CHAPTER 06

Linear Algebra

6.1 Introduction

Definition of Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m \times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

6.2 Special Types of Matrices

1. **Square Matrix:** An $m \times n$ matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. The elements a_{11}, a_{22}.... are called DIAGONAL ELEMENTS and the line along which they lie is called PRINCIPLE DIAGONAL of matrix.

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$ is an square Matrix

NOTE: A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A.

2. **Diagonal Matrix:** A square matrix in which all of non-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

Example: $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ is a diagonal matrix

The above matrix can also be written as A = diag [3, 5, 9]

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

Example: $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix.

5. **Null Matrix:** The m × n matrix whose elements are all zero is called null matrix. Null matrix is denoted by O.

Example: $O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

6. **Upper Triangular Matrix:** A upper triangular matrix is a square matrix whose lower off diagonal elements are zero, i.e. $q_{ij} = 0$ whenever $i < j$

It is denoted by U.

Example: $U = \begin{pmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{pmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose triangular elements are zero, ilel $q_{ij} = 0$ whenever $i > j$. It is denoted by L,

Example: $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{pmatrix}$

6.3 Algebra of Matrices

Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

(a) They are of same size.

(b) The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Addition of Matrices

Let A and B be two matrices of the same type $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$; $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$

Scalar Properties of Matrix Addition:

(a) Matrix addition is commutative A + B = B + A.

(b) Matrix addition is associative (A + B) + C = A + (B + C)

(c) Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, A + O = A = O + A for every $m \times n$ matrix A.

(d) Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.
Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by –A.
⇒ Matrix –A is additive inverse of A. Because (–A) + A = O = A + (–A). Here O is null matrix of order $m \times n$.

(e) Cancellation laws holds good in case of addition of matrices.
$A + X = B + X \Rightarrow A = B$
$X + A = X + B \Rightarrow A = B$

(f) The equation A + X = 0 has a unique solution in the set of all $m \times n$ matrices.

Substraction of Two Matrices

If A and B are two m × n matrices, then we define, A – B = A + (–B).

Thus the difference A – B is obtained by substracting from each element of A corresponding elements of B.

NOTE: Subtraction of matrices is neither commutative nor associative.

Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and k be any real number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

⇒ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{pmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{pmatrix}$ then, $3A = \begin{pmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{pmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

(a) Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.

(b) If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.

(c) If p and q are two matrices and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.

(d) If $A = [a_{ij}]_{m \times n}$ matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write $C = AB$.

Properties of Matrix Multiplication:

(a) Multiplication of matrices is not commutative. In fact if the product of AB exists, then it is not necessary that the product of BA will also exist.

(b) Matrix multiplication is associative if conformability is assured. i.e., A (BC) = (AB)C where A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.

(c) Multiplication of matrices is distributive with respect to addition of matrices. i.e., A (B+C) = AB + AC.

(d) The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix.

(e) In the case of matrix multiplication if AB = O then it is not necessarily imply that BA = O.

6.4 Properties of Matrices

Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by tr A.

Thus if $A = [a_{ij}]_{n \times n}$ then, $Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$.

Let,
$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{pmatrix}$$

Then, trace (A) = tr(A) = 1 + (–3) + 5 = 3

Properties: Let A and B be two square matrices of order n and λ be a scalar. Then,

(a) tr (λA) = λ tr A

(b) tr (A + B) = tr A + tr B

(c) tr (AB) = tr (BA)

Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A′ or A^T.

Let, $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{pmatrix}$ then, $A^T = A = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{pmatrix}$

Properties: If A′ and B′ be transposes of A and B respectively then,

(a) (A′)′ = A

(b) (A + B)′ = A′ + B′

(c) (kA)′ = kA′, k being any complex number

(d) (AB)′ = B′A′

Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$; $\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$

If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively. Then,

(a) $\overline{(\bar{A})} = A$

(b) $\overline{(A+B)} = \bar{A}+\bar{B}$

(c) $\overline{(kA)} = \bar{k}\bar{A}$, k being any complex number

(d) $\overline{(AB)} = \bar{A}\bar{B}$, A and B being conformable to multiplication.

Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or A^{*} or $(\bar{A})^{T}$. It is also called conjugate transpose of A.

Some properties: If A^{θ} and B^{θ} be the transposed conjugates of A and B respectively then,

(a) $(A^{\theta})^{\theta} = A$

(b) $(A + B)^{\theta} = A^{\theta} + B^{\theta}$

(c) $(kA)^{\theta} = \bar{k}A^{\theta}$, $k \to$ complex number

(d) $(AB)^{\theta} = B^{\theta}A^{\theta}$

Symmetric Matrix

A square matrix $A= [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i and j.

In a symmetric matrix, $A^T = A$

Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

Skew Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}$ $\forall$ i, j.

In a skew symmetric matrix $A^T = -A$.

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Orthogonal Matrix

A square matrix A is said be orthogonal if:
$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if
$AA^T = I = A^TA$.

Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i, j)^{th}$ element of A is equal to conjugate complex of $(j, i)^{th}$ element of A. i.e., if $a_{ij} = \bar{a}_{ji}\ \forall\ i\ \&\ j$

Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$

A necessary and sufficient condition for a matrix A to be Hermitian is that $A^{\theta} = A$.

Skew Hermition Matrix

A square matrix $A = [a_{ij}]$ is said to be skew Hermitian $\forall$ $(i, j)^{th}$ element of A is equal to negative of conjugate complex of $(j, i)^{th}$ element of A i.e.,

if $a_{ij} = a_{ij} = -\bar{a}_{ji}\ \forall\ i.j.$

Example: $A = \begin{bmatrix} 0 & -2+i \\ 2+i & 0 \end{bmatrix}_{2\times 2}$

A necessary and sufficient condition for a matrix to be skew Hermitian if $A^{\theta} = -A$.

Unitary Matrix

A square matrix A is said to be unitary if:

$A^{\theta} = A^{-1} \Rightarrow AA^{\theta} = AA^{-1} = I$

Thus A will be unitary matrix if

$AA^{\theta} = I = A^{\theta} A$

6.5 Determinants

Definition: Let $a_{11}, a_{12}, a_{21}, a_{22}$ be any four number. The symbol $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number $a_{11}, a_{12}, a_{21}, a_{22}$ are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

6.5.1 Minors and Cofactors

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

Cofactors

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij}. We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

$$\text{Cofactor of element } a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{by cofactor of element } a_{32} = A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Therefore, "in a determinant the sum of the products of the elements of any row or column with corresponding cofactors is equal to value of the determinant.

6.5.2 Determinant of Order n

A determinant of order n has n-row and n-columns. It has $n \times n$ elements.

A determinant of order n is a square array of $n \times n$ quantities enclosed between vertical bars.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & \cdots\cdots\cdots & a_{2n} \\ a_{n1} & a_{n2} \cdots\cdots & a_{nn} \end{vmatrix}$$

Cofactor of A_{ij} of elements a_{ij} in D is equal to $(-1)^{i+j}$ times the determinants of order $(n-1)$ obtained from D by leaving the row and column passing through element a_{ij}.

Example-6.1 Compute the determinant of each of the following matrices and determine which is singular.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

Solution:

(a) The determinant is simply the product of the diagonal running left ro right minus the product of the diagonal running from right to left. So, here is the determinant for this matrix. The only thing we need to worry about is paying attention to minus signs. It is easy to make a mistake with minus signs in these computations if you are not paying attention.

$$\det(A) = (3)(5) - (2)(-9) = 33$$

(b)
$$\det(B) = \begin{vmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{vmatrix} = 3 \times \begin{vmatrix} -1 & 8 \\ 1 & 7 \end{vmatrix} - 5 \times \begin{vmatrix} -2 & 8 \\ -11 & 7 \end{vmatrix} + 4 \times \begin{vmatrix} -2 & -1 \\ -11 & 1 \end{vmatrix} = -467$$

(c)
$$\det(C) = \begin{vmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} -8 & 3 \\ 1 & 1 \end{vmatrix} + 6 \times \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & -8 \\ -3 & 1 \end{vmatrix} = 0$$

Since only det(C) is zero, only C is singular.

Example-6.2 For the following matrix compute the co-factors C_{12}, C_{24} and C_{32}.

$$A = \begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix}$$

Solution:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

In order to compute the co-factors we will first need the minor associated with each co-factor. Remember that in order to compute the minor we will remove the i^{th} row and j^{th} column of A. So, to compute M_{12} (which we will need for C_{12}) we will need to compute the determinate of the matrix we get by removing the 1st row and 2nd column of A. Here is that work.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{12} = \begin{vmatrix} -1 & 3 & 9 \\ 5 & -1 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 160$$

We have marked out the row and column that we eliminated. Now we can get the co-factor,

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (160) = -160$$

Let's now move onto the second co-factor,

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{24} = \begin{vmatrix} 4 & 0 & 10 \\ 5 & -5 & -1 \\ 3 & 7 & 1 \end{vmatrix} = 508$$

The co-factor in this case is, $C_{24} = (-1)^{2+4} M_{24} = (-1)^6 (508) = 508$

For the final co-factor,

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{32} = \begin{vmatrix} 4 & 10 & 4 \\ -1 & 3 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 150$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (150) = -150$$

6.5.3 Properties of Determinants

1. The value of a determinant does not change when rows and columns are interchanged. i.e. $|A^T| = |A|$
2. If any row (or column) of a matrix A is completely zero, then $|A| = 0$.
   Such a row (or column) is called a zero row (or column).
   Also if any two rows (or columns) of a matrix A are indential, then $|A| = 0$.
3. If any two rows or two columns of a determinant are interchanged the value of determinant is multiplied by −1.
4. If all elements of the one row (or one column) of a determinant are multiplied by same number k the value of determinant is k times the value of given determinant.
5. If A be n-rowed square matrix, and k be any scalar, then $|kA| = k^n |A|$

6. (a) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of any row or column is equal to the determinant value.

(b) In determinant the sum of the products of the elements of any row (or column) with the cofactors of some other row or column is zero.

Example: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then,
$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$
$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$
$$a_1A_3 + b_1B_3 + c_1C_3 = 0$$
$$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$$
$$a_2A_1 + b_2B_1 + c_2C_1 = 0 \text{ etc}$$

where A_1, B_1, C_1 etc., be cofactors of the elements a_1, b_1, c_1 in D.

7. If to the elements of a row (or column) of a determinant are added m times the corresponding elements of another row (or column) the value of determinant thus obtained is equal to the value of original determinant.

8. $|AB| = |A| * |B|$ and $|A^n| = (|A|)^n$

9. Let A be a square matrix of order n then,

(a) $|\bar{A}| = |\bar{A}|$

(b) $|A^\theta| = |\bar{A}|$

6.6 Inverse of Matrix

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be any $n \times n$ matrix. The transpose B of the matrix $B = [A_{ij}]_{n \times n}$ where A_{ij} denotes the cofactor of element a_{ij} in the determinant $|A|$ is called the adjoint of matrix A and is denoted by symbol Adj A.

Results:

1. If A be any n-rowed square matrix, then (Adj A) A = A (Adj A) = $|A| I_n$.
2. Every invertible matrix possesses a unique inverse.
3. The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Non-Singular Matrix

A square matrix is said to be non-singular or singular as $|A| \neq 0$ or $|A| = 0$.

NOTE: If A and B be two n-rowed non-singular matrices, then AB is also non-singular then, $(AB)^{-1} = B^{-1}A^{-1}$ i.e., the inverse of a product is product of the inverse taken in the reverse order.

Results:

1. If A be an $n \times n$ non-singular matrix, then $(A')^{-1} = (A^{-1})'$.
2. If A be an $n \times n$ non-singular matrix then $(A^{-1})^\theta = (A^\theta)^{-1}$.
3. Formula to determine the inverse of matrix A is $A^{-1} = \frac{1}{|A|} \text{Adj} A$.

Rank of Matrix

Some Important Concepts:

1. **Submatrix of a Matrix:** Suppose A is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and some columns from A is called sub-matrix of A.
2. **Rank of a Matrix:** A number r is said to be the rank of a matrix A if it possesses the following properties:
   (a) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
   (b) If the matrix A contains any square sub-matrix of order $(r + 1)$, then the determinant of every square sub-matrix of A of order $(r + 1)$ should zero.

Important Points:

(a) The rank of a matrix is $\leq r$, if all $(r + 1)$ - rowed minors of the matrix vanish.
(b) The rank of a matrix is $\geq r$, if there is at least one r-rowed minor of the matrix which is not equal to zero.

NOTE: The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$

Elementry Matrices

A matrix obtained from unit matrix by a single elementry transformation is called an elementry matrix.

Results:

1. Elementry transformations do not change the rank of a matrix.
2. The rank of a product of two matrices cannot exceed the rank of either matrix. i.e. $r(AB) \leq r(A)$ and $r(AB) \leq r(B)$.
3. Rank of sum of two matrices cannot exceed the sum of their ranks $r(A + B) \leq r(A) + r(B)$.
4. If A, B are two n-rowed square matrices then Rank (AB) $\geq$ (Rank A) + (Rank B) – n.

6.7 System of Linear Equations

6.7.1 Homogenous Linear Equations

Suppose,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = 0 \end{array}\right\} \quad \dots(6.1)$$

is a system of m homogenous equations in n unknowns $x_1, x_2, \dots x_n$.

Let,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}_{m \times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

where A, X, O are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively. Then obviously we can write the system of equations in the form of a single matrix equation A X = O ...(6.2)

The matrix A is called coefficient matrix of the system of equation (1).

The set S = $\{x_1 = 0, x_2 = 0, \dots x_n = 0\}$ i.e., X = 0 is a solution of equation (1).

Again suppose X_1 and X_2 are two solutions of (2). Then their linear combination, $R_1X_1 + R_2X_2$ when R_1 and R_2 are any arbitrary numbers, is also solution of (2).

Important Results:

The number of linearly independent solutions of m homogenous linear equations in n variables, $AX = O$, is $(n-r)$, where r is rank of matrix A.

Some Important results regarding nature of solutions of equation $AX = O$:

Suppose there are m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be rank of matrix A. Obviously r cannot greater than n. Therefore we have either $r = n$ or $r < n$.

Case-1: If $r = n$; the equation $AX = O$ will have $n - n$ i.e., no linearly independent solution.

Case-2: If $r < n$ we shall have $n - r$ linearly independent solutions. Any linear combination of these $(n-r)$ solutions will also be a solution of $AX = O$. Thus in this case the equation $AX = O$ will have infinite solutions.

NOTE: That $r < n \Rightarrow |A| = 0$ i.e. A is a singular matrix.

Case-3: Suppose $m < n$ i.e., the number of equations is less than the number of unknowns. Since $r \le m$ therefore r is less than n. Hence in this case the given system of equation must possess a non zero solution.

$\Rightarrow$ In this case the number of solutions of the equation $AX = O$ will be infinite.

6.7.2 System of Linear Non-Homogeneous Equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = b_m \end{array}\right\} \quad \dots(6.3)$$

be a system of m non-homogenous equations in n unknown, $x_1, x_2 \dots x_n$

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & \dots\dots\dots\dots a_{2n} \\ \dots\dots & \dots\dots\dots\dots\dots \\ a_{m1} & a_{m2} \dots\dots\dots a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m \times 1}$$

where A, X, B are $m \times n$, $n \times 1$, and $m \times 1$ matrices respectively. The above equations can be written in the form of a single matrix equation $AX = B$.

"Any set of values of $x_1, x_2 \dots\dots\dots x_n$ which simultaneously satisfy all these equation is called a solutions of the system. When the system of equations has one or more solutions, the equation are said to be consistent otherwise they are said to be inconsistent".

$$\text{The matrix } [A \ \ B] = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} & b_1 \\ a_{21} & a_{22} \dots a_{2n} & b_2 \\ \dots\dots & \dots\dots\dots & \\ a_{m1} & a_{m2} \dots a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix of the given system of equations.

Condition for Consistency

The system of equations AX = B is consistent i.e., possess a solution iff the coefficient matrix A and the augmented matrix [A B] are of the same rank. i.e. ρ (A) = ρ (A, B).

If $\rho(a) \neq \rho(A, B)$ the system $A x = B$, has no solution. We say that such a system is inconsistent.

Now, when $\rho(A) = \rho(A|B) = r$.

We say, that the rank of the system is r. Now two cases arise.

Case-1: If $\rho(A) = \rho(A|B) = r = n$ (where n is the number of unknown variables of the system), then the system is not only consistent but also has a unique solution.

Case-2: If $\rho(A) = \rho(A|B) = r < n$, then the system is consistent, but has infinite number of solutions.

In summary we can say the following:

1. If $\rho(A) = \rho(A|B) = r = n$ (consistent and unique solution)
2. If $\rho(A) = \rho(A|B) = r < n$ (consistent and infinite solution)
3. If $\rho(A) \neq \rho(A|B)$ (Inconsistent and hence, no solution)

Result:

If A be an n-rowed non-singular matrix, X be an $n \times 1$ matrix, B be an $n \times 1$ matrix, the system of equations A X = B has a unique solution. i.e. if $|A| \neq 0$, then the system AX = B has a unique solution.

The rank of a system of equations as well as its solution (if it exists) can be obtained by a procedure called Gauss-Elimination method.

6.8 Solution of System of Linear Equation by LU Decomposition Method (Factorisation or Triangularisation Method)

This method is based on the fact that a square matrix A can be factorised into the form LU where L is unit lower triangular and U is a upper triangular, if all the principal minors of A are non singular i.e., it is a standard result of linear algebra that such a factorisation, when it exists, is unique.

We consider, for definiteness, the linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the form

$$AX = B \quad ...(6.1)$$

Let, $$A = LU \quad ...(6.2)$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad ...(6.3)$$

and $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad ...(6.4)$$

Equation (6.1) becomes, $LUX = B$ ...(6.5)

If we set $UX = Y$ ...(6.6)

then (6.5) way be written as $LY = B$ ...(6.7) which is equivalent to the system $y_1 = b_1$

$$\ell_{21}y_1 + y_2 = b_2,\ \ell_{31}y_1 + \ell_{32}y_2 + y_3 = b_3$$

and can be solved for y_1, y_2, y_3 by the forward substitution. When Y is known, the system (6.6) become

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

which can be solved by backward substitution we shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order 3. From the relation (2), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the watrices on the left and equating the corresponding claments of both sides we get

$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$

$$\ell_{21}u_{11} = a_{21} \text{ or } \ell_{21} = \frac{a_{21}}{u_{11}}$$

$$\ell_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - \ell_{21}u_{12}$$

$$\ell_{31}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - \ell_{21}u_{13}$$

$$\ell_{31}u_{11} = a_{31} \Rightarrow \ell_{31} = \frac{a_{31}}{u_{11}}$$

$$\ell_{31}u_{12} = \ell_{32}u_{22} = a_{32} \Rightarrow \ell_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Lastly, $\ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} = a_{33}$

$\Rightarrow$
$$u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

∴ the variables are solved in the following

order u_{11}, u_{12}, u_{13}

then $\ell_{21}, u_{22}, u_{23}$

lastly, $\ell_{31}, \ell_{32}, u_{33}$

Example-6.3 Solve the equations by the factorisation method.

$2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Clearly, $u_{11} = 2,\ u_{12} = 3,\ u_{13} = 1$

Also, $\ell_{21}u_{11} = 1$, so that $\ell_{21} = \frac{1}{2}$

$$l_{21}u_{12} + u_{22} = 2$$

$$\Rightarrow \quad u_{22} = 2 - l_{21}u_{12} = \frac{1}{2}$$

$$l_{21}u_{13} + u_{23} = 3$$

from which we obtain $u_{23} = \frac{5}{2}$

$$l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{2}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow l_{32} = -7$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18$$

It follows that,
$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

and hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

or as
$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

solving this system by forward substitution, we get

$$y_1 = 9, \ \frac{y_1}{2} + y_2 = 6 \Rightarrow y_2 = \frac{3}{2}$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \text{ or } y_3 = 5$$

Hence the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

which when solved by back substitution process.

$$x = \frac{35}{18}; \ y = \frac{29}{18}; \ z = \frac{5}{18}$$

6.9 Eigenvalues and Eigenvectors

Definitions: Let $A = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ is a scalar. The matrix $A - \lambda I$ is called characteristic matrix of A, where I is the unit matrix of order n. Also the determinant

$$|A - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22}-\lambda \ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots\ldots a_{nn}-\lambda \end{bmatrix}$$

which is ordinary polynomial in λ of degree n is called "characteristic polynomial of A". The equation $|A - \lambda I| = 0$ is called **"characteristic equation of A"** and the roots of this equation is called **"characteristic roots or characteristic values or latent roots or proper values"** of the matrix A. The set of eigenvalues of A is called the **"spectrum of A"**.

If I is a characteristic root of the matrix A, then $|A - \lambda I| = 0$ and the matrix $A - \lambda I$ is singular. Therefore there exist a non-zero vector X such that $(A - \lambda I)X = 0$ or $AX = \lambda X$

Characteristic Vectors

If I is a characteristic root of an $n \times n$ matrix A, then a non-zero vector X such that $AX = \lambda X$ is called characteristic vector or eigenvector of A corresponding to characteristic root λ.

Some Results:

1. λ is a characteristic root of a matrix A iff there exist a non-zero vector X such that $AX = \lambda X$.
2. If X is a characteristic vector of matrix A corresponding to characteristic value λ, then kX is also a characteristic vector of A corresponding to the same characteristic value λ where k is non-zero vector.
3. If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.
4. The characteristic roots of a hermitian matrix are real.
5. The characteristic roots of a real symmetric matrix are all real.
6. Characteristic roots of a skew Hermitian matrix are either pure imaginary or zero.
7. The characteristic roots of a real skew symmetric matrix are either pure imaginary or zero, for every such matrix is skew Hermitian.
8. The characteristic roots of a unitary matrix are of unit modulus. i.e., $|\lambda| = 1$.
9. **Theorem:** The maximum value of $x^T Ax$ where the maximum is taken over all x that are the unit eigen-vectors of A is the maximum eigen value of A.
10. The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of any symmetric positive definite matrix is 0.

6.9.1 Process of Finding the Eigenvalues and Eigenvectors of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n first we should write the characteristic equation of the matrix A. i.e., the equation $|A - \lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will be the n eigenvalues of the matrix A.

If I_1 is an eigenvalue of A, the corresponding eigenvectors of A will be given by the non-zero vectors $X = [x_1, x_2, \ldots\ldots\ldots\ldots x_n]'$ satisfying the equations $AX = \lambda_1 X$ or $(A - \lambda_1 X)X = 0$.

NOTE

- If $l_1, l_2 \ldots l_n$ are the eigenvalues of A, then $kl_1, kl_2 \ldots kl_n$ are eigenvalues of kA.
- the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

  i.e. if $\lambda_1, \lambda_2, \ldots \lambda_n$ are two eigenvalue of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen value of A^{-1}.
- If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of A^k.
- Eigen values of A = Eigen values of A^T.
- Number of eigen values = size of A.
- Sum of eigen values = Trace of A = Sum of diagonal elements.
- Product of eigen values = $|A|$.
- If a is a characteristic root of a non-singular matrix A, the $\frac{|A|}{\alpha}$ is characteristic root of AdjA.

Example-6.4 Find all the eigenvalues the for the given matrices.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$ (c) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution:

(a) We will do this one with a little more detail than we will do the other two. First we will need the matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} \lambda-6 & -16 \\ 1 & \lambda+4 \end{bmatrix}$$

Next we need the determinant of this matrix, which gives us the characteristic polynomial.

$$\det(\lambda I - A) = (\lambda - 6)(\lambda + 4) - (-16) = \lambda^2 - 2\lambda - 8$$

Now, set this equal to zero and solve for the eigenvalues.

$$\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 4$$

So, we have to eigenvalues and since they occur only once in the list they are both simple eigenvalues.

(b) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda+4 & -2 \\ -3 & \lambda+5 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^2 + 9\lambda + 14$$

We will leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues.

$$\lambda^2 + 9\lambda + 14 = (\lambda + 7)(\lambda + 2) = 0 \Rightarrow \lambda_1 = -7, \lambda_2 = -2$$

Again, we get two simple eigenvalues.

(c) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda-7 & 1 \\ -4 & \lambda-3 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^2 - 10\lambda + 25$$

Now, set the characteristic polynomial equal to zero and solve for the eigenvalues,

$$\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \Rightarrow \lambda_{1,2} = 5$$

In this case we have an eigenvalue of multiplicity two. Sometimes we call this kind of eigenvalue a double eigenvalue. Notice as well that we used the notation $\lambda_{1,2}$ to denote the fact that this was a double eigenvalue.

Example - 6.5 Find all the eigenvalues for the given matrices.

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:

(a) As with the previous example we will do this one in a little more detail than the remaining two parts. First, we will need $\lambda_{1,2}$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{bmatrix}$$

Now, let's take the determinant of this matrix and get the characteristic polynomial for A.

$$\det(\lambda I - A) = \left| \begin{matrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{matrix} \right| \begin{matrix} \lambda-4 & 0 \\ 1 & \lambda+6 \\ -5 & 0 \end{matrix}$$

$$= \lambda(\lambda-4)(\lambda+6) - 5(\lambda+6) = \lambda^3 + 2\lambda^2 - 29\lambda - 30$$

Next, set this equal to zero.

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = 0$$

Suppose we are trying to find the roots of an equation of the form,

$$\lambda_n + c_{n-1}\lambda_{n-1} + \ldots + c_1\lambda + c_0 = 0$$

where the c_1 are integer solutions to this (and there may NOT be) then we know that they may be divisors of c_0. This won't give us any integer solutions, but it will allow us to write down a list of possible integer solution. The list will be all possible divisors of c_0.

In this case the list of possible integer solutions is all possible divisors of –30.

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$$

Start with the smaller possible solutions and plug them in until you find one (i.e. until the polynomial is zero for one of them) and then stop. In this case the smallest one in the list that works is –1. This means that,

$$\lambda - (-1) = \lambda + 1$$

must be a factor in the characteristic polynomial. In other words, we can write the characteristic polynomial as,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + I)\, q(\lambda)$$

where $q(\lambda)$ is a quadratic polynomial. We find $q(\lambda)$ by performing long division on the characteristic polynomial. Doing this in this case gives,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + 1)(\lambda^2 + \lambda - 30)$$

At this point all we need to do is find the solutions to the quadratic and nicely enough for us that factors in this case. So, putting all this together gives,

$$(\lambda + 1)(\lambda + 6)(\lambda - 5) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -6, \lambda_3 = 5$$

So, this matrix has three simple eigenvalues.

(b) Here $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - \lambda^2 - 16\lambda - 20$$

Now, in this case the list of possible integer solutions to the characteristic polynomial are,

$$\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$$

Again, if we start with the smallest integers in the list we will find that -2 is the first integer solution. Therefore, $\lambda - (-2) = \lambda + 2$ must be a factor of the characteristic polynomial. Factoring this out of the characteristic polynomial gives,

$$\lambda^3 - \lambda^2 - 16\lambda - 20 = (\lambda + 2)(\lambda^2 - 3\lambda - 10)$$

Finally, factoring the quadratic and setting equal to zero gives us,

$$(\lambda + 2)^2 (\lambda - 5) = 0 \Rightarrow \lambda_{1,2} = -2, \lambda_3 = 5$$

So, we have one double eigenvalue ($\lambda_{1,2} = -2$) and one simple eigenvalue ($\lambda_3 = 5$).

(c) Here is $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 3\lambda - 2$$

We have a very small list of possible integer solutions for this characteristic polynomial.

$$\pm1, \pm2$$

The smallest integer that works in this case is -1 and the complete factored form is characteristic polynomial is, $\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2 (\lambda - 2)$

and so we can see that we have got two eigenvalues $\lambda_{1,2} = -1$ (a multiplicity 2 eigenvalue) and $\lambda_3 = 2$ (a simple eigenvalue)

(d) Here is $\lambda I - A$ and the characteristic polynomial for this matrix,

$$\lambda I - A = \begin{bmatrix} \lambda-4 & 0 & 1 \\ 0 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 27\lambda - 27$$

In this case the list of possible integer solutions is, $\pm1, \pm3, \pm9, \pm27$

The smallest integer that will work in this case is 3. The factored form of the characteristic polynomial is,

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = (\lambda - 3)^3$$

and so we can see that if we set this equal to zero and solve we will have on eigenvalue of muliplicity 3 (sometimes called a triple eigenvalue),

$$\lambda_{1,2,3} = 3$$

Example-6.6 **For each of the following matrices determine the eigenvectors corresponding to each eigenvalue and determine a basis of the eigenspace of the matrix corresponding to each eigenvalue.**

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution:

We determined the eigenvalues for each of these in example above so refer to that example for the details in finding them. For each eigenvalue we will need to solve the system,

$$(\lambda I - A)x = 0$$

to determine the general form of the eigenvector. Once we have that we can use the general form of the eigenvector to find a basis for the eigenspace.

(a) We know that the eigenvalues of this matrix are $\lambda_1 = -2$ and $\lambda_2 = 4$.

Let's first find the eigenvector(s) and eigenspace for $\lambda_1 = -2$. Referring to example 2 for the formula for $\lambda I - A$ and plugging $\lambda_1 = -2$ into this we can see that the system we need to solve is,

$$\begin{bmatrix} -8 & -16 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We will leave it to you to verify that the solution to this system is,

$$x_1 = -2t \qquad x_2 = t$$

Therefore, the general eigenvector corresponding to $\lambda_1 = -2$ is of the form,

$$x = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ t \end{bmatrix}$$

The eigenspaces is all vectors of this form and so we can see that a basis for the eigenspace corresponding to $\lambda_1 = -2$ is,

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now, let's find the eigenvector(s) and eigenspace for $\lambda_2 = 4$. Plugging $\lambda_2 = 4$ into the formula for $\lambda I - A$ from previous example gives the following system we need to solve,

$$\begin{bmatrix} -2 & -16 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is (you should verify this),

$$x_1 = -8t \qquad x_2 = t$$

The general eigenvector and a basis for the eivenspace corresponding to $\lambda_2 = 4$ is then,

$$x = \begin{bmatrix} -8t \\ t \end{bmatrix} = t\begin{bmatrix} -8 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note that if we wanted our hands on specific eigenvalues for each eigenvector the basis vector for each eigenspace would work. So, if we do that we could use the following eigenvectors (and their corresponding eigenvalues) if we'd like,

$$\lambda_1 = -2 \quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = 4 \quad v_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note as well that these eigenvectors are linearly independent vectors.

(b) From previous example we know that $\lambda_{1,2} = 5$ is a double eigenvalue and so there will be a single eigenspace to compute for this matrix. Using the formula for $\lambda I - A$ from example 2 and plugging $\lambda_{1,2}$ into this gives the following system that we will need to solve for the eigenvector and eigenspace.

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is,

$$x_1 = \frac{1}{2}t, \quad x_2 = t$$

The general eigenvector and a basis for the eigenspace corresponding $\lambda_{1,2} = 5$ is then,

$$x = \begin{bmatrix} 1/2t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ t \end{bmatrix} \text{ and } v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

In this case we get only a single eigenvector and so a good eigenvalue/eigenvector pair is,

$$\lambda_{1,2} = 5 \qquad v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

6.9.2 The Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation i.e., if for a square matrix A of order n.

Important Result:

If A and B are two square matrices of the same order then AB and BA are two square matrices of the same order then AB and BA have the same characteristic roots.

Any matrix polynomial in A of size $n \times n$ can be expressed as a polynomial of degree $n-1$ in A by using Cayley-Hamilton theorem.

Example: Process to express a polynomial of a 2 × 2 Matrix as a linear polynomial in A:

Example-6.7 Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then we have to express the $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Solution:

Step 1: First of all write the characteristic equation of A.

In this case,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) + 1$$

$$= \lambda^2 - 5\lambda + 7$$

Thus the characteristic equation of A is $|A - \lambda I| = 0$

i.e., is $\lambda^2 - 5\lambda + 7 = 0$ ...(1)

Step 2: By Cayley Hamilton theorem, matrix A satisfies the equation (1).

Therefore, putting $A = I$ in (1) we get

$$A^2 - 5A + 7 = 0$$

$$\Rightarrow \quad A^2 = 5A - 7I \qquad \text{...(2)}$$

Step 3: Find the A^5, A^4, A^3 with the help of (2). In this case,

$$A^3 = 5A^2 - 7A$$

$$\Rightarrow \quad A^4 = 5A^3 - 7A^2$$

$$\Rightarrow \quad A^4 = 5A^4 - 7A^3$$

$$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I$$

$$= 7[5A^3 - 7A^2) - 14A^3 + A^2 - 4I$$
$$= 21A^3 - 48A^2 - 4I$$
$$= 21(5A^2 - 7A) - 48A^2 - 4I$$
$$= 57A^2 - 147A - 4I$$
$$= 57[5A - 7I] - 147A - 4I$$
$$= 138A - 403I$$

which is a linear polynomial in A.

Summary

- A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix *A*.
- Subtraction of matrices is neither commutative nor associative.
- Properties of multiplication of a matrix by a scalar:
  (a) Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
  (b) If *p* and *q* are two scalars and *A* is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
  (c) If *p* and *q* are two matrices and $A = [a_{ij}]_{m\times n}$ then, $p(qA) = (pq)A$.
  (d) If $A = [a_{ij}]_{m\times n}$ matrix and *k* be any scalar then, $(-k)A = -(kA) = k(-A)$.
- If *A* and *B* be two *n*-rowed non-singular matrices, then AB is also non-singular then, $(AB)^{-1} = B^{-1}A^{-1}$ i.e., the inverse of a product is product of the inverse taken in the reverse order.
- The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$.

Student's Assignment

Q.1 The number of linearly independent solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$ is equal to

(a) 1
(b) 2
(c) 3
(d) 4

Q.2 The eigen values of the matrix $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ are

(a) 2, 4, 6
(b) −2, −4, −6
(c) 1, 0, 7
(d) −2 , 4, 6

Q.3 The number of linearly independent eigen vector(s) of the above matrix is

(a) 1
(b) 2
(c) 3
(d) None of these

Q.4 The nullity of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 3 & 4 \end{bmatrix}$ is

(a) 1
(b) 2
(c) 3
(d) 4

Q.5 If matrix A is of order 3 × 4 and matrix B is 4 × 5. The number of multiplication operation and addition operations needed to calculate the matrix product AB.

(a) 240, 60
(b) 60, 45
(c) 60, 60
(d) 240, 32

Q.6 If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$Q = PAP^{-1}$ and $X = P^{-1}Q^{2005}P$, then X is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Q.7 The rank of matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

Q.8 What could we say about the following system of linear equations?

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

(a) the system is consistent with a unique solution
(b) the system is consistent
(c) the system is inconsistent
(d) none of the above

Q.9 Find eigen values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

(a) 2 and –6 (b) 3 and 2
(c) 4 and –6 (d) 3 and –7

Q.10 $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 8 & 9 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $|A| = ?$

(a) 0 (b) 120
(c) 252 (d) 40

Q.11 $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ find A^{-1}

(a) $A = \begin{bmatrix} 3/2 & -2 \\ 5/2 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & -4 \\ -5 & -6 \end{bmatrix}$

(c) $A = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$ (d) A is not invertible

Q.12 $A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$ Compute $|A|$.

(a) 90 (b) 20
(c) 0 (d) 4

Q.13 Find the eigen values of the following matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

(a) 4, 0, –3 (b) 4, 1
(c) 1, –3 (d) 2, –1.5

Q.14 Consider $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -3 & 4 & -2 \end{bmatrix}$ which of the following statements is not true?

(a) A is invertible (b) A is non-invertible
(c) $|A| \neq 0$ (d) None of above

Q.15 Consider the following system of linear equations

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

Q.16 What is the index to prove matrix

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$ a nilpotent matrix?

(a) 2

(b) 1

(c) 3

(d) None of above

Common Data Questions (17 and 18):

Two eigen values of 3 × 3 matrix A are 2, 1 and $|A| = 12$.

Q.17 Find the third eigen value of matrix A.

(a) 2

(b) 6

(c) 12

(d) None of these

Q.18 The product of eigen values of A^3 and A^5 are

(a) 1728, 248832

(b) 248832, 1728

(c) 243288, 1827

(d) None of these

Q.19 For a system of m linear equations in n unknowns, the cramer's rule is applicable when

(a) $m = n$

(b) $m \neq n$

(c) $m = n$ and coefficient matrix is non singular

(d) $m = n$ and the coefficient matrix is singular

Q.20 If $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$ and $a, b, c \neq 0$, then

(a) $c = (a + b)/2$

(b) $a = b$ or $a = c$ or $b = c$

(c) $c = (a - b)/2$

(d) $a = 2b = 2c$

Q.21 Choose the incorrect statements:

1. The determinant of a matrix equals the sum of its eigen values.
2. A square matrix satisfies its characteristic equation.
3. The sum of principal diagonal elements of a matrix equals the sum of its eigen values.
4. If a row of a matrix is same as one of its columns, its determinant value is 0.

(a) 1 and 2

(b) 1 and 3

(c) 1 and 4

(d) 3 and 4

Q.22 The given equations are solved by using LU-decomposition method.

$$x_1 + 3x_2 - 8x_3 = 4$$
$$x_1 + 4x_2 + 3x_3 = -2$$
$$x_1 + 3x_2 + 4x_3 = 1$$

Find lower triangular matrix value.

(a) $\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 12 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

(d) None of above

Q.23 Find the U matrix from the previous question.

(a) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 11 \\ 0 & 0 & -8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & -8 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -8 \\ 0 & 11 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -8 & 11 & 1 \end{bmatrix}$

Answer Key:

1. (a)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (b)
6. (a)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (c)
11. (c)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (a)
16. (a)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (b)
21. (c)	**22.** (b)	**23.** (b)		

Student's Assignments **Explanations**

1. (a)

$$\begin{bmatrix}1 & 0 & 2\\1 & -1 & 0\\2 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix};$$

$$x_1 + 2x_3 = 0$$
$$x_1 - x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

$x_1 = x_2 = t$ and $x_3 = \frac{-x_1}{2} = \frac{-t}{2}$

The solution is $\begin{bmatrix}t\\t\\-t/2\end{bmatrix}$ for all values of t.

∴ The number of linearly independent solution is 1.

3. (c)

Characteristic equation is

$$\begin{vmatrix}5-\lambda & 0 & 1\\0 & -2-\lambda & 0\\1 & 0 & 5-\lambda\end{vmatrix} = 0$$

$$\lambda = -2, 4, 6$$

$[A - \lambda I]X = 0$ is the eigen value problem

$$\begin{bmatrix}5-\lambda & 0 & 1\\0 & -2-\lambda & 0\\1 & 0 & 5-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ...(i)$$

Putting $\lambda = -2$ in eqn (i)

$$\begin{bmatrix}7 & 0 & 1\\0 & 0 & 0\\1 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$7x_1 + x_3 = 0$$
$$x_1 + 7x_3 = 0$$

Solving which we get $x_1 = 0, x_3 = 0$

Putting $x_2 = t$

We get one eigen vector as $\begin{bmatrix}0\\t\\0\end{bmatrix}$

Putting, $\lambda = 4$ in equation ...(i)

$$\begin{bmatrix}1 & 0 & 1\\0 & -6 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving which we get $x_2 = 0$, $x_1 = -x_3$ putting $x_1 = t$, we get $x_3 = -t$

∴ Another eigen vector is $\begin{bmatrix}-t\\0\\t\end{bmatrix}$

Putting, $\lambda = 6$ in equation (i),

$$\begin{bmatrix}-1 & 0 & 1\\0 & -8 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving we get $x_1 = x_3$ and $x_2 = 0$

Putting $x_1 = t$ we get, $x_3 = t$

∴ Another eigen vector is $\begin{bmatrix}t\\0\\t\end{bmatrix}$

All three are linearly independent

∴ No of linearly independent eigen vectors = 3

4. (a)

$|A| = 0$. So rank is not 3.

The minor $\begin{bmatrix}5 & 0\\7 & 3\end{bmatrix} = 15 \neq 0$

So, Rank = 2

Nullity = number of columns – rank = 3 – 2 = 1

5. (b)

Let matrix A is of order $m \times n$

Let matrix B is of order $n \times P$

for matrix product AB,

The number of multiplication operations = mpn

The number of addition operations = $mp(n-1)$

Here $m = 3$, $n = 4$, $P = 5$

No. of multiplication operations = $3 \times 4 \times 5 = 60$

No. of addition operations = $3 \times 5\,(4-1) = 45$

6. (a)

$P^{-1}P = I$

$Q = PAP^{-1}$, so that

$X = P^{-1}Q^{2005}P = P^{-1}(PAP^{-1})^{2005}P$

Now, $(PAP^{-1})^2 = (PAP^{-1})(PAP^{-1})$

$= PA(P^{-1}P)AP^{-1} = PAIAP^{-1}$

$= PA^2P^{-1}$

Similarly, $(PAP^{-1})^3 = PA^3P^{-1}$ and so on

$(PAP^{-1})^n = PA^nP^{-1}$

$\therefore$ $X = P^{-1}(PAP^{-1})^{2005}P$

$\therefore$ $= P^{-1}(PA^{2005}P^{-1})P$

$= (P^{-1}P).A^{2005}.(P^{-1}P) = A^{2005}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

and so on

$$\therefore \quad A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

7. (c)

$$\begin{vmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{vmatrix} = 0$$

$\therefore$ rank $\neq 3$

and $\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$

Since 2×2 minor is not zero, rank = 2

8. (c)

Use gauss elimination on augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R(1,2)} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 5/2R_1} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right] \xrightarrow{R_3 + 1/2R_2}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right]$$

rank (A) = 2

rank (A | B) = 3

rank (A) $\neq$ rank (A | B)

This system is therefore inconsistent and has no solution.

9. (d)

The characteristic equation of matrix is

$$|A - tI| = 0$$

$$|A - \tau I| = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$(2-\tau)(-6-\tau) - (3)(3) = 0$$

$$\tau^2 + 4\tau + 21 = 0$$

$$(\tau - 3)(\tau + 7) = 0$$

which has roots 3 and –7, which are the eigen values.

10. (c)

In case of triangular matrix, the value of determinant is equal to multiplication of diagonal elements = 2*6*7*3 = 252

11. (c)

Suppose, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if $ad - bc \neq 0$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

substituting the values, $a = 3$, $b = 4$, $c = 5$ and $d = 6$,

we get, $A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$

12. **(b)**
Taking advantage of 0s in third column start cofactor expansion down the third column to obtain 3 × 3 matrix.

$$= (-1)^{(1+3)} * 2 * \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

again taking advantage of 0s in the first column, we expand using cofactors of column 1 elements.

$$= 2 * (-1)^{2+1} * -5 * \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

13. **(a)**
Eigen values of triangular matrix are entries in the diagonal: 4, 0 and –3.

14. **(b)**
The quick test for invertibility is the value of determinant of the matrix. If the determinant of a matrixis non-zero, then it is invertible. Since $|A| = -30 \neq 0$. The matrix is invertible.

15. **(a)**
Gauss elimination on argumented matrix gives rank (A) = rank (A|B) = 3 for all values of α.
∴ for all values of α, we get unique solution.
i.e. there is no value of α, which gives infinite solution for this system.

16. **(a)**
To prove a matrix A to be nilpotent, A should be a square matrix, and $A^n = 0$.

$n \rightarrow$ index (least positive integer that satisfies A^n) = 0

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$A^2 = A \,.\, A = A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^2 = 0$$

Therefore, index is 2.

17. **(b)**
Product of eigen values = |A|
∴ $2 \times 1 \times \tau = 12$
$\tau = 6$
Third eigen values = 6

18. **(a)**
Eigen values of $A^3 \Rightarrow 1^3, 2^3, 6^3$
Eigen values of $A^5 \Rightarrow 1^5, 2^5, 6^5$
Product of eigen values of
$A^3 \Rightarrow 1 \times 2^3 \times 6^3 = 1728$
Product of eigen values of
$A^5 \Rightarrow 1 \times 2^5 \times 6^5 = 248832$

20. **(b)**

Given $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} \rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$\rightarrow abc[(b-a)(c^2-a^2)-(c-a)(b^2-a^2)]$
$\rightarrow abc[(b-a)(c-a)(c+a)-(c-a)(b-a)(b+a)]$
$\rightarrow abc\{(b-a)(c-a)[(c+a)-(b+a)]\}$
$\rightarrow abc[(b-a)(c-a)(c-b)]$
Since the determinant is zero
$abc\,(b-a)(c-a)(c-b) = 0$
since a, b, $c \neq 0$ (given)
$b-a = 0$ or $c-a = 0$ or $c-b = 0$
$\Rightarrow a = b$ or $a = c$ or $b = c$

21. (c)
A is false since, determinant of a matrix is equal to product of eigen values.
B is true
C is true
D is false since determinate is zero, if a row is same as another row or if a column is same as another column.

22. (b)
The system may be written in matrix form as

$$\begin{bmatrix}1 & 3 & -8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}4\\ -2\\ 1\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 3 & -8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$$

$$LU = A$$

$$= \begin{bmatrix}1 & 3 & -8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$$

$\ell_{11} = 1, \ell_{21} = \ell, \ell_{31} = \ell$
$\ell_{11} U_{12} = 3 \Rightarrow U_{12} = 3,$
$\ell_{21} U_{12} + \ell_{22} = 4 \Rightarrow \ell_{22} = 4 - 1.3 = 1$
$\ell_{31} U_{12} + \ell_{32} = 3 \Rightarrow \ell_{32} = 3 - 1 \times 3 = 0$

$\ell_{11} U_{13} = -8 \Rightarrow U_{13} = \frac{-8}{1} = -8$

$\ell_{21} U_{13} + \ell_{22} U_{23} = 3 \Rightarrow U_{23} = 11$
$\ell_{31} U_{13} + \ell_{32} U_{23} + \ell_{33} = 4 \Rightarrow \ell_{33} = 12$

$$\therefore L = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 12\end{bmatrix} \text{ and } U = \begin{bmatrix}1 & 3 & -8\\ 0 & 1 & 11\\ 0 & 0 & 1\end{bmatrix}$$

23. (b)
See explanation of previous problem.