CHAPTER 07

Calculus

7.1 Limit

Definition: A number A is said to be limit of f at $x = a$ iff for any arbitrarily choosen positive integer $\in$, however small but not zero there exist a corresponding number d greater than zero such that: $|f(x) - A| < \in$ for all values of x for which $0 < |x - a| < \partial$ where $|x - a|$ means the absolute value of $(x - a)$ without any regard to sign.

Right and Left Hand Limits

If x approaches a from the right, that is, from larger value of x than a, the limit of f as defined before is called the right hand limit of $f(x)$ and is written as: $\underset{x \to a+0}{\text{Lt}} f(x)$ or $f(a + 0)$

Working rule for finding right hand limit is, put $a + h$ for x in $f(x)$ and make h approach zero.

In short, we have, $f(a + 0) = \underset{h \to 0}{\text{Limit}} f(a + h)$

Similarly if x approaches a from left, that is from smaller values of x than a, the limit of f is called the left hand limit and is written as: $\underset{x \to a-0}{\text{Lt}} f(x)$ or $f(a - 0)$

In this case, we have $f(a - 0) = \underset{h \to 0}{\text{Limit}} f(a - h)$

If both right hand and left hand limit of f, as $x \to a$ exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

Indeterminate Form

A fraction whose numerator and denominator both tend to zero as $x \to a$ is called indeterminate form 0/0. It has not definite values. Other indeterminate forms are: ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0, ∞^0.

L' Hospital Rule

If $f(x)$ and $\phi(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = \phi(a) = 0$ then $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{\phi'(x)}$ provided, the latter limit exists, finite or infinite.

Working Rule

If the limit of $\dfrac{f(x)}{\phi(x)}$ as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately with respect to x and obtain a new function $\dfrac{f'(x)}{\phi'(x)}$. Now as $x \to a$ if it again takes the form 0/0, differentiate the numerator and denominator again with respect to x and repeat the above process, till indeterminate form persists.

Caution

Before applying Hospital's rule at any stage be sure that the form is 0/0. Do not go on applying this rule even if the form is not 0/0.

Various Formulae

$$(1 + x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(x\log a)^2 + \frac{x^3}{3!}(x\log a)^3 + \ldots\ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \quad |x| < 1$$

$$\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\right) |x| < 1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\ldots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$$

$\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

$\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

REMEMBER: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

Some useful results:

(i) $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$

(ii) $\underset{x \to 0}{\text{Lt}} \cos x = 1$

(iii) $\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1$

(iv) $\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$

(v) $\underset{x \to 0}{\text{Lt}} (1+nx)^{\frac{1}{x}} = e^n$

(vi) $\underset{x \to \infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$

(vii) $\underset{x \to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x = e^a$

Form-II

∞/∞: If $f(x)$ and $\phi(x)$ be two functions such that, $\underset{x \to a}{\text{Limit}} f(x) = \infty$ and $\underset{x \to a}{\text{Limit}} \phi(x) = \infty$, then

$$\underset{x \to a}{\text{Limit}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Limit}} \frac{f'(x)}{\phi'(x)},$$ provided the limit exists.

Form-III

$0 \times \infty$: This form can be easily reduced to the form 0/0 or to the form ∞/∞.

Let $\underset{x \to a}{\text{Limit}} f(x) = 0$ and $\underset{x \to a}{\text{Limit}} \phi(x) = \infty$.

Then we can write

$$\underset{x \to a}{\text{Limit}} f(x) \cdot \phi(x) = \underset{x \to a}{\text{Limit}} \frac{f(x)}{1/\phi(x)} \text{ [form 0/0] or } \underset{x \to a}{\text{Limit}} \frac{\phi(x)}{1/f(x)} \text{ [form } \infty/\infty]$$

Thus $\underset{x \to a}{\text{Limit}} f(x) \cdot \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L' Hospital rule or otherwise.

Form-IV

0^0, 1^∞, ∞^0: Suppose $\underset{x \to a}{\text{Limit}} [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then let $y = \underset{x \to a}{\text{Limit}} [f(x)]^{\phi(x)}$

Taking log on both sides, we get

$\log y = \underset{x \to a}{\text{Limit}} \phi(x) \cdot \log f(x)$.

Now in any of these above cases $\log y$ takes the form $0 \times \infty$ which is changed to the form 0/0 or ∞/∞ then it can be evaluated by previous methods.

Example-7.1 Find the value of 'a'?

$$\text{If } \lim_{n \to 0} \frac{\sin 2x + a\sin x}{x^3} = \text{finite.}$$

Solution:

$$\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3} = \frac{0}{0} \quad \text{(Indeterminate form)}$$

Apply L-Hospital's rule

$$\Rightarrow \quad \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^3} = \frac{2+a}{0} = \text{finite (given)}$$

$$\therefore \quad 2 + a = 0 \Rightarrow a = -2$$

Example-7.2 Consider the following function given below:

$$f(x) = \begin{cases} \dfrac{\sin[x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [0] = 0 \end{cases}$$

The reason for $f(x)$ be discontinuous at $x = 0$ is

(a) $f(0)$ is not defined.

(b) $f(0)$ is defined but $\lim_{x \to 0} f(x)$ does not exist.

(c) $\lim_{x \to 0} f(x)$ exists, $f(0)$ is defined but $\lim_{x \to 0} f(0) \neq f(0)$

(d) $f(x)$ is continuous at $x = 0$

Solution:

$$f(x) = \begin{cases} \dfrac{\sin(-1)}{(-1)} & \text{for } -1 \le x < 0 \; [x] = 1 \\ 0 & \text{for } 0 \le x < 1 \; [x] = 0 \end{cases}$$

$$f(0) = 0$$

Left limit is sin1. Right limit is 0.

∴ Limit does not exists.

Example-7.3 Find the limits of the following function:

(a) $\lim_{x \to 0} \dfrac{1 - \cos 3x}{x \sin x}$

(b) $\lim_{x \to 0} \dfrac{\cos\left(\frac{\pi}{2} x\right)}{1 - \sqrt{x}}$

Solution:

(a)

$$\lim_{x \to 0} \left[\frac{1 - \cos 3x}{x \sin x}\right] = \frac{0}{0} \quad \text{(Indeterminate form)}$$

to overcome the indeterminate form we use L Hospital's rule.

$$\lim_{x \to 0} \left[\frac{3\sin 3x}{\sin 2x + 2x\cos x}\right] = \frac{0}{0} \quad \text{(This still in indeterminate form)}$$

Apply L Hospital's rule again

$$\lim_{x\to 0}\left[\frac{9\cos 3x}{2\cos 2x+2\cos 2x-4x\sin 2x}\right]=\frac{9(1)}{2(1)+2(1)-4(0)}=\frac{9}{4}$$

(b) $$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\left(\frac{0}{0}\right)\quad \text{(Indeterminate form)}$$

$$=\lim_{x\to 1}\frac{-\frac{\pi}{2}\sin\frac{\pi x}{2}}{-\frac{1}{2}\sqrt{x}}=\frac{\frac{\pi}{2}\sin\frac{\pi}{2}}{\frac{1}{2}}=\pi$$

7.2 Continuity

Definition: A function $f(x)$ is defined for $x = a$ is said to be continuous at $x = a$ if:

(*i*) $f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number

(*ii*) the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Otherwise the function is discontinuous at $x = a$.

A function is said to be continuous in an interval (a, b) if it is continuous at every point of interval (a, b).

Arithmetical Definition of Continuity: A function $f(x)$ is said to be continuous at $x = a$, if for any arbitrarily choosen positive number $\in$, however small (but not zero) $\exists$ a corresponding number δ such that, $|f(x) - f(a)| < \in$ for all values of x for which $|x - a| < \delta$.

NOTE: On comparing the definitions of limit and continuity we find that a function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x\to a} f(x) = f(a)$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a + 0) = f(a - 0) = f(a)$, otherwise it is discontinuous at $x = a$.

Continuity from Left and Continuity from Right

Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a if $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a if $\lim\limits_{x\to a+0} f(x)$ exists and is equal to $f(a)$.

Continuity in an Open Interval

A function f is said to be continuous in open interval $[a, b]$ if it is continuous at each point of interval.

Continuity in a Closed Interval

Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous at a if it is continuous from the right a and also that f is continuous at b if it is continuous from the left at b. Further f is said to be continuous on the closed interval $[a, b]$ if (i) it is continuous from the right and (ii) continuous from the lift at b and (iii) continuous on the open interval $[a, b]$.

7.3 Differentiability

Derivative at a point: Let I denote the open interval [a, b] in R and let $x_0 \in I$. Then a function $f: I \to R$ is said to be differentiable at x_0, iff:

$$\lim_{x \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ or } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ exist (finitely) and is denoted by $f'(x_0)$.

Progressive and Regressive Derivatives

The progressive derivative of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, h > 0$$ and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$

The regressive derivative of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}, h > 0$$ and is denoted by $Lf'(x_0)$ or by $f'(x_0 - 0)$

Differentiability in [a, b]

A function $f: [a, b] \to R$ is said to be differentiable at a iff $Rf'(a)$ exists, differentiable at b iff $Lf'(b)$ exists, f is said to be differentiable at every point [a, b].

Result:

Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.
i.e. differentiability $\Rightarrow$ continuity

But continuity $\not\Rightarrow$ differentiability

Example-7.4 What can be said about the continuity and differentiability of $f(x)$, where $f(x) = \frac{1}{1+|x|}$; at $x = 0$?

Solution:

$$f(x) = \begin{cases} \frac{1}{1+x} & \text{for } x > 0 \\ \frac{1}{1-x} & \text{for } x < 0 \\ 1 & \text{for } x = 0 \end{cases}$$

$$\left.\begin{array}{l} f(0) = 1, \text{ left limit } = \frac{1}{1+0} = 1 \\ \text{similarly right limit} = 1 \end{array}\right\} \text{continuous at } x = 0$$

$$\text{Left hand derivative} = \lim_{h \to 0} \left[\frac{\frac{1}{1-(-h)} - 1}{-h} \right] = \lim_{h \to 0} \frac{-h}{-h(1+h)} = 1$$

$$\text{Right hand derivative} = \lim_{h\to 0}\left[\frac{\frac{1}{1+h}-1}{h}\right] = \lim_{h\to 0}\frac{-h}{h(1+h)} = -1$$

∴ Left hand derivative = Right hand derivative

∴ It is continuous but not differentiable at $x = 0$

Example-7.5 Let $f(x) = x|x|$ where $x \in R$, then $f(x)$ at $x = 0$ is

(a) continuous and differentiable
(b) continuous but not differentiable
(c) differentiable but not continuous
(d) neither differentiable not continuous

Solution:

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$f(0) = 0$, Left limit = 0, Right limit = 0

$$\text{L.H.D.} = \lim_{h\to 0}\left[\frac{-(-h)^2 - 0}{-h}\right] = 0$$

$$\text{R.H.D.} = \lim_{h\to 0}\left[\frac{(+h)^2 - 0}{h}\right] = \lim_{h\to 0} h = 0$$

∴ It is both continuous and differentiable.

7.4 Mean Value Theorems

Rolle's Theorem

If a function $f(x)$ is such that:

(*i*) $f(x)$ is continuous in the closed interval $a \le x \le b$.

(*ii*) $f'(x)$ exists for every point in the open interval $a < x < b$.

(*iii*) $f(a) = f(b)$, then there exists at least one value of x, say c where $a < c < b$ such that $f'(c) = 0$

NOTE

Role's theorem will not hold good.

- If $f(x)$ is discontinuous at some point in the interval $a < x < b$
- If $f'(x)$ does not exist at some point in the interval $a \le x \le b$ or
- If $f(a) \neq f(b)$

Example-7.6 The Mean value C for the below function:

$f(x) = e^x [\sin x - \cos x]$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ is ______.

Solution:

$$f'(x) = e^x[\sin x - \cos x] + e^x[\cos x + \sin x] = 2e^x \sin x$$

$$f\left(\frac{\pi}{4}\right) = 0, \quad f\left(\frac{5\pi}{4}\right) = 0$$

By Rolle's theorem there exist $C \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Such that $f'(c) = 0$, $\quad 2e^C \sin C = 0 \Rightarrow \sin C = 0$

$$C = 0, \pm\pi, \pm 2\pi, \ldots$$

$\Rightarrow$

$$C = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Lagrange's Mean Value Theorem or First Mean Value Theorem

If a function $f(x)$ is

(*i*) continuous in closed interval $a \le x \le b$ and

(*ii*) differentiable in open interval (a, b) i.e., $a < x < b$, then there exist at least one value c of x lying in the open interval $a < x < b$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Some important deductions from mean value theorem:

(*i*) If a function $f(x)$ be such that $f'(x)$ is zero throughout the interval, then $f(x)$ must be constant throughout the interval.

(*ii*) If $f(x)$ and $\phi(x)$ be two functions such that $f'(x) = f'(x)$ throughout the interval (a, b), then $f(x)$ and $\phi(x)$ differ only by a constant.

(*iii*) If $f(x)$ is:

(a) continuous in closed interval $[a, b]$

(b) differentiable in open interval (a, b)

(c) $f(x)$ is –ve in $a < x < b$, then $f(x)$ is monotonically decreasing function in the closed interval $[a, b]$

Example-7.7 Verify Lagrange's mean mean value theorem for the following functions in the given interval and find '*c*' of this theorem.

(a) $f(x) = x^2 + 2x + 3$ in $[4, 6]$ (b) $f(x) = px^2 + qx + r, p \ne 0$, in $[a, b]$

Solution:

(a) Given $f(x) = x^2 + 2x + 3$ ... (*i*)

(*i*) $f(x)$ being a polynomial function is continuous in $[4, 6]$.

(*ii*) $f(x)$ being a polynomial function is derivable in $(4, 6)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(4, 6)$ such that

$$f'(c) = \frac{f(6) - f(4)}{6 - 4}$$

$$f(6) = 6^2 + 2.6 + 3 = 51, \; f(4) = 4^2 + 2.4 + 3 = 27$$

Differentiating (*i*) w.r.t. x, we get

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2$$

$$f'(c) = \frac{f(6) - f(4)}{6 - 4} \quad 2c + 2 = \frac{51 - 27}{2} \Rightarrow 2c + 2 = 12$$

$\therefore$ $\quad 2c = 10 \Rightarrow c = 5$

$\Rightarrow$

Thus, there exists $c = 5$ in $(4, 6)$ such that $f'(5) = \frac{f(6) - f(4)}{6 - 4}$

Hence, Lagrange's mean value theorem is verified and $c = 5$.

(b) Given $f(x) = px^2 + qx + r, p \neq 0$

(i) f being a polynomial function is continuous in $[a, b]$

(ii) f being a polynomial function is derivable in (a, b).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

$$f(b) = pb^2 + qb + r, f(a) = pa^2 + qa + r$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q$$

$$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{(pb^2+qb+r)-(pa^2+qa+r)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{p(b^2+a^2)+q(b-a)}{b-a} \Rightarrow 2pc = p(a+b)$$

$$\Rightarrow \quad c = \frac{a+b}{2} \text{ and } \frac{a+b}{2} \in (a, b)$$

Thus, there exist $c = \frac{a+b}{2}$ in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Hence Lagrange's mean value theorem is verified and $c = \frac{a+b}{2}$

Example-7.8 Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining (1, 1) and (3, 27).

Solution:

$$f(x) = x^3 \text{ in the interval } [1, 3]$$

(a) $f(x)$ being a polynomial is continuous in [1, 3].

(b) $f(x)$ being a polynomial is derivable in (1, 3).

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function $f(x)$ in [1, 3], therefore, there exists atleast one real number c in (1, 3) such that

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

Now

$$f'(c) = \frac{f(3)-f(1)}{3-1} \Rightarrow 3c^2 = \frac{27-1}{3-1} \Rightarrow 3c^2 = 13$$

$$\Rightarrow \quad c^2 = \frac{13}{3} = \frac{39}{9} \Rightarrow c = \pm\frac{\sqrt{39}}{3}$$

But, $c \in (1, 3) \Rightarrow c = \frac{\sqrt{39}}{3}$

When, $x = \frac{\sqrt{39}}{3}$, from (1) $y = \frac{\sqrt{39}}{3}$

Hence, there exists a point $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points (1, 1) and (3, 27).

7.5 Theorems of Integral Calculus

1. The integral of the product of a constant and a function is equal to be product of the constant and the integral of function.

   Thus if λ is constant, then $\int \lambda f(x) dx = \lambda \int f(x) dx$.

2. The integral of a sum of or difference of a finite number of functions is equal to sum or difference of integrals. Symbolically

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm ... \pm f_n(x)] dx = \int f_1(x) dx \pm \int f_2(x) dx \pm \int f_3(x) dx \pm ... \pm \int f_n(x) dx$$

Fundamental Formulae

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1}$

(ii) $\int \frac{1}{x} dx = \log x$

(iii) $\int \sin x \, dx = \cos x$

(iv) $\int \cos x \, dx = \sin x$

(v) $\int \sec^2 x \, dx = \tan x$

(vi) $\int \text{cosec}^2 x \, dx = -\cot x$

(vii) $\int \text{cosec} \, x \cot x = -\text{cosec} \, x$

(viii) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$

(ix) $\int \frac{1}{1+x^2} dx = \tan^{-1} x$

(x) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

(xi) $\int \cos hx = \sin hx$

(xii) $\int \sin hx \, dx = \cos hx$

(xiii) $\int \text{cosec} \, h^2 x \, dx = -\cot hx$

(xiv) $\int \sec hx \tan hx \, dx = -\sec hx$

(xv) $\int \text{cosec} \, hx \cot hx = -\text{cosec} \, hx$

7.6 Methods of Integration

There are various methods of integration by which we can reduce the given integral to one of fundamental known integral. There are four principal methods of integration.

7.6.1 Integration by Substitution

A change in the variable of integration often reduces an integral to one of fundamental integrals.

Let $I = \int f(x)dx$, then by differentiation w.r.to x we have $\frac{dI}{dx} = f(x)$. Now put,

$$x = \phi(t), \text{ so that } \frac{dx}{dt} = \phi'(t)$$

Then, $$\frac{dI}{dt} = \frac{dI}{dx} \cdot \frac{dx}{dt} = f(x) \cdot \phi'(t) = f\{\phi(t) \cdot \phi'(t)\} \text{ for } x = \phi(t)$$

This gives $$I = \int f\{\phi(t) \cdot \phi'(t)\}\, dt$$

Rule to Remember

To evaluate $\int f\{\phi(x) \cdot \phi'(x)\}\, dx$

Put $\phi(x) = t$ and $\phi'(x)dx = dt$

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ with respect to x.

Three forms of Integrals

1. $$\int \frac{f'(x)}{f(x)} dx = \log f(x)$$

Put $f(x) = t$ differentiating we get $f'(x) \cdot dx = dt$

$$\Rightarrow \quad \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Thus the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithmic of its denominator.

Example: $$\int \frac{4x^3}{1+x^4} dx = \log(1+x^4) \qquad \text{...(7.1)}$$

Because, if we put $(1 + x^4) = t$

$$\Rightarrow \quad 4x^3\, dx = dt$$

Equation (7.1) reduces to $\Rightarrow \int \frac{dt}{t} \Rightarrow \log t \Rightarrow \log(1 + x^4)$.

Some important formulae based on the above form

(i) $$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(-\sin x)}{\cos x} dx$$

$$= -\log \cos x = \log(\cos x)^{-1}$$

$$= \log \sec x$$

(ii) $$\int \cot x dx = \log \sin x$$

(iii) $$\int \operatorname{cosec} x dx = \log \tan\frac{x}{2}$$

2. $\int [f(x)]^n f'(x)dx = \dfrac{[f(x)]^{n+1}}{(n+1)}$ when $n \neq 1$: If the integrand consists of the product of a constant power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$, to obtain the integral we increase the index by unity and then divide by increased index. This is known as power formula.

Formulae:

(i) $\int f'(ax+b)dx = \dfrac{f(ax+b)}{a}$

(ii) $\int \dfrac{1}{\sqrt{a^2+x^2}}dx = \sin h^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2+a^2}\right]$

(iii) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$

(iv) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \log\left[x+\sqrt{x^2-a^2}\right] = \cos h^{-1}\left(\dfrac{x}{a}\right)$

(v) $\int \sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\sin h^{-1}\left(\dfrac{x}{a}\right)$ or $\dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\log\left\{x+\sqrt{x^2+a^2}\right\}$

(vi) $\int \sqrt{a^2-x^2}\,dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

7.6.2 Integral of the Product of Two Functions

Integration by parts: Let u and v be two functions of x. Then we have from differential calculus.

$$\frac{d}{dx}(uv) = u \times \frac{dv}{dx} + v \times \frac{du}{dx} \quad \text{...(7.2)}$$

Integrating both sides of (7.2) with respect to x, we have

$$uv = \int u \cdot \frac{dv}{dx}dx + \int v \cdot \frac{du}{dx}dx$$

$$\Rightarrow \quad \int u\frac{dv}{dx}dx = uv - \int v \cdot \frac{du}{dx} \cdot dx \quad \text{...(7.3)}$$

i.e. $\int u\,dv = uv - \int v \cdot du$

This can also be written as $\int uv\,dx = u\int v\,dx - \int [du \int v\,dx]dx$

Formulae based Upon Above Method

1. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

2. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

7.6.3 Integration by Partial Fractions

1.
$$I = \int \frac{1}{x^2 - a^2}, (x > a)$$

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\} = \frac{1}{2a} \log\frac{x-a}{x+a}$$

Thus, $$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\frac{x-a}{x+a}, x > a$$

2.
$$I = \int \frac{1}{a^2 - x^2} dx \; (x < a)$$

In this case $$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\frac{a+x}{a-x}, x < a$$

7.7 Definite Integrals

If $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of $f(x)$ between the limit of a and b. $b \rightarrow$ upper limit; $a \rightarrow$ lower limit.

Fundamental Properties of Definite Integrals

1. We have $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e., the value of a definite integral does not change with the change of variable of integration provided the limits of integration remain the same.

   Let, $\int f(x)dx = F(x)$ and $\int f(t)dt = F(t)$

   Now, $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

   $\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$

2. $\int_a^b f(x)dt = -\int_a^b f(x)dt$. Interchanging the limits of a definite integral does not change in the absolute value but change the sign of integrals.

3. We have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

NOTE: 1. This property also holds true even if the point c is exterior to the interval (a, b). 2. In place of one additional point c, we can take several points. Thus several points

Thus, $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \ldots + \int_{c_n}^{b} f(x)dx$

4. We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Let $I = \int_0^a f(x)dx$

Put $x = a - t \Rightarrow dx = -dt$ where $x = 0$, $t = a$ and when $x = a$, $t = 0$

$\Rightarrow$ $$I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt = \int_0^a f(a-x)dx$$

5. $\int_{-a}^{+a} f(x)dx = 0$ or $2\int_0^a f(x)dx$ according as $f(x)$ is an odd or even function of x.

Odd and Even function:

(*i*) An odd function of x if $f(-x) = -f(x)$

(*ii*) An even function of x if $f(-x) = f(x)$

6. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$ and $\int_0^{2a} f(x)dx = 0$ if $f(2a-x) = -f(x)$

Corollary: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Example-7.9 Evaluate the following definite integrals:

(a) $\int_{-5}^{5} |x+2|dx$ (b) $\int_1^4 (|x|+|x-3|)dx$

Solution:

(a) Since for $-5 \le x \le -2$, $x + 2 \le 0$

$\Rightarrow$ $|x+2| = -(x+2)$

and for $-2 \le x \le 5$, $x + 2 \ge 0$

$\Rightarrow$ $|x+2| = x+2$,

$\therefore$ $$\int_{-5}^{5} |x+2|dx = \int_{-5}^{-2} |x+2|dx + \int_{-2}^{5} |x+2|dx \quad \text{(Property 3)}$$

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx = \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= (-2+4) - \left(-\frac{25}{2} + 10\right) + \left(\frac{25}{2} + 10\right) - (2-4) = 29$$

(b) Since for $1 \le x \le 3$, $x \ge 0$, $x - 3 \le 0 \Rightarrow |x| = x$, $|x-3| = -(x-3)$

Also for $3 \le x \le 4$, $x \le 0$, $x - 3 \ge 0 \Rightarrow |x| = x$, $|x-3| = x - 3$

$\therefore$ $$\int_1^4 (|x|+|x-3|)dx = \int_1^3 (|x|+|x-3|)dx + \int_3^4 (|x|+|x-3|)dx \quad \text{(Property 3)}$$

$$= \int_1^3 (x-(x-3))dx + \int_2^4 (x+x-3)dx = \int_1^3 3dx + \int_3^4 (2x-3)dx$$

$$= 3[x]_1^3 + \left[2.\frac{x^2}{2} - 3x\right]_3^4$$

$$= 3(3-1) + (16-12) - (9-9) = 16 + 4 - 0 = 10$$

Example-7.10 Evaluate the following definite integrals:

(a) $\int_{-1}^{2} f(x)dx$ where $f(x) = \begin{cases} 2x+1, & x \le 1 \\ x-5, & x > 1 \end{cases}$ (b) $\int_{-1}^{1} \frac{|x|}{x} dx$

(c) $\int_0^1 [3x]dx$

Solution:

(a) First note that the given function is discontinuous at $x = 1$.

$$\therefore \quad \int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^2 f(x)dx$$

$$= \int_{-1}^{1} (2x+1)dx + \int_1^2 (x-5)dx = \left[x^2+x\right]_{-1}^{1} + \left[\frac{x^2}{2} - 5x\right]_1^{2}$$

$$= (1+1)-(1-1)+(2-10)-\left(\frac{1}{2}-5\right) = 2-0-8+\frac{9}{2} = -\frac{3}{2}$$

(b) First note that $\frac{|x|}{x}$ is discontinuous at $x = 0$.

$$\therefore \quad \int_{-1}^{1} \frac{|x|}{x}dx = \int_{-1}^{0} \frac{|x|}{x}dx + \int_0^1 \frac{|x|}{x}dx = \int_{-1}^{0} \frac{-x}{x}dx + \int_0^1 \frac{x}{x}dx$$

$(\because -1 \le x \le 0 \Rightarrow |x| = -x \text{ and } 0 \le x \le 1 \Rightarrow |x| = x)$

$$= \int_{-1}^{0} -1dx + \int_0^1 1dx = [-x]_{-1}^{0} + [x]_0^1$$

$$= -(0-(-1)) + (1-0) = -1 + 1 = 0$$

(c) First note that $[3x]$ is discontinuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$,

$$\therefore \quad \int_0^1 [3x]\,dx = \int_0^{1/3} [3x]\,dx + \int_{1/3}^{2/3} [3x]dx + \int_{2/3}^{1} [3x]dx$$

$$= \int_0^{1/3} 0\,dx + \int_{1/3}^{2/3} 1\,dx + \int_{2/3}^{1} 2\,dx = 0 + [x]_{1/3}^{2/3} + 2[x]_{2/3}^{1}$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right) + 2\left(1-\frac{2}{3}\right) = \frac{1}{3}+\frac{2}{3} = 1$$

Example-7.11 Evaluate the following definite integrals:

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

Solution:

Let, $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \text{... (i)}$$

Then, by using property 4*b*, we get

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Example-7.12 Evaluate the following definite integrals:

(a) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$ (b) $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

Solution:

(a) Let, $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{...(i)}$$

Then, by using property 4*b*, we get

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx = \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx = \int_0^1 -1.\log\left(\frac{1-x}{x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx = -I$$

$\Rightarrow$ $2I = 0$

$\Rightarrow$ $I = 0$

(b) Let $$I = \int_0^{\pi/2} \sin 2x \log(\tan x) dx \quad \text{...(i)}$$

Then, by using property 4*b*, we get

Let, $$I = \int_0^{\pi/2} \sin\left(2\left(\frac{\pi}{2} - x\right)\right) \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x)\log(\cot x) dx = \int_0^{\pi/2} \sin 2x \log\left((\tan x)^{-1}\right) dx$$

$$= \int_0^{\pi/2} \sin 2x(-1)\log(\tan x)dx = \int_0^{\pi/2} \sin 2x \log(\tan x)dx$$

$$= -I \qquad \text{[using (}i\text{)]}$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0$$

Example - 7.13 Evaluate the following definite integrals:

$$\int_0^{\pi} \log(1+\cos x)\, dx$$

Solution:

$$I = \int_0^{\pi} \log(1+\cos x)dx \qquad ...(i)$$

Then, by using property 4*b*, we get

$$I = \int_0^{\pi} \log(1+\cos(\pi-x))dx = \int_0^{\pi} \log(1-\cos x)dx \qquad ...(ii)$$

On adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} (\log(1+\cos x)+\log(1-\cos x))dx = \int_0^{\pi} \log\left(1-\cos^2 x\right)dx$$

$$= \int_0^{\pi} \log(\sin^2 x)dx = 2\int_0^{\pi} \log \sin x\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \log \sin x\, dx$$

Let $f(x) = \log \sin x \Rightarrow f(\pi - x) = \log(\sin(\pi - x)) = \log \sin x = f(x)$, therefore, by using property 6, we get

$$I = 2\int_0^{\pi/2} \log \sin x\, dx = 2.\left(-\frac{\pi}{2}\log 2\right) = -\pi \log 2.$$

7.8 Partial Derivatives

Definition of Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivative of a function of more than one independent variables is called Partial Differentiation.

The symbols $\partial/\partial x$, $\partial/\partial y$ etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$ etc., are respectively called partial differential coefficients of u with respect to x and y.

If $u = f(x, y, z)$ the partial differential coefficient of u with respect to x i.e., $\partial u/\partial x$ is obtained by differentiating u with respect to x keeping y and z as constants.

Second Order Partial Differential Coefficients

If $u = f(x, y)$ then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves function of x and y can be again differentiated partially.

We call $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right), \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$ as second order partial derivatives of u and these are respectively denoted by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}$.

NOTE

If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Example-7.14 Let $f = y^x$.

What is $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2, y = 1$?

(a) 0

(b) ln 2

(c) 1

(d) $\frac{1}{\ln 2}$

Solution: (c)

$$f = y^x$$

Treating x as constant, we get

$$\frac{\partial t}{\partial y} = xy^{x-1}$$

Now we treat y as a constant and get,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(y^{x-1}\, x\right) = y^{x-1} + xy^{x-1} \ln y$$

whose value at $x = 2$ and $y = 1$ is $= 1^{(2-1)}(1 + 2.\ \ln 1) = 1$

Example-7.15 If $z = xy \ln(xy)$, then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

(b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$

(d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

Solution: (c)

$$\frac{\partial z}{\partial x} = y\ln(xy) + \frac{xy}{xy}xy$$

$$\frac{\partial z}{\partial x} = y[\ln(xy) + 1] \quad ...(i)$$

$$\frac{\partial z}{\partial x} = x\ln(xy) + \frac{xy}{xy} \times x$$

$$\frac{\partial z}{\partial x} = x[\ln(xy)+1] \quad \text{...(ii)}$$

Here, $\boxed{x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}}$

Homogenous Functions

An expression in which every term is of the same degree is called homogenous function. Thus, $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n$ is a homogenous function of x and y of degree n. This can also be written as,

$$x^n\left\{a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right\}$$

or $x^n f\left(\frac{y}{x}\right)$, where $f\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

NOTE

- To test whether a given function $f(x, y)$ is homogenous or not we put tx for x and ty for y in it.
  If we get $f(tx, ty) = t^n f(x, y)$ the function $f(x, y)$ is homogenous of degree n otherwise $f(x, y)$ is not a homogenous function.
- If u is a homogenous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogenous function of x and y each being of degree $(n-1)$.

Euler's Theorem on Homogenous Functions

If u is a homogenous function of x and y of degree n, then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Euler's theorem can be extended to a homogenous function of any number of variables. Thus if $f(x_1, x_2, ... x_n)$ be a homogenous function of $x_1, x_2, ... x_n$ of degree n then, $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + ... + x_n\frac{\partial f}{\partial x_n} = nf$

7.9 Total Derivatives

If $u = f(x, y)$, where $x = \phi_1(t)$ and $y = \phi_2(t)$,

then, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Here $\frac{du}{dt}$ is called the total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u.

In the same way if $u = f(x, y, z)$ where x, y, z are all functions of some variable t, when

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

This result can be extended to any number of variables.

Corollary-1: If u be a function of x and y, where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

Corollary-2: If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

Corollary-3: If x and y are connected by an equation of the form $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{\partial f / \partial x}{\partial f / \partial y}$$

7.10 Maxima and Minima (of function of a single independent variable)

Definitions: A function $f(x)$ is said to be "maximum" at $x = a$, if there exist a positive number δ such that $f(a + h) < f(a)$ for all values of h other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be minimum at $x = a$, if there exists a positive number δ such that $f(a + h) > f(a)$ for all values of h, other than zero, in the interval $(-\delta, \delta)$.

Maximum and Minimum values of a function are also called extreme values or turning values and the points at which they are attained are called points of maxima and minima.

The points at which a function has extreme values are called Turning Points.

7.10.1 Properties of Maxima and Minima

1. At leastone maximum or one minimum must lie between two equal values of a function.
2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function $y = f(x)$ is maximum at $x = a$, if dy/dx changes sign from +ve to –ve as x passes through a.
5. A function $y = f(x)$ is minimum at $x = a$, if dy/dx changes sign from –ve and +ve as x passes through a.
6. If the sign of dy/dx does not change while x passes through a_1 then y is neither maximum nor minimum at $x = a$.

Conditions for Maximum or Minimum Values

The necessary condition that $f(x)$ should have a maximum or a minimum at $x = a$ is that $f'(a) = 0$.

Definition of Stationary Values

A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$.

Thus for a function $f(x)$ to be a maximum or minimum at $x = a$ it must stationary at $x = a$.

Sufficient Conditions of Maximum or Minimum Values

There is a maximum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is negative.

Similarly there is a minimum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ positive.

NOTE: If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Again, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f'(a)$ is positive there will be minimum at $x = a$.

In general if, $f'(a) = f''(a) = f'''(a) = \ldots f^{n-1}(a) = 0$ and $f^n(a) \neq 0$ then n must an even integer for maximum or minimum. Also for a maximum $f^n(a)$ must be negative and for a minimum $f^n(a)$ must be positive.

7.10.2 Working rule for Maxima and Minima of $f(x)$

1. Find $f'(x)$ and equate to zero.
2. Solve the resulting equation for x. Let its roots be $a_1, a_2, \ldots\ldots$ Then $f(x)$ is stationary at $x = a_1, a_2, \ldots\ldots\ldots\ldots$ Thus $x = a_1, a_2, \ldots\ldots\ldots$ are the only points at which $f(x)$ can be maximum or a minimum.
3. Find $f''(x)$ and substitute in it by terms $x = a_1, a_2, \ldots\ldots\ldots\ldots$
4. If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is –ve, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

Example-7.16 Prove that the function $f(x) = ax + b$ is strictly increasing iff $a > 0$.

Solution:

Given: $f(x) = ax + b$, $D_f = R$

Note that f is continuous and differentiable for all $x \in R$.

Differentiating the given function w.r.t. x, we get $f'(x) = a$.

Now the given function is strictly increasing iff $f'(x) > 0$ i.e. iff $a > 0$.

Hence, the given function is strictly increasing for all $x \in R$ iff $a > 0$.

Example-7.17 Prove that the function e^{2x} is strictly increasing on R.

Solution:

Let, $f(x) = e^{2x}$, $D_f = R$.

Differentiating w.r.t. x, we get

$$f'(x) = e^{2x}.\,2 > 0 \text{ for all } x \in R.$$

$\Rightarrow f(x)$ is strictly increasing on R.

Example-7.18 Prove that $(2/x) + 5$ is a strictly decreasing function.

Solution:

Let, $f(x) = \dfrac{2}{x} + 5$, $D_f = R - [0]$.

Differentiate it w.r.t. x, we get $f'(x) = 2.(-1.\,x^{-2}) + 0 = -\dfrac{2}{x^2}$

Since, $x^2 > 0$ for all $x \in R$, $x \neq 0$, therefore,

$f'(x) < 0$ for all $x \in R$, $x \neq 0$, i.e., for all $x \in D_f$

$\Rightarrow$ the given function is strictly decreasing.

Example-7.19 Prove that the function $f(x) = x^3 - 6x^2 + 15x - 18$ is strictly increasing on R.

Solution:

Given, $f(x) = x^3 - 6x^2 + 15x - 18,\ D_f = R$

Differentiate it w.r.t. x we get, $f'(x) = 3x^2 - 6.2x + 15.1 = 3(x^2 - 4x + 5)$

$$= 3[(x-2)^2 + 1] \geq 3 \quad (\because (x-2)^2 \geq 0 \text{ for all } x \in R)$$

$\Rightarrow$ $f'(x) > 0$ for all $x \in R$

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

Example-7.20 Find the intervals in which the following functions are strictly increasing or strictly decreasing

(a) $f(x) = 10 - 6x - 2x^2$

(b) $f(x) = x^2 - 12x^2 + 36x + 17$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Solution:

(a) Given, $f(x) = 10 - 6x - 2x^2,\ D_f = R$

Differentiating it w.r.t. x, we get

$$f'(x) = 0 - 6.1 - 2.\ 2x = -6 - 4x = -4\left(x + \frac{3}{2}\right)$$

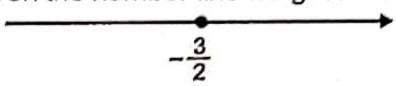
Putting, $f'(x) = 0$,

we get, $\dfrac{20 \pm \sqrt{400 - 156}}{2} = 0$

$\Rightarrow$ $x + \dfrac{3}{2} = 0 \Rightarrow x = -\dfrac{3}{2}$

So there is only one critical point which is $x = -\dfrac{3}{2}$

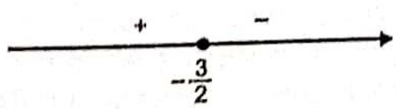
Plotting this critical point on the number line we get the following picture



So the critical point divides the real number line into two regions which are $x \in \left(-\infty, -\dfrac{3}{2}\right)$ and $x \in \left(-\dfrac{3}{2}, \infty\right)$.

Now we find $f'(0) = -6$ which is negative and so the region $x \in \left(-\dfrac{3}{2}, \infty\right)$ (which contains $x = 0$) is the region where the function is strictly decreasing.

Therefore in the other region i.e. $x \in \left(-\infty, -\dfrac{3}{2}\right)$ is the region in which the function is strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



(b) Given, $f(x) = x^3 - 12x^2 + 36x + 17, D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36$$
$$= 3(x^2 - 8x + 12) = 3(x-2)(x-6)$$

Putting, $f'(x) = 0$ i.e. $3(x-2)(x-6) = 0$

$\Rightarrow$ $(x-2)(x-6) = 0$

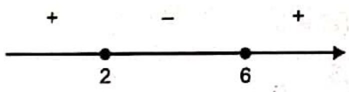
$\Rightarrow$ $x = 2$ or $x = 6$ are the two critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, 2)$ and $x \in (2, 6)$ and $x \in (6, \infty)$.

Now we find $f'(0) = 3(0-2)(0-6) = +36$ which is positive and so in the region $x \in (-\infty, 2)$ (which contains $x = 0$), the function is strictly increasing.

Therefore in the next region i.e. $x \in (2, 6)$, the function is strictly decreasing and in the next region $x \in (6, \infty)$, the function is again strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

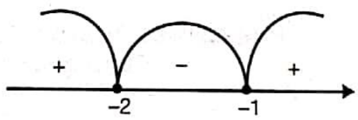


So the final region in which the function strictly increasing is $x \in (-\infty, 2) \cup (6, \infty)$ and the region in which the function is strictly decreasing is $x \in (2, 6)$.

(c) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1, D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x+2)(x+1)$$



Putting, $f'(x) = 0$ i.e. $-6(x+2)(x+1) = 0$

$\Rightarrow$ $(x+2)(x+1) = 0$

$\Rightarrow$ $x = -2$ and $x = -1$ are the critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, -2)$ and $x \in (-2, -1)$ and $x \in (-1, \infty)$.

Now we find $f'(0) = -6(0+2)(0+1) = -12$ which is negative and so in the region $x \in (-1, \infty)$. (which contains $x = 0$), the function is strictly decreasing.

Therefore in the next adjacent region on the left i.e. $x \in (-2, -1)$, the function is strictly increasing and in the next adjacent region on the left $x \in (-\infty, -2)$, the function is again strictly decreasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

So the final region in which the function strictly increasing is $x \in (-2, -1)$ and the region in which the function is strictly decreasing is $x \in (-\infty, -2) \cup (-1, \infty)$.

Example-7.21 The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1 (b) $\dfrac{\sqrt{3}}{2}$

(c) $\sqrt{3}$ (d) 2

Solution: (a)

Let the point be (x, y, z) on surface $z^2 = 1 + xy$

$$\text{Distance from origin} = l = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

$$l = \sqrt{x^2 + y^2 + 1 + xy} \quad [\text{since } z^2 = 1 + xy \text{ is given}]$$

This distance is shortest when l is minimum we need to find minima of $x^2 + y^2 + 1 + xy$

Let, $u = x^2 + y^2 + 1 + xy$

$$\frac{\partial u}{\partial x} = 2x + y$$

$$\frac{\partial u}{\partial y} = 2y + z$$

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ $2x + y = 0$ and $2y + x = 0$

Solving simultaneously, we get

$$x = 0 \text{ and } y = 0$$

is the only solution and so (0, 0) is the only stationary point.

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 2$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2$$

Since, $rt = 2 \times 2 = 4 > s^2 = 1$

We have case 1, i.e. either a maximum or minimum exists at (0, 0)

Now, since $r = 2 > 0$, so it is a minima at (0, 0).

Now at $x = 0, \quad y = 0, \quad z = \sqrt{1 + xy} = \sqrt{1+0} = 1$

So, the point nearest to the origin on surface $z^2 = 1 + xy$ is (0, 0, 1)

The distance, $l = \sqrt{0^2 + 0^2 + 1^2} = 1$

So, correct answer is choice (a).

7.10.3 Maxima and Minima (of function of a two independent variable)

Definitions: Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively.

The $f(a, b)$ is said to be maximum and a minimum value of $f(x, y)$ according as $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k. Positive negative, provided both of than are not equal to zero.

Necessary Conditions

The necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} = 0$$

Sufficient condition for Maxima or Minima

Let,
$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}};\ s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}};\ t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$$

Case-1: $f(x, y)$ will have a maximum or a minimum at $x = a$, $y = b$, if $rt > s^2$. Further, $f(x, y)$ is maximum or minimum according as r in negative or positive.

Case-2: $f(x, y)$ will have neither maximum or minimum at $x = a$, $y = b$ if $rt < s^2$. i.e. $x = a$, $y = b$ is a saddle point.

Case 3: If $rt = s^2$ this case is doubtful case and further investigation is needed to determine whetherr $f(x, y)$ is a maximum or minimum at $x = a$, $y = b$ or not.

Example-7.22 Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$.

The optimal value of $f(x, y)$

(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3

Solution: (a)

$$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$\frac{\partial f}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4y$$

Putting,
$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$8x - 8 = 0 \text{ and } 12y - 4y = 0$$

Given,
$$x = 1 \text{ and } y = \frac{1}{3}$$

$\left(1, \frac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{\left(1,\frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1,\frac{1}{3}\right)} = 12$$

Since, $rt = 8 \times 12 = 96$

$s^2 = 0$

Since, $rt > s^2$

we have either a maxima or minima at $\left(1,\frac{1}{3}\right)$

also since, $r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8 > 0$, the point $\left(1,\frac{1}{3}\right)$ is a point of minima.

The minimum value is

$$f\left(1,\frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$$

So the optimal value of $f(x, y)$ is a minimum equal to $\frac{10}{3}$.

Summary

- If $f(x)$ and $f(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = f(a) = 0$ then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}$ provided, the latter limit exists, finite or infinite.
- Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a if $\underset{x \to a}{\text{Limit}} f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a if $\underset{x \to a+0}{\text{Limit}} f(x)$ exists and is equal to $f(a)$.
- Continuity is a necessary but not a sufficient condition for the existence of a finite derivative. i.e. differentiability $\Rightarrow$ continuity. But continuity $\not\Rightarrow$ differentiability
- Role's theorem will not hold good:
  - (*i*) If $f(x)$ is discontinuous at some point in the interval $a < x < b$
  - (*ii*) If $f'(x)$ does not exist at some point in the interval $a \le x \le b$ or
  - (*iii*) If $f(a) \neq f(b)$
- If a function $f(x)$ is (*i*) continuous in closed interval $a \le x \le b$ and (*ii*) differentiable in open interval (a, b) i.e., $a < x < b$, then there exist at least one value c of x lying in the open interval $a < x < b$ such that, $\frac{f(b) - f(a)}{b - a} = f'(c)$

- If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
- To test whether a given function $f(x, y)$ is homogenous or not we put tx for x and ty for y in it. If we get $f(tx, ty) = t^n f(x, y)$ the function $f(x, y)$ is homogenous of degree n otherwise $f(x, y)$ is not a homogenous function.
- If u is a homogenous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogenous function of x and y each being of degree $(n-1)$.
- If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Again, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f'(a)$ is positive there will be minimum at $x = a$.

Student's Assignment

Q.1 Evaluate $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi a}{2x}}$

(a) $e^{2/\pi}$ (b) $2/e^{\pi}$
(c) $e^{\pi/2}$ (d) $e^{-2/\pi}$

Q.2 Check whether the following integrals and its values are true or false?

1. $\int_0^{\pi} \cos^{2n+1} x dx = 0$
2. $\int_0^{\pi} \cos^{2n} x dx = 2\int_0^{\pi/2} \cos^{2n} x dx$
3. $\int_0^{\pi} \sin^{2n} x dx = \int_0^{\pi/2} \sin^{2n} x dx$

For all values of n

(a) 1 and 3 are true (b) 1 and 2 are true
(c) 3 only true (d) 2 and 3 are true

Q.3 If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ and $I_n + n(n-1) I_{n-2} = n(\pi/2)^{n-1}$ then find I_2

(a) $2-\pi$ (b) $\pi-2$
(c) $\pi/2$ (d) $(\pi/2)^2$

Q.4 Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(a) 0 (b) 1
(c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x^2}{\sqrt{x^2-1}}$

Q.5 $\int \log x \, dx =$

(a) $(x \log x - 1)$ (b) $\log x - x$
(c) $x(\log x - 1)$ (d) None of these

Q.6 If $0 < x < 1$ then

(a) $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$

(b) $\frac{\sqrt{1-x}}{\sqrt{1+x}} > \frac{\log(1+x)}{\sin^{-1}x} > 1$

(c) $\frac{\sqrt{1-x}}{\sqrt{1+x}} > \frac{\log(1+x)}{\sin^{-1}x} < 1$

(d) $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} > 1$

Q.7 If $u = f(y/x)$ then

(a) $\frac{x\,du}{dx} - \frac{y\,du}{dy} = 0$

(b) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 0$

(c) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 2u$

(d) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 1$

Q.8 The function $f(x, y) = xy(a - x - y)$, $a > 0$ attains extreme values at $\left(\frac{a}{3}, \frac{a}{3}\right)$ and its value is

(a) minimum value, $\frac{a^2}{3}$

(b) minimum value, $\frac{a^3}{27}$

(c) maximum value, $\frac{a^3}{3}$

(d) maximum value, $\frac{a^3}{27}$

Q.9 Match List-I (Sequence) with List-II (Generating function) and select the correct answer using the codes given below the lists:

List-I	List-II
A. 1	1. $x(1 + x)(1 - x)^{-3}$
B. $(-1)^n$	2. $\frac{1}{1-x}$
C. n^2	3. $x(1 - x)^{-2}$
D. n	4. $\frac{1}{1+x}$

Codes:

	A	B	C	D
(a)	1	2	4	3
(b)	2	1	3	4
(c)	4	3	2	1
(d)	2	4	1	3

Q.10 The Domain of the function $\frac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(0, \infty)$

(c) $(0, x)$ (d) $(0, 1)$

Q.11 The function is continuous in [0, 1], such that $f(0) = -1$, $f(1/2) = 1$ and $f(1) = -1$.
We can conclude that

(a) f attains the value zero at least twice in [0, 1]

(b) f attains the value zero exactly once in [0, 1]

(c) f is non-zero in [0, 1]

(d) f attains the value zero exactly twice in [0, 1]

Q.12 The function $f(x) = |x| - |x + 1|$

(a) is less than 1, for all x

(b) equal $f(-x)$

(c) equals $1 - f(1/x)$

(d) none of the above

Q.13 $f(x)$ and $g(x)$ are two functions differentiable in [0, 1] such that $f(0) = 2$; $g(0) = 0$; $f(1) = 6$; and $g(1) = 2$. Then there must exist a constant C in

(a) (0, 1), such that $f'(c) = 2g'(c)$

(b) [0, 1], such that $f'(c) = g'(c)$

(c) (0, 1), such that $2f'(c) = g'(c)$

(d) [0, 1], such that $2f'(c) = g'(c)$

Answer Key:

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)
6. (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (a)
11. (a)	**12.** (d)	**13.** (a)		

Student's Assignments Explanations

1. (a)

$$\lim_{x \to a}\left(2 - \frac{a}{x}\right)^{\tan\left(\frac{\pi a}{2x}\right)}$$

Let, $f(x) = 2 - \frac{a}{x}$ and $g(x) = \tan\frac{\pi x}{2a}$

Since $\lim_{x \to a} f(x) = 1$

and $\lim_{x \to a} g(x) = \tan\frac{\pi}{2} = \infty$

$$\lim_{x \to a}[f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)\{f(x) - 1\}}$$

$$= e^{\lim_{x \to a} \tan\frac{\pi a}{2a}\left(1 - \frac{a}{x}\right)}$$

Now, $\lim_{x \to a} \tan\frac{\pi a}{2x}\left(1 - \frac{a}{x}\right) = \lim_{x \to a} \frac{1 - \frac{a}{x}}{\cot\frac{\pi a}{2x}}$

$$= \lim_{x\to a} \frac{1-\frac{a}{x}}{\tan\left(\frac{\pi}{2}-\frac{\pi a}{2x}\right)}$$

$$= \lim_{x\to a} \frac{1-\frac{a}{x}}{\tan\left[\frac{\pi}{2}\left(1-\frac{a}{x}\right)\right]}$$

$$= \frac{2}{\pi} \lim_{x\to a} \frac{\frac{\pi}{2}\left[1-\frac{a}{x}\right]}{\tan\frac{\pi}{2}\left[1-\frac{a}{x}\right]}$$

$$= \frac{2}{\pi} \lim_{x\to a} \frac{t}{\tan t}$$

Where, $t = \frac{\pi}{2}\left(1-\frac{a}{x}\right)$ as $x \to a,\ t \Rightarrow 0$

$$= \frac{2}{\pi} \cdot 1 = \frac{2}{\pi}$$

Now the required limit is

$$e^{\lim_{x\to a} \tan\frac{\pi a}{2x}\left(1-\frac{a}{x}\right)} = e^{2/\pi}$$

2. (b)

1. $\int_0^{\pi} \cos^{2n+1} x\, dx = 0$

$$f(x) = \cos^{2n+1} x$$

Since, $f(\pi - x) = \cos^{2n+1}(\pi - x)$

$$= (-\cos x)^{2n+1} = -f(x)$$

$\therefore \int_0^{\pi} \cos^{2n+1} x\, dx = 0$

1 is true.

2. $\int_0^{\pi} \cos^{2n} x\, dx = 2\int_0^{\pi/2} \cos^{2n} x\, dx$

$$f(x) = \cos^{2n}(x)$$
$$f(\pi - x) = \cos^{2n}(\pi - x)$$
$$= (-\cos x)^{2n}$$
$$= \cos^{2n} x = f(x)$$

$\therefore \int_0^{\pi} \cos^{2n} x\, dx = 2\int_0^{\pi/2} \cos^{2n} x\, dx$

2 is true.

3. $\int_0^{\pi} \sin^{2n} x\, dx = \int_0^{\pi/2} \sin^{2n} x\, dx$

$$f(x) = \sin^{2n}(x)$$
$$f(\pi - x) = (\sin(\pi - x))^{2n}$$
$$= (\sin x)^{2n} = f(x)$$

$\Rightarrow \int_0^{\pi} \sin^{2n} x\, dx = 2\int_0^{\pi/2} \sin^{2n} x\, dx$

$\therefore$ 3 is false.

3. (b)

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Put, $n = 2$

$$I_2 + 2(2-1)I_0 = 2\left(\frac{\pi}{2}\right)^{2-1}$$

$\therefore$ $I_2 = \pi - 2I_0$

$$I_0 = \int_0^{\pi/2} \dot{x} \sin x dx$$

$$= \int_0^{\pi/2} \sin x dx$$

$$= [-\cos x]_0^{\pi/2}$$

$$= -0 + 1 = 1$$

$\therefore$ $I_2 = \pi - 2(1) = \pi - 2$

4. (b)

Let, $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

putting $x = \tan\theta$

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{2} = 1$$

6. (c)
Use Integration by parts.

7. (a)
Consider the functions
$f(x) = \log(1+x)$ and $g(x) = \sin^{-1} x$
Both of these are continuous and differentiable for $0 \le x \le 1$.
∴ These satisfy the conditions of the Cauchy's Mean value theorem in interval $[0, x]$ when $x < 1$

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(C)}{g'(C)}$$

$$\frac{\log(1+x)-\log 1}{\sin^{-1}x-\sin^{-1}0} = \frac{1}{1+C}\sqrt{1-C^2}$$

$$\frac{\log(1+x)}{\sin^{-1}x} = \frac{\sqrt{1-C}}{\sqrt{1+C}}$$

Since $0 < C < x < 1$ we have

$$\frac{\sqrt{1-C}}{\sqrt{1+C}} < 1 \text{ and } \frac{\sqrt{1-C}}{\sqrt{1+C}} > \frac{\sqrt{1-x}}{\sqrt{1+x}}$$

Hence $$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\sqrt{1-C}}{\sqrt{1+C}} < 1$$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$$

8. (b)

$$u = f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right) * \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right) * \frac{1}{x}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x * f'\left(\frac{y}{x}\right) * \frac{-y}{x^2} + y * f'\left(\frac{y}{x}\right) * \frac{1}{x}$$
$= 0$

9. (d)

$$f(x, y) = xy(a-x-y)$$

$$\frac{\partial f}{\partial x} = xy(-1) + (a-x-y)\,y$$
$$= ay - 2xy - y^2$$

Similarly, $\frac{\partial f}{\partial y} = ax - 2xy - x^2$

Putting $\frac{\partial f}{\partial x} = 0$, we get

$ay - 2xy - y^2 = 0$
$\Rightarrow y = 0$ or $2x + y = a$

and putting $\frac{\partial f}{\partial y} = 0$, we get

$ax - 2xy - x^2 = 0$
$\Rightarrow x = 0$ or $2y + x = a$

Solving $2x + y = a$
and $2y + x = a$

we get $x = \frac{a}{3}, y = \frac{a}{3}$

∴ The function attains extreme values at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$

$\Rightarrow r$ value at $\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{-2a}{3}$

$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$$

$\Rightarrow s$ value at $\left(\frac{a}{3}, \frac{a}{3}\right) = -\frac{a}{3}$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

$\Rightarrow t$ value at $\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{-2a}{3}$

$$r \times t = \frac{-2a}{3} \times \frac{-2a}{3} = \frac{4a^2}{9}$$

$$s^2 = \frac{a^2}{9}$$

$\Rightarrow \quad rt > s^2$

This means that at $\left(\frac{a}{3}, \frac{a}{3}\right)$, this function reaches a maximum or minimum.

However, since $r = \frac{-2a}{3} < 0$, it is maximum

$$f_{max} = \frac{a}{3} \times \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}$$

10. (a)

f is defined if $|x| - x > 0$ i.e., $|x| > x$

If $x \geq 0$, $|x| = x$ so, $x > x$, which has no feasible solution

If $x < 0$, $|x| = -x$, so $-x > x$, which has solution $x < 0$.

$\therefore$ Domain = $(-\infty, 0)$

11. (a)

Whenever $f(a)$ and $f(b)$ are of different signs, f has an odd number of roots (at least one root) between a and b.

$\therefore$ There exists at least one root between 0 and $\frac{1}{2}$ and at least 1 more root between $\frac{1}{2}$ and 1

i.e. $f(x)$ has at least 2 zeroes between 0 and 1.

12. (d)

$$f(x) = |x| - |x+1|$$
$$f(-2) = |-2| - |-2+1| = 1$$

$\therefore$ (a) is false.

$$f(2) = |2| - |2+1| = -1$$
$$f(2) \neq f(-2).$$

So (b) is false.

$$f\left(\frac{1}{2}\right) = -1$$

$$1 - f\left(\frac{1}{2}\right) = 1 - (-1) = 2$$

$$f(2) = -1$$

So, $\quad f(2) \neq 1 - f\left(\frac{1}{2}\right)$

(c) is false.

So "none of the above" is the answer.

13. (a)

Consider the function $\phi(x) = f(x) - 2g(x)$

$$\phi(0) = \phi(1) = 2,$$

So, $f(x)$ satisfies the conditions of Roll's theorem in [0, 1] so,

$$\phi'(x) = f'(x) - 2g'(x)$$

has atleast one 0 at (in (0, 1)

i.e., $\quad \phi'(C) = 0$

$\Rightarrow \quad f'(C) - 2g'(C) = 0$

$\Rightarrow \quad f'(C) = 2g'(C)$