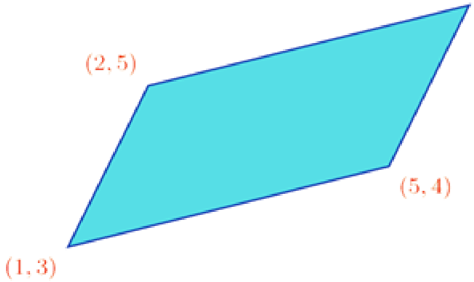
9. Geometry and Algebra

Questions Pg-215

1. Question

What are the coordinates of the fourth vertex of the parallelogram shown on the right?



Answer

Let the coordinate of fourth vertex be (x_1, y_1) and the point of intersection of diagonals be (x_2, y_2)

Diagonals of a parallelogram bisects each other

According to the section formula for mid points

$$x_2 = \frac{2 + 5}{2}$$

$\Rightarrow x_2 = 3.5$

$$y_2 = \frac{5 + 4}{2}$$

$\Rightarrow y_2 = 4.5$

The point of intersection is (3.5, 4.5)

Again using the section formula for mid points

$$3.5 = \frac{1 + x_1}{2}$$

$\Rightarrow 1 + x_1 = 7$

$\Rightarrow x_1 = 6$

$$4.5 = \frac{1 + y_1}{2}$$

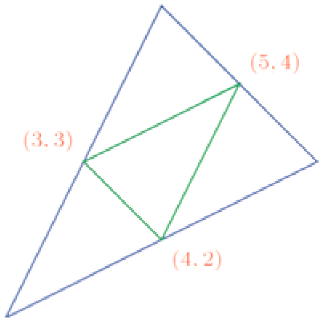
$\Rightarrow 1 + y_1 = 9$

$\Rightarrow y_1 = 8$

The fourth vertex of parallelogram is (6, 8)

2. Question

In this picture, the mid points of the sides of the large triangle are joined to make a small triangle inside.



Calculate the coordinates of the vertices of the large triangle.

Answer

Let the three coordinates of the larger triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3)

The three mid points on the three sides are (3, 3) , (4, 2) , (5, 4)

According to the section formula for mid points

$$3 = \frac{x_1 + x_2}{2}$$

$\Rightarrow x_1 + x_2 = 6$ ...Equation(i)

$$4 = \frac{x_2 + x_3}{2}$$

$\Rightarrow x_2 + x_3 = 8$ ...Equation (ii)

$$5 = \frac{x_3 + x_1}{2}$$

$\Rightarrow x_3 + x_1 = 10$ ...Equation(iii)

Solving Equation (i) ,(ii) and (iii)

$x_2 = 6 - x_1$

Putting this value in equation (ii) we get

$x_3 - x_1 = 2$ ...Equation (iv)

Solving Equation (iii) & (iv)

$2x_3 = 12$

$\Rightarrow x_3 = 6$

Putting in Equation (iv)

$6 - x_1 = 2$

$\Rightarrow x_1 = 4$

Putting this value in equation (i)

$4 + x_2 = 6$

$\Rightarrow x_2 = 2$

$$3 = \frac{y_1 + y_2}{2}$$

$\Rightarrow y_1 + y_2 = 6$ ...Equation (v)

$$2 = \frac{y_2 + y_3}{2}$$

$\Rightarrow y_2 + y_3 = 4$ ...Equation (vi)

$$4 = \frac{y_3 + y_1}{2}$$

$\Rightarrow y_3 + y_1 = 8$ ...Equation (vii)

Solving Equation (v) ,(vi) and (vii)

$y_2 = 6 - y_1$

Putting this value in equation (vi) we get

$y_3 - y_1 = -2$ ...Equation (viii)

Solving Equation (vii) & (viii)

$2y_3 = 6$

$\Rightarrow y_3 = 3$

Putting in Equation (viii)

$3 - y_1 = -2$

$\Rightarrow y_1 = 5$

Putting this value in equation (v)

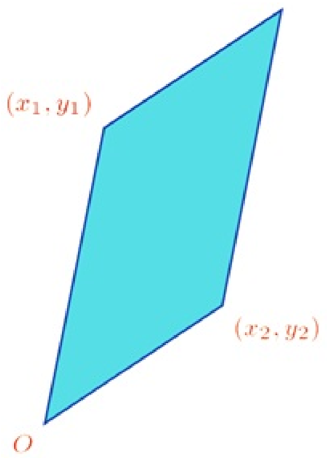
$5 + y_2 = 6$

$\Rightarrow y_2 = 1$

The Vertices of the larger triangle are (4,5) , (2,1) , (6,3)

3. Question

A parallelogram is drawn with the lines joining (x_1, y_1) and (x_2, y_2) to the origin as adjacent sides. What are the coordinates of the fourth vertex?



Answer

Let the coordinate of fourth vertex be (a_1, b_1) and the point of intersection of diagonals be (a_2, b_2)

Diagonals of a parallelogram bisects each other

According to the section formula for mid points

$$a_2 = \frac{x_1 + x_2}{2}$$

$\Rightarrow a_2 = 0.5(x_1 + x_2)$

$$b_2 = \frac{y_1 + y_2}{2}$$

$\Rightarrow b_2 = 0.5(y_1 + y_2)$

The point of intersection is $(0.5(x_1 + x_2), 0.5(y_1 + y_2))$

Again using the section formula for mid points

$$0.5(x_1 + x_2) = \frac{a_1}{2}$$

$\Rightarrow a_1 = x_1 + x_2$

$$0.5(y_1 + y_2) = \frac{b_1}{2}$$

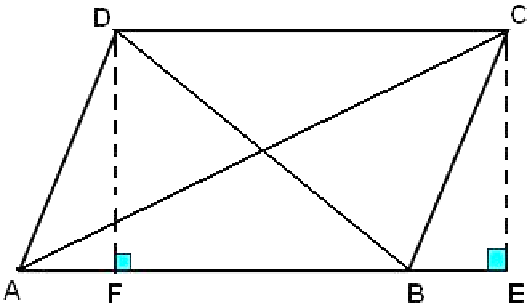
$\Rightarrow b_1 = y_1 + y_2$

The fourth vertex of parallelogram is $(x_1 + x_2, y_1 + y_2)$

4. Question

Prove that in any parallelogram, the sum of the square of all sides is equal to the sum of the squares of the diagonals.

Answer



In the parallelogram ABCD, AB = CD & AD = BC

Let DF and CE be two perpendiculars drawn on AB

In ΔAEC using Pythagoras theorem we can say

$AC^2 = AE^2 + CE^2$

Since AE = AB + BE so we can say

$\Rightarrow AC^2 = (AB + BE)^2 + CE^2$

$AC^2 = AB^2 + BE^2 + 2 \times AB \times BE + CE^2$ ...Equation (i)

In ΔDBF using Pythagoras theorem we can say

$DB^2 = DF^2 + BF^2$

Since BF = AB - AF so we can say

$\Rightarrow DB^2 = (AB - AF)^2 + DF^2$

$DB^2 = AB^2 + AF^2 - 2 \times AB \times AF + DF^2$ ...Equation (ii)

In ΔDAF & ΔCBE

DA = CB (Opposite sides of a parallelogram)

DF = CE (DCEF is a rectangle)

∠DFA = ∠ CEB (Perpendiculars)

So ΔDAF & ΔCBE are congruent by S.A.S. axiom of congruency

AF = BE (Corresponding Parts of Congruent Triangle)

Adding Equation (i) and (ii)

$AC^2 + DB^2 = AB^2 + AF^2 - 2 \times AB \times AF + DF^2 + AB^2 + BE^2 + 2 \times AB \times BE + CE^2$

$\Rightarrow AC^2 + DB^2 = AB^2 + AF^2 + DF^2 + AB^2 + BE^2 + CE^2$

(Since AF = BE)

Since AB = CD (Opposite side of parallelogram)

$\Rightarrow AC^2 + DB^2 = AB^2 + AF^2 + DF^2 + CD^2 + BE^2 + CE^2$

Using Pythagoras theorem

$\Rightarrow AC^2 + DB^2 = AB^2 + AD^2 + CD^2 + BC^2$

Hence Proved

Questions Pg-220

1. Question

The coordinates of two points A, B are (3, 2) and (8, 7).

i) Calculate the coordinates of the point P on AB such that AP : PB = 2 : 3

ii) Calculate the coordinates of the point Q on AB such that AQ : QB = 3 : 2

Answer

(i) The points are A(3,2) , B(8,7)

AP : PB = 2 : 3(Given)

$$P(x,y) = \left(\frac{2 \times 8 + 3 \times 3}{2 + 3}, \frac{2 \times 7 + 3 \times 2}{2 + 3}\right)$$

P(x,y) = (5,4)

(ii) The points are A(3,2) , B(8,7)

AQ : QB = 3 : 2(Given)

$$Q(x,y) = \left(\frac{3 \times 8 + 2 \times 3}{2 + 3}, \frac{3 \times 7 + 2 \times 2}{2 + 3}\right)$$

Q(x,y) = (6,5)

2. Question

The coordinates of the vertices of a quadrilateral are (2, 1), (5, 3), (8, 7), (4, 9) in order.

i) Find the coordinates of the midpoints of all sides.

ii) Prove that the quadrilateral with these midpoints as vertices is a parallelogram.

Answer

Let the mid points be A, B, C, D

Using the section formula for mid points

$$A(x, y) = \left(\frac{2+5}{2}, \frac{1+3}{2}\right)$$

$$A(x, y) = \left(\frac{7}{2}, \frac{4}{2}\right)$$

A(x, y) = (3.5,2)

B(x, y) = $\left(\frac{8+5}{2}, \frac{7+3}{2}\right)$

B(x, y) = $\left(\frac{13}{2}, \frac{10}{2}\right)$

B(x, y) = (6.5,5)

C(x, y) = $\left(\frac{8+4}{2}, \frac{7+9}{2}\right)$

C(x, y) = $\left(\frac{12}{2}, \frac{16}{2}\right)$

C(x, y) = (6,8)

D(x, y) = $\left(\frac{2+4}{2}, \frac{1+9}{2}\right)$

D(x, y) = $\left(\frac{6}{2}, \frac{10}{2}\right)$

D(x, y) = (3,5)

(ii) Length AB = $\sqrt{(6.5 - 3.5)^2 + (5 - 2)^2}$

Length AB = $3\sqrt{2}$ units

Length BC = $\sqrt{(6.5 - 6)^2 + (5 - 8)^2}$

Length BC = $\sqrt{9.25}$ units

Length CD = $\sqrt{(6 - 3)^2 + (8 - 5)^2}$

Length CD = $3\sqrt{2}$ units

Length DA = $\sqrt{(3.5 - 3)^2 + (2 - 5)^2}$

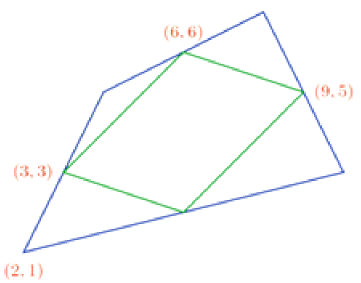
Length DA = $\sqrt{9.25}$ units

Since the lengths of opposite sides are equal hence it forms a parallelogram.

Hence the quadrilateral forms a parallelogram by joining the mid points .

3. Question

In the picture, the midpoints of the sides of the large quadrilateral are joined to draw the small quadrilateral inside.



i) Find the coordinates of the fourth vertex of the small quadrilateral.

ii) Find the coordinates of the other three vertices of the large quadrilateral.

Answer

Let the three vertices of larger quadrilateral be A, B ,C

$$3 = \frac{2 + A(x)}{2}$$

⇒ A(x) = 4

$$3 = \frac{1 + A(y)}{2}$$

⇒ A(y) = 5

A(4 ,5)

$$6 = \frac{4 + B(x)}{2}$$

⇒ B(x) = 8

$$6 = \frac{5 + B(y)}{2}$$

⇒ B(y) = 7

B(8,7)

$$9 = \frac{4 + C(x)}{2}$$

⇒ C(x) = 14

$$5 = \frac{5 + C(y)}{2}$$

⇒ C(y) = 5

C(14,5)

Let the fourth vertex of the smaller quadrilateral be (a,b)

$$a = \frac{2 + 14}{2}$$

⇒ a = 8

$$b = \frac{5 + 1}{2}$$

⇒ b = 3

Three vertices of larger quadrilateral are A(4 ,5) , B(8,7) & C(14,5) and the fourth vertex of the smaller quadrilateral is (8,3)

4. Question

The vertices of a triangle are the points with coordinates (3, 5), (9, 13), (10, 6). Prove that the triangle is isosceles. Calculate is area.

Answer

The three vertices are (3, 5), (9, 13), (10, 6)

Length of first side = $\sqrt{(9 - 3)^2 + (13 - 5)^2}$

⇒ Length of first side = $\sqrt{6^2 + 8^2}$

⇒ Length of first side = 10 units

Length of second side = $\sqrt{(10 - 9)^2 + (6 - 13)^2}$

⇒ Length of second side = $\sqrt{1^2 + 7^2}$

⇒ Length of second side = √50 units

Length of third side = $\sqrt{(10 - 3)^2 + (6 - 5)^2}$

⇒ Length of third side = $\sqrt{7^2 + 1^2}$

⇒ Length of third side = √50 units

Since the length of second and third sides are equal so the triangle is isosceles

Length of height = $\sqrt{(50) + \left(\frac{10}{2}\right)^2}$

Length of height = √75 units

Area = 0.5 × base × height

⇒ Area = 12.5 × 1.732

⇒ Area = 12.5 × 1.732 = 10.768 sq. units

5. Question

The coordinates of the vertices of a triangle are (–1, 5), (3, 7), (3, 1). Find the coordinates of its centroid.

Answer

The three coordinates are (- 1, 5) , (3, 7) , (3, 1)

Let the centroid be C(x_1, y_1)

$$C(x_1) = \frac{-1 + 3 + 3}{3}$$

$$\Rightarrow C(x_1) = \frac{5}{3}$$

$$C(y_1) = \frac{5 + 7 + 1}{3}$$

$$\Rightarrow C(y_1) = \frac{13}{3}$$

The Coordinates of centroid is $\left(\frac{5}{3}, \frac{13}{3}\right)$

6. Question

The centre of circle is (1, 2) and (3, 2) is a point on it. Find the coordinates of the other end of the diameter through this point.

Answer

Centre of circle is (1, 2)

Point (3, 2)

Let the coordinates of the other end be (x_1,y_1)

Using the section formula for mid points

$$1 = \frac{3 + x_1}{2}$$

⇒ 3 + x_1 = 2

⇒ x_1 = - 1

$$2 = \frac{2 + y_1}{2}$$

⇒ 2 + y_1 = 4

⇒ y_1 = 2

Answer : The Other point is (- 1 , 2)

Questions Pg-226

1. Question

Prove that the points (1, 8), (2, 5), (3, 7) are on the same line.

Answer

The three points are (1, 8) ,(2, 5) ,(3, 7)

Slope of First two points $= \frac{5-8}{2-1}$

Slope of First two points = - 3

The Equation of line between first two points :

y - 8 = - 3(x - 1)

⇒ y - 8 = - 3x + 3

⇒ 3x + y = 11

We put the point (3, 7) in the above equation.

On solving we will find the point doesn’t satisfy the equation.

Hence the three points doesn’t lie on a straight line.

2. Question

Find the coordinates of two other points on the line joining (–1, 4) and (1, 2).

Answer

The two points are (- 1, 4) ,(1,2)

Slope of two points $= \frac{4-2}{-1-1}$

Slope of two points = - 1

The Equation of line between two points :

y - 2 = - 1 (x–1)

⇒ x + y - 3 = 0

Putting x = 0 in the above equation we get y = 3

Putting y = 0 in the above equation we get x = 3

So the other two points are (0,3) (3,0)

3. Question

$x_1, x_2, x_3,\ldots$ and $y_1, y_2, y_3,\ldots$ are arithmetic sequences. Prove that all points with coordinates in the sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3),\ldots$ are on the same line.

Answer

Let the common difference between the x - coordinate be d_x , between the y - coordinate be d_y

The three points can be written in terms of their common difference as $(x_1, y_1), (x_1 + d_x, y_1 + d_y), (x_1 + 2d_x, y_1 + 2d_y)$

Slope of first two points $= \frac{(y_1 + d_y)-y_1}{(x_1 + d_x)-x_1}$

⇒ Slope of first two points $= \frac{d_y}{d_x}$

Slope of last two points = $\frac{(y_1 + 2d_y)-(y_1 + d_y)}{(x_1 + 2d_x)-(x_1 + d_x)}$

⇒ Slope of last two points = $\frac{d_y}{d_x}$

Since the slope of first two points and last two points are same hence they are on the same line.

Hence proved

4. Question

Prove that if the point (x_1, y_1), (x_2, y_2), (x_3, y_3) are on a line, so are $(3x_1 + 2y_1, 3x_1 - 2y_1)$, $(3x_2 + 2y_2, 3x_2 - 2y_2)$, $(3x_3 + 2y_3, 3x_3 - 2y_3)$. Would this be true if we take other numbers instead of 3 and 2?

Answer

The three points are say $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

Since they lie on a line so slope of any two points are always equal

$\frac{y_1-y_2}{x_1-x_2} = \frac{y_2-y_3}{x_2-x_3}$ ...Equation (i)

The other set of three points are say $P(3x_1 + 2y_1, 3x_1 - 2y_1)$, $Q(3x_2 + 2y_2, 3x_2 - 2y_2)$, $R(3x_3 + 2y_3, 3x_3 - 2y_3)$

Since they also lie on a line so slope between any two points is always equal

$$\frac{(3x_1 - 2y_1) - (3x_2 - 2y_2)}{(3x_1 + 2y_1) - (3x_2 + 2y_2)} = \frac{(3x_2 - 2y_2) - (3x_3 - 2y_3)}{(3x_2 + 2y_2) - (3x_3 + 2y_3)}$$

$$\Rightarrow \frac{3(x_1 - x_2) - 2(y_1 - y_2)}{3(x_1 - x_2) + 2(y_1 - y_2)} = \frac{3(x_2 - x_3) - 2(y_2 - y_3)}{3(x_2 - x_3) + 2(y_2 - y_3)}$$

Applying Componendo and dividendo we get

$$\Rightarrow \frac{6(x_1 - x_2)}{-2(y_1 - y_2)} = \frac{6(x_2 - x_3)}{-2(y_2 - y_3)}$$

$$\Rightarrow \frac{(x_1 - x_2)}{(y_1 - y_2)} = \frac{(x_2 - x_3)}{(y_2 - y_3)}$$

Applying Invertendo we get

$\frac{y_1-y_2}{x_1-x_2} = \frac{y_2-y_3}{x_2-x_3}$ ...Equation (ii)

Since Equation (i) & Equation (ii) are similar so the points P,Q and R lie on the line joining A,B & C

Hence Proved

Yes it is possible if we take multiples of 2 and 3

Questions Pg-230

1. Question

Find the equation of the line joining (1, 2) and (2, 4). In this, find the sequence of y coordinates of those points with the consecutive natural numbers 3, 4, 5, ... as the x coordinates.

Answer

Slope of the line = $\frac{4-2}{2-1}$

⇒ Slope of the line = 2

Equation of line :

y - 2 = 2(x - 1)

$\Rightarrow 2x - y = 0$

$\Rightarrow y = 2x$

Value of y for consecutive natural numbers :

x	3	4	5	6	7	8	9	10	11
y	6	8	10	12	14	16	18	20	22

2. Question

Find the equation of the line joining (–1, 3), (2, 5). Prove that if (x, y) is a point on this line, so is (x + 3, y + 2).

Answer

Slope of the line $= \frac{5-3}{2-(-1)}$

$\Rightarrow$ Slope of the line $= \frac{2}{3}$

Equation of line :

$$y - 5 = \frac{2}{3}(x - 2)$$

$\Rightarrow 3(y - 5) = 2(x - 2)$

$\Rightarrow 2x - 3y + 11 = 0$ ...Equation (i)

Putting x = x + 3 and y = y + 2 in the above equation we get

$2(x + 3) - 3(y + 2) + 11 = 0$

$\Rightarrow 2x - 3y + 11 = 0$

Since it again gives the same equation as (i) so it is a point on this line.

Hence Proved

3. Question

Prove that for any number x, the point (x, 2x + 3) is on the line joining (–1, 1), (2, 7).

Answer

Slope of the line $= \frac{7-1}{2-(-1)}$

$\Rightarrow$ Slope of the line = 2

Equation of line :

$y - 2 = 2(x - 7)$

$\Rightarrow 2x - y - 12 = 0$

Putting x = x and y = 2x + 3 in the above equation we get

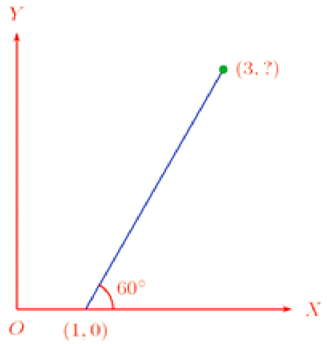
$2x - (2x + 3) - 12 = 0$

which satisfies the above equation

Hence Proved

4. Question

The x coordinate of a point on the slanted (blue) line in the picture is 3.



i) What is its y coordinate?

ii) What is the slope of the line?

iii) Write the equation of the line.

Answer

(i) Let the y coordinate be a

tan $60^0 = \frac{a-0}{3-1}$

⇒ a = 2√3

The y coordinate is 2√3

(ii) Slope = tan 60^0

⇒ Slope of line = √3

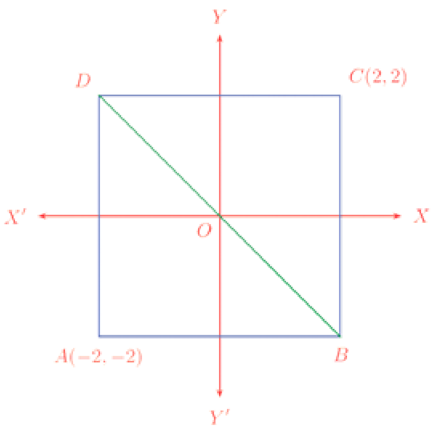
(iii) Equation of line:

y - 0 = √3(x - 1)

⇒ √3x - y - √3 = 0

5. Question

In the picture, ABCD is a square. Prove that for any point on the diagonal BD, the sum of the x and y coordinates is zero.



Answer

Slope of diagonal AC $= \frac{2-(-2)}{2-(-2)}$

⇒ Slope of diagonal AC = 1

Since diagonals of a square are perpendicular to each other

Slope of diagonal AC × Slope of diagonal BD = - 1

⇒ Slope of diagonal BD = - 1

Origin (0,0) is a point on the diagonal

Equation of diagonal BD:

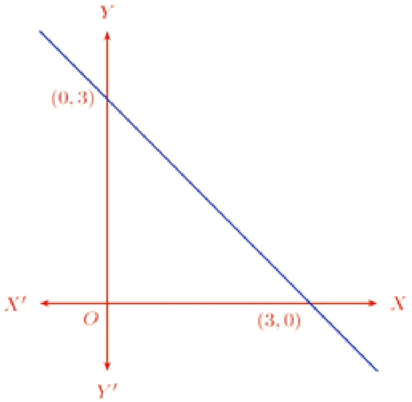
y = - x

⇒ x + y = 0

Hence Proved

6. Question

Prove that for any point on the line intersecting the axes in this picture, the sum of the x and y coordinates is 3.



Answer

Length of x intercept (a) = 3 units

Length of y intercept (b) = 3 units

According to slope intercept form , the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \frac{x}{3} + \frac{y}{3} = 1$$

⇒ x + y = 3

Proved

7. Question

Find the equation of the circle with centre at the org in and radius 5. Write the coordinates of eight points on this circle.

Answer

Radius = 5 units

Equation of circle:

$x^2 + y^2 = 25$

Consider x = 0

⇒ $y^2 = 25$

⇒ y = ±5

Again Consider y = 0

⇒ $x^2 = 25$

⇒ x = ±5

Consider x = y

$2x^2 = 25$

$$\Rightarrow x = \pm\frac{5}{\sqrt{2}}$$

$$\Rightarrow y = \pm\frac{5}{\sqrt{2}}$$

Consider x = - y

$2x^2 = 25$

$$\Rightarrow x = \pm\frac{5}{\sqrt{2}}$$

$$\Rightarrow y = \mp\frac{5}{\sqrt{2}}$$

Hence the Eight points are:

$$(5,0),(-5,0),(0,5),(0,-5),\left(\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right),\left(-\frac{5}{\sqrt{2}},-\frac{5}{\sqrt{2}}\right),\left(\frac{5}{\sqrt{2}},-\frac{5}{\sqrt{2}}\right),\left(-\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)$$

8. Question

Let (x, y) be a point on the circle with the line joining (0, 1) and (2, 3) as diameter. Prove that $x^2 + y^2 – 2x – 4y + 3 = 0$. Find the coordinates of the points where this circle cuts the x axis.

Answer

Two diametrically Opposite points are (0,1),(2,3)

Equation of circle for two Diametrically Opposite points:

$(x - 0)(x - 2) + (y - 1)(y - 3) = 0$

$\Rightarrow x^2 - 2x + y^2 - 4y + 3 = 0$

When the circle cuts the x axis the y coordinate is 0

$\Rightarrow x^2 - 2x + 3 = 0$

Discriminant = $(- 2)^2 - 4 \times 1 \times 3$

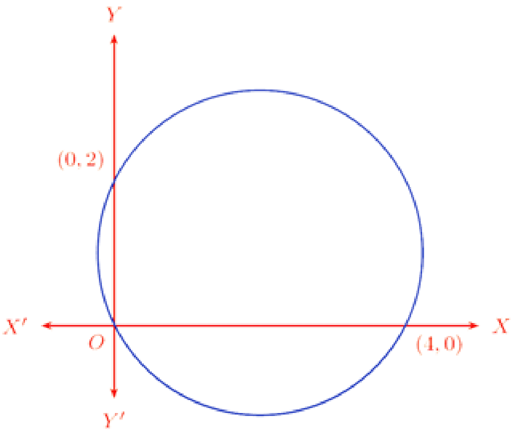
Discriminant = - 8

Since Discriminant is negative so the roots are imaginary

Hence the circle doesn’t cut the x - axis at any point.

9. Question

What is the equation of the circle in the picture below?



Answer

The X and Y axis meets the circle at point O

The line joining the two points (0,2) and (4,0) forms the diameter of the circle.

Let the centre of the circle be C(x, y)

Using Section Formula for mid points

$$C(x) = \frac{0 + 4}{2}$$

⇒ C(x) = 2

$$C(y) = \frac{2 + 0}{2}$$

⇒ C(y) = 1

The coordinate of Center is (2, 1)

Radius of Circle = $\sqrt{(4 - 2)^2 + (0 - 1)^2}$

⇒ Radius = $\sqrt{5}$ units

So the equation of circle is:

$(x - 2)^2 + (y - 1)^2 = 5$

⇒ $x^2 + y^2 - 4x - 2y + 5 = 5$

$x^2 + y^2 - 4x - 2y = 0$