6x log x

= x(5 + 6 log x).

Exercise 1.5 | Q 1.5 | Page 60

Find the second order derivatives of the following : log(logx)

SOLUTION

Let y = log(logx)

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\log(\log x)]$

= $\frac{1}{\mathrm{logx}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(\log x)$

= $\frac{1}{\mathrm{logx}} \times \frac{1}{x} = \frac{1}{\mathrm{xlogx}}$

and

$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}(x \log x)^{-1}$

= $(-1)(x \log x)^{-2} \cdot \frac{\mathrm{d}}{\mathrm{dx}}(x \log x)$

= $\frac{-1}{(x \log x)^2} \cdot \left[x \frac{\mathrm{d}}{\mathrm{dx}}(\log x) + (\log x) \cdot \frac{\mathrm{d}}{\mathrm{dx}}(x)\right]$

= $\frac{-1}{(x \log x)^2} \cdot \left[x \times \frac{1}{x} + (\log x) \times 1\right]$

= $-\frac{1 + \log x}{(x \log x)^2}$.

Exercise 1.5 | Q 2.1 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : x = a(θ – sin θ), y = a(1 – cos θ)

SOLUTION

x = a(θ – sin θ), y = a(1 – cos θ)

DIfferentiating x and y w.r.t. θ,, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta)$$

= a(1 – cos θ) ...(1)

and

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}(1 - \cos\theta)$$

= a[0 – (– sin θ)]

= a sin θ

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{a \sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right).\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\cot\left(\frac{\theta}{2}\right)\right]$$

$$= \frac{d}{dx}\left[\cot\left(\frac{\theta}{2}\right)\right].\frac{d\theta}{dx}\Bigg]$$

$$= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right).\frac{d}{d\theta}\left(\frac{\theta}{2}\right) \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \times \frac{1}{2} \times \frac{1}{a(1-\cos\theta)}$$ ...[by (1)]

$$= -\frac{1}{2a}\text{cosec}^2\left(\frac{\theta}{2}\right) \times \frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{1}{4a}.\ \text{cosec}^4\left(\frac{\theta}{2}\right).$$

Exercise 1.5 | Q 2.2 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : x = 2at^2, y = 4at

SOLUTION

x = 2at^2, y = 4at

Differentiating x and y w.r.t. t, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(2at^2\right)$$

$$= 2a.\frac{\mathrm{d}}{\mathrm{dt}}\left(t^2\right)$$

= 2a x 2t = 4at ...(1)

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(4at)$$

$$= 4a\frac{\mathrm{d}}{\mathrm{dt}}(t)$$

= 4a x 1 = 4a

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{4a}{4at} = \frac{1}{t}$$

and

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{t}\right)$$

$= \frac{d}{dt}\left(t^{-1}\right) \times \frac{dt}{dx}$

$= 1(t)^{-2} \times \frac{1}{\left(\frac{dx}{dt}\right)}$

$= \frac{1}{t^2} \times \frac{1}{4at}$ ...[By (1)]

$= -\frac{1}{4at^3}.$

Exercise 1.5 | Q 2.3 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : x = sinθ, y = $\sin^3\theta$ at θ = $\frac{\pi}{2}$

SOLUTION

x = sinθ, y = $\sin^3\theta$

Differentiating x and y w.r.t. θ, we get

$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta)$ = cosθ ...(1)

and

$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta)^3$

$= 3(\sin\theta)^2 . \frac{d}{d\theta}(\sin\theta)$

= $3\sin^2\theta.\cos\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$= \frac{3\sin^2\theta\cos\theta}{\cos\theta}$

= $3\sin^2\theta$

and

$$\frac{d^2y}{dx^2} = 3\frac{\mathrm{d}}{\mathrm{dx}}(\sin\theta)^2$$

$$= 3\frac{\mathrm{d}}{\mathrm{d\theta}}(\sin\theta)^2 \times \frac{\mathrm{d\theta}}{\mathrm{dx}}$$

$$= 3 \times 2\sin\theta\frac{\mathrm{d}}{\mathrm{d\theta}}(\sin\theta) \times \frac{1}{\left(\frac{\mathrm{dx}}{\mathrm{d\theta}}\right)}$$

$$= 6\sin\theta.\cos\theta \times \frac{1}{\cos\theta}$$ ...[By (1)]

= 6sinθ

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at}\theta = \frac{\pi}{2}}$$

$$= 6\sin\ \frac{\pi}{2}$$

= 6 x 1

= 6.

Alternative Method :

x = sinθ, y = $\sin^3\theta$

$\therefore$ y = x^3

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(x^3) = 3x^2$$

$$\therefore \frac{d^2y}{dx^2} = 3\frac{\mathrm{d}}{\mathrm{dx}}(x^2)$$

= 3 x 2x

= 6x

If θ = $\frac{\pi}{2}$, then $x = \sin\ \frac{\pi}{2}$ = 1

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at}\theta = \frac{\pi}{2}}$$

$= \left(\frac{d^2y}{dx^2}\right)_{\text{at} x = 1}$

= 6(1)

= 6.

Exercise 1.5 | Q 2.4 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : x = a cos θ, y = b sin θ at θ = $\frac{\pi}{4}$.

SOLUTION

x = a cos θ, y = b sin θ

DIfferentiating x and y w.r.t. θ, we get

$\frac{\mathrm{dx}}{\mathrm{d\theta}} = \frac{\mathrm{d}}{\mathrm{d\theta}}(a\cos\theta)$

$= a\frac{\mathrm{d}}{\mathrm{d\theta}}(\cos\theta)$

= a(− sin θ)

= − a sin θ ...(1)

and

$\frac{\mathrm{dy}}{\mathrm{d\theta}} = \frac{\mathrm{d}}{\mathrm{d\theta}}(b\sin\theta)$

$= b\frac{\mathrm{d}}{\mathrm{dx}}(\sin\theta)$

= b cos θ

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{d\theta}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{d\theta}}\right)}$

$= \frac{\mathrm{bcos\theta}}{\mathrm{-asin\theta}}$

$= \left(-\frac{b}{a}\right)\cot\theta$

and

$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\left(-\frac{b}{a}\right)\cot\theta\right]$

$= -\frac{b}{a}.\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot\theta) \times \frac{\mathrm{d}\theta}{\mathrm{dx}}$

$= \left(-\frac{b}{a}\right)\left(-\mathrm{cosec}^2\theta\right) \times \frac{1}{\left(\frac{\mathrm{dx}}{\mathrm{d}\theta}\right)}$

$= \left(\frac{b}{a}\right)\mathrm{cosec}^2\theta \times \frac{1}{-a\sin\theta}$ ..[By (1)]

$= \left(-\frac{b}{a^2}\right)\mathrm{cosec}^3\theta$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{\mathrm{at}\theta=\frac{\pi}{4}}$

$= \left(-\frac{b}{a^2}\right)\mathrm{cosec}^3\frac{\pi}{4}$

$= \frac{-b}{a^2} \times \left(\sqrt{2}\right)^3$

$= -\frac{2\sqrt{2\mathrm{b}}}{a^2}.$

Exercise 1.5 | Q 3.01 | Page 60

If x = at^2 and y = 2at, then show that $xy\frac{d^2y}{dx^2} + a = 0$.

SOLUTION

$x = at^2$ and $y = 2at$ ...(1)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(at^2\right)$$

$$= a\frac{d}{dt}\left(t^2\right)$$

$= a \times 2t$

$= 2at$ ...(2)

and

$$\frac{dy}{dt} = \frac{d}{dt}(2at)$$

$$= 2a\frac{d}{dt}(t)$$

$= 2a \times 1$

$= 2a$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2a}{2at} = \frac{1}{t}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right)$$

$$= \frac{d}{dt}\left(t^{-1}\right).\frac{dt}{dx}$$

$$= (-1)t^{-2}.\frac{1}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{-1}{t^2} \times \frac{1}{2at}$$ ...[By (2)]

$= \dfrac{1}{2at^3}.$

$\therefore \dfrac{d^2y}{dx^2} = -\dfrac{1}{(at^2)(2\text{at}) \times a}$

$= -\dfrac{a}{\text{xy}}$ ...[By (1)]

$\therefore xy\dfrac{d^2y}{dx^2} = -$ a

$\therefore xy\dfrac{d^2y}{dx^2} + a = 0.$

Exercise 1.5 | Q 3.02 | Page 60

If y = $e^{m\tan^{-1}x}$, show that $\left(1+x^2\right)\dfrac{d^2y}{dx^2} + (2x-m)\dfrac{\text{dy}}{\text{dx}} = 0.$

SOLUTION

y = $e^{m\tan^{-1}x}$ ...(1)

$\therefore \dfrac{\text{dy}}{\text{dx}} = \dfrac{\text{d}}{\text{dx}}\left(e^{m\tan^{-1}x}\right)$

$= e^{m\tan^{-1}x}.\dfrac{\text{d}}{\text{dx}}\left(m\tan^{-1}x\right)$

$= e^{m\tan^{-1}x} \times m \times \dfrac{1}{1+x^2}$

$\therefore \left(1+x^2\right)\dfrac{\text{dy}}{\text{dx}} =$ my ...[By (1)]

Differentiaitng again w.r.t. x, we get

$$\left(1+x^2\right).\frac{\text{d}}{\text{dx}}\left(\frac{\text{dy}}{\text{dx}}\right) + \frac{\text{dy}}{\text{dx}}.\frac{\text{d}}{\text{dx}}\left(1+x^2\right) = m\frac{\text{dy}}{\text{dx}}$$

$$\therefore \left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{\text{dy}}{\text{dx}}(0+2x) = m\frac{\text{dy}}{\text{dx}}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + 2x.\frac{\mathrm{dy}}{\mathrm{dx}} = m\frac{\mathrm{dy}}{\mathrm{dx}}.$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + (2x-m)\frac{\mathrm{dy}}{\mathrm{dx}} = 0.$$

Exercise 1.5 | Q 3.03 | Page 60

If x = cos t, y = e^{mt}, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{\mathrm{dy}}{\mathrm{dx}} - m^2y = 0.$

SOLUTION

x = cos t, y = e^{mt}

$\therefore$ t = $\cos^{-1}$x and y = $e^{m\cos^{-1}x}$ ...(1)

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{m\cos^{-1}x}\right)$$

$$= e^{m\cos^{-1}x}.\frac{\mathrm{d}}{\mathrm{dx}}\left(m\cos^{-1}x\right)$$

$$= e^{m\cos^{-1}x} \times m \times \frac{-1}{\sqrt{1-x^2}}$$

$\therefore \sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}$ = – my ...[By (1)]

$\therefore \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2$ = m^2y2

Differentiating again w.r.t. x, we get

$$\left(1-x^2\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(1-x^2\right) = m^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(y^2\right)$$

$$\therefore \left(1-x^2\right).2\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(0-2x) = m^2 \times 2y\frac{\mathrm{dy}}{\mathrm{dx}}$$

Cancelling $2\frac{\mathrm{dy}}{\mathrm{dx}}$ throughtout, we get

$\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{\text{dy}}{\text{dx}}$ = m^2y

$\therefore \left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{\text{dy}}{\text{dx}}-m^2y$ = 0.

Exercise 1.5 | Q 3.04 | Page 60

If y = x + tan x, show that $\cos^2 x.\dfrac{d^2y}{dx^2}-2y+2x$ = 0.

SOLUTION

y = x + tan x

$\therefore \dfrac{\text{dy}}{\text{dx}}=\dfrac{\text{d}}{\text{dx}}(x+\tan x)$

= 1 + $\sec^2$x

and

$\dfrac{d^2y}{dx^2}=\dfrac{\text{d}}{\text{dx}}(1+\sec x)^2$

= $\dfrac{\text{d}}{\text{dx}}(1)+\dfrac{\text{d}}{\text{dx}}(\sec x)^2$

= 2sec x . sec x tan x

= 2 $\sec^2$x tan x

$\therefore \cos^2 x.\dfrac{d^2y}{dx^2}-2y+2x$

= $\cos^2 x\left(2\sec^2 x\tan x\right)-2(x+\tan x)+2x$

= $\cos^2 x\times\dfrac{2}{\cos^2 x}\times\tan x-2x-2\tan x+2x$

= 2 tan x – 2 tan x

$\therefore \cos^2 x.\dfrac{d^2y}{dx^2}-2y+2x$ = 0.

Exercise 1.5 | Q 3.05 | Page 60

If $y = e^{ax}.\sin(bx)$, show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

SOLUTION

$y = e^{ax}.\sin(bx)$ ...(1)

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[e^{ax}.\sin(bx)]$

$= e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}[\sin(bx)] + \sin(bx).\frac{\mathrm{d}}{\mathrm{dx}}(e^{ax})$

$= e^{ax}.\cos(bx).\frac{\mathrm{d}}{\mathrm{dx}}(bx) + \sin(bx) \times e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}(ax)$

$= e^{ax}.\cos(bx) \times b + e^{ax}.\sin(bx) \times a$

$\therefore y_1 = e^{ax}[b\cos(bx) + a\sin(bx)]$ ...(2)

Differentiating again w.r.t. x, we get

$\frac{\mathrm{dy_1}}{\mathrm{dx}} = \frac{\mathrm{d}}{,} dx[e^{ax}\{b\ \cos(bx) + a\sin(bx)\}]$

$= e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}[b\cos(bx) + a\sin(bx)] + [b\cos(bx) + a\sin(bx)].\frac{\mathrm{d}}{\mathrm{dx}}(e^a x)$

$= e^{ax}.\left[b\{-\sin(bx)\}.\frac{\mathrm{dy}}{\mathrm{dx}}(bx) + a\cos(bx).\frac{\mathrm{dy}}{\mathrm{dx}}(bx)\right] + [b\cos(bx) + a\sin(bx)] \times e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}(ax)$

$= e^{ax}$ [– b sin (bx) x b + a cos (bx) x b} + [b cos (bx) + a sin (bx)e^{ax} x a

$= e^{ax}[-b^2\sin(bx) + ab\cos(bx) + a^2\sin(bx)]$

$\therefore y_2 = e^{ax}[-b^2\sin(bx) + 2ab\cos(bx) + a^2\sin(bx)]$ ...(3)

$\therefore y^2 - 2ay_1 + (a^2 + b^2)y$

$= e^{ax}[-b^2\sin(bx) + 2ab\cos(bx) + a^2\sin(bx)] - 2a.e^{ax}[b\cos(bx) + a\sin(bx)] + (a^2 + b^2)e^{ax}\sin(bx)$ ...[By (1), (2) and (3)]

$= e^{ax}[-b^2\sin bx + 2ab\cos(bx) + a^2\sin(bx) - 2ab\cos(bx) - 2a^2\sin(bx) + a^2\sin(bx) + b^2\sin(bx)]$

$= e^{ax} \times 0$

$\therefore y_2 - 2ay_1 + (a^2 + b^2)y = 0.$

Exercise 1.5 | Q 3.06 | Page 60

If $\sec^{-1}\left(\frac{7x^3 - 5y^3}{7^3 + 5y^3}\right) = m, \text{show } \frac{d^2y}{dx^2}$ = 0.

SOLUTION

$\sec^{-1}\left(\frac{7x^3 - 5y^3}{7^3 + 5y^3}\right)$ = m

$\therefore \frac{7x^3 - 5y^3}{7^3 + 5y^3}$ = sec m =k ...(Say)

$\therefore$ $7x^3 - 5y^3 = 7kx^3 + 5ky^3$

$\therefore$ $(5k + 5)y^3 = (7 - 7k)x^2$

$\therefore \frac{y^3}{x^3} = \frac{7 - 7k}{5k + 5}$

$\therefore \frac{y}{x} = \left(\frac{7 - 7k}{5k + 5}\right)^{\frac{1}{3}}$ = p, where p is a constant

$\therefore \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x}\right) = \frac{\mathrm{d}}{\mathrm{dx}}(p)$

$\therefore \frac{x\frac{\mathrm{dy}}{\mathrm{dx}} - y\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$ = 0

$\therefore x\frac{\mathrm{dy}}{\mathrm{dx}} - y \times 1$ = 0

$\therefore x\frac{\mathrm{dy}}{\mathrm{dx}}$ = y

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$ ...(1)

$\therefore \frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x}\right)$

= $\frac{x\frac{\mathrm{dy}}{\mathrm{dx}} - y\frac{\mathrm{d}}{\mathrm{dx}}(x)}{x^2}$

= $\frac{x\left(\frac{y}{x}\right) - y \times 1}{x^2}$ ...[By (1)]

= $\frac{y - y}{x^2}$

= $\frac{0}{x^2}$

= 0

Note : $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x}$. where $\frac{y}{x}$ = p.

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}$ = p, where p is a constant.

$\therefore \frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}(p)$ = 0.

Exercise 1.5 | Q 3.07 | Page 60

If 2y = $\sqrt{x+1} + \sqrt{x-1}$, show that $4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

SOLUTION

2y = $\sqrt{x+1} + \sqrt{x-1}$ ...(1)

Differentiating both sides w.r.t. x, x we get

$\therefore 2\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x+1}\right) + \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x+1}\right)$

= $\frac{1}{2\sqrt{x+1}}(1+0) + \frac{1}{2\sqrt{x-1}}(1-0)$

$\therefore 2\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x+1}}$

= $\frac{\sqrt{x+1}+\sqrt{x+1}}{2\sqrt{x+1}.\sqrt{x-1}}$

= $\frac{2y}{2\sqrt{x^2-1}}$ ...[By (1)]

$\therefore 2\sqrt{x^2-1}\frac{\mathrm{dy}}{\mathrm{dx}}$ = y

$\therefore 4(x^2-1).\left(\frac{dy}{dx}\right)^2$ = y^2

DIfferentiating both sides w.r.t. x, we get

$$4(x^2-1)\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2.\frac{\mathrm{d}}{\mathrm{dx}}[4(x^2-1)] = 2y\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore 4(x^2-1).2\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{d2y}{dx^2} + \left(\frac{dy}{dx}\right)^2.4(2x) = 2y\left(\frac{dy}{dx}\right)$$

Cancelling $2\frac{\mathrm{dy}}{\mathrm{dx}}$ on both sides, we get

$4(x^2-1)\frac{d^2y}{dx^2} + 4x\frac{\mathrm{dy}}{\mathrm{dx}}$ = y

$\therefore 4(x^2-1)\frac{d^2y}{dx^2} + 4x\frac{\mathrm{dy}}{\mathrm{dx}} - y$ = 0

$\therefore 4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

Exercise 1.5 | Q 3.08 | Page 60

If y = $\log\left(x+\sqrt{x^2+a^2}\right)^m$, show that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{\mathrm{d}}{\mathrm{dx}}$ = 0.

SOLUTION

y = $\log\left(x+\sqrt{x^2+a^2}\right)^m$

= $m\log\left(x+\sqrt{x^2+a^2}\right)$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = m\frac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(x+\sqrt{x^2+a^2}\right)\right]$

= $m \times \frac{1}{x+\sqrt{x^2+a^2}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x+\sqrt{x^2+a^2}\right)$

= $\frac{m}{x+\sqrt{x^2+a^2}} \times \left[1+\frac{1}{2\sqrt{x^2+a^2}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+a^2\right)\right]$

= $\frac{m}{x+\sqrt{x^2+a^2}} \times \left[1+\frac{1}{2\sqrt{x^2+a^2}}.(2x+0)\right]$

= $\frac{m}{x+\sqrt{x^2+a^2}} \times \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{m}{\sqrt{x^2+a^2}}$

$\therefore \sqrt{x^2+a^2}\frac{\mathrm{dy}}{\mathrm{dx}}$ = m

$\therefore \left(x^2+a^2\right)\left(\frac{dy}{dx}\right)^2$ = m^2

Differentiating both sides w.r.t. x, we get

$$\left(x^2+a^2\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2+a^2\right)=\frac{\mathrm{d}}{\mathrm{dx}}\left(m^2\right)$$

$\therefore \left(x^2+a^2\right) \times 2\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2 \times (2x+0)$ = 0

$\therefore \left(x^2+a^2\right).2\frac{\mathrm{dy}}{\mathrm{dx}}\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2$ = 0

Cancelling $2\frac{\mathrm{dy}}{\mathrm{dx}}$ throughtout, we get

$$\left(x^2+a^2\right)\frac{d^2y}{dx^2}+x\frac{\mathrm{dy}}{,}dx = 0.$$

Exercise 1.5 | Q 3.09 | Page 60

If y = sin (m cos^{-1}x), then show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{\mathrm{dy}}{\mathrm{dx}}+m^2y = 0.$

SOLUTION

y = sin (m cos^{-1}x)

∴ sin^{-1} y = m cos^{-1} x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-y^2}}\cdot\frac{\mathrm{dy}}{\mathrm{dx}}=m\times\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore\ \sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}=-2\sqrt{1-y^2}$$

$$\therefore\ \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=m^2\left(1-y^2\right)$$

$$\therefore\ \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2 - m^2y^2$$

Differentiating both sides w.r.t. x, we get

$$\left(1-x^2\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(1-x^2\right)=0-m^2.\frac{\mathrm{d}}{\mathrm{dx}}\left(y^2\right)$$

$$\therefore\ \left(1-x^2\right).2\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{\mathrm{dy}}{\mathrm{dx}}$$

Cancelling $2\frac{\mathrm{dy}}{\mathrm{dx}}$ throughtout, we get

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{\mathrm{dy}}{\mathrm{dx}} = -m^2y$$

$$\therefore \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{\mathrm{dy}}{\mathrm{dx}}+m^2y = 0.$$

Exercise 1.5 | Q 3.1 | Page 60

If y = log (log 2x), show that $xy_2 + y_1 (1 + xy_1) = 0$.

SOLUTION

y = log (log 2x)

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}[\log(\log 2x)]$$

$$= \frac{1}{\mathrm{log2x}}.\frac{\mathrm{d}}{\mathrm{dx}}(\log 2x)$$

$$= \frac{1}{\mathrm{log2x}}\times\frac{1}{2x}.\frac{\mathrm{d}}{\mathrm{dx}}(2x)$$

$$= \frac{1}{\mathrm{log2x}}\times\frac{1}{2x}\times 2$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{1}{x\log 2x}$$

$$\therefore (\log 2x).\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{1}{x} \qquad \text{...(1)}$$

Differentiating both sides w.r.t. x, we get

$$(\log 2x).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dx}{dy}\right)+\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{\mathrm{d}}{\mathrm{dx}}(\log 2x)=\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{x}\right)$$

$$\therefore (\log 2x).\frac{d^2y}{dx^2}+\frac{\mathrm{dy}}{\mathrm{dx}}.\frac{1}{2x}.\frac{\mathrm{d}}{\mathrm{dx}}(2x)=-\frac{1}{x^2}$$

$$\therefore (\log 2x).\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}}.\frac{1}{2x} \times 2 = -\frac{1}{x^2}$$

$$\therefore (\log 2x).\frac{d^2y}{dx^2} + \frac{1}{x}.\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x}.\frac{1}{x}$$

$$\therefore (\log 2x).\frac{d^2y}{dx^2} + \left[(\log 2x).\frac{\mathrm{dy}}{\mathrm{dx}}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{1}{x}\left[(\log 2x).\frac{\mathrm{dy}}{\mathrm{dx}}\right] \quad \text{...[By (1)]}$$

Dividing throughout by log 2x, we get

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{1}{x}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = -\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}} + x\left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}}\left(1 + x\frac{dy}{dx}\right) = 0$$

$\therefore xy_2 + y_1 (1 + xy_1) = 0.$

Exercise 1.5 | Q 3.11 | Page 60

If $x^2 + 6xy + y^2 = 10$, show that $\dfrac{d^2y}{dx^2} = \dfrac{80}{(3x+y)^3}$.

SOLUTION

$x^2 + 6xy + y^2 = 10$ ...(1)

Differentiating both sides w.r.t. x, we get

$$2x + 6\left[x\frac{\mathrm{dy}}{\mathrm{dx}} + y.\frac{\mathrm{d}}{\mathrm{dx}}(x)\right] + 2y\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore 2x + 6x\frac{\mathrm{dy}}{\mathrm{dx}} + 6y \times 1 + 2y\frac{\mathrm{dy}}{\mathrm{dx}} = 0$$

$$\therefore (6x + 2y)\frac{\mathrm{dy}}{\mathrm{dx}} = -2x - 6y$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-2(x+3y)}{2(3x+y)} = -\left(\frac{x+3y}{3x+y}\right) \qquad \text{...(2)}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x+3y}{3x+y}\right)$$

$$= -\left[\frac{(3x+y).\frac{\mathrm{d}}{\mathrm{dx}}(x+3y) - (x+3y).\frac{\mathrm{d}}{\mathrm{dx}}(3x+y)}{(3x+y)^2}\right]$$

$$= -\left[\frac{(3x+y)\left(1+3\frac{\mathrm{dy}}{\mathrm{dx}}\right) - (x+3y)\left(3+\frac{\mathrm{dy}}{\mathrm{dx}}\right)}{(3x+y)^2}\right]$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left\{1-\frac{3(x+3y)}{3x+y}\right\} + (x+3y)\left(3-\frac{x+3y}{3x+y}\right)\right] \qquad \text{...[By (2)]}$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left[-(3x+y)\left(\frac{3x+y-3x-9y}{3x+y}\right) + (x+3y)\left(\frac{9x+3y-x-3y}{3x+y}\right)\right]\right.$$

$$= \frac{1}{(3x+y)^2}\left[8y + \frac{(x+3y)(8x)}{3x+y}\right]$$

$$= \frac{1}{(3x+y)^2}\left[\frac{(8y(3x+y)+(x+3y)8x)}{3x+y}\right]$$

$$= \frac{24xy + 8y^2 + 8x^2 + 24xy}{(3x+y)^2}$$

$$= \frac{8x^2 + 48xy + 8y^2}{(3x+y)^3}$$

$$= \frac{8\left(x^2+6xy+y^2\right)}{(3x+y)^3}$$

$$= \frac{8(10)}{(3x+y)^3} \qquad \text{...[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}.$$

Exercise 1.5 | Q 3.12 | Page 60

If x = a sin t – b cos t, y = a cos t + b sin t, show that $\frac{d^2y}{dx^2} = -\frac{x^3+y^2}{y^3}$.

SOLUTION

x = a sin t – b cos t, y = a cos t + b sin t

Differentiating x and y w.r.t. t, we get

$$\frac{\mathrm{dx}}{\mathrm{dt}} = a\frac{\mathrm{d}}{\mathrm{dx}}(\sin t) - b\frac{\mathrm{d}}{\mathrm{dt}}(\cos t)$$

= a cos t – b(– sin t)

= a cos t + b sin t

and

$$\frac{\mathrm{dy}}{\mathrm{dt}} = a\frac{\mathrm{d}}{\mathrm{dx}}(\cos t) - b\frac{\mathrm{d}}{\mathrm{dt}}(\sin t)$$

= a(– sin t) + b cos t

= – a sin t + b cos t

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)}$$

$$= \frac{-a\sin t + b\cos t}{a\cos t + b\sin t}$$

$$= -\left(\frac{a\sin t - b\cos t}{a\cos t + b\sin t}\right)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{x}{y} \qquad \text{...(1)}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x}{y}\right)$$

$$= -\left[\frac{y\frac{\mathrm{d}}{\mathrm{dx}}(x) - x\frac{\mathrm{dy}}{\mathrm{dx}}}{y^2}\right]$$

$$= -\left[\frac{y \times 1 - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \text{...[By (1)]}$$

$$= -\left[\frac{y^2 + x^2}{y^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^2}.$$

Exercise 1.5 | Q 4.01 | Page 60

Find the n^{th} derivative of the following : $(ax + b)^m$

SOLUTION

Let $y = (ax + b)^m$

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(ax + b)^m$

$= m(ax + b)^{m-1}. \frac{\mathrm{d}}{\mathrm{dx}}(ax + b)$

$= m(ax + b)^{m-1} \times (a \times 1 + 0)$

$= am(ax + b)^{m-1}$

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left[am(ax + b)^{m-1}\right]$$

$$= am\frac{\mathrm{d}}{\mathrm{dx}}(ax + b)^{m-1}$$

= $am(m-1)(ax+b)^{m-2}.\frac{d}{dx}(ax+b)$

= am(m – 1)(ax + b)$^{m-2}$ x (a x 1 + 0)

= a^{2}m(m –1) (ax + b)$^{m-2}$

$$\frac{d^2y}{dx^3}=\frac{d}{dx}\left[a^2m(m-1)\frac{d}{dx}(ax+b)^{m-2}\right]$$

= $a^2m(m-1)\frac{d}{dx}(ax+b)^{m-2}$

= $a^m(m-1)(m-2)(ax+b)^{m-3}\frac{d}{dx}(ax+b)$

= a^{2}m(m – 1)(m – 2)(ax + b)$^{m-3}$ x (a x 1 + 0)

= a^{3}m(m – 1)(m – 2)(ax + b)$^{m-3}$

In general, the nth order derivative is given by

$\frac{d^ny}{dx^n}=a^nm(m-1)(m-2)$ ...(m – n + 1)(ax + b)$^{m-n}$

Case (i) : if m > 0, m > n, then

$$\frac{d^ny}{dx^n}=\frac{(a^n.m(m-1)(m-2)...(m-n+1)(m-n)...3.2.1)}{(m-n)(m-n-1)...3.2.1}\times(ax+b)^{m-n}$$

$\therefore\frac{d^2y}{dx^n}=\frac{(a^n.m!(ax+b)^{m-n})}{(m-n)!}$,If m > 0, m > n.

Case (ii) : if m > 0 and m < n, then its mth order derivative is a constant and every derivatives after mth order are zero.

$\therefore\frac{d^ny}{dx^n}$ = 0, if m > 0, m = n.

Case (iii) : If m > 0, m = n, then

$$\frac{d^ny}{dx^n}=a^n.n(n-1)(n-2)...(n-n+1)(ax+b)^{n-n}$$

= a^n.n(n – 1)(n – 2) ... 1.(ax + b)^0

$\therefore\frac{d^ny}{dx^n}$ = an . n!, if m > 0, m = n.

Exercise 1.5 | Q 4.02 | Page 60

Find the n^{th} derivative of the following : $\dfrac{1}{x}$

SOLUTION

Let y = $\dfrac{1}{x}$

Then $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{1}{x}\right)$

= $\dfrac{1}{x^2}$

= $\dfrac{(-1)^1 1!}{x^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left(-\dfrac{1}{x^2}\right)$

= $1\dfrac{\mathrm{d}}{\mathrm{dx}}\left(x^{-2}\right)$

= $(-1)(-2)x^{-3}$

= $\dfrac{(-1)^{2.1}.2}{x^3}$

= $\dfrac{(-1)^2 2!}{x^3}$

$\dfrac{d^3y}{dx^3} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\dfrac{(-1)^2.\,2!}{x^3}\right]$

= $(-1)^2.\ 2!\frac{\mathrm{d}}{\mathrm{dx}}\left(x^{-3}\right)$

= $(-1)^2.\ 2!.(-3)x^{-4}$

= $\frac{(-1)^3 \times 3.2!}{x^4}$

= $\frac{(-1)^3.3!}{x^4}$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^n.\ n!}{x^{n+1}}.$$

Exercise 1.5 | Q 4.03 | Page 60

Find the n^{th} derivative of the following : e^{ax+b}

SOLUTION

Let y = e^{ax+b}

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(e^{ax+b}\right)$

= $e^{ax+b}.\ \frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

= $e^{ax+b} \times (a \times 1 + 0)$

= ae^{ax+b}

$$\frac{d^2y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left(ae^{ax+b}\right)$$

= $a.\ \frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

$= ae^{ax+b} \times (a \times 1 + 0)$

$= a^2.e^{ax+b}$

$$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[a^2e^{ax+b}\right]$$

$$= a^2\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{ax+b}\right)$$

$$= a^2e^{ax+b}.\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$$

$= a^2e^{ax+b} \times (a \times 1 + 0)$

$= a^3.e^{ax+b}$

In genaral, the nth order derivative is given by

$\frac{d^ny}{dx^n} = a^n . e^{ax+b}.$

Exercise 1.5 | Q 4.04 | Page 60

Find the nth derivative of the following : a^{px+q}

SOLUTION

Let $y = a^{px+q}$

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left(a^{px+q}\right)$

$$= a^{px+q}\log a.\frac{\mathrm{d}}{\mathrm{dx}}(px+q)$$

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left[p\log a.\,a^{px+q}\right]$$

$$= p\log a.\frac{\mathrm{d}}{\mathrm{dx}}\left(a^{px+q}\right)$$

= $p\log a.\, a^{px+q}.\log a.\,\frac{\mathrm{d}}{\mathrm{dx}}(px+q)$

= $p\log a.\, a^{px+q}.\log a\times(p\times 1+0)$

= $p^2.\,(\log a)^2.\,a^{px+q}$

$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[p^2.\,(\log a)^2.\,a^{px+q}\right]$

= $p^2.\,(\log a)^2.\,\frac{\mathrm{d}}{\mathrm{dx}}\left(a^{px+q}\right)$

= $p^2.\,(\log a)^2.\,a^{px+q}.\log a.\,\frac{\mathrm{d}}{\mathrm{dx}}(px+q)$

= $p^2.\,(\log a)^3.\,a^{px+q}\times(p\times 1+0)$

= $p^3.\,(\log a)^3.\,a^{px+q}$

In general, the nth order derivative is given by

$\frac{d^ny}{dx^n} = p^n.\,(\log a)^n.\,a^{px+q}.$

Exercise 1.5 | Q 4.05 | Page 60

Find the n$^{\text{th}}$ derivative of the following : log (ax + b)

Let y = log (ax + b)

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\log(ax+b)]$

= $\frac{1}{ax+b}.\,\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

= $\frac{1}{ax+b}\times(a\times 1+0)$

= $\frac{a}{\mathrm{ax+b}}$

$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{a}{ax+b}\right)$

$= a\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)^{-1}$

$= a(-1)(ax+b)^{-2}.\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

$= \frac{(-1)a}{(ax+b)^2} \times (a \times 1 + 0)$

$= \frac{(-1)a}{(ax+b)^2}$

$$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{(-1)^1 a^2}{(ax+b)^2}\right]$$

$= (-1)^1 a^2.\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)^{-2}$

$= (-1)^1 a^2.(-2)(ax+b)^{-3}.\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

$= \frac{(-1)^2.1.2.a^2}{(ax+b)^3} \times (a \times 1 + 0)$

$= \frac{(-1)\cdot 2.2!a^3}{(ax+b)^3}$

In general, the nth order derivative is given by

$$\frac{d^ny}{dx^2} = \frac{(-1)^{n-1}.(n-1)!a^n}{(ax+b)^n}$$

Exercise 1.5 | Q 4.06 | Page 60

Find the nth derivative of the following : cos x

SOLUTION

Let y = cos x

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}(\cos x)$

$= -\sin x$

$= \cos\left(\frac{\pi}{2} + x\right)$

$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}(-\sin x)$

$= -\cos x$

= cos(π + x)

$= \cos\left(\frac{2\pi}{2} + x\right)$

$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}(-\cos x)$

$= -\frac{\mathrm{d}}{\mathrm{dx}}(\cos x)$

= − (− sin x)

= sin x

$= \cos\left(\frac{3\pi}{2} + x\right)$

In general, the nth order derivative is given by

$\frac{d^ny}{dx^n} = \cos\left(\frac{n\pi}{2} + x\right).$

Exercise 1.5 | Q 4.07 | Page 60

Find the nth derivative of the following : sin (ax + b)

SOLUTION

Let y = sin (ax + b)

Then $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{d}}{\mathrm{dx}}[\sin(ax+b)]$

= $\cos(ax+b).\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

= cos (ax + b) x (a x 1 + 0)

= $a\sin\left[\frac{\pi}{2}+(ax+b)\right]$

$\frac{d^2y}{dx^2}=\frac{\mathrm{d}}{\mathrm{dx}}[a\cos(ax+b)]$

= $a\frac{\mathrm{d}}{\mathrm{dx}}[\cos(ax+b)]$

= $a[-\sin(ax+b)].\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

= a[− sin (ax + b)] x (a x 1 + 0)

= a^2 . sin[π + (ax + b)]

= $a^2.\sin\left[\frac{2\pi}{2}+(ax+b)\right]$

$\frac{d^3y}{dx^3}=\frac{\mathrm{d}}{\mathrm{dx}}\left[-a^2\sin(ax+b)\right]$

= $-a^2\frac{\mathrm{d}}{\mathrm{dx}}[\sin(ax+b)]$

= $-a^2.\cos(ax+b).\frac{\mathrm{d}}{\mathrm{dx}}(ax+b)$

= − a^2. cos(ax + b) x (a x 1 + 0)

= $a^3.\sin\left[\frac{3\pi}{2}+(ax+b)\right]$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = a^n . \sin\left[\frac{n\pi}{2} + (ax + b)\right].$$

Exercise 1.5 | Q 4.08 | Page 60

Find the n^{th} derivative of the following : cos (3 – 2x)

SOLUTION

Let y = cos (3 – 2x)

Then $\frac{dy}{dx} = \frac{d}{dx}[\cos(3 - 2x)]$

= $\cos(3 - 2x). \frac{d}{dx}(3 - 2x)$

= cos (3 – 2x) x (a x 1 + 0)

= $a \cos\left[\frac{\pi}{2} + (3 - 2x)\right]$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[a \cos(3 - 2x)]$$

= $a \frac{d}{dx}[\cos(3 - 2x)]$

= $a[- \cos(3 - 2x)]. \frac{d}{dx}(3 - 2x)$

= a[– cos(3 – 2x)] x (a x 1 + 0)

= a^2 . cos[π + (3 – 2x)]

= $a^2 . \cos\left[\frac{2\pi}{2} + (3 - 2x)\right]$

$$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[-a^2\cos(ax+b)\right]$$

$$= -a^2\frac{\mathrm{d}}{\mathrm{dx}}[\cos(3-2x)]$$

$$= -a^2.\cos(3-2x).\frac{\mathrm{d}}{\mathrm{dx}}(3-2x)$$

= − a^2. cos(3 − 2x) x (a x 1 + 0)

$$= a^3.\cos\left[\frac{3\pi}{2}+(3-2x)\right]$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = (-2)^n\cos\left[\frac{n\pi}{2}+(3-2x)\right].$$

Exercise 1.5 | Q 4.09 | Page 60

Find the n^{th} derivative of the following : log (2x + 3)

SOLUTION

Let y = log (2x + 3)

Then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\log(2x+3)]$

$$= \frac{1}{2x+3}.\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)$$

$$= \frac{1}{2x+3}\times(a\times 1+0)$$

$$= \frac{a}{2\mathrm{x}+3}$$

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{a}{2x+3}\right)$$

$$= a\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)^{-1}$$

= $a(-1)(2x+3)^{-2}.\ \dfrac{\mathrm{d}}{\mathrm{dx}}(2x+3)$

= $\dfrac{(-1)a}{(2x+3)^2} \times (a \times 1 + 0)$

= $\dfrac{(-1)a}{(2x+3)^2}$

$$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{(-1)^1 a^2}{(2x+3)^2}\right]$$

= $(-1)^1 a^2.\ \dfrac{\mathrm{d}}{\mathrm{dx}}(2x+3)^{-2}$

= $(-1)^1 a^2.\,(-2)(2x+3)^{-3}.\ \dfrac{\mathrm{d}}{\mathrm{dx}}(2x+3)$

= $\dfrac{(-1)^2.\,1.2.\,a^2}{(2x+3)^3} \times (a \times 1 + 0)$

= $\dfrac{(-1)_{\cdot}2.2!a^3}{(2x+3)^3}$

In general, the n$^{\text{th}}$ order derivative is given by

$$\frac{d^ny}{dx^2} = \frac{(-1)^{n-1}.\,(n-1)!2^n}{(2x+3)^n}.$$

Exercise 1.5 | Q 4.1 | Page 60

Find the n$^{\text{th}}$ derivative of the following : $\dfrac{1}{3x-5}$

SOLUTION

Let y = $\dfrac{1}{3x-5}$

Then $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)$

= $-1(3x-5)^{-2}.\dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)$

= $\dfrac{-1}{(3x-5)^2} \times (3 \times 1 - 0)$

= $\dfrac{(-1)^1.3}{(3x-5)^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\dfrac{(-1)^1.3}{(3x-5)^2}\right]$

= $(-1)^1.3\dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)^{-2}$

= $(-1)^{-1}.3.(-2)(3x-5)^{-3}.\dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)$

= $\dfrac{(-1)^2.3.2}{(3x-5)^3} \times (3 \times 1 - 0)$

$\dfrac{d^3y}{dx^3} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[\dfrac{(-1)^2.2!.3^2}{(3x-5)^3}\right]$

= $(-1)^2.2!.3^2.\dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)^{-3}$

= $(-1)^2.2!.3^2.(-3)(3x-5)^{-4}.\dfrac{\mathrm{d}}{\mathrm{dx}}(3x-5)$

= $\dfrac{(-1)^3 \times 3.2! \times 3^2}{(3x-5)^4} \times (3 \times 1 - 0)$

$= \dfrac{(-1)^3 \times 3! \times 3^3}{(3x-5)^4}$

In general, the nth order derivative is goven by

$\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n . n! . 3^n}{(3x-5)^{n+1}}.$

Exercise 1.5 | Q 4.11 | Page 60

Find the n^{th} derivative of the following : y = e^{ax} . cos (bx + c)

SOLUTION

y = e^{ax} . cos (bx + c)

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}[e^{ax}.\cos[bx+c)]$

$= e^{ax}.\dfrac{\mathrm{d}}{\mathrm{dx}}[\cos(bx+c)] + \cos(bx+c).\dfrac{\mathrm{d}}{\mathrm{dx}}(e^{ax})$

$= e^{ax}.[-\sin(bx+c)].\dfrac{\mathrm{d}}{\mathrm{dx}}(bx+c) + \cos(bx+c).e^{ax}.\dfrac{\mathrm{d}}{\mathrm{dx}}(ax)$

= – e^{ax} sin (bx + c) x (b x 1 + 0) + e^{ax}cos(bx + c) x a x 1

= e^{ax} [a cos (bx + c) – b sin (bx + c)]

$= e^{ax}.\sqrt{a^2+b^2}\left[\dfrac{a}{\sqrt{a^2+b^2}}\cos(bx+c) - \dfrac{b}{\sqrt{a^2+b^2}}\sin(bx+c)\right]$

Let $\dfrac{a}{\sqrt{a^2+b^2}} = \cos x$ and $\dfrac{b}{\sqrt{a^2+b^2}} = \sin x$

Then tan ∞ = $\dfrac{b}{a}$

∴ ∞ = $\tan^{-1}\left(\dfrac{b}{a}\right)$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = e^{ax}.\sqrt{a^2+b^2}[\cos\infty.\cos(bx+c) - \sin\infty.\sin(bx+c)]$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\cos(bx+c+x)$$

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left[e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\cos(bx+c+\infty)\right]$$

$$= \left(a^2+b^2\right)^{\frac{1}{2}}.\frac{\mathrm{d}}{\mathrm{dx}}[e^{ax}.\cos(bx+c+\infty)]$$

$$= \left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}\{\cos(bx+c+\infty)\} + \cos(bx+c+\infty).\frac{\mathrm{d}}{\mathrm{dx}}(e^{ax})\right]$$

$$= \left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\{-\sin(bx+c+\infty)\}.\frac{\mathrm{d}}{\mathrm{dx}}(bx+c+\infty) + \cos(bx+c+\infty).e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}(ax)\right]$$

$$= \left(a^2+b^2\right)^{\frac{1}{2}}[-e^{ax}\sin(bx+c+\infty)\times(b\times 1+0+0) + \cos(bx+c+\infty).e^{ax}\times a\times 1]$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}[a\cos(bx+c+\infty) - b\sin(bx+c+\infty)]$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(bx+c+\infty) = \frac{b}{\sqrt{a^2+b^2}}\sin(bx+c+\infty)\right]$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}[\cos\infty.\cos(bx+c+\infty) - \sin\infty.\sin(bx+c+\infty)$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(bx+c+\infty+\infty)$$

$$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(bx+c+2\infty)$$

Similarly.

$$\frac{d^3y}{dx^3} = e^{ax}.\left(a^2+b^2\right)^{\frac{3}{2}}.\cos(bx+c+3\infty)$$

In general, the nth order derivative is given by

$$\frac{d^ny}{dx^n} = e^{ax}.\left(a^2+b^2\right)^{\frac{n}{2}}.\cos(bx+c+n\infty),$$

Where ∞ = $\tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore \frac{d^ny}{dx^n} = e^{ax}.\left(a^2+b^2\right)^{\frac{n}{2}}.\cos\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right].$$

Exercise 1.5 | Q 4.12 | Page 60

Find the n^{th} derivative of the following : $y = e^{8x} . \cos(6x + 7)$

SOLUTION

$y = e^{8x} . \cos(6x + 7)$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}[e^{ax}.\cos(6x+7)]$

$= e^{ax}. \dfrac{\mathrm{d}}{\mathrm{dx}}[\cos(6x+7)] + \cos(6x+7). \dfrac{\mathrm{d}}{\mathrm{dx}}(e^{ax})$

$= e^{ax}. [-\sin(6x+7)]. \dfrac{\mathrm{d}}{\mathrm{dx}}(6x+7) + \cos(6x+7). e^{ax}. \dfrac{\mathrm{d}}{\mathrm{dx}}(ax)$

$= - e^{ax} \sin(6x + 7) \times (b \times 1 + 0) + e^{ax}\cos(6x + 7) \times a \times 1$

$= e^{ax} [a \cos(6x + 7) - b \sin(6x + 7)]$

$= e^{ax}. \sqrt{a^2+b^2}\left[\dfrac{a}{\sqrt{a^2+b^2}}\cos(6x+7) - \dfrac{b}{\sqrt{a^2+b^2}}\sin(6x+7)\right]$

Let $\dfrac{a}{\sqrt{a^2+b^2}} = \cos x$ and $\dfrac{b}{\sqrt{a^2+b^2}} = \sin x$

Then $\tan \infty = \dfrac{b}{a}$

$\therefore \infty = \tan^{-1}\left(\dfrac{b}{a}\right)$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = e^{ax}. \sqrt{a^2+b^2}[\cos\infty. \cos(bx+c) - \sin\infty. \sin(bx+c)]$

$= e^{ax}. \left(a^2+b^2\right)^{\frac{1}{2}}. \cos(6x+7+x)$

$\dfrac{d^2y}{dx^2} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left[e^{ax}. \left(a^2+b^2\right)^{\frac{1}{2}}. \cos(6x+7+\infty)\right]$

$= \left(a^2+b^2\right)^{\frac{1}{2}}.\frac{\mathrm{d}}{\mathrm{dx}}[e^{ax}.\cos(6x+7+\infty)]$

$= \left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}\{\cos(6x+7+\infty)\}+\cos(6x+7+\infty).\frac{\mathrm{d}}{\mathrm{dx}}(e^{ax})\right]$

$= \left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\{-\sin(6x+7+\infty)\}.\frac{\mathrm{d}}{\mathrm{dx}}(6x+7+\infty)+\cos(6x+7+\infty).e^{ax}.\frac{\mathrm{d}}{\mathrm{dx}}(ax)\right]$

$= \left(a^2+b^2\right)^{\frac{1}{2}}[-e^{ax}\sin(6x+7+\infty)\times(b\times1+0+0)+\cos(6x+7+\infty).e^{ax}\times a\times1]$

$= e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}[a\cos(6x+7+\infty)-b\sin(6x+7+\infty)]$

$= e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(6x+7+\infty)=\frac{b}{\sqrt{a^2+b^2}}\sin(6x+7+\infty)\right]$

$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}[\cos\infty.\cos(6x+7+\infty)-\sin\infty.\sin(6x+7+\infty)$

$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+\infty+\infty)$

$= e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+2\infty)$

Similarly.

$\frac{d^3y}{dx^3}=e^{ax}.\left(a^2+b^2\right)^{\frac{3}{2}}.\cos(6x+7+3\infty)$

In general, the n$^{\text{th}}$ order derivative is given by

$\frac{d^ny}{dx^n}=e^{ax}.\left(a^2+b^2\right)^{\frac{n}{2}}.\cos(6x+7+n\infty),$

Where ∞ = $\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore\ \frac{d^ny}{dx^n}=e^{8x}.(10)^n.\cos\left[6x+7+n\tan^{-1}\left(\frac{3}{4}\right)\right].$

MISCELLANEOUS EXERCISE 1 [PAGES 61 - 63]

Miscellaneous Exercise 1 | Q 1 | Page 61

Choose the correct option from the given alternatives :

Let $f(1)=3, f\prime(1)=-\frac{1}{3}, g(1)=-4$ and $g\prime(1)=-\frac{8}{3}$. The derivative of $\sqrt{[f(x)]^2+[g(x)]^2}$ w.r.t. x at x = 1 is

$-\frac{29}{15}$

$\frac{7}{3}$

$\frac{31}{15}$

$\frac{29}{15}$

SOLUTION

$\frac{29}{15}$

[Hint : Let y = $\sqrt{[f(x)]^2 + [g(x)]^2}$

Then $\frac{\text{dy}}{\text{dx}} = \frac{1}{\sqrt{[f(x)]^2 + [g(x)]^2}} . [2f(x). f\prime(x) + 2g(x). g\prime(x)]$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at}x=1} = \frac{1}{\sqrt{[f(x)]^2 + [g(x)]^2}} . [2f(1). f\prime(1) + 2g(1). g\prime(1)]$

= $\frac{1}{2\sqrt{9+16}} \times \left[2(3)\left(-\frac{1}{3}\right) + 2(-4)\left(-\frac{8}{3}\right) = \frac{29}{15}\right]$.

Miscellaneous Exercise 1 | Q 2 | Page 62

Choose the correct option from the given alternatives :

If y = sec (tan $^{-1}$x), then $\frac{\text{dy}}{\text{dx}}$ at x = 1, is equal to

$\frac{1}{2}$

1

$\frac{1}{\sqrt{2}}$

$\sqrt{2}$

SOLUTION

$$\frac{1}{\sqrt{2}}$$

[Hint : $\frac{\mathrm{dy}}{\mathrm{dx}} = \sec\left(\tan^{-1} x\right).\tan\left(\tan^{-1} x\right) \times \frac{1}{1+x^2}$

$\therefore \left(\frac{dy}{dx}\right)_{\mathrm{at}x=1} = \sec\left(\tan^{-1} 1\right) \times 1 \times \frac{1}{1+1^2}$

$= \sec\ \frac{\pi}{4} \times \frac{1}{2} = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}$].

Miscellaneous Exercise 1 | Q 3 | Page 62

Choose the correct option from the given alternatives :

If f(x) = $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$, which of the following is not the derivative of f(x)?

- $\frac{2.4^x.\log 4}{1+4^{2x}}$
- $\frac{4^{x+1}.\log 2}{1+4^{2x}}$
- $\frac{4^{x+1}.\log 4}{1+4^{4x}}$
- $\frac{2^{2(x+1)}.\log 2}{1+2^{4x}}$

SOLUTION

$$\frac{4^{x+1}\log 4}{1+4^{4x}}$$

[Hint : Put 4^x = tanθ. Thenθ = $\tan^{-1}(4^x)$

$\therefore$ f(x) = $\sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}(4^x)$

$\therefore f'(x) = 2 \times \dfrac{1}{1+(4^x)^2} \times 4^x \log 4$

$= \dfrac{2.4^x.\log 4}{1+4^{2x}}$ ... (a)

$= \dfrac{2.4^x.2\log 2}{1+4^{2x}}$

$= \dfrac{4^{x+1}.\log 2}{1+4^{2x}}$ ...(b)

$= \dfrac{\left(2^2\right)^{x+1}.\log 2}{1+2^{4x}}$

$= \dfrac{2^{2(x+1)}.\log 2}{1+2^{4x}}$ ...(d)]

Miscellaneous Exercise 1 | Q 4 | Page 62

Choose the correct option from the given alternatives :

If $x^y = y^x$, then $\dfrac{dy}{dx}$ = ...

$\dfrac{x(x\log y - y)}{y(y\log x - x)}$

$\dfrac{y(x\log y - y)}{x(y\log x - x)}$

$\dfrac{y^2(1-\log x)}{x^2(1-\log y)}$

$\dfrac{y(1-\log x)}{x(1-\log y)}$

SOLUTION

$$\frac{y(x\log y - y)}{x(y\log x - x)}$$

$x^y = y^x$ $\therefore$ y log x = x log y

$$\therefore y \times \frac{1}{x} + (\log x)\frac{\mathrm{dy}}{\mathrm{dx}} = x \times \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} + \log y$$

$$\therefore \left(\log x - \frac{x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \log y - \frac{y}{x}$$

$$\therefore \left(\frac{y\log x - x}{y}\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x\log y - y}{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y(x\log y - y)}{x(y\log x - x)}.$$

Miscellaneous Exercise 1 | Q 5 | Page 62

Choose the correct option from the given alternatives :

If y = sin ($2\sin^{-1}$ x), then $_{\mathrm{dx}}$ = ...

$$\frac{2 - 4x^2}{\sqrt{1 - x^2}}$$

$$\frac{2 + 4x^2}{\sqrt{1 - x^2}}$$

$$\frac{4x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

SOLUTION

$$\frac{2-4x^2}{\sqrt{1-x^2}}$$

[Hint : y = $\sin(2\sin^{-1} x)$

Put x = $\sin\theta$. Then $\theta = \sin^{-1}x$

$\therefore$ y = $\sin 2\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2\cos 2\theta \times \frac{1}{\sqrt{1-x^2}}$

= $\frac{2\left(1-2\sin^2\theta\right)}{\sqrt{1-x^2}} = \frac{2-4x^2}{\sqrt{1-x^2}}$].

Miscellaneous Exercise 1 | Q 6 | Page 62

Choose the correct option from the given alternatives :

If y = $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$ then $\frac{dy}{dx}$ = ...

$\frac{x}{\sqrt{1-x^2}}$

$\frac{1-2x}{\sqrt{1-x^2}}$

$\frac{1-2x}{2\sqrt{1-x^2}}$

$\frac{1-2x^2}{\sqrt{1-x^2}}$

SOLUTION

$$\frac{1-2x}{2\sqrt{1-x^2}}$$

$$y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)+\sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$$

Put x = cos θ. Then θ = $\cos^{-1}$x

$$\therefore y=\tan^{-1}\left(\frac{\cos\theta}{1+\sqrt{1-\cos^2\theta}}\right)+\sin\left[2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

$$=\tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)+\sin\left[2\tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}\right]$$

$$=\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}\right]+\sin\left[2\tan^{-1}\left(\tan\ \frac{\theta}{2}\right)\right]$$

$$=\tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right).\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]+\sin\left(2\times\frac{\theta}{2}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\sin\theta\right]$$

$$=\frac{\pi}{4}-\frac{\theta}{2}+\sqrt{1-\cos^2\theta}$$

$$=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x+\sqrt{1-x^2}$$

$$\therefore\frac{dy}{dx}=0-\frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{1-x^2}}\times(-2x)$$

$$= \frac{1}{2\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1-2x}{2\sqrt{1-x^2}}.$$

Miscellaneous Exercise 1 | Q 7 | Page 62

Choose the correct option from the given alternatives :

If y is a function of x and log (x + y) = 2xy, then the value of y'(0) = ...

- 2
- 0
- −1
- **1**

SOLUTION

1

[Hint : log(x + y) = 2xy ...(1)

$$\therefore \frac{1}{\mathrm{x}+\mathrm{y}} \cdot \left(1 + \frac{\mathrm{dy}}{\mathrm{dx}}\right) = 2x\frac{\mathrm{dy}}{\mathrm{dx}} + 2y$$

$$\therefore \left(\frac{1}{x+y} - 2x\right)\frac{\mathrm{dy}}{\mathrm{dx}} = 2y - \frac{1}{\mathrm{x}+\mathrm{y}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{2y(x+y)-1}{1-2x(x+y)}$$

If x = 0, then from (1),

log y = 0 = log 1

∴ y = 1

$$\therefore \text{y}'(0) = \frac{(2(1)(0+1)-1)}{1-2(0)(0+1)} = 1\Big].$$

Miscellaneous Exercise 1 | Q 8 | Page 62

Choose the correct option from the given alternatives :

If g is the inverse of function f and f'(x) = $\dfrac{1}{1+x}$, then the value of g'(x) is equal to :

1 + x^7

$\dfrac{1}{1+[g(x)]^7}$

1 + $[g(x)]^7$

$7x^6$

SOLUTION

1 + $[g(x)]^7$

[Hint : Since g is the inverse of f, $f^{-1}(x) = g(x)$

∴ $f[f^{-1}(x)] = f[g(x)] = x$

∴ $f\prime[g(x)]. \dfrac{\mathrm{d}}{\mathrm{dx}}[g(x)]$ = 1

∴ f'[g(x)] x g'(x) = 1

∴ g'(x) = $\dfrac{1}{f\prime[g(x)]}, \text{where} f\prime(x) = \dfrac{1}{1+x^7}$

∴ g'(x) = 1 + $[g(x)^7]$.

Miscellaneous Exercise 1 | Q 9 | Page 62

Choose the correct option from the given alternatives :

If $x\sqrt{y+1} + y\sqrt{x+1} = 0$ and $x \neq y$, then $\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = ...

$\dfrac{1}{(1+x)^2}$

$-\dfrac{1}{(1+x)^2}$

(1 + x)^2

$-\dfrac{x}{x+1}$

SOLUTION

$-\dfrac{1}{(1+x)^2}$

[Hint : $x\sqrt{y-1}=-y\sqrt{x+1}$

∴ $x^2(y + 1) = y^2(x + 1)$

∴ $x^2y + x^2 = xy^2 + y^2$

∴ $x^2 - y^2 = xy^2 - x^2y$

∴ $(x - y)(x + y) = - xy(x - y)$

∴ $x + y = - xy$ ...[∵ $x \neq y$] ...(1)

DIfferentiating both sides w.r.t. x, we get

$1+\dfrac{dy}{dx}=-x\dfrac{dy}{dx}-y$

∴ $(1+x)\dfrac{dy}{dx}$ = − 1 − y

∴ $(1+x)\dfrac{dy}{dx}$

= − (1 + x)(1 + y)

= − (1 + x + y + xy)

= − (1 − xy + xy) ...[By (1)]

= − 1

∴ $\dfrac{dy}{dx}=-\dfrac{1}{(1+x)^2}$].

Miscellaneous Exercise 1 | Q 10 | Page 62

Choose the correct option from the given alternatives :

If y $\tan^{-1}\left(\sqrt{\dfrac{a-x}{a+x}}\right)$, where – a < x < a, then $\dfrac{\mathrm{dy}}{\mathrm{dx}}$ = ...

- $\dfrac{x}{\sqrt{a^2-x^2}}$
- $\dfrac{a}{\sqrt{a^2-x^2}}$
- $-\dfrac{1}{2\sqrt{a^2-x^2}}$
- $\dfrac{1}{2\sqrt{a^2-x^2}}$

SOLUTION

$$-\frac{1}{2\sqrt{a^2-x^2}}$$

[Hint : Put x =a cos θ].

Miscellaneous Exercise 1 | Q 11 | Page 63

Choose the correct option from the given alternatives :

If x = a(cosθ + θ sinθ), y = a(sinθ – θ cosθ), then $\left(\dfrac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}}$ = ...

- $\dfrac{8\sqrt{2}}{a\pi}$
- $-\dfrac{8\sqrt{2}}{a\pi}$
- $\dfrac{a\pi}{8\sqrt{2}}$
- $\dfrac{4\sqrt{2}}{a\pi}$

SOLUTION

$$\frac{8\sqrt{2}}{a\pi}$$

[Hint : $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{d\theta}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{d\theta}}\right)}$

$= \frac{a(\cos\theta + \theta\sin\theta - \cos\theta)}{a(-\sin\theta + \theta\cos\theta + \sin\theta)}$

$= \frac{a\theta\sin\theta}{a\theta\cos\theta}$

= $\tan\theta$

and

$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}(\tan\theta)$

$= \frac{\mathrm{d}}{\mathrm{d\theta}}(\tan\theta) \times \frac{\mathrm{d\theta}}{\mathrm{dx}}$

$= \sec^2\theta \times \frac{1}{a\theta\cos\theta}$

$= \frac{1}{a\theta}.\sec^3\theta$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{\mathrm{at}\theta=\frac{\pi}{4}} = \frac{1}{a\left(\frac{\pi}{4}\right)}\left(\sec\frac{\pi}{4}\right)^3$

$= \frac{4}{a\pi} \times \left(\sqrt{2}\right)^3$

$= \frac{8\sqrt{2}}{a\pi}$].

Miscellaneous Exercise 1 | Q 12 | Page 63

Choose the correct option from the given alternatives :

If y = $a\cos(\log x)$ and $\mathrm{A}\dfrac{d^2y}{dx^2} + \mathrm{B}\dfrac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{C}$ = 0, then the values of A, B, C are

- $x^2, -x, -y$
- **x^2, x, y**
- $x^2, x, -y$
- $x^2, -x, y$

SOLUTION

x^2, x, y

[Hint : y = a cos (log x) ...(1)

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = a[-\sin(\log x)] \times \frac{1}{x}$$

$$\therefore x\frac{\mathrm{dy}}{\mathrm{dx}} = -a\sin(\log x)$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}} = -a\cos(\log x) \times \frac{1}{x}$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}} = -y \qquad \text{...[By (1)]}$$

$$\therefore x2\frac{d^2y}{dx^2} + x\frac{\mathrm{dy}}{\mathrm{dx}} + y = 0$$

Comparing this with $\mathrm{A}\dfrac{d^2y}{dx^2} + \mathrm{B}\dfrac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{C}$ = 0, we get A = x^2, B = x, C = y.

MISCELLANEOUS EXERCISE 1 [PAGES 63 - 64]

Miscellaneous Exercise 1 | Q 1 | Page 63

Solve the following :

f(x) = –x, for – 2 ≤ x < 0

= 2x, for 0 ≤ x < 2

= $\frac{18-x}{4}$, for 2 < x ≤ 7

g(x) = 6 – 3x, for 0 ≤ x < 2

= $\frac{2x-4}{3}$, for 2 < x ≤ 7

Let u (x) = f[g(x)], v(x) = g[f(x)] and w(x) = g[g(x)]. Find each derivative at x = 1, if it exists i.e. find u'(1), v' (1) and w'(1). If it doesn't exist, then explain why?

SOLUTION

u(x) = f[g(x)]

$\therefore u\prime(x) = \frac{\text{d}}{\text{dx}}\{f[g(x)\}$

= $f\prime[g(x)].\frac{\text{d}}{\text{dx}}[g(x)]$

= f'[gx)] x g'(x)

∴ u'(1) = f'[g(1)] x g'(1)

= f'(3) x g'(1) ...(1)

...[∵ g(x) = 6 – 3x, 0 ≤ x ≤ 2]

Now, f(x) = $\frac{18-x}{4}$, for 2 < x ≤ 7

and g(x) = 6 – 3x, for 0 < x ≤ 2

∴ f'(x) = $\frac{1}{4}(0-1) = -\frac{1}{4}$, for 2 < x ≤ 7

and g'(x) = 0 – 3(1) = – 3, for 0 < x ≤ 2

$\therefore f\prime(3) = -\frac{1}{4}$ and $g\prime(1)$ = – 3

∴ from (1),

u'(1) = $-\frac{1}{4}(-3) = \frac{3}{4}$

Now, v(x) = g[f(x)]

∴ v'(x) = $\frac{\mathrm{d}}{\mathrm{dx}}\{g[f(x)]\}$

= $g\prime[f(x)].\frac{\mathrm{d}}{\mathrm{dx}}[f(x)]$

= g'[f(x)] x f'(x)

∴ v'(1) = g'[f(1)] x f'(x)

= g'(2) x f'(1) ...(2)

...[∵ f(x) = 2x, 0 ≤ x ≤ 2]

Now, g(x) = 6 – 3x, for 0 ≤ x ≤ 2

= $\frac{2x-4}{3}$, for 2 < x ≤ 7

∴ g''(x) = 0 – 3 x 1 = – 3, for 0 ≤ x ≤ 2

and g'(x) = $\frac{1}{3}(2\times1-0)=\frac{2}{3}$, for 2 < x ≤ 7

∴ Lg'(2) ≠ Rg'(2)

∴ g'(2) does not exist

∴ from (2),

v'(1) does not exist

Also, w(x) = g[g(x)]

∴ w'(x) = $\frac{\mathrm{d}}{\mathrm{dx}}\{g[g(x)]\}$

= $g\prime[g(x)].\frac{\mathrm{d}}{\mathrm{dx}}[g(x)]$

= g'[g(x)] x g'(x)

∴ w'(1) = g'[g(1)] x g'(x)

= g'(3) x g'(1) ...(3)

...[∵ g(x) = 6 – 3x, 0 ≤ x ≤ 2]

Now, g(x) = 6 –3x, for 0 ≤ x ≤ 2

= $\frac{2x-4}{3}$, for 2 < x ≤ 7

$\therefore$ g'(x) = 0 – 3 x 1 = – 3, for 0 ≤ x ≤ 2

and g'(x) = $\frac{1}{3}(2 \times 1 - 0) = \frac{2}{3}$, for 2 ≤ x ≤ 7

$\therefore$ g(3) = $\frac{2}{3}$ and $g\prime(1)$ = – 3

$\therefore$ from (3),

w'(1) = $\frac{2}{3}(-3)$ = – 2.

Hence, u'(1) = $\frac{3}{4}$, v'(1) does not exist and w'(1) = – 2.

Miscellaneous Exercise 1 | Q 2 | Page 63

Solve the following :

The values of f(x), g(x), f'(x) and g'(x) are given in the following table :

x	f(x)	g(x)	f'(x)	fg'(x)
– 1	3	2	– 3	4
2	2	– 1	– 5	– 4

Match the following :

A Group – Function	B Group – Derivative
(A) $\frac{\mathrm{d}}{\mathrm{dx}}[f(g(x))]\mathrm{at}x = -1$	1. – 16
(B) $\frac{\mathrm{d}}{\mathrm{dx}}[g(f(x) - 1)]\mathrm{at}x = -1$	2. 20
(C) $\frac{\mathrm{d}}{\mathrm{dx}}[f(f(x) - 3)]\mathrm{at}x = 2$	3. – 20
(D) $\frac{\mathrm{d}}{\mathrm{dx}}[g(g(x))]\mathrm{at}x = 2$	5. 12

SOLUTION

(A) $\frac{\mathrm{d}}{\mathrm{dx}}[f(g(x))]$

= $f\prime(g(x)).\frac{\mathrm{d}}{\mathrm{dx}}(g(x))$

= f'(g(x)) xg'(x)

∴ $\frac{\mathrm{d}}{\mathrm{dx}}[f(g(x))]$ at x = – 1

= f'(g(– 1)) x g'(– 1)

= f'(2) x g'(– 1) ...[∵ g(x) = 2, when x = – 1]

= – 5 x 4

= – 20

(B) $\frac{\mathrm{d}}{\mathrm{dx}}[g(f(x)-1)]$

= $g\prime(f(x)-1).\frac{\mathrm{d}}{\mathrm{dx}}[f(x)-1]$

= g'(f(x) – 1) x [f'(x) – 0]

∴ $\frac{\mathrm{d}}{\mathrm{dx}}[gf(x)-1]$ at x = – 1

= g'(f(– 1)– 1) xx f'(–1)

= g'(3 – 1) x f'(– 1) ...[∵ f(x) 33, when x = – 1]

= g'(2) x f'(– 1)

= (– 4)(– 3)

= 12

(C) $\frac{\mathrm{d}}{\mathrm{dx}}[f(f(x)-3)]$

= $f\prime(f(x)-3).\frac{\mathrm{d}}{\mathrm{dx}}[f(x)-3]$

= f'(f(x) – 3) x [f'(x) – 0]

$\therefore \dfrac{\mathrm{d}}{\mathrm{dx}}[f(f(x) - 3)]$ at = 2

= f"(f(2) – 3) x f'(2)

= f'(2 – 3) x f'(2) ...[∵ f(x) = 2, when x = 2]

= f'(– 1) x f'(2)

= (– 3)(– 5)

= 15

(D) $\dfrac{\mathrm{d}}{\mathrm{dx}}[g(g(x))]$

= $g\prime(g(x)). \dfrac{\mathrm{d}}{\mathrm{dx}}[g(x)]$

= g'(g(x)) x g'(x)

$\therefore \dfrac{\mathrm{d}}{\mathrm{dx}}[g(g(x))]$at x = 2

= g'(g(2)) x g'(2)

= g'(– 1) c g'(2) ...[∵ g(x) = – 1at x = 2]

= 4(– 4)

= – 16

Hence, (A) →3, (B) → 5, (C) → 4, (D) → 1.

Miscellaneous Exercise 1 | Q 3 | Page 63

Suppose that the functions f and g and their derivatives with respect to x have the following values at x = 0 and x = 1:

x	f(x)	g(x)	f')x)	g'(x)
0	1		5	$\frac{1}{3}$
1	3	– 4	$-\frac{1}{3}$	$-\frac{8}{3}$

(i) The derivative of f[g(x)] w.r.t. x at x = 0 is ......

(ii) The derivative of g[f(x)] w.r.t. x at x = 0 is ......

(iii) The value of $\left[\frac{d}{dx}\left[x^{10}+f(x)\right]^{-2}\right]_{x=1}$ is ....

(iv) The derivative of f[(x + g(x))] w.r.t. x at x = 0 is ...

SOLUTION

(i) $\frac{d}{dx}\{f[g(x)]\}$

= $f\prime[g(x)].\frac{d}{dx}[g(x)]$

= $f\prime[g(x)]xg\prime(x)$

∴ $\frac{d}{dx}\{f[g(x)]\}$ at x = 0

= $f\prime[g(0)]xg\prime(0)$

= $f\prime(1)xg\prime(0)$ ...[∵ g(x) = 1 at x = 0]

= $-\frac{1}{3}\times\frac{1}{3}$

= $-\frac{1}{9}$.

(ii) $\frac{d}{dx}\{g[f(x)]\}$

= $g\prime[f(x)].\frac{d}{dx}[f(x)]$

= $g\prime[f(x)]xf\prime(x)$

∴ $\frac{d}{dx}\{f[g(x)]\}$ at x = 0

$= g\prime(1) x f\prime(0)$ ...[∵ f(x) = 1 at x = 0]

$= -\frac{8}{3} \times 5$

$= -\frac{40}{3}.$

(iii) $\frac{d}{dx}\left[x^{10} + f(x)\right]^{-2}$

$= 2\left[x^{10} + f(x)\right]^{-3}. \frac{d}{dx}\left[x^{10} + f(x)\right]$

$= -2\left[x^{10} + f(x)\right]^{-3} \times \left[10x^{9} + f\prime(x)\right]$

$\therefore \left\{\frac{d}{dx}\left[x^{10}\ f(x)\right]^{-2}\right\}_{\text{at} x=1}$

$= -2\left[1^{10} + f(1)\right]^{-3} \times \left[10(1)^{9} + f\prime(1)\right]$

$= \frac{-2}{(1+3)^{3}} \times \left[10 + \left(-\frac{1}{3}\right)\right]$ ...[∵ f(x) = 3 at x = 1]

$= \frac{-2}{64} \times \frac{29}{3}$

$= -\frac{29}{96}.$

(iv) $\frac{d}{dx}[f(x + g(x))]$

$= f\prime(x + g(x)). \frac{d}{dx}[x + g(x)]$

$= f\prime(x + g(x)) \times [1 + g\prime(x)]$

$\therefore \left\{\frac{d}{dx}[f(x + g(x))]\right\}_{\text{at} x=0}$

$= f\prime(0 + g(0)) x [1 + g\prime(0)]$

$= f\prime(1). [1 + g\prime(0)]$ ...[∵ g(x) = 1 at x = 0]

$= -\frac{1}{3}\left[1 + \frac{1}{3}\right]$

$= -\frac{1}{3} \times \frac{4}{3}$

$= -\frac{4}{9}$.

Miscellaneous Exercise 1 | Q 4.1 | Page 64

Differentiate the following w.r.t. x : $\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$

SOLUTION

Let y = $\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$

Put x = cosθ. Thenθ = $\cos^{-1}$x and

$\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$= \sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}$

$= \sqrt{\tan^2\left(\frac{\theta}{2}\right)}$

$= \tan\left(\frac{\theta}{2}\right)$

$\therefore \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$

$= \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]$

$= \frac{\theta}{2}$

$= \frac{1}{2}\cos^{-1} x$

$\therefore$ y = $\sin\left[2 \times \frac{1}{2}\cos^{-1} x\right]$

= sin ($\cos^{-1}$x)

$\therefore \frac{\text{dy}}{\text{dx}} = \frac{\text{d}}{\text{dx}}[\sin(\cos{-1}x)]$

$= \cos\left(\cos^{-1} x\right). \frac{\text{d}}{\text{dx}}\left(\cos^{-1} x\right)$

$= x \times \frac{-1}{\sqrt{1-x^2}}$

$= \frac{-x}{\sqrt{1-x^2}}.$

Miscellaneous Exercise 1 | Q 4.2 | Page 64

Differentiate the following w.r.t. x : $\sin^2\left[\cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)\right]$

SOLUTION

Let y = $\sin^2\left[\cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)\right]$

Put x = cosθ. Thenθ = $\cos^{-1}$x and

$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

= $\sqrt{\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}}$

= $\sqrt{\cot^2\left(\frac{\theta}{2}\right)}$

= $\cot\left(\frac{\theta}{2}\right)$

$\therefore$ $\cot^{-1}\sqrt{\frac{1+x}{1-x}}$

= $\cot^{-1}\left[\cot\left(\frac{\theta}{2}\right)\right]$

= $\frac{\theta}{2}$

= $\frac{1}{2}\cos^{-1}x$

$\therefore$ y = $\sin^2\left(\frac{1}{2}\cos^{-1}x\right)$

$\therefore$ $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\sin\left(\frac{1}{2}\cos^{-1}x\right)\right]^2$

= $2\sin\left(\frac{1}{2}\cos^{-1}x\right).\frac{\mathrm{d}}{\mathrm{dx}}\sin\left(\frac{1}{2}\cos^{-1}x\right)$

$= 2\sin\left(\frac{1}{2}\cos^{-1}x\right).\cos\left(\frac{1}{2}\cos^{-1}x\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{2}\cos^{-1}x\right)$

$= \sin\left[2\left(\frac{1}{2}\cos^{-1}x\right)\right] \times \frac{1}{2}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\cos^{-1}x\right)$

$= \sin\left(\cos^{-1}x\right) \times \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}}$

$= \sin\left(\sin^{-1}\sqrt{1-x^2}\right) \times \frac{-1}{2\sqrt{1-x^2}}$ ...$\left[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$

$= \sqrt{1-x^2} \times \frac{-1}{2\sqrt{1-x^2}}$

$= -\frac{1}{2}.$

Miscellaneous Exercise 1 | Q 4.3 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$

$= \tan^{-1}\left[\frac{3\sqrt{x}-x\sqrt{x}}{1-3x}\right]$

Put $\sqrt{x} = \tan\theta$. Thenθ = $\tan^{-1}\left(\sqrt{x}\right)$

∴ y = $\tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

= $\tan^{-1}(\tan 3\theta)$

= 3θ

$= 3\tan^{-1}(\sqrt{x})$

$\therefore \frac{dy}{dx} = 3\frac{d}{dx}[\tan^{-1}(\sqrt{x})]$

$= 3 \times \frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x})$

$= \frac{3}{1+x} \times \frac{1}{2\sqrt{x}}$

$= \frac{3}{2\sqrt{x}(1+x)}.$

Miscellaneous Exercise 1 | Q 4.4 | Page 64

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

SOLUTION

Let y = $\cos^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

Put x = cosθ. Thenθ = $\cos^{-1}$x and

$\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

$= \left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{2}\right)$

$= \frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{2}$

$$= \frac{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{\sqrt{2}}$$

$$= \left[\cos\ \frac{\theta}{2}\right]\left(\frac{1}{\sqrt{2}}\right) - \left[\sin\ \frac{\theta}{2}\right]\left(\frac{1}{\sqrt{2}}\right)$$

$$= \cos\ \frac{\theta}{2}.\cos\ \frac{\pi}{4} - \sin\ \frac{\theta}{2}.\sin\ \frac{\pi}{4}$$

$$= \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$$

$$\therefore y = \cos^{-1}\left[\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right]$$

$$= \frac{\theta}{2} + \frac{\pi}{4}$$

$$= \frac{1}{2}\cos^{-1} x + \frac{\pi}{4}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\cos^{-1} x\right) + \frac{d}{dx}\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}} + 0$$

$$= \frac{-1}{2\sqrt{1-x^2}}.$$

Miscellaneous Exercise 1 | Q 4.5 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{x}{1+6x^2}\right) + \cot^{-1}\left(\frac{1-10x^2}{7x}\right)$

SOLUTION

Let y = $\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\cot^{-1}\left(\frac{1-10x^2}{7x}\right)$

= $\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\cot^{-1}\left(\frac{7x}{1-10x^2}\right)$ ...$\left[\because \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\right]$

= $\tan^{-1}\left[\frac{3x-2x}{1+(3)(2x)}\right]+\tan^{-1}\left[\frac{5x+2x}{1-(5x)(2x)}\right]$

= $\tan^{-1}3x - \tan^{-1}2x + \tan^{-1}5x + \tan^{-1}2x$

= $\tan^{-1}3x + \tan^{-1}5x$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}\left[\tan^{-1}3x+\tan^{-1}5x\right]$

= $\frac{\mathrm{d}}{\mathrm{dx}}\left(\tan^{-1}3x\right)+\frac{\mathrm{d}}{\mathrm{dx}}\left(\tan^{-1}5x\right)$

= $\frac{1}{1+(3x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(3x)+\frac{1}{1+(5x)^2}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(5x)$

= $\frac{1}{1+9x^2}\times 3\times 1+\frac{1}{1+25x^2}\times 5\times 1$

= $\frac{3}{1+9x^2}+\frac{5}{1+25x^2}$.

Miscellaneous Exercise 1 | Q 4.6 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$

SOLUTION

Let y = $\tan^{-1}\left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$

Put x = tanθ. Thenθ = $\tan^{-1}$x

$\therefore \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} = \frac{\sqrt{1+\tan^2\theta}+\tan\theta}{\sqrt{1+\tan^2\theta}-\tan\theta}$

= $\frac{\sec\theta+\tan\theta}{\sec\theta - \tan\theta}$

= $\frac{\left(\frac{1}{\cos\theta}\right)+\left(\frac{\sin\theta}{\cos\theta}\right)}{\left(\frac{1}{\cos\theta}\right)-\left(\frac{\sin\theta}{\cos\theta}\right)}$

= $\frac{1+\sin\theta}{1-\sin\theta}$

= $\frac{1-\cos\left(\frac{\pi}{2}+\theta\right)}{1+\cos\left(\frac{\pi}{2}+\theta\right)}$

= $\frac{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$

= $\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\therefore \sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$

= $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\therefore$ y = $\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

= $\frac{\pi}{4}+\frac{\theta}{2}$

= $\frac{\pi}{4}+\frac{1}{2}\tan^{-1}x$

$\therefore \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{1}{2}\frac{d}{dx}\left(\tan^{-1}x\right)$

= $0+\frac{1}{2}\times\frac{1}{1+x^2}$

= $\frac{1}{2(1+x^2)}$.

Miscellaneous Exercise 1 | Q 5.1 | Page 64

If $\sqrt{y+x}+\sqrt{y-x}$ = c, show that $\frac{dy}{dx}=\frac{y}{x}-\sqrt{\frac{y^2}{x^2}-1}$.

SOLUTION

$\sqrt{y+x}+\sqrt{y-x}$ = c

Differentiating both sides w.r.t. x, we get

$\frac{1}{2\sqrt{y+x}}\cdot\frac{d}{dx}(y+x)+\frac{1}{2\sqrt{y-x}}\cdot\frac{d}{dx}(y-x)$ = 0

$\therefore \frac{1}{\sqrt{y+x}}\cdot\left(\frac{dy}{dx}+1\right)+\frac{1}{\sqrt{y-x}}\cdot\left(\frac{dy}{dx}-1\right)$ = 0

$\therefore \frac{1}{\sqrt{y+x}}\cdot\frac{dy}{dx}+\frac{1}{\sqrt{y+x}}+\frac{1}{\sqrt{y-x}}\cdot\frac{dy}{dx}-\frac{1}{\sqrt{y-x}}$ = 0

$\therefore \left(\frac{1}{\sqrt{y+x}}+\frac{1}{\sqrt{y-x}}\right)\frac{dy}{dx}=\frac{1}{\sqrt{y-x}}-\frac{1}{\sqrt{y+x}}$

$$\therefore \left[\frac{\sqrt{y-x}+\sqrt{y+x}}{\sqrt{y+x}.\sqrt{y-x}}\right]\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\sqrt{y+x}+\sqrt{y-x}}{\sqrt{y-x}.\sqrt{y+x}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\sqrt{y+x}+\sqrt{y-x}}{\sqrt{y+x}.\sqrt{y-x}}$$

$$= = \frac{\sqrt{y+x}+\sqrt{y-x}}{\sqrt{y+x}+\sqrt{y-x}} \times \frac{\sqrt{y+x}+\sqrt{y-x}}{\sqrt{y+x}-\sqrt{y-x}}$$

$$= \frac{\left(\sqrt{y+x}-\sqrt{y-x}^2\right)}{(y+x)-(y-x)}$$

$$= \frac{y+x+y-x-2\sqrt{y+x}.\sqrt{y-x}}{y+x-y+x}$$

$$= \frac{2y-2\sqrt{y^2-x^2}}{2x}$$

$$= \frac{2y}{2x} - \frac{2\sqrt{y^2-x^2}}{2x}$$

$$= \frac{y}{x} - \sqrt{\frac{y^2-x^2}{x^2}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2}-1}.$$

Miscellaneous Exercise 1 | Q 5.2 | Page 64

If $x\sqrt{1-y^2}+y\sqrt{1-x^2}$ = 1, then show that $\frac{\mathrm{dy}}{\mathrm{dx}} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

SOLUTION

$x\sqrt{1-y^2}+y\sqrt{1-x^2}$ = 1

$\therefore y\sqrt{1-x^2}+x\sqrt{1-y^2}$ = 1

Differentiating both sides w.r.t. x, we get

$y.\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}+x.\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{1-y^2}\right)+\sqrt{1-y^2}.\frac{\mathrm{d}}{\mathrm{dx}}(x)$ = 0

$\therefore y\times\frac{1}{2\sqrt{1-x^2}}.\frac{\mathrm{d}}{\mathrm{dx}}(1-x^2)+\sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}+x\times\frac{1}{2\sqrt{1-y^2}}.\frac{\mathrm{d}}{\mathrm{dx}}(1-y^2)+\sqrt{1-y^2}\times 1$ = 0

$\therefore \frac{y}{2\sqrt{1-x^2}}\times(0-2x)+\sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}+\frac{x}{2\sqrt{1-y^2}}\times\left(0-2y\frac{\mathrm{dy}}{\mathrm{dx}}\right)+\sqrt{1-y^2}$ = 0

$\therefore \frac{-xy}{\sqrt{1-x^2}}+\sqrt{1-x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}-\frac{\mathrm{xy}}{\sqrt{1-y^2}}.\frac{\mathrm{dy}}{\mathrm{dx}}+\sqrt{1-y^2}$ = 0

$\therefore \left(\sqrt{1-x^2}-\frac{\mathrm{xy}}{\sqrt{1-y^2}}\right)\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{xy}}{\sqrt{1-x^2}}-\sqrt{1-y^2}$

$\therefore \left[\frac{\sqrt{1-x^2}.\sqrt{1-y^2}-xy}{\sqrt{1-y^2}}\right]\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{xy-\sqrt{1-x^2}.\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\therefore \frac{1}{\sqrt{1-x^2}}.\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{-1}{\sqrt{1-x^2}}$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=-\sqrt{\frac{1-y^2}{1-x^2}}.$

Miscellaneous Exercise 1 | Q 5.3 | Page 64

If x sin (a + y) + sin a . cos (a + y) = 0, then show that $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sin^2(a+y)}{\sin a}$.

SOLUTION

x sin (a + y) + sin a . cos (a + y) = 0 ...(1)

Differentiating w.r.t. x, we get

$$x\frac{\mathrm{d}}{\mathrm{dx}}[\sin(a+y)]+\sin(a+y).\,\frac{\mathrm{d}}{\mathrm{dx}}(x)+(\sin a).\,\frac{\mathrm{d}}{\mathrm{dx}}[\cos(a+y)]=0$$

$$\therefore x\cos(a+y).\,\frac{\mathrm{d}}{\mathrm{dx}}(a+y)+\sin(a+y)\times 1+(\sin a)\,[-\sin(a+y)].\,\frac{\mathrm{d}}{\mathrm{dx}}(a+y)=0$$

$$\therefore x\cos(a+y).\left(0+\frac{dy}{dx}\right)+\sin(a+y)-\sin a.\sin(a+y)\left(0+\frac{dy}{dx}\right)=0$$

$$\therefore x\cos(a+y)\frac{\mathrm{dy}}{\mathrm{dx}}+\sin(a+y)-\sin a.\sin(a+y)\frac{\mathrm{dy}}{\mathrm{dx}}=0$$

$$\therefore \sin a.\sin(a+y)\frac{\mathrm{dy}}{\mathrm{dx}}-x\cos(a+y)\frac{\mathrm{dy}}{\mathrm{dx}}=\sin(a+y)$$

$$\therefore [\sin a.\sin(a+y)-x\cos(a+y)]\frac{\mathrm{dy}}{\mathrm{dx}}=\sin(a+y)$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sin(a+y)}{\sin a.\sin(a+y)-x\cos(a+y)}$$

From (1),

$$x=\frac{-\sin a.\cos(a+y)}{\sin(a+y)}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sin(a+y)}{\sin a.\sin(a+y)+\frac{\sin a.\cos(a+y)}{\sin(a+y)}.\cos(a+y)}$$

$$=\frac{\sin^2(a+y)}{\sin a.\sin^2(a+y)+\sin a.\cos^2(a+y)}$$

$$=\frac{\sin^2(a+y)}{\sin a\left[\sin^2(a+y)+\cos^2(a+y)\right]}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sin^2(a+y)}{\sin a}.$$

Miscellaneous Exercise 1 | Q 5.4 | Page 64

If sin y = x sin (a + y), then show that $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\sin^2(a+y)}{\sin a}$.

SOLUTION

sin y = x sin(a + y)

$\Rightarrow x = \dfrac{\sin y}{\sin(a+y)}$ ....(i)

Differentiating (i) w.r.t.x,

$\Rightarrow 1 = \dfrac{\sin(a+y).\left(\frac{d}{dx}\sin y\right) - \sin y.\left(\frac{d}{dx}\sin(a+y)\right)}{\sin^2(a+y)}$

$\Rightarrow \sin(a+y).\cos y - \dfrac{d}{dx} - \sin y.\cos(a+y).\dfrac{d}{dx} = \sin^2(a+y)]$

$\Rightarrow \dfrac{d}{dx}[\sin(a+y).\cos y - \sin y.\cos(a+y)] = \sin^2(a+y)$

$\Rightarrow \dfrac{dy}{dx}[\sin(a+y-y)] = \sin^2(a+y)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

Hence proved.

Miscellaneous Exercise 1 | Q 5.5 | Page 64

If $x = e^{\frac{x}{y}}$, then show that $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{x-y}{x\log x}$

SOLUTION

$x = e^{\frac{x}{y}}$

$\therefore \dfrac{x}{y}$ = log x ...(1)

$\therefore$ y = $\dfrac{x}{\log x}$

$\therefore \dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{\mathrm{d}}{\mathrm{dx}}\left(\dfrac{x}{\log x}\right)$

$$= \frac{(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(x) - x.\frac{\mathrm{d}}{\mathrm{dx}}(\log x)}{(\log x)}$$

$$= \frac{(\log x) \times 1 - x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{\log x - 1}{(\log x)(\log x)}$$

$$= \frac{\frac{x}{y} - 1}{\left(\frac{x}{y}\right)(\log x)} \qquad \text{...[By (1)]}$$

$$= \frac{x - y}{x \log x}.$$

Miscellaneous Exercise 1 | Q 5.6 | Page 64

If y = f(x) is a differentiable function of x, then show that $\frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3}.\frac{d^2y}{dx^2}$.

SOLUTION

If y = f(x) is a differentiable function of x such that inverse function x = f^{-1}(y) exists, then

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{1}{\left(\frac{dy}{dx}\right)}, \text{where} \frac{\mathrm{dy}}{\mathrm{dx}} \neq 0$$

$$\therefore \frac{d^2x}{dy^2} = \frac{\mathrm{d}}{\mathrm{dy}}\left(\frac{dx}{dy}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{dy}}\left[\frac{1}{\left(\frac{dy}{dx}\right)}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{Dx}}\left(\frac{dy}{dx}\right)^{-1} \times \frac{\mathrm{dx}}{dy}$$

$$= -1\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right) \times \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx}\right)^{-1}$$

$$\therefore \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3} \cdot \frac{d^2y}{dx^2}.$$

Miscellaneous Exercise 1 | Q 6.1 | Page 64

DIfferentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) w.r.t. \tan^{-1}\left(\sqrt{\frac{2x\sqrt{1+x^2}}{1-2x^2}}\right).$

SOLUTION

Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

and

v = $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right).$

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$

u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put x = tanθ.

Thenθ = $\tan^{-1}$ x

and

$$\frac{\sqrt{1++x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \frac{\sec\theta - 1}{\tan\theta}$$

$$= \frac{\frac{1}{\cos\theta} - 1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$= \frac{1 - \cos\theta}{\sin\theta}$$

$$= \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

$$\therefore u = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

$$\therefore \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{1}{2} \times \frac{1}{1 + x^2}$$

$$= \frac{1}{2(1 + x^2)}$$

$$v = \tan^{-1}\left(\frac{2x\sqrt{1 - x^2}}{1 - 2x^2}\right)$$

Put $x = \sin\theta$.

Thenθ = $\sin^{-1}x$

and

$$\frac{2x\sqrt{1-x^2}}{1-2x^2}$$

$$= \frac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

= tan2θ

$\therefore$ v = $\tan^{-1}(\tan 2\theta)$

= 2θ

= $2\sin^{-1}x$

$$\therefore \frac{dv}{dx} = 2\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left[\frac{1}{2(1+x^2)}\right]}{\left(\frac{2}{\sqrt{1-x^2}}\right)}$$

$$= \frac{1}{2(1+x^2)} \times \frac{\sqrt{1-x^2}}{2}$$

$$= \frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

Miscellaneous Exercise 1 | Q 6.2 | Page 64

Differentiate log $\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right]$ w.r.t. cos (log x).

SOLUTION

Let y = log $\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right]$ and v = cos (log x)

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$.

u = $\log\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\times\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right)$

= $\log\left[\frac{\left(\sqrt{1+x^2}+x\right)^2}{1+x^2-x^2}\right]$

= $2\log\left(\sqrt{1+x^2}+x\right)$

$\therefore\ \frac{\mathrm{du}}{\mathrm{dx}}=2\frac{\mathrm{d}}{\mathrm{dx}}\left[\log\left(\sqrt{1+x^2}+x\right)\right]$

= $\frac{2}{\sqrt{1+x^2}+x}\cdot\frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{1+x^2}+x\right)$

= $\frac{2}{\sqrt{1+x^2}+x}\cdot\left[\frac{1}{2\sqrt{1+x^2}}\cdot\frac{\mathrm{d}}{\mathrm{dx}}\left(1+x^2\right)+1\right]$

$$= \frac{2}{\sqrt{1+x^2}+x} \cdot \left[\frac{2x}{2\sqrt{1+x^2}}+1\right]$$

$$= \frac{2}{\sqrt{1+x^2}+x}\left(\frac{x}{\sqrt{1+x^2}}+1\right)$$

$$= \frac{2\left(x+\sqrt{1+x^2}\right)}{\left(\sqrt{1+x^2}+x\right)\sqrt{1+x^2}}$$

$$= \frac{2}{\sqrt{1+x^2}}$$

$$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}[\cos(\log x)]$$

$$= -\sin(\log x)\frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$= [-\sin(\log x)] \times \frac{1}{x}$$

$$= \frac{-\sin(\log x)}{x}$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dv}}\right)}$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dv}}\right)}$$

$$= \frac{\left(\frac{2}{\sqrt{1+x^2}}\right)}{\left[\frac{(-\sin(\log x))}{\mathrm{x}}\right]}$$

$$= \frac{-2x}{\sqrt{1+x^2.\sin(\log x)}}.$$

Miscellaneous Exercise 1 | Q 6.3 | Page 64

DIfferentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) w.r.t. \cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$.

SOLUTION

Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$

Then we want to find $\frac{\mathrm{du}}{\mathrm{dv}}$.

Put s = tanθ. Thenθ = $\tan^{-1}$x.

Also, $\frac{\sqrt{1+x^2}-1}{x}$

= $\frac{\sqrt{1+\tan^2\theta-1}}{\tan\theta}$

= $\frac{\sec\theta-1}{\tan\theta}$

= $\frac{\frac{1}{\cos\theta}-1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$

= $\frac{1-\cos\theta}{\sin\theta}$

$$= \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right).\cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

and

$$\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}$$

$$= \frac{1+\sqrt{1+\tan^2\theta}}{2\sqrt{1+\tan^2\theta}}$$

$$= \frac{1+\sec\theta}{2\sec\theta}$$

$$= \frac{1+\frac{1}{\cos\theta}}{\left(\frac{2}{\cos\theta}\right)}$$

$$= \frac{1+\cos\theta}{2}$$

$$= \frac{2\cos^2\left(\frac{\theta}{2}\right)}{2}$$

$$= \cos^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}} = \cos\left(\frac{\theta}{2}\right)$$

$$\therefore u = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] \text{ and } v = \cos^{-1}\left[\cos\left(\frac{\theta}{2}\right)\right)\right]$$

$$\therefore u = \left(\frac{\theta}{2}\right) \text{ and } v = \left(\frac{\theta}{2}\right)$$

$\therefore$ u = $\dfrac{1}{2}\tan^{-1}x$ and $v = \dfrac{1}{2}\tan^{-1}x$

Differentiating u and v w.r.t. x, we get

$$\frac{\mathrm{du}}{\mathrm{dx}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}\left(\tan^{-1}x\right)$$

$$= \frac{1}{2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

and

$$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}\left(\tan^{-1}x\right)$$

$$= \frac{1}{2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)}{\left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)}$$

$$= \frac{\frac{1}{2(1+x^2)}}{\frac{1}{2(1+x^2)}} = 1.$$

Remark : $u = \dfrac{1}{2}\tan^{-1}x$ and $v = \dfrac{1}{2}\tan^{-1}x$

$\therefore$ u = v

$$\therefore \frac{\mathrm{du}}{\mathrm{dv}} = \frac{\mathrm{d}}{\mathrm{dv}}(v) = 1.$$

Miscellaneous Exercise 1 | Q 7.1 | Page 64

If $y^2 = a^2\cos^2x + b^2\sin^2x$, show that $y + \dfrac{d^2y}{dx^2} = \dfrac{a^2b^2}{y^3}$

SOLUTION

$y^2 = a^2\cos^2x + b^2\sin^2x$ ...(1)

Differentiating both sides w.r.t. x, we get

$$2y\frac{\mathrm{dy}}{\mathrm{dx}} = a^2\frac{\mathrm{d}}{\mathrm{dx}}(\cos x)^2 + b^2\frac{\mathrm{d}}{\mathrm{dx}}(\sin x)^2$$

$$= a^2 \times 2\cos x.\frac{\mathrm{d}}{\mathrm{dx}}(\cos x) + b^2 \times 2\sin x.\frac{\mathrm{d}}{,}dx(\sin x)$$

$= a^2 \times 2\cos x(-\sin x) + b^2 \times 2\sin x\cos x$

$= (b^2 - a^2)\sin 2x$

$$\therefore y\frac{\mathrm{dy}}{\mathrm{dx}} = \left(\frac{b^2-a^2}{2}\right)\sin 2x \quad ...(2)$$

Differentiating again w.r.t. x, we get

$$y.\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right) + \frac{\mathrm{dy}}{\mathrm{dx}}.\frac{\mathrm{dy}}{\mathrm{dx}} = \left(\frac{b^2-a^2}{2}\right).\frac{\mathrm{d}}{\mathrm{dx}}(\sin 2x)$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \left(\frac{b^2-a^2}{2}\right) \times \cos 2x \times 2$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = (b^2-a^2)\cos 2x$$

$$\therefore y^3\frac{d^2y}{dx^2} + y^2\left(\frac{dy}{dx}\right)^2 = y^2(b^2-a^2)\cos 2x$$

$$\therefore y^3\frac{d^2y}{dx^2} = y^2(b^2-a^2)\cos 2x - y^2\left(\frac{dy}{dx}\right)^2$$

$$\therefore y^4 + y^3\frac{d^2y}{dx^2} = y^2(b^2-a^2)\cos 2x - y^2\left(\frac{dy}{dx}\right)^2 + y^4$$

$= (a^2\cos^2x + b^2\sin^2x)(b^2 - a^2)(\cos^2x - \sin^2x) - [(b^2 - a^2)\sin x\cos x]^2 + (a^2\cos^2x + b^2\sin^2x)^2$ ...[By (1) and (2)]

= ($a^2b^2\cos^2x - a^4\cos^2x + b^4\sin^2x - a^2b^2\sin^2x$) x ($\cos^2x - \sin^2x$) – ($b^4\sin^2x\cos^2x + a^4\sin^2x\cos^2x - 2a^2b^2\sin^2x\cos^2x$) + ($a^4\cos^4x + b^4\sin^4x + b^4\sin^4x + 2a^2b^2\sin^2x\cos^2x$)

= $a^2b^2\cos 4x - a^2b^2\sin^2x\cos^2x - a^4\cos^4x + a^4\sin^2x\cos^2x + b^4\sin^2x\cos^2x - b^4\sin^2x\cos^2x - a^4\sin^2x\cos^2x + 2a^2b^2\sin^2x\cos^2x + a^4\cos^4x + b^4x + b^4\sin^4x + 2a^2b^2\sin^2x\cos^2x$

= $a^2b^2\cos^4x + 2a^2b^2\sin^2x\cos^2x + a^2b^2\sin^4x$
= $a^2b^2(\sin^4x + 2\sin^2x\cos^2x + \cos^4x)$

$$\therefore y^4 + y^3\frac{d^2y}{dx^2} = a^2b^2 \quad ...[\because \sin^2x + \cos^2x = 1]$$

$$\therefore y^3\left(y + \frac{d^2y}{dx^2}\right) = a^2b^2$$

$$\therefore y + \frac{d^2y}{dx^2} = \frac{a^2b^2}{y^3}.$$

Miscellaneous Exercise 1 | Q 7.2 | Page 64

If log y = log (sin x) – x^2, show that $\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2 + 3)y = 0.$

SOLUTION

log y = log (sin x) – x^2

$\therefore$ log y = $\log(\sin x) - \log e^{x^2}$

$\therefore$ log y = $\log\left(\frac{\sin x}{e^{x^2}}\right)$

$\therefore$ y = $\frac{\sin x}{e^{x^2}}$

$\therefore e^{x^2}.y$ = sin x ...(1)

Differentiating both sides w.r.t. x, we get

$$e^{x^2}.\frac{dy}{dx} + y.\frac{d}{dx}e^{x^2} = \frac{d}{dx}(\sin x)$$

$$\therefore e^{x^2}.\frac{dy}{dx} + y.e^{x^2}.\frac{d}{dx}(x^2) = \cos x$$

$$\therefore e^{x^2}.\frac{\mathrm{dy}}{\mathrm{dx}} + y.e^{x^2} \times 2x = \cos x$$

$$\therefore e^{x^2}\left(\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy\right) = \cos x$$

DIfferentiating again w.r.t. x, we get

$$e^{x^2}.\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy\right) + \left(\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy\right).\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{x^2}\right) = \frac{\mathrm{d}}{\mathrm{dx}}(\cos x)$$

$$\therefore e^{x^2}\left[\frac{d^2y}{dx^2} + 2\left(x\frac{\mathrm{dy}}{\mathrm{dx}} + y \times 1\right)\right] + \left(\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy\right).e^{x^2}.\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\right) = -\sin x$$

$$\therefore e^{x^2}\left[\frac{d^2y}{dx^2} + 2x\frac{\mathrm{dy}}{\mathrm{dx}} + y\right] + \left(\frac{\mathrm{dy}}{\mathrm{dx}} + 2xy\right).e^{x^2} \times 2x = -\sin x$$

$$\therefore e^{x^2}\left[\frac{d^2y}{dx^2} + 2x\frac{\mathrm{dy}}{\mathrm{dx}} + 2y + 2x\frac{\mathrm{dy}}{\mathrm{dx}} + 4x^2y\right]$$

$$= -e^{x^2}.y \qquad \text{...[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} + 4x\frac{\mathrm{dy}}{\mathrm{dx}} + 4x^2y + y = -y$$

$$\therefore \frac{d^2y}{dx^2} + 4x\frac{\mathrm{dy}}{\mathrm{dx}} + \left(4x^2 + 3\right)y = 0.$$

Miscellaneous Exercise 1 | Q 7.3 | Page 64

If x= a cos θ, y = b sin θ, show that $a^2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + b^2 = 0.$

SOLUTION

x= a cos θ, y = b sin θ

Differentiating x and y w.r.t. θ, we get

$$\frac{\mathrm{dx}}{\mathrm{d\theta}} = a\frac{\mathrm{d}}{\mathrm{d\theta}}(\cos\theta) = a(-\sin\theta) = -\sin\theta \quad \text{...(1)}$$

and

$$\frac{\mathrm{dy}}{\mathrm{d\theta}} = \frac{\left(\frac{\mathrm{dy}}{\mathrm{d\theta}}\right)}{\left(\frac{\mathrm{dx}}{\mathrm{d\theta}}\right)}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= \left(-\frac{b}{a}\right)\cot\theta$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\left(-\frac{b}{a}\right)\cot\theta\right]$$

$$= \left(-\frac{b}{a}\right).\frac{\mathrm{d}}{\mathrm{d\theta}}(\cot\theta).\frac{\mathrm{d\theta}}{\mathrm{dx}}$$

$$= \left(-\frac{b}{a}\right).\left(-\mathrm{cosec}^2\theta\right) \times \frac{1}{\left(\frac{\mathrm{dx}}{\mathrm{d\theta}}\right)}$$

$$= \left(-\frac{b}{a}\right)\mathrm{cosec}^2\theta \times \frac{1}{-a\sin\theta} \qquad \text{...[By (1)]}$$

$$= \left(-\frac{b}{a^2}\right)\mathrm{cosec}^3\theta$$

$$\therefore a^2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + b^2$$

$$= a^2\left[b\sin\theta.\left(-\frac{b}{a^2}\right)\mathrm{cosec}^3\theta + \left\{\left(-\frac{b}{a}\right)\cot\theta\right\}^2\right] + b^2$$

$$= a^2\left[-\frac{b^2}{a^2}\mathrm{cosec}^2\theta + \frac{b^2}{a^2}\cot^2\theta\right] + b^2$$

$$= a^2\left(-\frac{b^2}{a^2}\right)\left(\mathrm{cosec}^2\theta - \cot^2\theta\right) + b^2$$

$= -b^2 + b^2$ ...[$\because$ $\text{cosec}^2\theta - \cot^2\theta = 1$]

$$\therefore a^2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + b^2 = 0.$$

Miscellaneous Exercise 1 | Q 7.4 | Page 64

If y = A cos (log x) + B sin (log x), show that $x^2y^2 + xy_1 + y = 0$.

SOLUTION

y = A cos (log x) + B sin (log x) ...(1)

Differentiating both sides w.r.t. x, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{A}\frac{\mathrm{d}}{\mathrm{dx}}[\cos(\log x)] + \mathrm{B}\frac{\mathrm{d}}{\mathrm{dx}}[\sin(\log x)]$$

$$= \mathrm{A}[-\sin(\log x)].\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + \mathrm{B}\cos(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$= \mathrm{A}\sin(\log x) \times \frac{1}{x}\mathrm{B}\cos(\log x) \times \frac{1}{x}$$

$$\therefore x\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{dy}{dx}\right) + \frac{\mathrm{dy}}{\mathrm{dx}}.\frac{\mathrm{d}}{\mathrm{dx}}(x) = -\mathrm{A}\frac{\mathrm{d}}{\mathrm{dx}}[\sin(\log x)] + \mathrm{B}\frac{\mathrm{d}}{\mathrm{dx}}[\cos(\log x)]$$

$$\therefore x\frac{d^2y}{dx2} + \frac{\mathrm{dy}}{\mathrm{dx}} \times 1 = -\mathrm{A}\cos(\log x).\frac{\mathrm{d}}{\mathrm{dx}}(\log x) + \mathrm{B}[-\sin(\log x)].\frac{\mathrm{d}}{\mathrm{dx}}(\log x)$$

$$\therefore xy_2 + y_1 = -\mathrm{A}\cos(\log x) \times \frac{1}{x} - \mathrm{B}\sin(\log x) \times \frac{1}{x}$$

$\therefore x^2y_2 + xy_1 = -$ [A cos (log x) + B sin (log x)] ...[By (1)]

$\therefore x^2y_2 + xy_1 + y = 0.$

Miscellaneous Exercise 1 | Q 7.5 | Page 64

If $y = Ae^{mx} + Be^{nx}$, show that $y_2 - (m + n)y_1 + mny = 0$.

SOLUTION

$y = Ae^{mx} + Be^{nx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = A\frac{d}{dx}(e^{mx}) + B\frac{d}{dx}(e^{mx})$$

$$= Ae^{mx}.\frac{d}{dx}(mx) + Be^{nx}.\frac{d}{dx}(nx)$$

$= Ae^{mx}\,.\,m + Be^{nx}\,.\,n$

$= y_1 = mAe^{mx} + nBe^{nx}$ ...(2)

Differentiating again w.r.t. x, we get

$$y_2 = mA\frac{d}{dx}(e^{mx}) + nB\frac{d}{dx}(e^{nx})$$

$$= mAe^{mx}.\frac{d}{dx}(mx) + nBe^{nx}.\frac{d}{dx}(nx)$$

$= mAe^{mx}\,.\,m + nBe^{nx}\,.\,n$

$\therefore y_2 = m^2Ae^{mx} + n^2Be^{nx}$ ...(3)

$\therefore y_2 - (m + n)y_1 + mny = (m^2Ae^{mx} + n^2Be^{nx}) - (m + n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$ ...[By (1), (2), (3)]

$= m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx} - mnBe^{mx} - n^2Be^{nx} + mnAe^{mx} + mnBe^{nx}$

$= 0$

$\therefore y_2 - (m + n)y_1 + mny = 0.$