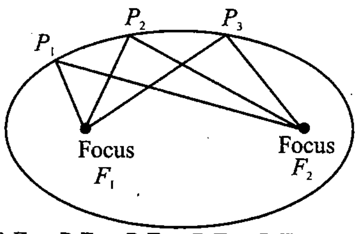
CHAPTER

4

Ellipse

ELLIPSE: DEFINITION 1



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$

Fig. 4.1

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. The two fixed points are called the foci (plural of 'focus') of the ellipse. The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points. The midpoint of the line segment joining the foci is called the centre of the ellipse. The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis. The end points of the major axis are called the vertices of the ellipse.

Standard Equation of an Ellipse

Let the foci of an ellipse be $(\pm c, 0)$, then its centre is $(0, 0)$.

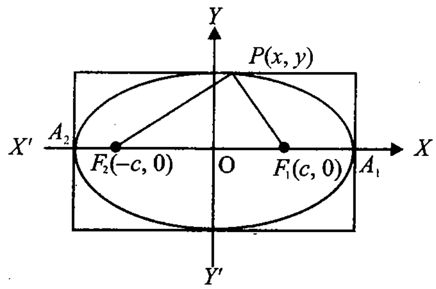


Fig. 4.2

According to the definition of an ellipse,

$$PF_1 + PF_2 = 2a \text{ (constant)}$$

$$\Rightarrow \quad \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a \; (2a > 2c) \qquad \text{(i)}$$

$$\text{Now}[(x-c)^2+y^2] - [(x+c)^2+y^2] = -4cx \qquad \text{(ii)}$$

Dividing (ii) by (i), we get

$$\sqrt{(x^2-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = -\frac{2cx}{a} \qquad \text{(iii)}$$

$$\Rightarrow \quad \sqrt{(x-c)^2+y^2} = a - \frac{cx}{a}$$

Simplifying, we get $\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

Let $\quad a^2 - c^2 = b^2$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad$ (iv)

where $\quad a > b$

It is a 2nd degree equation with powers of x and y both even and hence is symmetric about both the x-and y-axis. The entire curve is confined within the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$.

Eccentricity

Degree of flatness of an ellipse is defined as

$$\text{Eccentricity } (e) = \frac{c}{a} = \frac{OF_1}{OA_1}$$

$$= \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

$$\therefore \quad e^2 = \frac{c^2}{a^2} = \frac{a^2-b^2}{a^2} = 1 - \frac{b^2}{a^2} < 1$$

Now if $c \to 0$ (i.e. the two foci come closer and coalesce to form the centre), then,

$e \to 0 \Rightarrow b \to a$.

Hence, the ellipse gets thicker and $\to$ circle.

Again if $c \to a$ (i.e., the two foci tend to coincide with the vertex of the ellipse),

we have $e \to 1 \Rightarrow b \to 0$.

Hence, the ellipse gets thinner and thinner and tends to a line segment between the two foci.

Equation of an ellipse in terms of eccentricity becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$

where $b^2 = a^2(1-e^2)$ or $a^2e^2 = a^2 - b^2$ $(a > b)$.

Note:

1. ***Two ellipses are said to be similar if they have the same value of eccentricity.***
2. ***Distance of every focus from the extremity of minor axis is equal to a, as $b^2 + a^2e^2 = a^2$.***

Directrix

It is possible to define two lines $x = \pm \frac{a}{e}$ corresponding to each focus, which satisfy the focal directrix property of the ellipse, i.e., $PF_1 = ePM_1$ and $PF_2 = ePM_2$.

Hence, for any point P on the ellipse $\frac{PF_1}{PM_1} = e$ (constant)

Focal Distance

The sum of the focal radii of any point on the ellipse is equal to the length of the major axis.

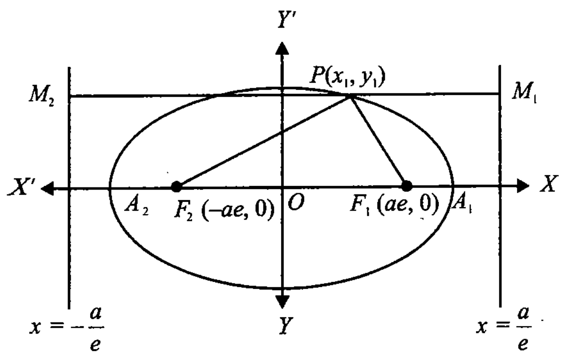


Fig. 4.3

We have

$$PF_1 = ePM_1 = e\left(\frac{a}{e} - x_1\right) = a - ex_1 \quad \text{(i)}$$

$$PF_2 = ePM_2 = e\left(\frac{a}{e} + x_1\right) = a + ex_1 \quad \text{(ii)}$$

(i) + (ii) gives $PF_1 + PF_2 = 2a$

Equation of an Ellipse Whose Axes are Parallel to Coordinate Axis and Centre is (*h*, *k*)

Equation of such an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, (a > b)$

Foci: $(h \pm ae, k)$

Directrix: $x = h \pm \frac{a}{e}$

Definition and Basic Terminology of Ellipse

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

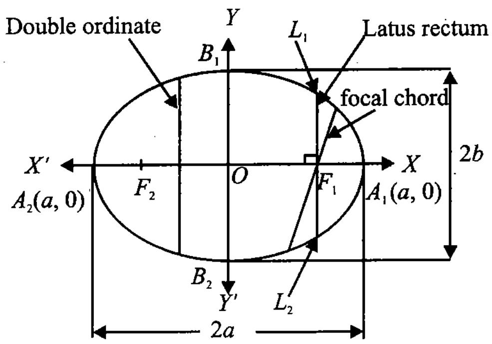


Fig. 4.4

- The line containing the two fixed points (called foci) is called the focal axis (major axis) and points of intersection of the curve with focal axis are called the vertices of the ellipse, i.e., A $(a, 0)$ and A_2 $(-a, 0)$. The distance between F_1 and F_2 is called the focal length. The distance between the two vertices, i.e., $2a$ is called the major axis.

  The distance $2b$, i.e., B_1B_2 is called the minor axes.
- Point of intersection of the major and minor axes is called the centre of the ellipse. Any chord of the ellipse passing through it gets bisected by it and is called the diameter. Major and minor axes together are known as principal axes of the ellipse.
- Any chord through focus is called a focal chord and any chord perpendicular to the focal axis is called double ordinate.
- A particular double ordinate through focus and perpendicular to focal axis is called its *latus rectum*.

Latus-rectum Length

The two foci are $(\pm ae, 0)$.

Putting $x = ae$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - e^2 = 1 - \left(1 - \frac{b^2}{a^2}\right) = \frac{b^2}{a^2};$$

$$\therefore \quad y = \pm\frac{b^2}{a}$$

$\therefore$ Coordinate of the extremities of the latus rectum $= \left(\pm ae, \pm\frac{b^2}{a}\right)$

Length of latus rectum $= \frac{2b^2}{a} = 2a(1 - e^2)$

$$= \frac{4b^2}{2a} = \frac{(\text{Minor axis})^2}{\text{Major axis}}$$

Also $\quad L_1L_2 = 2a(1 - e^2) = 2e\left(\frac{a}{e} - ae\right)$

$= 2e$ (distance between focus and corresponding foot of the directrix).

Tracing Out Ellipse

Method 1

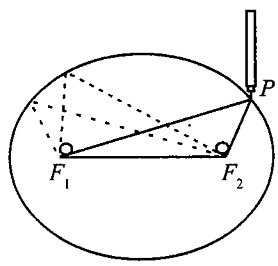


Fig. 4.5

We know that $\quad PF_1 + PF_2 = \text{constant}$

Hence $\quad PF_1 + PF_2 + F_1F_2 = \text{constant}$

Stick two drawing-pins into a board (though not pressed too far in) and slip the loop of thread over them. Insert a pencil point in the loop and position it so that the thread is tight. Move the pencil round the pins, always keeping the thread tight and thus trace out an ellipse.

Method 2

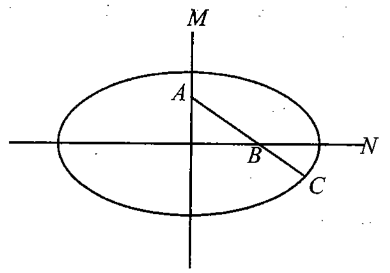


Fig 4.6

Draw two perpendicular lines M, N on the paper; these will be the major and minor axes of the ellipse. Mark three points A, B, C on the ruler. With one hand, move the ruler onto the paper, turning and sliding it so as to keep point A always on line M, and B on line N. With the other hand, keep the pencil's tip on the paper, following point C of the ruler. The tip will trace out an ellipse.

Position of a Point (h, k) with Respect to an Ellipse

Let an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $\qquad P \equiv (h, k)$

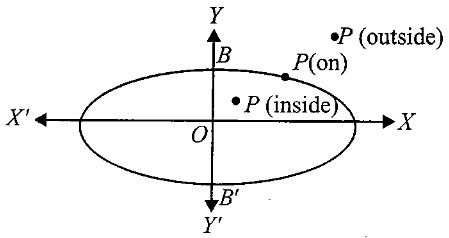
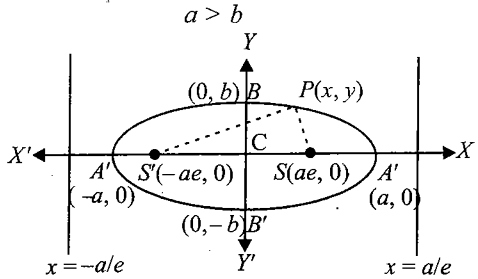


Fig. 4.7

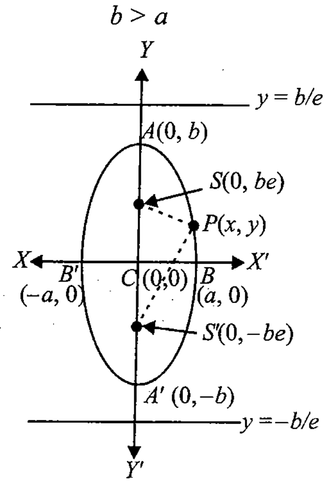
Now P will lie outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to as $\frac{h^2}{a^2}+\frac{k^2}{b^2}-1>,=,<0$

Comparison of Standard Equation of an Ellipse when $a > b$ and $a < b$

Equation of ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



(a)



(b)

Fig. 4.8

	$a > b$	$a < b$
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of the latus rectum	$2b^2/a$	$2a^2/b$
Ends of latus rectum	$(\pm ae, \pm b^2/a)$	$(\pm a^2/b, \pm be)$
Focal radii of any point $P(x_1, y_1)$ on ellipse	$PS = a - ex_1$ and $PS' = a + ex_1$	$PS = b - ey_1$ and $PS' = b + ey_1$
Sum of focal radii $SP + S'P$	$2a$	$2b$

Example 4.1 **A ladder 12 units long slides in a vertical plane with its ends in contact with a vertical wall and a horizontal floor along x-axis. Find the locus of a point on the ladder 4 units from its foot.**

Sol.

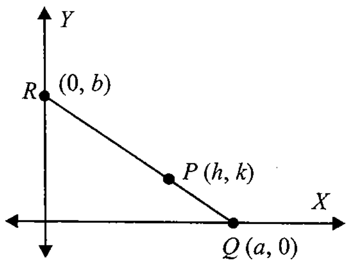


Fig. 4.9

Given $a^2 + b^2 = 12^2 = 144$ (i)

also $\frac{PR}{PQ} = \frac{8}{4} = \frac{2}{1}$

$\Rightarrow$ $h = \frac{2a}{3}$ and $k = \frac{b}{3}$

$\Rightarrow$ $a = \frac{3h}{2}$ and $b = 3k$

From (i), we get $\frac{9h^2}{4} + 9k^2 = 144$

$\Rightarrow$ $\frac{x^2}{64} + \frac{y^2}{16} = 1$

which is an ellipse.

Example 4.2 **Two circles are given such that one is completely lying inside other without touching. Prove that the locus of the centre of variable circle which touches smaller circle from outside and bigger circle from inside is an ellipse.**

Sol.

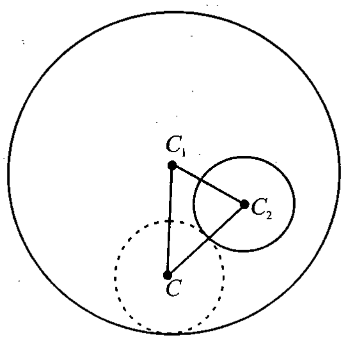


Fig. 4.10

In the figure, circles with hard line are given circles with centres C_1 and C_2 and radius r_1 and r_2 respectively. Let the circle with dotted line is a variable circle which touches given two circles as explained in the question which has centre C and radius r.

Now $CC_2 = r + r_2$ and $CC_1 = r_1 - r$

Hence, $CC_1 + CC_2 = r_1 + r_2 = \text{(constant)}$

Then locus of C is ellipse whose foci are C_1 and C_2.

Example 4.3 **Coordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0), respectively. The vertex A is moving in such a way that $4 \tan\frac{B}{2} \tan\frac{C}{2} = 1$. Then find the locus of A.**

Sol. $4 \tan\frac{B}{2} \tan\frac{C}{2} = 1$

$\Rightarrow \sqrt{\frac{(s-c)(s-a)}{s(s-b)} \times \frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{4}$

$\Rightarrow \frac{s-a}{s} = \frac{1}{4}$

$\Rightarrow \frac{2s-a}{a} = \frac{5}{3}$

$\Rightarrow b + c = \frac{5}{3} \times 6 = 10$

$(\because a = BC = 6)$

Thus, sum of distance of variable point A from two given fixed points B and C is always 10, therefore equation of locus of A is an ellipse. Also centre is midpoint of BC, which is (5, 0).

$2ae = BC = 6$ and sum of focal distance for any point on the ellipse is 10.

Hence, $e = \frac{6}{10} = \frac{3}{5}$.

Length of semi-major axis = 5.

Length of semi-minor axis = 4.

Hence, equation of ellipse is $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$.

Example 4.4 **Find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose latus rectum is half of its major axis.**

Sol. Let $a > b$, then latus rectum of the ellipse is $\frac{2b^2}{a}$ and semi-major axis is a.

Given $\frac{2b^2}{a} = a$

$\Rightarrow$ $2b^2 = a^2$

Also for the ellipse $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $2a^2(1 - e^2) = a^2$

$\Rightarrow$ $e = \frac{1}{\sqrt{2}}$

Example 4.5 **Find the equation of the ellipse (referred to its axes as the axes of x and y, respectively) whose foci are (± 2, 0) and eccentricity $\frac{1}{2}$.**

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given $e = \frac{1}{2}$.

Also foci of the ellipse are $(\pm ae, 0) \equiv (\pm 2, 0)$

$\Rightarrow$ $ae = 2 \Rightarrow a = 4$.

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 12$

Thus, the required ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

Example 4.6 **If $P(x, y)$ be any point of the ellipse $16x^2 + 25y^2 = 400$ and $F_1 = (3, 0)$, $F_2 = (-3, 0)$, then find the value of $PF_1 + PF_2$.**

Sol. We have $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow \quad a^2 = 25$ and $b^2 = 16$

Then, eccentricity is given by $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{16}{25} = \frac{9}{25}$

$\Rightarrow \quad e = \frac{3}{5}$

So, the coordinates of the foci are ($\pm ae$, 0) or ($\pm$ 3, 0).

Thus, F_1 and F_2 are the foci of the ellipse.

Since, the sum of the focal distances of a point on an ellipse is equal to its major axis, $PF_1 + PF_2 = 2a = 10$.

Example 4.7 **If the focal distance of an end of the minor axis of an ellipse (referred to its axes as the axes of x and y, respectively) is k and distance between its foci is $2h$, then find its equation.**

Sol. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It is given that distance between foci is $2h$

$\Rightarrow \quad 2ae = 2h$

$\Rightarrow \quad ae = h \qquad$ (i)

Focal distance of one end of minor axis is $a = k$

$\Rightarrow \quad b^2 = a^2 - a^2e^2 = k^2 - h^2$

So, the equation of the ellipse is $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$.

Example 4.8 **If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then find the eccentricity of the ellipse.**

Sol. If two foci are S (5, 12) and S' (24, 7) and the ellipse passes through origin O.

Then $\quad SO = \sqrt{25 + 144} = 13;$

$S'O = \sqrt{576 + 49} = 25$ and $SS' = \sqrt{386}$

If the conic is an ellipse, then $SO + S'O = 2a$ and

$SS' = 2ae$

$\therefore \quad e = \frac{SS'}{S'O + SO} = \frac{\sqrt{386}}{38}$

Example 4.9 **Find the centre, foci, the length of the axes and the eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.**

Sol. The given equation can be rewritten as

$$2(x^2 - 2x) + 3(y^2 - 4y) + 13 = 0$$

$$\Rightarrow \quad 2(x - 1)^2 + 3(y - 2)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{(1/\sqrt{2})^2} + \frac{(y-2)^2}{(1/\sqrt{3})^2} = 1,$$

$$\Rightarrow \quad \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$\Rightarrow$ Centre $\quad X = 0, Y = 0$, i.e. (1, 2)

Length of major axis $= 2a = \sqrt{2}$,

Length of minor axis $= 2b = \frac{2}{\sqrt{3}}$

and $\quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{3}}$

$ae = \frac{1}{\sqrt{6}}$

Then foci are $\left(1 \pm \frac{1}{\sqrt{6}}, 2\right)$.

Concept Application Exercise 4.1

1. P is a variable on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with AA' as the major axis. Find the maximum area of the triangle APA'.
2. Prove that the curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$ $t \in R$ is an ellipse.
3. An arc of a bridge is semi-elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal, then prove that the best approximation of the height of the arch 2 m from the centre of the base is $\frac{8}{3}$ m.
4. An ellipse has OB as a semi-minor axis, F, F' as its foci and the angle $\angle FBF'$ is a right angle. Then, find the eccentricity of the ellipse
5. Find the equation of an ellipse whose axes are x- and y-axis and whose one focus is at (4, 0) and eccentricity is $\frac{4}{5}$.
6. If $P(\alpha, \beta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S and S' and eccentricity e, then prove that area be of $\Delta SPS'$ is be $\sqrt{a^2 - \alpha^2}$.
7. An ellipse is drawn with major and minor axes of lengths 10 and 8, respectively. Using one focus as centre a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Then find the radius of the circle.
8. Find the foci of the ellipse $25(x + 1)^2 + 9(y + 2)^2 = 225$.
9. Find the sum of the focal distances of any point on the ellipse $9x^2 + 16y^2 = 144$.
10. If C is the centre of the ellipse $9x^2 + 16y^2 = 144$ and S is a focus. Then find the ratio of CS to semi-major axis.
11. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of eccentricity e. If A, A' are the vertices and S, S' are the foci of the ellipse, then find the ratio area $\Delta PSS'$: area $\Delta APA'$.
12. If the foci of an ellipse are (0, $\pm$ 1) and minor axis is of unit length. Then find the equation of the ellipse. Axis of ellipse are coordinate axes.

ELLIPSE: DEFINITION 2

From the discussions in the previous sections we can also define an ellipse with respect to one fixed point and fixed line. An ellipse is the locus of a point which moves in a plane such that the ratio of its distances from a fixed point (i.e., focus) and the fixed line (i.e., directrix) is constant and less than 1. This ratio is called eccentricity and is denoted by e. For an ellipse $e < 1$.

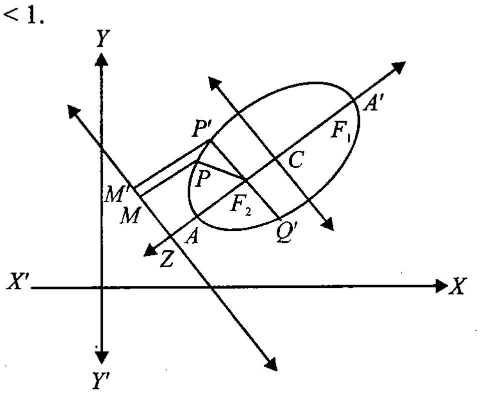


Fig. 4.11

From the diagram, for any point P on the curve, we have by definition,

$\frac{F_2P}{PM} = e$, or $F_2P = ePM$ (focal length or focal radius of point P)

Also A and A' divide F_2Z in the ratio e: 1 internally and externally, respectively.

If the focus F_2 has coordinates (α, β) and equation of directrix ZM is $lx + my + n = 0$, then equation of the ellipse is

$$\sqrt{(x-\alpha) + (y-\beta)^2} = e\frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

which is of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$,

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and $h^2 < ab$.

From the diagram, length of latus rectum.

$= P'Q'$

$= 2F_2P'$

$= 2(eP'M')$

$= 2(eF_2Z)$

$= 2(e)$(distance of focus from corresponding directrix)

Example 4.10 **Find the equation of the ellipse whose focus is $S(-1, 1)$, the corresponding directrix is $x - y + 3 = 0$ and the eccentricity is $\frac{1}{2}$. Also find its centre, the second focus, the equation of the second directrix and the length of latus rectum.**

Sol.

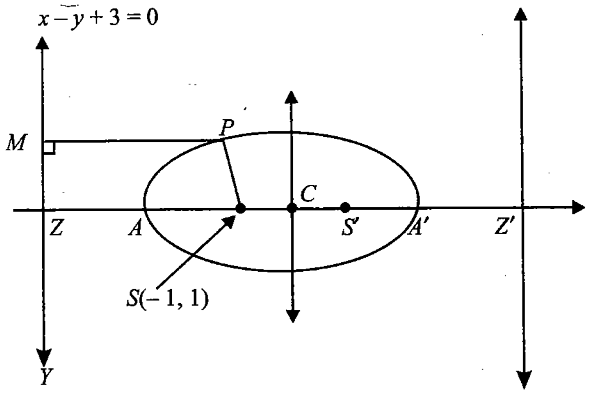


Fig. 4.12

Let $P(x, y)$ be any point on the ellipse and PM be perpendicular to the directrix, then $PS = ePM$ gives

$$(x+1)^2 + (y-1)^2 = \frac{1}{4}[(x - y + 3)/\sqrt{2}]^2$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy + 10x - 10y - 7 = 0 \qquad \text{(i)}$$

The major axis passes through $S(-1, 1)$ and is perpendicular to the directrix.

So the equation of the major axis is $x + y = 0$ (ii)

Axis meets the directrix at Z, then Z is $(-\frac{3}{2}, \frac{3}{2})$

A and A' divide ZS in the ratio 1: e, i.e., 1: $\frac{1}{2}$

or in 2: 1 internally and externally, respectively.

Therefore, we have

$$A = \left(-\frac{7}{6}, \frac{7}{6}\right) \text{ and } A' = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Thus, the centre C (midpoint of AA') $\equiv \left(-\frac{5}{6}, \frac{5}{6}\right)$ (iii)

Let other focus S' be (h, k).

Then $\quad \frac{1}{2}(h-1) = -\frac{5}{6}$ and $\frac{1}{2}(k+1) = \frac{5}{6}$

so $\quad h = -\frac{2}{3}$, and $k = \frac{2}{3}$

$\therefore$ S' is $\left(-\frac{2}{3}, \frac{2}{3}\right)$ (iv)

If major axis meets the other directrix at $Z'(\alpha, \beta)$ then since C is the midpoint of ZZ', we have

$$\frac{1}{2}(\alpha - \tfrac{3}{2}) = -\frac{5}{6}, \frac{1}{2}(\beta + \tfrac{3}{2}) = \frac{5}{6} \Rightarrow Z'(-\tfrac{1}{6}, \tfrac{1}{6})$$

The second directrix is the line perpendicular to the axis passing through $Z'(-\frac{1}{6}, \frac{1}{6})$

$\therefore$ Equation of other directrix is $x - y + \frac{1}{3} = 0$ ...(v)

Also length of latus rectum $= 2(e)$(distance of $(-1, 1)$ from the line $x - y + 3 = 0$)

$$= 2 \times \frac{1}{2}\frac{|-1 - 1 + 3|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Example 4.11 **If the equation $(5x - 1)^2 + (5y - 2)^2 = (\lambda^2 - 2\lambda + 1)(3x + 4y - 1)^2$ represents an ellipse, then find values of λ.**

Sol. Given equation of an ellipse is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=(\lambda-1)^2\left[\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right]^2$$

or $\sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=|\lambda-1|\frac{|3x+4y-1|}{\sqrt{3^2+4^2}}$

For ellipse $SP = ePM$, where $0 < e < 1$

$\Rightarrow \quad 0 < |\lambda - 1| < 1$

$\Rightarrow \quad \lambda \in (0, 2) - \{1\}$

Equation of an Ellipse Referred to Two Perpendicular Lines

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in the figure.

Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.

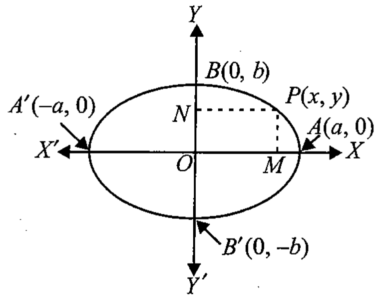


Fig. 4.13

$$\Rightarrow \quad \frac{PN^2}{a^2}+\frac{PM^2}{b^2}=1$$

It follows from this, that if perpendicular distances p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight line $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv b_1x - a_1y + c_2 = 0$, respectively, satisfy the equation

$$\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}=1,$$

i.e. $$\frac{\left(\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right)^2}{a^2}+\frac{\left(\frac{b_1x-a_1y+c_2}{\sqrt{b_1^2+a_1^2}}\right)^2}{b^2}=1$$

then locus of the point P is an ellipse in the plane of the given lines such that

i. The centre of the ellipse is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$

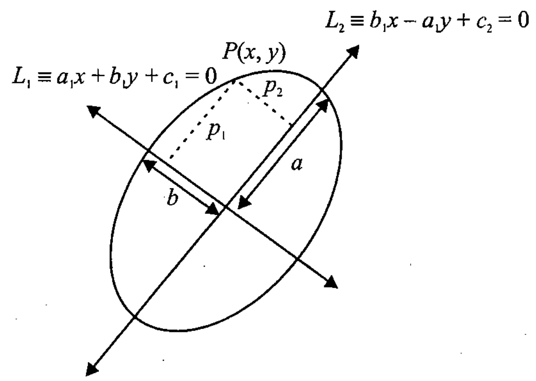


Fig. 4.14

ii. The major axis lies along $L_2 = 0$ and the minor axis lies along $L_1 = 0$, if $a > b$

Example 4.12 **Find the equation of the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.**

Sol. Equation of the ellipse is $\left(\frac{\frac{x-3y+3}{\sqrt{1+9}}}{\sqrt{6}}\right)^2+\left(\frac{\frac{3x+y-1}{\sqrt{9+1}}}{3}\right)^2=1$

$$\Rightarrow \quad \frac{(x-3y+3)^2}{60}+\frac{(3x+y-1)^2}{90}=1$$

$$\Rightarrow \quad 3(x-3y+3)^2+2(3x+y-1)^2=180$$

$$\Rightarrow \quad 21x^2-6xy+29y^2+6x-58y-151=0$$

Concept Application Exercise 4.2

1. Find the eccentricity, one of the foci, directrix and length of the latus rectum for the conic $(3x-12)^2+(3y+15)^2=\frac{(3x-4y+5)^2}{25}$.
2. Find the length of major axis, minor axis, eccentricity of the ellipse $\frac{(3x-4y+2)^2}{16}+\frac{(4x+3y-5)^2}{9}=1$.

AUXILIARY CIRCLE AND ECCENTRIC ANGLE

Definition

A circle described on the major axis as diameter is called the auxiliary circle. For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$ it has equation

$$x^2+y^2=a^2 \qquad \text{(i)}$$

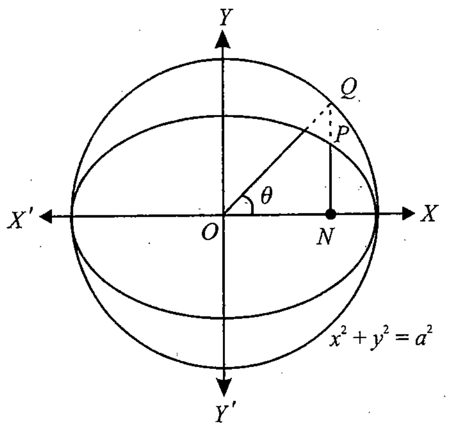


Fig. 4.15

$$P \equiv (a \cos \theta, b \sin \theta)$$

$$Q \equiv (a \cos \theta, a \sin \theta) \quad 0 \le \theta < 2\pi.$$

Here θ is called eccentric angle of point P. P and Q are corresponding points and θ is called the eccentric angle of the point P.

We have $\dfrac{PN}{PQ} = \dfrac{b \sin \theta}{a \sin \theta - b \sin \theta} = \dfrac{b}{a-b}$

= constant

Hence, if from each point on a circle, perpendiculars are drawn on a fixed diameter then the locus of a point P dividing these perpendiculars in a constant ratio is an ellipse whose auxiliary circle is the original circle.

Example 4.13 **Find the equation of the curve whose parametric equations are $x = 1 + 4 \cos \theta$, $y = 2 + 3 \sin \theta$, $\theta \in R$.**

Sol. We have $x = 1 + 4 \cos \theta$, $y = 2 + 3 \sin \theta$

$$\therefore \quad \left(\frac{x-1}{4}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

which is an ellipse.

Example 4.14 **Prove that any point on the ellipse whose foci are (–1, 0) and (7, 0) and eccentricity 1/2 is $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$, $\theta \in R$.**

Sol. Foci are (–1, 0) and (7, 0).

Distance between foci is $2ae = 8$.

$\Rightarrow \quad ae = 4$ and since $e = \frac{1}{2}$, $a = 8$.

Now, $\quad b^2 = a^2(1 - e^2) \Rightarrow b^2 = 48$

$\Rightarrow \quad b = 4\sqrt{3}$.

The centre of the ellipse is the midpoint of the line joining two foci, therefore the coordinates of the centre are (3, 0).

Hence its equation is $\dfrac{(x-3)^2}{8^2} + \dfrac{(y-0)^2}{(4\sqrt{3})^2} = 1 \qquad$ (i)

Thus, the parametric coordinates of a point on (i) are $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$.

Example 4.15 **Find the eccentric angle of a point on the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{2} = 1$ whose distance from the centre of the ellipse is $\sqrt{5}$.**

Sol. Any point on the ellipse is $P(\sqrt{6} \cos \theta, \sqrt{2} \sin \theta)$

Now $CP = \sqrt{6 \cos^2 \theta + 2 \sin^2 \theta} = \sqrt{5}$ where C is a centre

$\Rightarrow \quad 6(1 - \sin^2 \theta) + 2 \sin^2 \theta = 5$

$\Rightarrow \quad 4 \sin^2 \theta = 1$

$\Rightarrow \quad \sin \theta = \pm \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Example 4.16 **Find the area of the greatest rectangle that can be inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.**

Sol. Let $PQRS$ be a rectangle, where P is $(a \cos \theta, b \sin \theta)$

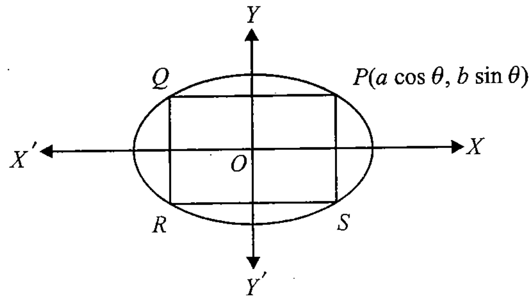


Fig. 4.16

$\therefore \quad$ Area of the rectangle $= 4(a \cos \theta)(b \sin \theta)$

$= 2ab \sin 2\theta$

This is max. when $\sin 2\theta = 1$

Hence, max. area $= 2ab(1) = 2ab$

Example 4.17 **If the line $lx + my + n = 0$ cuts the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at points whose eccentric angles differ by $\dfrac{\pi}{2}$, then find the value of $\dfrac{a^2l^2 + b^2m^2}{n^2}$.**

Sol. Let the points of intersection of the line and the ellipse be $(a \cos \theta, b \sin \theta)$ and $\left(a \cos\left(\frac{\pi}{2} + \theta\right), b \sin\left(\frac{\pi}{2} + \theta\right)\right)$. Since they lie on the given line $lx + my + n = 0$

$la \cos \theta + mb \sin \theta + n = 0$

$\Rightarrow \quad la \cos \theta + mb \sin \theta = -n$

and $\quad -la \sin \theta + mb \cos \theta + n = 0$

$\Rightarrow \quad la \sin \theta - mb \cos \theta = n$

Squaring and adding, we get

$$a^2l^2 + b^2m^2 = 2n^2$$

$$\Rightarrow \quad \frac{a^2l^2 + b^2m^2}{n^2} = 2$$

Example 4.18 **Find the area of the greatest isosceles triangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having its vertex coincident with one extremity of major axis.**

Sol. Let APQ be the isosceles Δ inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with one vertex A at the extremity $(a, 0)$ of the major axis. If coordinates of P are $(a\cos\theta, b\sin\theta)$, then those of Q will be $(a\cos\theta, -b\sin\theta)$.

So area of ΔAPQ is given by

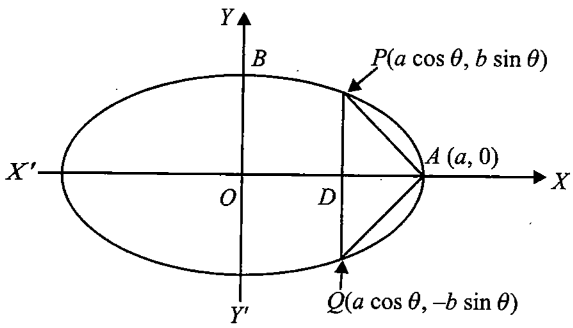


Fig. 4.17

$$\Delta = \frac{1}{2} PQ.AD$$
$$= PD.AD$$
$$= PD\,(OA - OD)$$
$$= b\sin\theta\,(a - a\cos\theta)$$
$$= \frac{1}{2} ab\,(2\sin\theta - \sin 2\theta),$$

$$0 < \theta < \pi$$

$$\therefore \quad \frac{d\Delta}{d\theta} = ab\,(\cos\theta - \cos 2\theta),$$

and $$\frac{d^2\Delta}{d\theta^2} = ab\,(-\sin\theta + 2\sin 2\theta)$$

For max. or min. of A, $\frac{d\Delta}{d\theta} = 0$

$\Rightarrow \quad \cos\theta - \cos 2\theta = 0$

$\Rightarrow \quad 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta + 1)(\cos\theta - 1) = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2},$

$(\because \cos\theta \neq 1 \text{ as } \theta \neq 0)$

$\Rightarrow \quad \theta = 2\pi/3,$

when $\quad \theta = 2\pi/3,$

$$\frac{d^2\Delta}{d\theta^2} = -\frac{1}{2}(3\sqrt{3})\,ab, \ (-\text{ve})$$

$\therefore \quad \Delta$ has max. when $\theta = 2\pi/3$

$\therefore$ Max. area, $\quad \Delta = \frac{1}{2} ab\,[2\sin(2\pi/3) - \sin(4\pi/3)]$

$$= \frac{1}{4}(3\sqrt{3})\,ab$$

Concept Application Exercise 4.3

1. Find the eccentric angles of the extremities of the latus recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then prove that the eccentricity of the ellipse is $\frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$.
3. If the chord joining points $P(\alpha)$ and $Q(\beta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtends a right angle at the vertex $A(a, 0)$, then prove that $\tan(\alpha/2)\tan(\beta/2) = -\frac{b^2}{a^2}$.

Some Important Properties of Ellipse

1. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab

Proof :

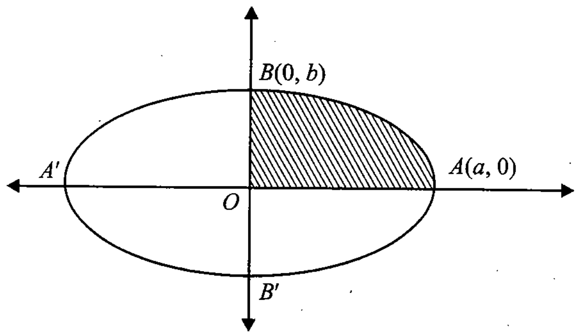


Fig. 4.18

$$\text{Area} = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[4\int_0^a \sqrt{a^2 - x^2}\,dx\right]$$

$$= \frac{b}{a}\,[\text{area of circle having radius } a]$$

$$= \frac{b}{a}\pi a^2$$

$$= \pi ab$$

2. Ratio of area of any triangle PQR inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and that of triangle formed by corresponding points on the auxiliary circle is b/a.

Proof :

Let the three point on the ellipse be $P\,(a\cos\alpha, b\sin\alpha)$, $Q\,(a\cos\beta, b\sin\beta)$ and $R\,(a\cos\gamma, b\sin\gamma)$.

Then corresponding points on the auxiliary circle are $A\,(a\cos\alpha, a\sin\alpha)$, $B\,(a\cos\beta, a\sin\beta)$ and $C\,(a\cos\gamma, a\sin\gamma)$.

$$\text{Now } \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}{a^2\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{b}{a}$$

3. Semi latus rectum is harmonic mean of segments of focal chord or $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$ $(a > b)$ (where PQ is focal chord through focus S)
4. Circle described on focal length as diameter always touches auxiliary circle.

Proof : Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

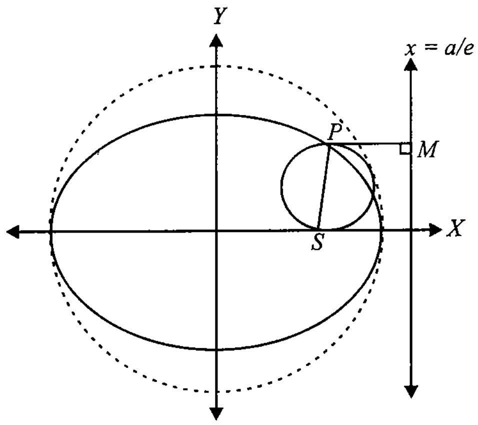


Fig. 4.19

Let P be $(a\cos\theta, b\sin\theta)$

$$SP = ePM = e\left(\frac{a}{e} - a\cos\theta\right)$$
$$= a\,(1 - e\cos\theta)$$

Auxiliary circle is $x^2 + y^2 = a^2$ having centre $C_1(0, 0)$ and $r_1 = a$.

Circle having SP as a diameter has centre

$C_2\left(\frac{ae + a\cos\theta}{2}, \frac{b\sin\theta}{2}\right)$ and radius

$$r_2 = \frac{SP}{2} = \frac{a(1 - e\cos\theta)}{2}$$

$$\text{Now} \quad r_1 - r_2 = \frac{a(1 + e\cos\theta)}{2}$$

$$\text{and} \quad C_1C_2 = \sqrt{\frac{a^2\,(e + \cos\theta)^2}{4} + \frac{b^2\sin^2\theta}{4}}$$
$$= \frac{a}{2}\sqrt{e^2 + 2e\cos\theta + \cos^2\theta + \frac{b^2}{a^2}\sin^2\theta}$$
$$= \frac{a}{2}\sqrt{e^2 + 2e\cos\theta + \cos^2\theta + (1 - e^2)\sin^2\theta}$$
$$= \frac{a}{2}\sqrt{e^2\cos^2\theta + 2e\cos\theta + 1}$$
$$= \frac{a}{2}(e\cos\theta + 1)$$

Hence, circle on SP as a diameter touches the auxiliary circle internally.

Example 4.19 **If PSQ is a focal chord of the ellipse $16x^2 + 25y^2 = 400$ such that $SP = 8$, then find length of SQ.**

Sol. Given ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

then $\frac{1}{8} + \frac{1}{SQ} = \frac{2(5)}{16} = \frac{5}{8} \Rightarrow SQ = 2$

Example 4.20 **The ratio of area of triangle inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to that of triangle formed by the corresponding points on the auxiliary circle is 0.5, then find the eccentricity of ellipse.**

Sol. The given ratio is $\frac{b}{a} = \frac{1}{2}$

Now $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$.

Example 4.21 **If S and S' are two foci of the ellipse $16x^2 + 25y^2 = 400$ and PSQ is a focal chord such that $SP = 16$, then find $S'Q$.**

Sol. We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

Then for given ellipse $\frac{1}{16} + \frac{1}{SQ} = 2\times\frac{5}{16} = \frac{5}{8} \Rightarrow \frac{1}{SQ} = \frac{9}{16}$

$$\text{Now} \quad SQ + SQ' = 2a = 10 \Rightarrow SQ'$$
$$= 10 - \frac{16}{9} = \frac{74}{9}$$

Example 4.22 ***AOB* is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in which $OA = a$, $OB = b$. Then find the area between the arc AB and the chord AB of the ellipse.**

Sol.

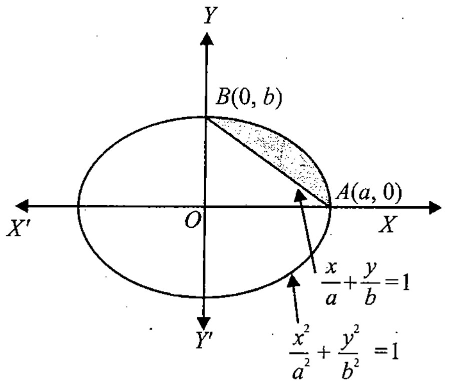


Fig. 4.20

Area of ellipse is πab. Then area of ellipse in first quadrant is $\frac{1}{4}\pi ab$ sq. units.

Now area of triangle $OAB = \frac{1}{2}ab$ sq. units

Hence, the required area is $\frac{1}{4}\pi ab - \frac{1}{2}ab = \frac{ab}{4}(\pi - 2)$ sq. units

Example 4.23 **Prove that area bounded by the circle $x^2 + y^2 = a^2$ and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the area of another ellipse having semi-axis $a - b$ and b $(a > b)$.**

Sol. Area bounded by the given circle and ellipse

= area of circle – area of ellipse

$= \pi a^2 - \pi ab$

$= \pi a(a - b)$

= area of ellipse having semi-axis $a - b$ and b

INTERSECTION OF A LINE AND AN ELLIPSE

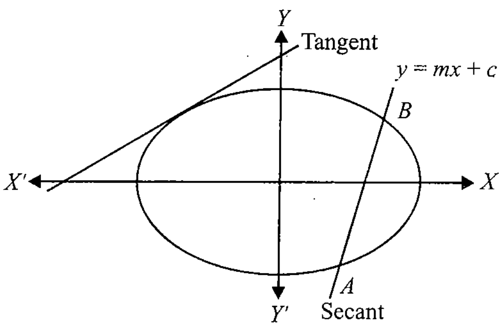


Fig. 4.21

Line $\quad y = mx + c \quad$ (i)

Ellipse: $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad$ (ii)

Solving Eqs. (i) and (ii), we get

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

i.e., $(a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$

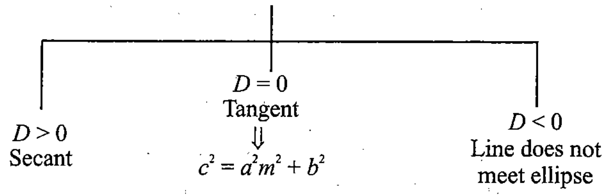


Fig. 4.22

Hence, $y = mx \pm \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all $m \in R$.

Note there are two parallel tangents for a given m.

If it passes through (h, k), then $k = mh \pm \sqrt{a^2m^2 + b^2}$

or $\quad (k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow (h^2 - a^2)m^2 - 2khm + k^2 - b^2 = 0 \quad$ (iii)

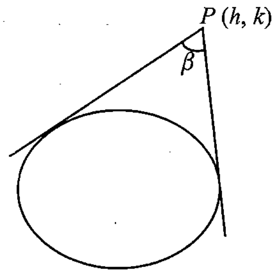


Fig. 4.23

Hence, passing through a given point there can be a maximum of two tangents.

Equation (iii) can be used to determine the locus of the point of intersection of two tangents enclosing an angle β.

If $\quad \beta = 90°$, then $m_1m_2 = -1$

$\Rightarrow \quad k^2 - b^2 = a^2 - h^2$

$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$

which is known as *director circle* of the ellipse.

Hence, *director circle* of an ellipse is a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axes.

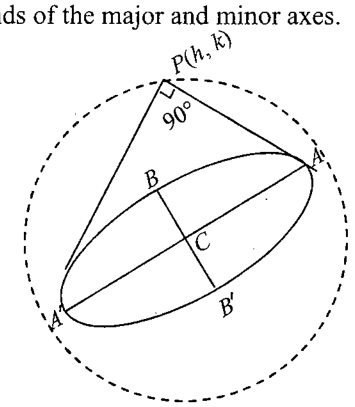


Fig. 4.24

Equation of Tangent to the Ellipse at Point (x_1, y_1)

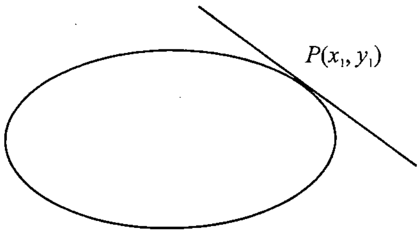


Fig. 4.25

Differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ w.r.t. x, we have

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-\frac{b^2x_1}{a^2y_1}$$

Hence, equation of the tangent is $y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$

$$\text{or} \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

But (x_1, y_1) lies on the ellipse $\Rightarrow \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Hence, equation of the tangent is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \text{(i)}$$

$$\text{or} \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0 \text{ or } T=0$$

$$\text{where} \quad T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

Equation of Tangent at Point $(a\cos\theta, b\sin\theta)$

Putting $x_1 = a\cos\theta$ and $y_1 = b\sin\theta$ in (1), we get

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \text{(ii)}$$

Point of Contact where Line $y = mx + c$ Touches the Ellipse

Line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when $c=\pm\sqrt{a^2m^2+b^2}$

Comparing lines $y=mx\pm\sqrt{a^2m^2+b^2}$ with (i), we get

$$\frac{\frac{x_1}{a^2}}{m}=\frac{\frac{y_1}{b^2}}{-1}=\frac{1}{\mp\sqrt{a^2m^2+b^2}}$$

$$\Rightarrow (x_1, y_1)\equiv\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

or $\left(\pm\frac{a^2m}{c}, \mp\frac{b^2}{c}\right)$, where $c=\sqrt{a^2m^2+b^2}$

Example 4.24 **Find the equations of the tangents drawn from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.**

Sol. Let the equation of the tangent is $y = mx \pm \sqrt{16m^2+9}$

It passes through the point (2, 3)

$$\Rightarrow \quad 3 = 2m+\sqrt{16m^2+9}$$

$$\Rightarrow \quad (3-2m)^2 = 16m^2+9$$

$$\Rightarrow \quad 12m^2+12m=0$$

$$\Rightarrow \quad m=0, -1$$

$\Rightarrow$ Tangents are $\quad y = 3$ or $y = -x + 5$

Draw the diagram and verify that both tangents have + ve y-intercept.

Example 4.25 **If $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then find its eccentric angle θ of point of contact.**

Sol. Let θ be the eccentric angle of the point of contact, then tangent at this point is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

Also $\quad \frac{x}{a\sqrt{2}}+\frac{y}{b\sqrt{2}}=1$ is the tangent

$$\therefore \quad \frac{\cos\theta}{\frac{1}{\sqrt{2}}}+\frac{\sin\theta}{\frac{1}{\sqrt{2}}}=1$$

$$\Rightarrow \quad \cos\theta=\frac{1}{\sqrt{2}}; \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta=45°$$

Example 4.26 **Find the locus of the middle point of the portion of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ included between the axes.**

Sol.

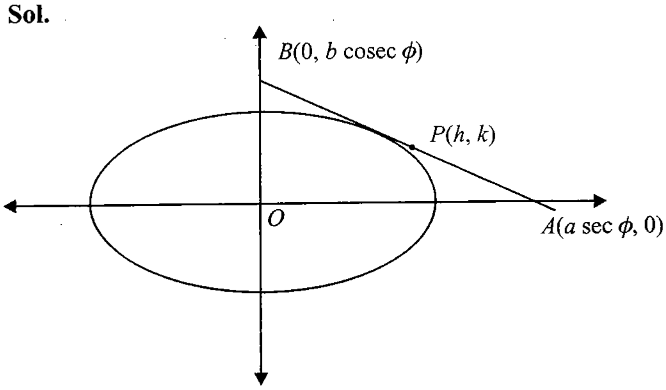


Fig. 4.26

The equation of the tangent at any point ϕ of the ellipse is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

It meets the axes at the point $A(a \sec \phi, 0)$ and $B(0, b \operatorname{cosec} \phi)$

Let $P(h, k)$ be the midpoint of AB,

then $$h = \frac{a \sec\phi}{2}, k = \frac{b \operatorname{cosec} \phi}{2}$$

$\therefore$ $$\cos \phi = \frac{a}{2h} \text{ and } \sin \phi = \frac{b}{2k}$$

$\Rightarrow$ $$\frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1 \text{ (eliminating } \phi)$$

Thus, the locus of (h, k) is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$

Example 4.27 **If the line $3x + 4y = \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$, then find the point of contact.**

Sol. Let the given line touches the ellipse at point $P(\theta)$.

The equation of the tangent at P is

$\sqrt{3}\, x \cos \theta + 2y \sin \theta = 1$ (i)

Comparing (i) with the given equation of the line $3x + 4y = \sqrt{7}$, we get

$$\frac{\cos \theta}{\sqrt{3}} = \frac{\sin \theta}{2} = \frac{1}{\sqrt{7}}$$

The point of the contact $(a \cos \theta, b \sin \theta)$ is $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$.

Example 4.28 **Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangent at each point makes equal angles with the axes.**

Sol. Let P be $(a \cos \theta, b \sin \theta)$, then the equation of tangent at P is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

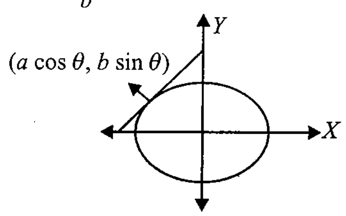


Fig. 4.27

Slope of tangent $= \frac{-\cos \theta}{\sin \theta} \frac{b}{a} = \pm \tan 45° = \pm 1$

$\Rightarrow$ $$\cot \theta = \pm \frac{a}{b}$$

$\Rightarrow$ $$\cos \theta = \pm \frac{a}{\sqrt{a^2 + b^2}} \text{ and}$$

$$\sin \theta = \pm \frac{b}{\sqrt{a^2 + b^2}}$$

$\therefore$ Coordinates of the required points are

$$\left[\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}}\right]$$

Example 4.29 **An ellipse passes through the point (4, –1) and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with coordinate axes.**

Sol. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (i)

It passes through (4, –1) so $a^2 + 16b^2 = a^2b^2$ (ii)

Also $x + 4y - 10 = 0$ touches the ellipse

$\Rightarrow$ $$y = (-1/4)x + (5/2)$$

$\Rightarrow$ $$\frac{25}{4} = \frac{a^2}{16} + b^2$$

$\Rightarrow$ $$a^2 + 16b^2 = 100$$ (iii)

From (ii) and (iii), we get

$$a^2b^2 = 100 \text{ or } ab = 10$$

Solving (ii) and (iii), we have (iv)

$(a = 4\sqrt{5}, b = \sqrt{5}/2)$ or $(a = 2\sqrt{5}, b = \sqrt{5})$

Hence, there are two ellipses satisfying the given conditions, i.e.,

$$\frac{x^2}{80} + \frac{4y^2}{5} = 1 \text{ and } \frac{x^2}{20} + \frac{y^2}{5} = 1$$

Example 4.30 **An ellipse slides between two perpendicular straight lines. Then identify the locus of its centre.**

Sol. Clearly, P is the point of intersection of perpendicular tangents.

So, P lies on the director circle of the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (say).

This means that the centre of the ellipse will always remain at a constant distance $\sqrt{a^2 + b^2}$ from P.

Hence, the locus of C is a circle.

Example 4.31 **Find the locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol. Let the foot of perpendicular is $P(h, k)$

Then the slope of OP is $\frac{k}{h}$ (O is centre)

$\Rightarrow$ Slope of the tangent is $-\frac{h}{k}$

$\Rightarrow$ Equation of the tangent is $y - k = -\frac{h}{k}(x - h)$

or $$y = -\frac{h}{k} x + \frac{h^2 + k^2}{k}$$

This touches the ellipse, then $\left(\frac{h^2 + k^2}{k}\right)^2 = a^2\left(\frac{h}{k}\right)^2 + b^2$

$\Rightarrow$ $$a^2x^2 + b^2y^2 = (x^2 + y^2)^2$$

Equation of Pair of Tangents from Point (*h*, *k*)

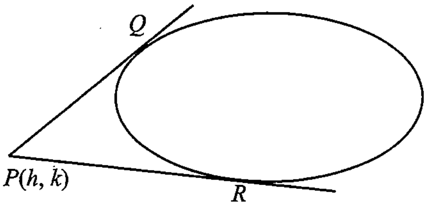


Fig. 4.28

Combined equation pair of tangents PQ and PR is given by $T^2 = SS_1$

where $T = \frac{hx}{a^2} + \frac{ky}{b^2} - 1, S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

and $S_1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

Example 4.32 **Find the angle between the pair of tangents from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$.**

Sol. The combined equation of the pair of tangents drawn from (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is

$(3x^2 + 2y^2 - 5)(3 + 8 - 5) = (3x + 4y - 5)^2$

$\Rightarrow 9x^2 - 24xy - 4y^2 + ... = 0$ [Using $SS' = T^2$]

If the angle between these lines is θ, then

$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$, where $a = 9, h = -12, b = -4$

$\Rightarrow \quad \tan\theta = \frac{12}{\sqrt{5}}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$

IMPORTANT PROPERTIES RELATED TO TANGENT

1. Locus of feet of perpendiculars from foci upon any tangent is an auxiliary circle.

Proof: For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the equation of tangent at any point θ, i.e., at $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Equation of perpendiculars from foci $(\pm ae, 0)$ to tangent (i) is

$\frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \pm\frac{ae\sin\theta}{b}$ (ii)

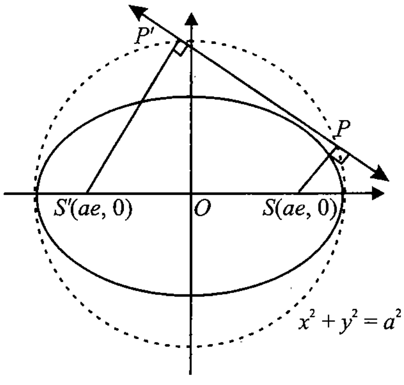


Fig. 4.29

Locus of the feet of perpendicular, i.e., of points of intersetion of (i) and (ii) is obtained by eliminating θ.

Squaring (i) and (ii) and adding, we get

$$x^2\left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right] + y^2\left[\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right]$$

$$= 1 + \frac{a^2e^2\sin^2\theta}{b^2}$$

$$= a^2\left[\frac{b^2 + a^2\sin^2\theta - b^2\sin^2\theta}{a^2b^2}\right]$$

$$= a^2\left[\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}\right]$$

$[\because a^2e^2 = a^2 - b^2]$

$$= a^2\left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right]$$

Hence, cancelling $[\cos^2\theta/a^2 + \sin^2\theta/b^2]$,

the locus is $x^2 + y^2 = a^2$, which is an auxiliary circle.

2. Product of perpendiculars from foci upon any tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is b^2.

Proof: Equation of tangent having slope m is

$$mx - y + \sqrt{a^2m^2 + b^2} = 0$$

$$P_1P_2 = \left[\frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\ \frac{-mea + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right]$$

$$= \left|\frac{(a^2m^2 + b^2 - m^2a^2e^2)}{1 + m^2}\right|$$

$$= \left|\frac{a^2m^2 + b^2 - m^2(a^2 - b^2)}{1 + m^2}\right|$$

$$= \left|\frac{b^2(1 + m^2)}{1 + m^2}\right| = b^2$$

3. Tangents at the extremities of latus rectum pass through the corresponding foot of directrix on major axis.
4. Length of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.

Example 4.33 **If F_1 and F_2 are the feet of the perpendiculars from the foci S_1 and S_2 of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ on the tangent at any point P on the ellipse, then prove that $S_1F_1 + S_2F_2 \geq 8$.**

Sol. We known that the product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of the semi-minor axis.

Then $\quad (S_1F_1)(S_2F_2) = 16$

Now A.M. ≥ G.M.

$$\Rightarrow \quad \frac{S_1F_1 + S_2F_2}{2} \geq \sqrt{S_1F_1 \cdot S_2F_2}$$

$$\Rightarrow \quad S_1F_1 + S_2F_2 \geq 8$$

Concept Application Exercise 4.4

1. If the line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.
2. Find the slope of a common tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a concentric circle of radius r.
3. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then find l.
4. If the tangent to the ellipse $x^2 + 2y^2 = 1$ at point $P\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ meets the auxiliary circle at point R and Q. Then find point of intersection of tangents to circle at Q and R.
5. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32}$ intersects the major and minor axes in points A and B, respectively. If C is the centre of the ellipse, then find the area of the triangle ABC.
6. Find the two perpendicular tangents drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ intersect on the curve.
7. If the tangent at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact, then show that the eccentricity of the ellipse is given by $e = \frac{\cos \beta}{\cos \alpha}$.

EQUATION OF NORMAL

Equation of Normal to the Ellipse at Point (x_1, y_1)

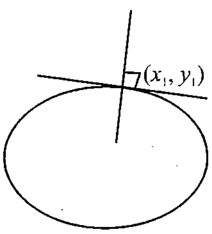


Fig. 4.30

Slope of normal at point (x_1, y_1) is $\frac{a^2 y_1}{b^2 x_1}$. (i)

Hence, equation of normal is $(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{(iii)}$$

Normal at Point $(a \cos \theta, b \sin \theta)$

Putting $x_1 = a \cos \theta$ and $y_1 = b \sin \theta$ in (iii), we get

$$\Rightarrow \quad \frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad \text{(iv)}$$

Note:

1. ***Normal other than major axis never passes through the focus.***
2. ***Normal at point P bisects the angle SPS′***

$$SP = a - ex,$$

$$S'P = a + ex,$$

Normal NP is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

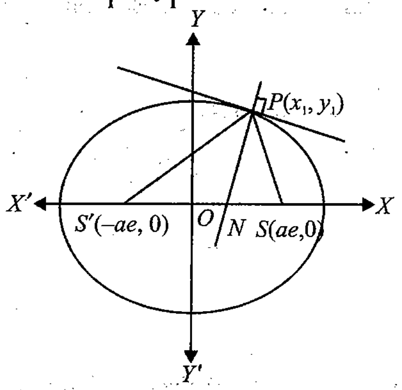


Fig. 4.31

For point N, putting y = 0, we get

$$\Rightarrow \quad x = x_1\left(\frac{a^2 - b^2}{a^2}\right) = x_1\left(1 - \frac{b^2}{a^2}\right)$$

$$= x_1 e^2$$

$$\Rightarrow \quad SN = ae - x_1 e^2$$

$$= e(a - ex_1) = eSP$$

and $$S'N = ae + x_1 e^2$$

$$= e(a + ex_1) = eS'P$$

$$\Rightarrow \quad \frac{SP}{S'P} = \frac{SN}{S'N}$$

$\Rightarrow$ ***PN bisect the*** $\angle SPS'$

Thus, incident ray from focus S after reflection by ellipse at point P passes through other focus S′.

Co-normal Points

From any point in the plane maximum four normals can be drawn to ellipse.

Four feet of normals on the ellipse are called co-normal points. The condition for their eccentric angles is $\alpha + \beta + \gamma + \delta = (2n+1)\pi, n \in Z$.

Proof:

Normal at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

If it passes through point (h, k), then

$$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow \frac{ah\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}} - \frac{bk\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}} = a^2 - b^2$$

$$\Rightarrow \frac{(1+t^2)\,ah}{(1-t^2)} - \frac{(1+t^2)\,bk}{2t} = a^2 - b^2 \text{ where } t = \tan\frac{\theta}{2}$$

$$\Rightarrow 2t(1+t^2)\,ah - (1-t^4)\,bk = 2t(1-t^2)(a^2-b^2)$$

$$\Rightarrow bkt^4 + [2ah - 2(a^2-b^2)]\,t^3 + [2ah - 2(a^2-b^2)]t - bk = 0$$

This equation has four roots, i.e., $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$ $\tan\frac{\gamma}{2}$, $\tan\frac{\delta}{2}$, where $\alpha, \beta, \gamma, \delta$ are eccentric angles of feet of normals on the ellipse.

Now $s_1 = \Sigma \tan\frac{\alpha}{2}$

$$s_2 = \Sigma \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 0$$

$$s_3 = \Sigma \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$s_4 = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = -1$$

Now $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4}$

But $1 - s_2 + s_4 = 0$

$$\Rightarrow \left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right) = (2n+1)\frac{\pi}{2}, n \in Z$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = (2n+1)\pi, n \in Z$$

Example 4.34 **Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{9}$, on which the normals are parallel to the line $2x - y = 1$**

Sol. Normal at $P(2\cos\theta, 3\sin\theta)$ is $\frac{2x}{\cos\theta} - \frac{3y}{\sin\theta} = -5$

Now this normal is parallel to $2x - y = 1$, then $\dfrac{\frac{2}{\cos\theta}}{\frac{3}{\sin\theta}} = 2$

$$\Rightarrow \tan\theta = \frac{3}{1}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{10}} \text{ and } \sin\theta = \pm\frac{3}{\sqrt{10}}$$

Hence, one of the points is $\left(\pm\frac{2}{\sqrt{10}}, \pm\frac{9}{\sqrt{10}}\right)$.

Example 4.35 **If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in G and g, respectively, then find the ratio $PG:Pg$.**

Sol. Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the equation of the normal at P is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

It meets the axes at $G\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$ and $g\left(0, -\frac{a^2-b^2}{a}\sin\theta\right)$

$$\therefore PG^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta = \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

and $$Pg^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$\therefore PG:Pg = b^2:a^2$$

Example 4.36 **P is the point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9}$ and Q is the corresponding point on the auxiliary circle of the ellipse. If the line joining centre C to Q meets the normal at P with respect to the given ellipse at K, then find the value of CK.**

Sol.

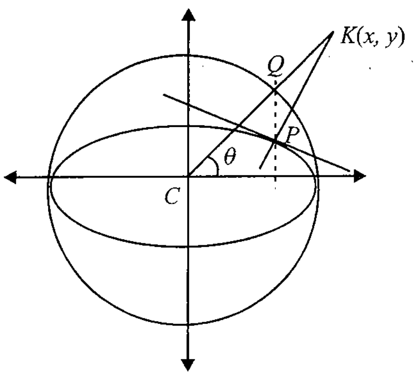


Fig. 4.32

Equation of normal at P is $\frac{4x}{\cos\theta} - \frac{3y}{\sin\theta} = 7$

It passes through K

$$\Rightarrow 4CK - 3CK = 7$$

$$\Rightarrow CK = 7$$

Example 4.37 **If normal at $P\left(2, \frac{3\sqrt{3}}{2}\right)$ meets the major axis of ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at Q and S and S' are foci of given ellipse, then find the ratio $SQ:S'Q$.**

Sol. Equation of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

$$9 = 16(1-e^2)$$

$$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$$

Hence, foci are $S(\sqrt{7}, 0)$ and $S'(-\sqrt{7}, 0)$

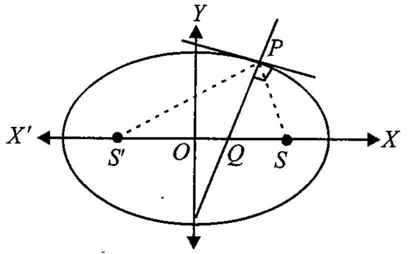


Fig. 4.33

Normal at P is a bisector of angle between $S'P$ and SP

Hence, $\quad \frac{SQ}{S'Q}=\frac{SP}{S'P}=\frac{8-\sqrt{7}}{8+\sqrt{7}}$

Concept Application Exercise 4.5

1. The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.
2. Find the equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the positive end of the latus rectum.
3. If the normal at one end of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the one end of the mirror axis, then prove that eccentricity is constant.
4. If the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are concurrent, then prove that

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0.$$

CHORD OF CONTACT

Let PQ and PR be tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from any external point $P(h, k)$.

Then QR is called chord of contact of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $Q(x_1, y_1)$ and $R(x_2, y_2)$

∴ Equations of tangent PQ and PR are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \text{(i)}$$

and $$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \qquad \text{(ii)}$$

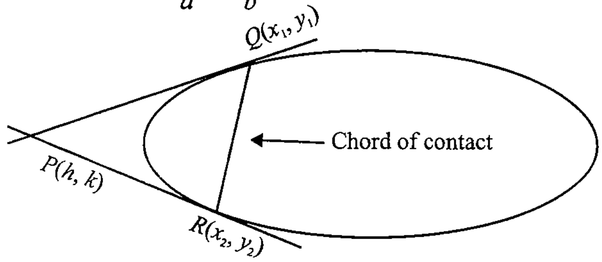


Fig. 4.34

Since (i) and (ii) pass through $P(h, k)$ then

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=1 \qquad \text{(iii)}$$

and $$\frac{hx_2}{a^2}+\frac{ky_2}{b^2}=1 \qquad \text{(iv)}$$

Hence, it is clear that $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

or $\quad T = 0$ which is chord of contact QR.

where $\quad T=\frac{hx}{a^2}+\frac{ky}{b^2}-1$

Example 4.38 **If from a point P tangents PQ and PR are drawn to the ellipse $\frac{x^2}{2}+y^2=1$ so that equation of QR is $x+3y=1$, then find the coordinates of P.**

Sol. Let coordinates of P be (h, k), then equation of QR is

$$\frac{hx}{2}+ky=1 \qquad \text{(i)}$$

but is given as $\quad x+3y=1 \qquad$ (ii)

∴ (i) and (ii) are identical

$$\therefore \quad \frac{h}{2}=\frac{k}{3}=1 \Rightarrow h=2 \text{ and } k=3$$

∴ Coordinates of P are (2, 3).

Example 4.39 **Tangents are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$. Then all the chords of contact pass through a fixed point, find its coordinates.**

Sol. Any point on the line $x-y-5=0$ will be of the form $(t, t-5)$ where $t \in R$.

Chord of contact of this point with respect to curve $x^2 + 4y^2 = 4$ is

$$tx + 4(t-5)y - 4 = 0$$

or $(-20y-4)+t(x+4y) = 0$ which is a family of straight lines, each member of this family passes through the point of intersection of straight lines $-20y - 4 = 0$ and $x + 4y = 0$ which is $(4/5, -1/5)$.

Example 4.40 **Find the locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ forms a triangle of constant area with the coordinate axes.**

Sol. The chord of contact of tangents from (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$.

It meets the axes at the points $\left(\frac{a^2}{x_1}, 0\right)$ and $\left(0, \frac{b^2}{y_1}\right)$.

$$\text{Area of the triangle} = \frac{1}{2}\frac{a^2}{x_1}\frac{b^2}{y_1} = k \text{ (constant)}$$

$$\Rightarrow \quad x_1y_1 = \frac{a^2b^2}{2k} = c^2 \text{ (c is constant)}$$

$$\Rightarrow \quad xy = c^2, \text{ which is a hyperbola}$$

Example 4.41 **Prove that the chord of contact of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ with respect to any point on the directrix is a focal chord.**

Sol. Let any point on the directrix is $(a/e, k)$.

Chord of contact with respect to this point is $\frac{(a/e)x}{a^2}+\frac{ky}{b^2} = 1$

Chearly focus $S(ae, 0)$ satisfies the above line.

Hence, proved.

EQUATION OF CHORD JOINING POINTS $P(\alpha)$ AND $Q(\beta)$

Equation of chord passing through the points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$

$$\text{Then its equation is } \begin{vmatrix} x & y & 1 \\ a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \end{vmatrix} = 0$$

$$\Rightarrow bx(\sin\alpha - \sin\beta) - ay(\cos\alpha - \cos\beta) + ab\sin(\beta - \alpha) = 0$$

$$\Rightarrow \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Example 4.42 **Find the equation of a chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.**

Sol. Equation of chord is

$$\frac{x}{5}\cos\left(\frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2}\right)+\frac{y}{4}\sin\left(\frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2}\right) = \cos\left(\frac{\frac{\pi}{4}-\frac{5\pi}{4}}{2}\right)$$

$$\Rightarrow \frac{x}{5}\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}\sin\left(\frac{3\pi}{4}\right) = 0$$

$$\Rightarrow \quad -\frac{x}{5}+\frac{y}{5} = 0$$

$$\Rightarrow \quad y = x$$

POINT OF INTERSECTION OF TANGENTS AT POINTS $P(\alpha)$ AND $Q(\beta)$

Point of intersection of the tangents at points α and β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $Q(\alpha)$ and $R(\beta)$ with chord of contact of the pair of tangents from (x_1, y_1) on the ellipse.

Proof:

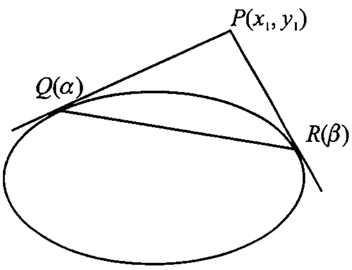


Fig. 4.35

Equation of chord of contact QR with respect to point P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1 \quad \text{(i)}$$

Also equation of chord PQ is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii),

$$\Rightarrow \quad \frac{\frac{x_1}{a^2}}{\frac{a}{\cos\frac{\alpha+\beta}{2}}} = \frac{\frac{y_1}{b^2}}{\frac{b}{\sin\frac{\alpha+\beta}{2}}} = \frac{1}{\cos\frac{\alpha-\beta}{2}}$$

$$\therefore \quad x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}};$$

$$\text{and} \quad y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

Example 4.43 **Find the locus of the point of intersection of tangents to the ellipse if the difference of the eccentric angle of the points is $\frac{2\pi}{3}$.**

Sol. $|\alpha - \beta| = \frac{2\pi}{3}$

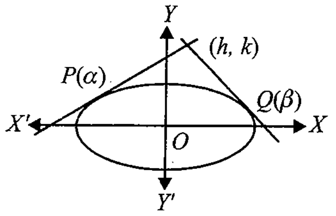


Fig. 4.36

$$h = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = 2a\cos\frac{\alpha+\beta}{2}$$

and $$k = 2b\sin\frac{\alpha+\beta}{2}$$

$$\Rightarrow \quad \frac{h^2}{4a^2} + \frac{k^2}{4b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$$

EQUATION OF THE CHORD OF THE ELLIPSE WHOSE MIDPOINT IS (x_1, y_1)

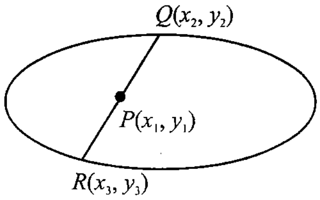


Fig. 4.37

Let the slope of the chord be $\tan\theta$, then any point on the chord at distance r from the point (x_1, y_1) is $(x_1 + r\cos\theta, y_1 + r\sin\theta)$

If this point lies on the ellipse, then

$$\frac{(x_1 + r\cos\theta)^2}{a^2} + \frac{(y_1 + r\sin\theta)^2}{b^2} = 1 \quad \text{(i)}$$

Since line cuts the ellipse in two point Q and R, this is quadratic in r, whose roots are $r_1 = PQ$ and $r_2 = -QR$

Hence, sum of roots, $r_1 + r_2 = 0$ (as $PQ = QR$)

Then from (i), coefficient of $r = 0$

$$\Rightarrow \quad \frac{2x_1\cos\theta}{a^2} + \frac{2y_1\sin\theta}{b^2} = 0$$

$$\Rightarrow \quad \tan\theta = -\frac{b^2x_1}{a^2y_1}$$

Hence, equation of chord is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

or $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

or $$T = S_1, \text{ where } S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

Example 4.44 **Tangents are drawn from the point (3, 2) to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the middle point of this chord of contact.**

Sol. $x^2 + 4y^2 = 9$

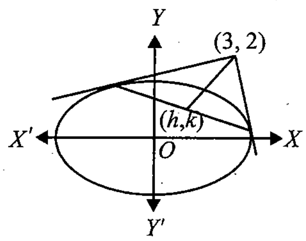


Fig. 4.38

Equation of the chord of contact of the pair of tangents from (3, 2)

$$3x + 8y = 9 \quad \text{(i)}$$

This must be the same as chord whose middle point is (h, k).

$$T = S_1$$

$$\frac{hx}{9} + \frac{ky}{9/4} = \frac{h^2}{9} + \frac{k^2}{9/4}$$

$$\Rightarrow \quad hx + 4ky = h^2 + 4k^2 \quad \text{(ii)}$$

Equations (i) and (ii) represent same straight lines.

Comparing coefficient of (i) and (ii), we get

$$\frac{h}{3} = \frac{4k}{8} = \frac{h^2 + 4k^2}{9}$$

$$\Rightarrow \quad 2h = 3k \text{ and } 3h = h^2 + 4k^2$$

$$\Rightarrow \quad 3h = h^2 + 4 \times \frac{4h^2}{9}$$

$$\Rightarrow \quad \frac{25h^2}{9} = 3h$$

$$\Rightarrow \quad h = \frac{27}{25} \text{ and } k = \frac{18}{25}$$

Example 4.45 **Find the locus of the midpoints of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol. Let the midpoint of the focal chord of the given ellipse be (h, k).

Then its equation is $\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$ [Using $T=S_1$]

Since this passes though $(ae, 0)$

$$\therefore \quad \frac{hae}{a^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$$\Rightarrow \quad \frac{he}{a}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$\therefore$ Locus of (h, k) is $\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Example 4.46 **Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose middle point is (1/2, 2/5).**

Sol. Equation of the chord having (1/2, 2/5) as midpoint is

$$\frac{1/4}{25}+\frac{4/25}{16}-1=\frac{(1/2)x}{25}+\frac{(2/5)y}{16}-1 \quad [T=S_1]$$

$$\Rightarrow \quad 4x+5y=4$$

$$\Rightarrow \quad 5y=4(1-x) \quad \text{(i)}$$

Solving with ellipse, we get

$$16x^2+16(1-x)^2=400$$

$$\Rightarrow \quad x^2-x-12=0$$

$$\Rightarrow \quad x=4, -3$$

$$\text{for } x=4, y=-\frac{12}{5},$$

$$\text{For } x=-3, y=\frac{16}{5}$$

Therefore, length of the chord $=\sqrt{\left\{7^2+\left(\frac{28}{5}\right)^2\right\}}=7\sqrt{\frac{41}{25}}$

Concept Application Exercise 4.6

1. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find the value of $\frac{x_1x_2}{y_1y_2}$.
2. From the point $A(4, 3)$, tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ to touch the ellipse at B and C. EF is a tangent to the ellipse parallel to the line BC and towards the point A. Then find the distance of A from EF.
3. Find the locus of the middle points of all chords of $\frac{x^2}{4}+\frac{y^2}{9}=1$, which are at a distance of 2 units from the vertex of parabola $y^2=-8ax$.
4. Tangents PQ and PR are drawn at the extremities of the chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, which get bisected at point $P(1, 1)$, then find the point of intersection of tangents.
5. Chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are drawn through the positive end of the minor axis. Then prove that their midpoint lies on the ellipse.

CONCYCLIC POINTS ON ELLIPSE

Let the circle $x^2+y^2+2gx+2fy+c=0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in four points P, Q, R, S.

Solving circle and ellipse ($x=a\cos\theta$, $y=b\sin\theta$), we have

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2bf\sin\theta+c=0$$

$$\Rightarrow a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ag\left(\frac{1-t^2}{1+t^2}\right)+2bf\left(\frac{2t}{1+t^2}\right)+c=0, \text{ where } t=\tan\frac{\theta}{2}$$

$$\Rightarrow a^2(1-t^2)^2+4b^2t^2+2ag(1-t^2)(1+t^2)+4bft(1+t^2)+c(1+t^2)^2=0$$

$$\Rightarrow (a^2-2ag+c)t^4+4bft^3+(-2a^2+4b^2+2c)t^2+4bft+(a^2+2ag+c)=0$$

Roots of the equation are $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}, \tan\frac{\gamma}{2}\tan\frac{\delta}{2}$, where $\alpha, \beta, \gamma, \delta$ are eccentric angles of P, Q, R, S, respectively.

Also $\qquad s_1=s_3$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}=0$$

$$\Rightarrow \quad \frac{\alpha+\beta+\gamma+\delta}{2}=n\pi, n\in Z$$

$$\Rightarrow \quad \alpha+\beta+\gamma+\delta=2n\pi, n\in Z$$

EXERCISES

Subjective Type *Solutions on page 4.34*

1. A circle which is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and passes through the foci F_1 and F_2 of the ellipse. The two curves intersect at four points. Let P be any point of intersection. If the major axis of the ellipse is 15 and area of the triangle PF_1F_2 is 26, then find the value of $4a^2-4b^2$.

2. Find the values of α for which three distinct chords drawn from $(\alpha, 0)$ to the ellipse $x^2 + 2y^2 = 1$ are bisected by the parabola $y^2 = 4x$.

3. Prove that if any tangent to the ellipse is cut by the tangents at the ends points of the major axis in T and T', then the circle whose diameter is TT' will pass through the foci of the ellipse.

4. Let P be a point on an ellipse with eccentricity $\frac{1}{2}$, such that $\angle PS_1S_2 = \alpha$, $\angle PS_2S_1 = \beta$ and $\angle S_1PS_2 = \gamma$ where S_1 and S_2 are foci of the ellipse. Then prove that $\cot\frac{\alpha}{2}$, $\cot\frac{\gamma}{2}$ and $\cot\frac{\beta}{2}$ are in A.P.

5. Find the range of eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ such that the line segment joining the foci does not subtend a right angle at any point on the ellipse.

6. From any point on the line $y = x + 4$, tangent are drawn to the auxiliary circle of the ellipse $x^2 + 4y^2 = 4$. If P, Q are the points of contact and A, B are the corresponding points of P and Q on the ellipse respectively, then find the locus of the midpoint of AB.

7. If a triangle is inscribed is an ellipse and two of its sides are parallel to the given straight lines, then prove that third side touches the fixed ellipse.

8. The tangent at a point P $(a\cos\phi, b\sin\phi)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre. Find the eccentricity of the ellipse.

9. Tangents are drawn to the ellipse from the point

$$\left(\frac{a^2}{\sqrt{a^2 - b^2}}, \sqrt{a^2 + b^2}\right).$$

Prove that the tangents intercept on the ordinate through the nearer focus a distance equal to the major axis.

10. From any point on any directrix of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ a pair of tangents is drawn to the auxiliary circle. Show that chord of contact will pass through the corresponding focus of the ellipse.

11. A tangent is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to cut the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ at the points P and Q. If tangents at P and Q to the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ intersect at right angle, then prove that $\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$.

12. Origin O is the centre of two concentric circles whose radii are a and b, respectively, $a < b$. A line OPQ is drawn to cut the inner circle at P and the outer circle at Q. PR is drawn parallel to the y-axis and QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the foci of this ellipse lie on the inner circle, find the ratio of inner: outer radii and also find the eccentricity of the ellipse.

13. The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

14. Find the locus of point P such that tangents drawn from it to the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the coordinate axes in concyclic points.

15. Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum.

Objective Type

Solutions on page 4.38

Each question has four choices a, b, c, d, out of which *only one* answer is correct. Find the correct answer.

1. P and Q are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and B is an end of the minor axis. If PBQ is an equilateral triangle, then eccentricity of the ellipse is

   a. $\frac{1}{\sqrt{2}}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{\sqrt{3}}{2}$

2. An ellipse having foci at (3, 3) and (– 4, 4) and passing through the origin has eccentricity equal to

   a. $\frac{3}{7}$ **b.** $\frac{2}{7}$ **c.** $\frac{5}{7}$ **d.** $\frac{3}{5}$

3. If the eccentricity of the ellipse $\frac{x^2}{a^2+1} + \frac{y^2}{a^2+2} = 1$ is $\frac{1}{\sqrt{6}}$, then latus rectum of ellipse is

   a. $\frac{5}{\sqrt{6}}$ **b.** $\frac{10}{\sqrt{6}}$ **c.** $\frac{8}{\sqrt{6}}$ **d.** none of these

4. If PQR is an equilateral triangle inscribed in the auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ and $P'Q'R'$ is corresponding triangle inscribed within the ellipse then centroid of the triangle $P'Q'R'$ lies at

   a. centre of ellipse
   b. focus of ellipse
   c. between focus and centre on major axis
   d. none of these

5. S_1, S_2 are foci of an ellipse of major axis of length 10 units and P is any point on the ellipse such that perimeter of triangle PS_1S_2 is 15. Then eccentricity of the ellipse is

   a. 0.5 **b.** 0.25 **c.** 0.28 **d.** 0.75

6. If the ellipse $\frac{x^2}{4} + y^2 = 1$ meets the ellipse $x^2 + \frac{y^2}{a^2} = 1$ in four distinct points and $a = b^2 - 5b + 7$, then b does not lie in

   a. [4, 5] **b.** $(-\infty, 2) \cup (3, \infty)$
   c. $(-\infty, 0)$ **d.** [2, 3]

7. With a given point and line as focus and directrix, a series of ellipses are described, the locus of the extremities of their minor axis is

a. ellipse **b.** parabola
c. hyperbola **d.** none of these

8. A line of fixed length $a + b$ moves so that its ends are always on two fixed perpendicular straight lines, then the locus of a point, which divides this line into portions of lengths a and b is a/an

a. ellipse **b.** parabola
c. straight line **d.** none of these

9. The length of the major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x - 4y + 7)^2}{4}$ is

a. 10 **b.** $\frac{20}{3}$ **c.** $\frac{20}{7}$ **d.** 4

10. Angle subtended by common tangents of two ellipses $4(x - 4)^2 + 25y^2 = 100$ and $4(x + 1)^2 + y^2 = 4$ at origin is

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{6}$ **d.** $\frac{\pi}{2}$

11. A circle has the same centre as an ellipse and passes through the focii F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of the triangle PF_1F_2 is 30, then the distance between the focii is

a. 13 **b.** 10 **c.** 11 **d.** None of these

12. The line $x = t^2$ meets the ellipse $x^2 + \frac{y^2}{9} = 1$ in the real and distinct points if and only if

a. $|t| < 2$ **b.** $|t| < 1$
c. $|t| > 1$ **d.** None of these

13. The eccentric angle of a point on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at a distance of $\frac{5}{4}$ units from the focus on the positive x-axis, is

a. $\cos^{-1}\left(\frac{3}{4}\right)$ **b.** $\pi - \cos^{-1}\left(\frac{3}{4}\right)$
c. $\pi + \cos^{-1}\left(\frac{3}{4}\right)$ **d.** None of these

14. If S and S' are the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, and P is any point on it then range of values of $SP \cdot S'P$ is

a. $9 \le f(\theta) \le 16$ **b.** $9 \le f(\theta) \le 25$
c. $16 \le f(\theta) \le 25$ **d.** $1 \le f(\theta) \le 16$.

15. Let d_1 and d_2 be the lengths of the perpendiculars drawn from foci S and S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at any point P on the ellipse. Then, $SP{:}S'P$ =

a. $d_1{:}d_2$ **b.** $d_2{:}d_1$ **c.** $d_1^2 : d_2^2$ **d.** $\sqrt{d_1}/\sqrt{d_2}$

16. The auxiliary circle of a family of ellipse passes through origin and makes intercept of 8 and 6 units on the x-axis and the y-axis, respectively. If eccentricity of all such family of ellipse is $\frac{1}{2}$, then locus of the focus will be

a. $\frac{x^2}{16} + \frac{y^2}{9} = 25$ **b.** $4x^2 + 4y^2 - 32x - 24y + 75 = 0$
c. $\frac{x^2}{16} + \frac{y^2}{9} = 25$ **d.** none of these

17. A man running round a race course notes that the sum of the distances of two flagposts from him is always 10 m and the distance between the flag posts is 8 m. Then the area of the path he encloses in square metres is

a. 15π **b.** 20π **c.** 27π **d.** 30π

18. There are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose distance from its centre is the same and is equal to $\frac{\sqrt{a^2 + 2b^2}}{2}$. Then the eccentricity of the ellipse is

a. $\frac{1}{2}$ **b.** $\frac{1}{\sqrt{2}}$ **c.** $\frac{1}{3}$ **d.** $\frac{1}{3\sqrt{2}}$

19. The eccentricity of locus of point $(3h + 2, k)$ where (h, k) lies on the circle $x^2 + y^2 = 1$ is

a. $\frac{1}{3}$ **b.** $\frac{\sqrt{2}}{3}$ **c.** $\frac{2\sqrt{2}}{3}$ **d.** $\frac{1}{\sqrt{3}}$

20. Let S and S' be two foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If a circle described on SS' as diameter intersects the ellipse in real and distinct points, then the eccentricity e of the ellipse satisfies

a. $e = 1\sqrt{2}$ **b.** $e \in (1/\sqrt{2}, 1)$
c. $e \in (0, 1/\sqrt{2})$ **d.** none of these

21. If the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{a^2} + y^2 = 1$ for suitable value of a cut on four concyclic points, the equation of the circle passing through these four points is

a. $x^2 + y^2 = 2$ **b.** $x^2 + y^2 = 1$
c. $x^2 + y^2 = 4$ **d.** none of these

22. From any point P lying in first quadrant on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, PN is drawn perpendicular to the major axis and produced at Q so that NQ equals to PS, where S is a focus. Then the locus of Q is

a. $5y - 3x - 25 = 0$
b. $3x + 5y + 25 = 0$
c. $3x - 5y - 25 = 0$
d. none of these

23. Locus of the point which divides double ordinates of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the ratio 1:2 internally is

a. $\frac{x^2}{a^2}+\frac{9y^2}{b^2}=1$ **b.** $\frac{x^2}{a^2}+\frac{9y^2}{b^2}=\frac{1}{9}$

c. $\frac{9x^2}{a^2}+\frac{9y^2}{b^2}=1$ **d.** none of these

24. The slopes of the common tangents of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ and the circle $x^2+y^2=3$ are

a. ± 1 **b.** $\pm\sqrt{2}$

c. $\pm\sqrt{3}$ **d.** none of these

25. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$ and the circle $x^2+y^2=a^2$ at the points where a common ordinate cuts them (on the same side of the x-axis). Then the greatest acute angle between these tangents is given by

a. $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ **b.** $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$

c. $\tan^{-1}\left(\frac{2ab}{\sqrt{a-b}}\right)$ **d.** $\tan^{-1}\left(\frac{2ab}{\sqrt{a+b}}\right)$

26. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxiliary circle meet on the line

a. $x=a/e$ **b.** $x=0$

c. $y=0$ **d.** none of these

27. If the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles α and β with the major axis such that $\tan\alpha+\tan\beta=\lambda$, then the locus of their point of intersection is

a. $x^2+y^2=a^2$ **b.** $x^2+y^2=b^2$

c. $x^2-a^2=2\lambda xy$ **d.** $\lambda(x^2-a^2)=2xy$

28. If $\alpha-\beta=$ constant, then the locus of the point of intersection of tangents at P $(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

a. a circle **b.** a straight line

c. an ellipse **d.** a parabola

29. The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant, is a/an

a. parabola **b.** circle

c. ellipse **d.** straight line

30. The sum of the squares of the perpendiculars on any tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from two points on the minor axis each at a distance ae from the centre is

a. $2a^2$ **b.** $2b^2$ **c.** a^2+b^2 **d.** a^2-b^2

31. A tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ at any point P meets the line $x=0$ at a point Q. Let R be the image of Q in the line $y=x$, then the circle whose extremities of a diameter are Q and R passes through a fixed point. The fixed point is

a. (3, 0) **b.** (5, 0) **c.** (0, 0) **d.** (4, 0)

32. For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangent drawn at the point P in the first quadrant meets the y-axis at Q and the chord $A'P$ meets the y-axis at M. If O is the origin, then OQ^2-MQ^2 equals to

a. 9 **b.** 13 **c.** 4 **d.** 5

33. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes at points A and B respectively. If C is the centre of the ellipses, then the area of the triangle ABC is

a. 12 sq. units **b.** 24 sq. units

c. 36 sq. units **d.** 48 sq. units

34. Let P be any point on a directrix of an ellipse of eccentricity e. S be the corresponding focus and C the centre of the ellipse. The line PC meets the ellipse at A. The angle between PS and tangent at A is α, then α is equal to

a. $\tan^{-1}e$ **b.** $\frac{\pi}{2}$

c. $\tan^{-1}(1-e^2)$ **d.** none of these

35. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is normal to the circle $x^2+y^2+4x+1=0$, then the maximum value of ab is

a. 4 **b.** 2

c. 1 **d.** none of these

36. If $(\sqrt{3})\,bx+ay=2ab$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the eccentric angle of the point of contact is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

37. If the ellipse $\frac{x^2}{a^2-7}+\frac{y^2}{13-5a}=1$ is inscribed in a square of side length $\sqrt{2}a$, then a is equal to

a. $\frac{6}{5}$

b. $(-\infty,-\sqrt{7})\cup(\sqrt{7}, 13/5)$

c. $(-\infty,-\sqrt{7})\cup(13/5, \sqrt{7})$

d. no such a exists

38. Locus of the point of intersection of the tangent at the end points of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(b<a)$ is a/an

a. circle **b.** ellipse

c. hyperbola **d.** pair of straight lines

39. The normal at a variable point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse at Q and R, then the locus of the midpoint of QR is a conic with an eccentricity e' such that

a. e' is independent of e

b. $e'=1$

c. $e'=e$

d. $e'=1/e$

40. Any ordinate MP of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ meets the auxiliary circle at Q, then locus of the point of intersection of normals at P and Q to the respective curves is

a. $x^2+y^2=8$ **b.** $x^2+y^2=34$

c. $x^2+y^2=64$ **d.** $x^2+y^2=15$

41. Number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ from the point $P(0, 6)$ is

a. one **b.** two **c.** three **d.** four

42. An ellipse has the points $(1, -1)$ and $(2, -1)$ as its foci and $x+y-5=0$ as one of its tangents. Then the point where this line touches the ellipse from origin is

a. $\left(\frac{32}{9}, \frac{22}{9}\right)$ **b.** $\left(\frac{23}{9}, \frac{2}{9}\right)$

c. $\left(\frac{34}{9}, \frac{11}{9}\right)$ **d.** none of these

43. If tangents PQ and PR are drawn from a point on the circle $x^2+y^2=25$ to the ellipse $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, $(b<4)$, so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of triangle PQR, then eccentricity of the ellipse is

a. $\frac{\sqrt{5}}{4}$ **b.** $\frac{\sqrt{7}}{3}$

c. $\frac{\sqrt{7}}{4}$ **d.** $\frac{\sqrt{5}}{3}$

44. If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is inscribed in a rectangle whose length to breadth ratio is 2:1 then the area of the rectangle is

a. $4\frac{a^2+b^2}{7}$ **b.** $4\frac{a^2+b^2}{3}$

c. $12\frac{a^2+b^2}{5}$ **d.** $8\frac{a^2+b^2}{5}$

45. If the normals at $P(\theta)$ and $Q(\pi/2+\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the major axis at G and g, respectively, then $PG^2+Qg^2=$

a. $b^2(1-e^2)(2-e^2)$

b. $a^2(e^4-e^2+2)$

c. $a^2(1+e^2)(2+e^2)$

d. $b^2(1+e^2)(2+e^2)$

46. The line $y=mx-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m belongs to

a. $(0, 1)$ **b.** $(0, \infty)$

c. R **d.** none of these

47. The length of the sides of square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{7}+\frac{2y^2}{11}=1$ is

a. 10 units **b.** 8 units **c.** 6 units **d.** 5 units

48. From point $P(8, 27)$, tangent PQ and PR are drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. Then the angle subtended by QR at origin is

a. $\tan^{-1}\frac{\sqrt{6}}{65}$ **b.** $\tan^{-1}\frac{4\sqrt{6}}{65}$

c. $\tan^{-1}\frac{8\sqrt{2}}{65}$ **d.** $\tan^{-1}\frac{48\sqrt{6}}{455}$

49. Let P be any point on any directrix of an ellipse. Then chords of contact of point P with respect to the ellipse and its auxiliary circle intersect at

a. some point on the major axis depending upon the position of point P

b. midpoint of the line segment joining the centre to the corresponding focus

c. corresponding focus

d. none of these

50. The equation of the line passing through the centre and bisecting the chord $7x+y-1=0$ of the ellipse $\frac{x^2}{1}+\frac{y^2}{7}=1$ is

a. $x=y$ **b.** $2x=y$ **c.** $x=2y$ **d.** $x+y=0$

51. Equation of the chord of contact of pair of tangents drawn to the ellipse $4x^2+9y^2=36$ from the point (m, n) where $m.n=m+n$, m, n being non-zero positive integers is

a. $2x+9y=18$ **b.** $2x+2y=1$

c. $4x+9y=18$ **d.** none of these

52. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms a triangle of constant area with the coordinate axes is

a. a straight line b. a hyperbola
c. an ellipse d. a circle

53. An ellipse is sliding along the co-ordinate axes. If the foci of the ellipse are (1, 1) and (3, 3), then area of the director circle of the ellipse (in sq. units) is

a. 2π b. 4π c. 6π d. 8π

54. The equation of the ellipse whose axes are coincident with the co-ordinates axes and which touches the straight lines $3x-2y-20=0$ and $x+6y-20=0$ is

a. $\frac{x^2}{40}+\frac{y^2}{10}=1$ b. $\frac{x^2}{5}+\frac{y^2}{8}=1$
c. $\frac{x^2}{10}+\frac{y^2}{40}=1$ d. $\frac{x^2}{40}+\frac{y^2}{30}=1$

55. An ellipse with major and minor axes length as $2a$ and $2b$ touches coordinate axes in first quadrant and having foci (x_1, y_1) and (x_2, y_2) then the value of x_1x_2 and y_1y_2 is

a. a^2 b. b^2 c. a^2b^2 d. a^2+b^2

56. Let P_i and P_i' be the feet of the perpendiculars drawn from foci S, S' on a tangent T_i to an ellipse whose length of semi-major axis is 20, if $\sum_{i=1}^{10}(SP_i)(S'P_i')=2560$, then the value of eccentricity is

a. $\frac{1}{5}$ b. $\frac{2}{5}$ c. $\frac{3}{5}$ d. $\frac{4}{5}$

57. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

a. 0 b. 2 c. 1 d. 4

58. A parabola is drawn with focus is at one of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b$) and directrix passing through the other focus and perpendicular to the major axes of the ellipse. If latus rectum of the ellipse and the parabola are same, then the eccentricity of the ellipse is

a. $1-\frac{1}{\sqrt{2}}$ b. $2\sqrt{2}-2$
c. $\sqrt{2}-1$ d. None of these

59. If maximum distance of any point on the ellipse $x^2+2y^2+2xy=1$ from its centre be r, then r is equal to

a. $3+\sqrt{3}$ b. $2+\sqrt{2}$
c. $\frac{\sqrt{2}}{\sqrt{3-\sqrt{5}}}$ d. $\sqrt{2-\sqrt{2}}$

60. The set of values of m for which it is possible to draw the chord $y=\sqrt{m}\,x+1$ to the curve $x^2+2xy+(2+\sin^2\alpha)y^2=1$, which subtends a right angle at the origin for some value of α is

a. [2, 3] b. [0, 1] c. [1, 3] d. none of these

Multiple Correct Answers Type

Solutions on page 4.48

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. $\frac{x^2}{r^2-r-6}+\frac{y^2}{r^2-6r+5}=1$ will represents the ellipse, if r lies in the interval

a. $(-\infty,-2)$ b. $(3,\infty)$ c. $(5,\infty)$ d. $(1,\infty)$

2. On the x-y plane, the eccentricity of an ellipse is fixed (in size and position) by

a. both foci
b. both directrices
c. one focus and corresponding directrix
d. length of major axis

3. The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2. Then the eccentric angle of the point is

a. $\frac{\pi}{4}$ b. $\frac{3\pi}{4}$ c. $\frac{5\pi}{6}$ d. $\pi/6$

4. If the equation of the ellipse is $3x^2+2y^2+6x-8y+5=0$, then which of the following is/are true?

a. $e=\frac{1}{\sqrt{3}}$
b. centre is $(-1, 2)$
c. foci are $(-1, 1)$ and $(-1, 3)$
d. directrices are $y=2\pm\sqrt{3}$

5. If the tangent at the point $P(\theta)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x=15$, then $\theta=$

a. $\frac{2\pi}{3}$ b. $\frac{4\pi}{3}$ c. $\frac{5\pi}{3}$ d. $\frac{\pi}{3}$

6. The co-ordinates (2, 3) and (1, 5) are the foci of an ellipse which passes through the origin, then the equation of

a. tangent at the origin is $(3\sqrt{2}-5)x+(1-2\sqrt{2})y=0$
b. tangent at the origin is $(3\sqrt{2}+5)x+(1+2\sqrt{2}y)=0$
c. normal at the origin is $(3\sqrt{2}+5)x-(2\sqrt{2}+1)y=0$
d. normal at the origin is $x(3\sqrt{2}-5)-y(1-2\sqrt{2})=0$

7. If the chord through the points whose eccentric angles are θ and ϕ on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ passes through a focus, then the value of $\tan(\theta/2)\tan(\phi/2)$ is

a. $\frac{1}{9}$ b. -9 c. $-\frac{1}{9}$ d. 9

8. The equation $3x^2 + 4y^2 - 18x + 16y + 43 = k$
   a. represents empty set, if $k < 0$
   b. represents an ellipse, if $k > 0$
   c. a point, if $k = 0$
   d. cannot represent a real pair of straight lines for any value of k

9. If a pair of variable straight lines $x^2 + 4y^2 + \alpha xy = 0$ (where α is a real parameter) cut the ellipse $x^2 + 4y^2 = 4$ at two points A and B, then the locus of the point of intersection of tangents at A and B is
   a. $x - 2y = 0$ b. $2x - y = 0$
   c. $x + 2y = 0$ d. $2x + y = 0$

10. Which of the following is/are true?
   a. There are infinite positive integral values of a for which $(13x - 1)^2 + (13y - 2)^2 = \left(\frac{5x + 12y - 1}{a}\right)^2$ represents an ellipse
   b. The minimum distance of a point (1, 2) from the ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ is 1
   c. If from a point $P(0, \alpha)$ two normals other than axes are drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $|\alpha| < \frac{9}{4}$
   d. If the length of latus rectum of an ellipse is one-third of its major axis, then its eccentricity is equal to $\frac{1}{\sqrt{3}}$

11. Which of the following is/are true about the ellipse $x^2 + 4y^2 - 2x - 16y + 13 = 0$?
   a. The latus rectum of the ellipse is 1
   b. Distance between foci of the ellipse is $4\sqrt{3}$
   c. Sum of the focal distances of a point $P(x, y)$ on the ellipse is 4
   d. $y = 3$ meets the tangents drawn at the vertices of the ellipse at points P and Q then PQ subtends a right angle at any of its foci

12. A point on the ellipse $x^2 + 3y^2 = 37$ where the normal is parallel to the line $6x - 5y = 2$ is
   a. (5, −2) b. (5, 2) c. (−5, 2) d. (−5, −2)

13. The locus of the image of the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ $(a > b)$ with respect to any of the tangents to the ellipse is
   a. $(x + 4)^2 + y^2 = 100$ b. $(x + 2)^2 + y^2 = 50$
   c. $(x - 4)^2 + y^2 = 100$ d. $(x - 2)^2 + y^2 = 50$

14. Let E_1 and E_2 be two ellipses $\frac{x^2}{a^2} + y^2 = 1$ and $x^2 + \frac{y^2}{a^2} = 1$ (where a is a parameter). Then the locus of the points of intersection of the ellipses E_1 and E_2 is a set of curves comprising
   a. two straight lines b. one straight line
   c. one circle d. one parabola

15. Consider the ellipse $\frac{x^2}{f(k^2 + 2k + 5)} + \frac{y^2}{f(k + 11)} = 1$ and $f(x)$ is a positive decreasing function, then
   a. the set of values of k, for which the major axis is x-axis is $(-3, 2)$
   b. the set of values of k, for which the major axis is y-axis is $(-\infty, 2)$
   c. the set of values of k, for which the major axis is y-axis is $(-\infty, -3) \cup (2, \infty)$
   d. the set of values of k, for which the major axis is y-axis is $(-3, \infty)$

16. If two concentric ellipses are such that the foci of one are on the other and their major axes are equal. Let e and e' be their eccentricities, then
   a. the quadrilateral formed by joining the foci of the two ellipses is a parallelogram
   b. the angle θ between their axes is given by $\theta = \cos^{-1}\sqrt{\frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2e'^2}}$
   c. if $e^2 + e'^2 = 1$, then the angle between the axes of the two ellipses is 90°
   d. none of these

17. If the tangent drawn at point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Then
   a. $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ b. $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$
   c. $t = -\frac{2}{\sqrt{5}}$ d. $t = -\frac{1}{\sqrt{5}}$

Reasoning Type

Solutions on page 4.52

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

   a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.
   b. Both the statements are True but Statement 2 is NOT the correct explanation of Statement 1.
   c. Statement 1 is True and Statement 2 is False.
   d. Statement 1 is False and Statement 2 is True.

1. **Statement 1:** The locus of a moving point (x, y) satisfying $\sqrt{(x-2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4$ is ellipse.

   Statement 2: Distance between (−2, 0) and (2, 0) is 4.

2. **Statement 1:** In a triangle ABC, if base BC is fixed and perimeter of the triangle is constant, then vertex A moves on an ellipse.

   Statement 2: If the sum of distances of a point P from two fixed points is constant, then locus of P is a real ellipse.

3. **Statement 1:** In an ellipse the sum of the distances between foci is always less than the sum of focal distances of any point on it.

   Statement 2: The eccentricity of any ellipse is less than 1.

4. **Statement 1:** The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$ is $y = 0, y = 6$.

   Statement 2: The equation of the tangent drawn at the ends of major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is always parallel to y-axis.

5. **Statement 1:** There can be maximum two points on the line $px + qy + r = 0$, from which perpendicular tangents can be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Statement 2: Circle $x^2 + y^2 = a^2 + b^2$ and the given line can intersect in maximum two distinct points.

6. **Statement 1:** Circle $x^2 + y^2 = 9$, and the circle $(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$ touches each other internally.

   Statement 2: Circle described on the focal distance as diameter of the ellipse $4x^2 + 9y^2 = 36$ touches the auxiliary circle $x^2 + y^2 = 9$ internally.

7. **Statement 1:** Locus of the centre of a variable circle touching two circles $(x - 1)^2 + (y - 2)^2 = 25$ and $(x - 2)^2 + (y - 1)^2 = 16$ is an ellipse.

   Statement 2: If a circle $S_2 = 0$ lies completely inside the circle $S_1 = 0$, then locus of the centre of a variable circle $S = 0$ that touches both the circles is an ellipse.

8. **Statement 1:** For the ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ the product of the perpendiculars drawn from foci on any tangent is 3.

   Statement 2: For ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$, the foot of the perpendiculars drawn from foci on any tangent lies on the circle $x^2 + y^2 = 5$ which is an auxiliary circle of the ellipse.

9. **Statement 1:** If there is exactly one point on the line $3x + 4y + 5\sqrt{5} = 0$, from which perpendicular tangents can be drawn to the ellipse $\frac{x^2}{a^2} + y^2 = 1$ $(a > 1)$, then the eccentricity of the ellipse is $\frac{1}{3}$.

   Statement 2: For the condition given in statement 1, given line must touch the circle $x^2 + y^2 = a^2 + 1$.

10. **Statement 1:** Any chord of the conic $x^2 + y^2 + xy = 1$ through (0, 0) is bisected at (0, 0).

    Statement 2: The centre of a conic is a point through which every chord is bisected.

11. **Statement 1:** The area of the ellipse $2x^2 + 3y^2 = 6$ is more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

    Statement 2: the length of semi-major axes of an ellipse is more than the radius of the circle.

12. **Statement 1:** If line $x + y = 3$ is a tangent to an ellipse with foci (4, 3) and (6, y) at the point (1, 2), then $y = 17$.

    Statement 2: Tangent and normal to the ellipse at any point bisects the angle subtended by foci at that point.

13. **Statement 1:** Diagonals of any parallelogram inscribed in an ellipse always intersect at the centre of the ellipse

    Statement 2: Centre of the ellipse is the point at which chord passing through the centre of the ellipse gets bisected at the centre.

14. **Statement 1:** A triangle ABC right angled at A moves so that its perpendicular sides touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ all the time. Then loci of the points A, B and C are circle.

    Statement 2: Locus of point of intersection of two perpendicular tangents to the curve is a director circle.

15. **Statement 1:** Tangents are drawn to the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at the points, where it is intersected by the line $2x + 3y = 1$. Point of intersection of these tangents is (8, 6).

    Statement 2: Equation of the chord of contact to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from an external point is given by $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.

16. **Statement 1:** If tangent at point P (in first quadrant) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$, meets corresponding directrix $x = a/e$ at point Q, then circle with minimum radius having PQ as chord passes through the corresponding focus.

    Statement 2: PQ subtends right angle at corresponding focus.

17. **Statement 1:** If a, b are real numbers and $c > 0$, then the locus represented by the equation $|ay - bx| = c\sqrt{(x - a)^2 + (y - b)^2}$ is an ellipse.

    Statement 2: An ellipse is the locus of a point which moves in a plane such that ratio of its distances from a fixed point (i.e., focus) to the fixed line (i.e., directrix) is constant and less than 1.

Linked Comprehension Type

Solutions on page 4.53

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

An ellipse (E) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centred at point O have AB and CD as its major and minor axes, respectively, let the S_1 be one of the foci of the ellipse, radius of incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units. Then

1. perimeter of ΔOCS_1 is
   a. 20 units **b.** 10 units **c.** 15 units **d.** 25 units
2. Equation of director the director circle of ellipse (E) is
   a. $x^2 + y^2 = (48.5)$ **b.** $x^2 + y^2 = \sqrt{97}$
   c. $x^2 + y^2 = 97$ **d.** $x^2 + y^2 = \sqrt{48.5}$
3. Area of ellipse (E) is
   a. $\frac{65\pi}{4}$ **b.** $\frac{64\pi}{5}$ **c.** 64π **d.** 65π

For Problems 4–6

Consider the ellipse whose major and minor axes are x-axis and y-axis, respectively. If ϕ is the angle between the CP and the normal at point P on the ellipse, and the greatest value tan ϕ is $\frac{3}{2}$ (where C is the centre of the ellipse). Also semi-major axis is 10 units.

4. The eccentricity of the ellipse is
   a. $\frac{1}{2}$ **b.** $\frac{1}{3}$
   c. $\frac{\sqrt{3}}{2}$ **d.** none of these
5. A rectangle is inscribed in the ellipse whose sides are parallel to the co-ordinates axes, then maximum area of rectangle is
   a. 50 units **b.** 100 units
   c. 25 units **d.** none of these
6. Locus of the point of intersection of perpendicular tangents to the ellipse is
   a. $x^2 + y^2 = 125$ **b.** $x^2 + y^2 = 150$
   c. $x^2 + y^2 = 200$ **d.** none of these

For Problems 7–9

A curve is represented by $C = 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

7. Eccentricity of curve is
   a. 1/3 **b.** $1/\sqrt{3}$ **c.** 2/3 **d.** $2/\sqrt{5}$
8. The lengths of axes
   a. $6, 2\sqrt{6}$ **b.** $5, 2\sqrt{5}$
   c. $4, 4\sqrt{5}$ **d.** none of these
9. The centre of the conic C is
   a. (1, 0) **b.** (0,0)
   c. (0, 1) **d.** none of these

For Problems 10–12

For all real p, the line $2px + y\sqrt{1-p^2} = 1$ touches a fixed ellipse whose axes are coordinate axes.

10. The eccentricity of the ellipse is
    a. $\frac{2}{3}$ **b.** $\frac{\sqrt{3}}{2}$ **c.** $\frac{1}{\sqrt{3}}$ **d.** $\frac{1}{2}$
11. The foci of ellipse are
    a. $(0, \pm\sqrt{3})$ **b.** $(0, \pm 2/3)$
    c. $(\pm\sqrt{3}/2, 0)$ **d.** none of these
12. The locus of point of intersection of perpendicular tangents is
    a. $x^2 + y^2 = \frac{5}{4}$ **b.** $x^2 + y^2 = \frac{3}{2}$
    c. $x^2 + y^2 = 2$ **d.** none of these

For Problems 13–15

Let S, S' be the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentricity is e. P is a variable point on the ellipse. Consider the locus of the incentre of the $\Delta PSS'$.

13. The locus of incentre is
    a. ellipse **b.** hyperbola
    c. parabola **d.** circle
14. The eccentricity of locus of P is
    a. $\sqrt{\frac{2e}{1-e}}$ **b.** $\sqrt{\frac{2e}{1+e}}$
    c. 1 **d.** none of these
15. Maximum area of rectangle inscribed in the locus is
    a. $\frac{2abe^2}{1+e}$ **b.** $\frac{2abe}{1-e}$
    c. $\frac{abe}{1+e}$ **d.** none of these

For Problems 16–18

$C_1: x^2 + y^2 = r^2$ and $C_2: \frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four distinct points A, B, C and D. Their common tangents form a parallelogram $A'B'C'D'$.

16. If $ABCD$ is a square then r is equal to
    a. $\frac{12}{5}\sqrt{2}$ **b.** $\frac{12}{5}$
    c. $\frac{12}{5\sqrt{5}}$ **d.** none of these
17. If A', B', C', D' is a square then r is equal to
    a. $\sqrt{20}$ **b.** $\sqrt{12}$
    c. $\sqrt{15}$ **d.** none of these
18. If $A'B'C'D'$ is a square, then the ratio of area of the circle C_1 to the area of the circumcircle of $\Delta A'B'C'$ is
    a. $\frac{9}{16}$ **b.** $\frac{3}{4}$
    c. $\frac{1}{2}$ **d.** none of these

For Problems 19–21

A coplanar beam of light emerging from a point source has the equation $\lambda x - y + 2(1 + \lambda) = 0$, $\lambda \in R$, the rays of the beam strike an elliptical surface and get reflected. The reflected rays form another convergent beam having equation $\mu x - y + 2(1 - \mu) = 0$, $\mu \in R$. Further it is found that the foot of the perpendicular from the point (2, 2) upon any tangent to the ellipse lies on the circle $x^2 + y^2 - 4y - 5 = 0$.

19. The eccentricity of the ellipse is equal to

a. $\frac{1}{3}$ **b.** $\frac{1}{\sqrt{3}}$ **c.** $\frac{2}{3}$ **d.** $\frac{1}{2}$

20. The area of the largest triangle that an incident ray and the corresponding reflected ray can enclose with axis of the ellipse is equal to

a. $4\sqrt{5}$ **b.** $2\sqrt{5}$

c. $\sqrt{5}$ **d.** none of these

21. Total distance travelled by an incident ray and the corresponding reflected ray is the least if the point of incidence coincides with

a. an end of the minor with

b. an end of the major axis

c. an end of this latus rectum

d. none of these

For Problems 22–24

The tangent at any point P of the circle $x^2 + y^2 = 16$ meets the tangent at a fixed point A at T, and T is joined to B, the other end of the diameter through A.

22. The locus of the intersection of AP and BT is conic whose eccentricity is

a. $\frac{1}{2}$ **b.** $\frac{1}{\sqrt{2}}$ **c.** $\frac{1}{3}$ **d.** $\frac{1}{\sqrt{3}}$

23. Sum of focal distances of any point on the curve is

a. 12 **b.** 16 **c.** 20 **d.** 8

24. Which of the following does not change by changing the radius of the circle?

a. coordinates of focii

b. length of major axis

c. eccentricity

d. length of minor axis

For Problems 25–27

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that it has the least area but contains the circle $(x - 1)^2 + y^2 = 1$.

25. The eccentricity of the ellipse is

a. $\sqrt{\frac{2}{3}}$ **b.** $\frac{1}{\sqrt{3}}$

c. $\frac{1}{2}$ **d.** none of these

26. Equation of auxiliary circle of ellipse is

a. $x^2 + y^4 = 6.5$ **b.** $x^2 + y^4 = 5$

c. $x^2 + y^4 = 45$ **d.** none of these

27. Length of latus-rectum of the ellipse is

a. 2 units **b.** 1 unit **c.** 3 unit **d.** 2.5 unit

Matrix-Match Type

Solutions on page 4.57

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, , b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

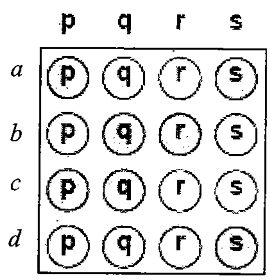


Fig. 4.35

1.

Column I	Column II
a. Distance between the points on the curve $4x^2 + 9y^2 = 1$, where tangent is parallel to the line $8x = 9y$, is less than	*p.* 1
b. Sum of distance between the foci of the curve $25(x + 1)^2 + 9(y + 2)^2 = 225$ from (–1, 0) is more than	*q.* 4
c. Sum of distances from the x-axis of the points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4}$, where the normal is parallel to the line $2x + y = 1$, is less than	*r.* 7
d. Tangents are drawn from points on the line $x - y + 2 = 0$ to the ellipse $x^2 + 2y^2 = 2$, then all the chords of contact pass through the point whose distance from (2, 1/2) is more than	*s.* 5

2. The tangents drawn from a point P to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make angle α and β with the major axis.

Column I	Column II
a. If $\alpha + \beta = \frac{c\pi}{2} (c \in N)$	*p.* circle
b. If $\tan\alpha \tan\beta = c$ {where $c \in R$}, then locus of P can be	*q.* ellipse
c. If $\tan\alpha + \tan\beta = c$ {where $c \in R$}, then locus of P can be	*r.* hyperbola
d. If $\cot\alpha + \cot\beta = c$ {where $c \in R$}, then locus of P can be	*s.* pair of straight lines

3.

Column I	Column II
a. If the tangent to the ellipse $x^2 + 4y^2 = 16$ at the point $P(\phi)$ is a normal to the circle $x^2 + y^2 - 8x - 4y = 0$, then $\frac{\phi}{2}$ may be	*p.* 0
b. The eccentric angle(s) of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is/are	*q.* $\cos^{-1}\left(-\frac{2}{3}\right)$
c. The eccentric angle of intersection of the ellipse $x^2 + 4y^2 = 4$ and the parabola $x^2 + 1 = y$ is	*r.* $\frac{\pi}{4}$
d. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point $Q(2\theta)$, then θ is	*s.* $\frac{5\pi}{4}$

4.

Column I	Column II
a. A stick of length 10 m slides on co-ordinate axes, then locus of a point dividing this stick from x-axis in the ratio 6 : 4 is a curve whose eccentrictly is e, then $3e$ is equal to	*p.* $\sqrt{6}$
b. AA' is a major axis of an ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ and P is a variable point on it, then greatest area of triangle APA' is	*q.* $2\sqrt{7}$
c. Distance between foci of the curve represented by the equation $x = 1 + 4\cos\theta$, $y = 2 + 3\sin\theta$ is	*r.* $\frac{128}{3}$
d. Tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at end points of the latus rectum. The area of equadrilateral so formed is	*s.* $\sqrt{5}$

5.

Column I	Column II
a. An ellipse passing through the origin has its foci (3, 4) and (6, 8), then length of its minor axis is	*p.* 8
b. If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which passes through $S \equiv (3, 0)$ and $PS = 2$, then length of chord PQ is	*q.* $10\sqrt{2}$
c. If the line $y = x + K$ touches the ellipse $9x^2 + 16y^2 = 144$, then the difference of values of K is	*r.* 10
d. Sum of distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ from the foci	*s.* 12

6.

Column I	Column II
a. If vertices of a rectangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are extremities of latus rectum. Then eccentricity of ellipse is	*p.* $\frac{2}{\sqrt{5}}$
b. If extremities of diameter of the circle $x^2 + y^2 = 16$ are foci of a ellipse, then eccentricity of the ellipse, if its size is just sufficient to contain the circle, is	*q.* $\frac{1}{\sqrt{2}}$
c. If normal at point (6, 2) to the ellipse passes through its nearest focus (5, 2), having centre at (4, 2) then its eccentricity is	*r.* $\frac{1}{3}$
d. If extremities of latus rectum of the parabola $y^2 = 24x$ are foci of ellipse and if ellipse passes through the vertex of the parabola, then its eccentricity is	*s.* $\frac{1}{2}$

Integer type

Solutions on page 4.60

1. If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, then the difference between the largest and smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$ is
2. The value of a for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, if the extremities of the latus rectum of the ellipse having positive ordinate lies on the parabola $x^2 = -2(y - 2)$, is

3. If the variable line $y = kx + 2h$ is tangent to an ellipse $2x^2 + 3y^2 = 6$, then locus of $P(h, k)$ is a conic C whose eccentricity is e then the value of $3e^2$ is

4. Tangents drawn from the point $P(2, 3)$ to the circle $x^2 + y^2 - 8x + 6y + 1 = 0$ touch the circle at the points A and B. The circumcircle of the ΔPAB cuts the director circle of ellipse $\frac{(x+5)^2}{9} + \frac{(y-3)^2}{b^2} = 1$ orthogonally. Then the value of $b^2/6$ is

5. If from a point $P(0, \alpha)$ two normals other than axes are drawn to ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, such that $|\alpha| < k$, then the value of $4k$ is

6. An ellipse passing through the origin has its foci (3, 4) and (6, 8) and length of its semi-minor axis is b, then the value of $b/\sqrt{2}$ is

7. If the mid point of a chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is (0, 3), and length of the chord is $\frac{4k}{5}$, then k is

8. Let the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$. If the length of the minor axis is k, then $\sqrt{3}\,k/2$ is

9. Consider an ellipse (E) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centered at point '$O$' and having AB and CD as its major and minor axes respectively if S_1 be one of the foci of the ellipse, radius of incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units, then the value of $(a - b)/2$ is

10. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

11. Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$, then the maximum value of $(4x - 9y)/2$ is

12. Rectangle $ABCD$ has area 200. An ellipse with area 200π passes through A and C and has foci at B and D. If the perimeter of the rectangle is P, then the value of $P/20$ is

Archives

Solutions on page 4.62

Subjective Type

1. Let 'd' be the perpendicular distance from the centre of the ellipse to any tangent to ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$. **(IIT-JEE, 1995)**

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **(IIT-JEE, 1997)**

3. Consider the family of circle $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the co-ordiante axes at A and B, then find the equation of the locus of the midpoint of AB. **(IIT-JEE, 1999)**

4. Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the triangle PON is maximum, where O denotes the origin and N be the foot of the perpendicular from O to the tangent at P. **(IIT-JEE, 1999)**

5. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendicular from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ meets the ellipse, respectively, at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C, respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. **(IIT-JEE, 2000)**

6. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : RQ = r : s$ as P varies over the ellipse. **(IIT-JEE, 2001)**

7. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **(IIT-JEE, 2002)**

8. From a point, common tangents are drawn to the curve $x^2 + y^2 = 16$ and $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Find the slope of common tangent in 1st quadrant and also find the length of intercept between coordinate axes. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. An ellipse has OB as a semi-minor axis, F, F' as its foci and the angle $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is.

Multiple choice questions with one correct answer

1. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}$, and having its centre (0, 3) is

   a. 4 b. 3 c. $\sqrt{12}$ d. $\frac{7}{2}$

   (IIT-JEE, 1995)

2. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4}+\frac{y^2}{1}=1$ is

   a. 0 b. 1 c. 2 d. infinite

   (IIT-JEE, 1998)

3. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

   a. 8 b. 6 c. 10 d. 12

   (IIT-JEE, 1998)

4. The area of the quadrilateral formed by the tangents at the end point of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is

   a. 27/4 sq. units b. 9 sq. units
   c. 27/2 sq. units d. 27 sq. units

   (IIT-JEE, 2003)

5. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the midpoint of the intercept made by the tangents between the coordinate axes is

   a. $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ b. $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

   c. $\frac{x^2}{2}+\frac{y^2}{4}=1$ d. $\frac{x^2}{4}+\frac{y^2}{2}=1$

   (IIT-JEE, 2004)

6. The minimum area of a triangle formed by the tangent to the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes is

   a. ab sq. units b. $\frac{a^2+b^2}{2}$ sq. units

   c. $\frac{(a+b)^2}{2}$ sq. units d. $\frac{a^2+ab+b^2}{3}$ sq. units

   (IIT-JEE, 2004)

7. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

   a. $\frac{31}{10}$ b. $\frac{29}{10}$ c. $\frac{21}{10}$ d. $\frac{27}{10}$

   (IIT-JEE, 2009)

8. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

   a. $(\pm(3\sqrt{5})/2, \pm 2/7)$ b. $(\pm(3\sqrt{5})/2, \pm\sqrt{19}/7)$
   c. $(\pm 2\sqrt{3}, \pm 1/7)$ d. $(\pm 2\sqrt{3}, \pm 4\sqrt{3}/7)$

   (IIT-JEE, 2009)

9. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then

   a. Q lies inside C but outside E
   b. Q lies outside both C and E
   c. P lies inside both C and E
   d. P lies inside C but outside E

   (IIT-JEE, 1994)

Multiple choice questions with one or more than one correct answer

1. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

   (IIT-JEE, 1999)

   a. $\left(\frac{2}{5},\frac{1}{5}\right)$ b. $\left(-\frac{2}{5},\frac{1}{5}\right)$

   c. $\left(-\frac{2}{5},-\frac{1}{5}\right)$ d. $-\left(\frac{2}{5},-\frac{1}{5}\right)$

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are

   a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
   b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
   d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

   (IIT-JEE, 2008)

3. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then **(IIT-JEE, 2009)**

   a. $b + c = 4a$
   b. $b + c = 2a$
   c. locus of point A is an ellipse
   d. locus of point A is a pair of straight line

Comprehension type

Tangents are drawn from the point $P(3,4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

1. The coordinate of A and B are

   a. (3,0) and (0,2)

b. $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

c. $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0,2)

d. (3,0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$ **(IIT-JEE, 2010)**

2. The orthocenter of the triangle PAB is

a. $\left(5, \frac{8}{7}\right)$ b. $\left(\frac{7}{5}, \frac{25}{8}\right)$ c. $\left(\frac{11}{5}, \frac{8}{5}\right)$ d. $\left(\frac{8}{25}, \frac{7}{5}\right)$

(IIT-JEE, 2010)

3. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

a. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

b. $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

c. $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

d. $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1.

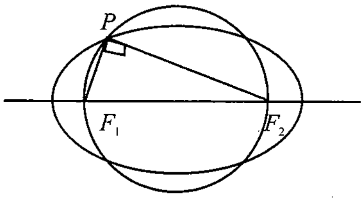


Fig. 4.39

$PF_1 + PF_2 = 15$

$PF_1 \times PF_2 = 52$

$(F_1F_2)^2 = (PF_1 + PF_2)^2 - 2PF_1 \times PF_2$

$= 225 - 104 = 121$

$\Rightarrow \quad F_1F_2 = 11$

$\Rightarrow \quad 2ae = 11$

$\Rightarrow \quad ae = \frac{11}{2}$

Also $\quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2$

$\Rightarrow \quad a^2 - b^2 = (ae)^2$

$\Rightarrow \quad 4(a^2 - b^2) = 4\left(\frac{11}{2}\right)^2 = 121$

2. Let the middle point of chord be $(t^2, 2t)$

Midpoint of chord must lie inside the ellipse

$\Rightarrow \quad t^4 + 8t^2 - 1 < 0$

$\Rightarrow \quad t^2 \in (0, -4 + \sqrt{17})$ (i)

Also equation of the chord is $T = S_1$

$\Rightarrow \quad t^2x + 4ty = t^4 + 8t^2$

Let this passes through $(\alpha, 0)$

$\Rightarrow \quad \alpha t^2 = t^4 + 8t^2$

$\Rightarrow \quad t^4 + (8 - \alpha)t^2 = 0$

$\Rightarrow \quad t^2 = 0$ or $t^2 = \alpha - 8$

$\Rightarrow \quad \alpha = t^2 + 8$

$\Rightarrow \quad \alpha \in (8, 4 + \sqrt{17})$

3. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Any tangent to the ellipse be

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Tangents at the vertices are $x = a$ and $x = -a$.

Solving with (i), we get the points T and T' as

$T\left[a, \frac{b(1 - \cos\theta)}{\sin\theta}\right], T'\left[-a, \frac{b(1 + \cos\theta)}{\sin\theta}\right]$

i.e., $T\left[a, b\tan\frac{\theta}{2}\right], T'\left[-a, b\cot\frac{\theta}{2}\right]$

circle on TT' as diameter is

$(x - a)(x + a) + (y - b\tan\frac{\theta}{2})(y - b\cot\frac{\theta}{2}) = 0$

$\Rightarrow \quad x^2 - a^2 + y^2 + b^2 - b\left[\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right]y = 0$

It will pass through the foci (± ae, 0)

If $\quad a^2e^2 - a^2 + b^2 = 0$

$\Rightarrow \quad b^2 = a^2(1 - e^2)$, which is true.

4.

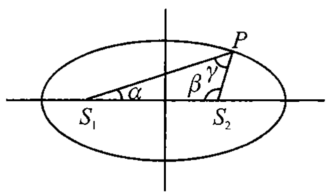


Fig. 4.40

We know that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}$$

In triangle ABC, we know that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$\Rightarrow \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=3\cot\frac{\gamma}{2}$$

$$\Rightarrow \quad 2\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}$$

$\Rightarrow \cot\frac{\alpha}{2}, \cot\frac{\gamma}{2}, \cot\frac{\beta}{2}$ in A.P.

5.

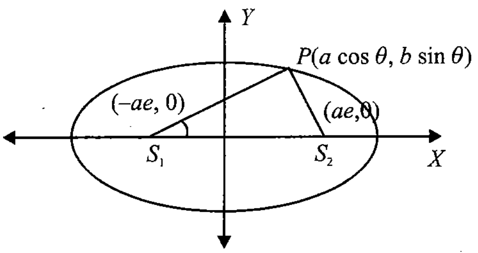


Fig. 4.41

Let any point P on the ellipse be $P(a\cos\theta, b\sin\theta)$.

If $\angle S_1PS_2=\frac{\pi}{2}$, then P lies on the circle having S_1S_2 as its diameter.

$\Rightarrow$ Equation of the circle drawn on S_1S_2 as diameter is

$$x^2+y^2=a^2e^2=a^2-b^2$$

Since the point P should not lie on the ellipse.

$\Rightarrow$ There should not be any point on intersection of

$$x^2+y^2=a^2-b^2 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \quad a^2-b^2<b^2$$

$$\Rightarrow \quad 2b^2>a^2$$

$$\Rightarrow \quad \frac{b^2}{a^2}>\frac{1}{2}$$

$$\Rightarrow \quad 1-e^2>\frac{1}{2}$$

$$\Rightarrow \quad e^2<\frac{1}{2}$$

$$\Rightarrow \quad e\in\left(0,\frac{1}{\sqrt{2}}\right)$$

6. Let $M(h, k)$ be the midpoint of AB.

Let $R(t, t+4)$, $t\in R$ be the point on the line $y=x+4$ and points be $P(x_1, y_1)$, $Q(x_2, y_2)$.

$A(x_1', y_1')$, $B(x_2', y_2')$

Now, $\quad x_1=x_1'$ and $x_2=x_2'$

and $\quad y_1=\frac{a}{b}y_1', y_2=\frac{a}{b}y_2'$.

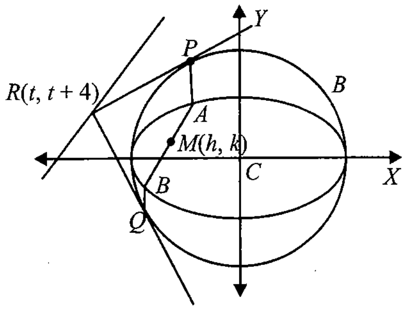


Fig. 4.42

Also $\quad h=\frac{x_1'+x_2'}{2}$ and $k=\frac{y_1'+y_2'}{2}$

Let N be the midpoint of PQ. So coordinates corresponding to N are

$$\frac{x_1+x_2}{2}=h,\ \frac{y_1+y_2}{2}=\frac{2(y_1'+y_2')}{2}$$

$$=2k \quad (\text{since } a/b=2)$$

So equation of PQ with $N(h, 2k)$ as midpoint is

$$xh+2yk=h^2+4k^2 \quad \text{(i)}$$

Also PQ is chord of contact with respect to R; So equation of PQ is

$$xt+y(t+4)=4 \quad \text{(ii)}$$

Comparing the coefficients of (i) and (ii), we get

$$\frac{h}{t}=\frac{2k}{t+4}=\frac{h^2+4k^2}{4}$$

$$\Rightarrow \quad t=\frac{4h}{h^2+4k^2} \text{ and}$$

$$8k=(h^2+4k^2)(t+4)$$

Eliminating t, we have

$$8k=(h^2+4k^2)\left(\frac{4h}{h^2+4k^2}+4\right)$$

$$\Rightarrow \quad 8k=4h+4h^2+16k^2.$$

$$\Rightarrow \quad 4k^2+h^2+h-2k=0$$

Hence, locus is

$$4y^2+x^2+x-2y=0.$$

7. Let the eccentric angles of the vertices P, Q, R of ΔPQR be $\theta_1, \theta_2, \theta_3$.

Then the equations of PQ and PR are

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}$$

and $\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2}+\frac{y}{b}\sin\frac{\theta_2+\theta_3}{2}=\cos\frac{\theta_2-\theta_3}{2}$, respectively

If PQ and PR are parallel to given straight lines, then we have

$$\theta_1+\theta_2=\text{constant}=2\alpha \text{ (say)}$$

and $\quad \theta_1+\theta_3=\text{constant}=2\beta$

Hence, $\theta_2 - \theta_3 = 2(\alpha - \beta)$ (i)

Now, the equation of QR is

$$\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2}+\frac{y}{b}\sin\frac{\theta_2+\theta_3}{2}=\cos\frac{\theta_2-\theta_3}{2} \quad \text{(ii)}$$

or $$\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2}+\frac{y}{b}\sin\frac{\theta_2+\theta_3}{2}=\cos(\alpha-\beta) \quad \text{(iii)}$$

which shows that the line (ii), for different values of $\theta_2 + \theta_3$, is tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2(\alpha-\beta)$$

8. Equation of the auxiliary circle is

$$x^2 + y^2 = a^2 \quad \text{(i)}$$

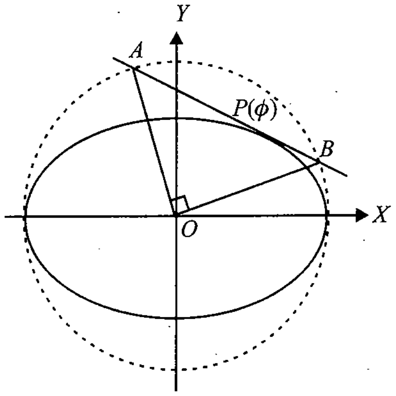


Fig. 4.43

Therefore, equation of tangent at a point $p(a\cos\phi, b\sin\phi)$ is

$$\left(\frac{x}{a}\right)\cos\phi+\left(\frac{y}{b}\right)\sin\phi=1 \quad \text{(ii)}$$

which meets the auxiliary circle at points A and B.

Therefore, equation of the pair of lines OA and OB is obtained by making Eq. (i) homogeneous with the help of Eq. (ii).

$$\Rightarrow \quad x^2+y^2 = a^2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)^2$$

But $\angle AOB = 90°$

$\therefore$ Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow 1-\cos^2\phi+1-\frac{a^2}{b^2}\sin^2\phi = 0$$

$$\Rightarrow \quad \sin^2\phi\left(1-\frac{a^2}{b^2}\right)+1=0$$

$$\Rightarrow \quad (a^2-b^2)\sin^2\phi = b^2$$

$$\Rightarrow \quad a^2e^2\sin^2\phi = a^2(1-e^2)$$

$$\Rightarrow \quad (1+\sin^2\phi)e^2 = 1$$

$$\Rightarrow \quad e=\frac{1}{\sqrt{1+\sin^2\theta}}$$

9. Equation of pair of tangents from point P is $SS_1 = T^2$

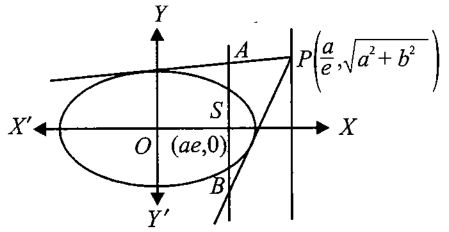


Fig. 4.44

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{1}{e^2}+\frac{a^2+b^2}{b^2}-1\right)$$

$$=\left(\frac{xa}{ea^2}+\frac{y\sqrt{a^2+b^2}}{b^2}-1\right)^2$$

Solving this with $x = ae$, we get

$$\left(e^2+\frac{y^2}{b^2}-1\right)\left(\frac{1}{e^2}+\frac{a^2}{b^2}\right)=\frac{y^2(a^2+b^2)}{b^2\,b^2}$$

$$\Rightarrow \left(\frac{y^2}{b^2}-\frac{b^2}{a^2}\right)\left(\frac{b^2+a^2-b^2}{b^2e^2}\right)=\frac{y^2(a^2+b^2)}{b^2\,b^2}$$

(using $e^2 = 1-\frac{b^2}{a^2}$)

$$\Rightarrow \left(\frac{a^2y^2-b^4}{a^2b^2}\right)\left(\frac{a^2}{b^2e^2}\right)=\frac{y^2(a^2+b^2)}{b^4}$$

$$\Rightarrow \quad a^2(a^2y^2-b^4) = y^2(a^4-b^4)$$

[using $(a^2e^2 = a^2 - b^2)$]

$$\Rightarrow \quad a^2b^4 = y^2b^4$$

$$\Rightarrow \quad y^2 = a^2$$

$$\Rightarrow \quad y = \pm a$$

$$\Rightarrow \quad AB = 2a$$

10.

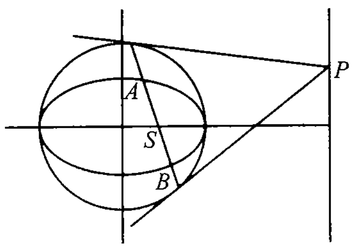


Fig. 4.45

Let the point on directrix be $\left(\frac{a}{e}, k\right)$.

Then equation of AB will be,

$$x\left(\frac{a}{e}\right)+yk=a^2.$$

Now it will clearly pass through focus of the ellipse as

$$(ae)\left(\frac{a}{e}\right)+k.0=a^2,$$

so AB is a focal chord.

11. Equation of any tangent PQ to the ellipse

$$\frac{x^2}{y^2}+\frac{y^2}{b^2}=1 \text{ be}$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \text{(i)}$$

This tangent cuts the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=a$ at the points P and Q.

Let tangents at P and Q intersect the point $R(h, k)$.

Then PQ becomes chord of contact with respect to the point R for the ellpse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$, i.e., equation of PQ

$$\frac{hx}{c^2}+\frac{ky}{d^2}=1 \qquad \text{(ii)}$$

Equations (i) and (ii) represent same straight lines.

$$\Rightarrow \quad \frac{\frac{\cos\theta}{a}}{\frac{h}{c^2}}=\frac{\frac{\sin\theta}{a}}{\frac{k}{d^2}}=1$$

$$\Rightarrow \quad \cos\theta=\frac{ah}{c^2}\sin\theta=\frac{bk}{d^2}$$

Squaring and adding, we get

$$\frac{a^2h^2}{c^4}+\frac{b^2k^2}{d^4}=1$$

$$\Rightarrow \quad \frac{a^2x^2}{c^4}+\frac{b^2y^2}{d^4}=1 \qquad \text{(iii)}$$

which is the locus of the point $R(h, k)$.

If $R(h, k)$ is the point of intersection of two perpendicular tangents, then locus of R should be the director circle of the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$,

i.e,. $\quad x^2+y^2=c^2+d^2,$

i.e,. $\quad \frac{x^2}{c^2+d^2}+\frac{y^2}{c^2+d^2}=1 \qquad \text{(iv)}$

Equations (iii) and (iv) represent the same locus

$$\Rightarrow \quad \frac{a^2}{c^4}=\frac{1}{c^2+d^2},$$

and $\quad \frac{b^2}{d^4}=\frac{1}{c^2+d^2}$

$$\Rightarrow \quad \frac{a^2}{c^2}+\frac{b^2}{d^2}=1$$

12. Let line OPQ makes an angle θ with x-axis so

$$P\equiv(a\cos\theta, a\sin\theta),$$
$$Q\,(b\cos\theta, b\sin\theta)$$

AQ and let $R\,(x, y)$

So $\quad x=a\cos\theta, y=b\sin\theta$

Eliminating θ, we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, locus of R is an ellipse.

Also $a<b$ so vertices are $(0, b)$ and $(0, -b)$ and extremities of minor axis are $(\pm a, 0)$.

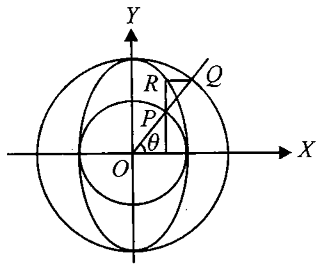


Fig. 4.46

So the ellipse touches both inner circle and outer circle if foci are $(0, \pm a)$

$\Rightarrow \quad a=be$, i.e., $e=\frac{a}{b}$

Also $\quad e=\sqrt{1-e^2}$

$\Rightarrow \quad e^2=1-e^2$

$\Rightarrow \quad e=\frac{1}{\sqrt{2}}$

and ratio of radii is $\quad \frac{a}{b}=e=\frac{1}{\sqrt{2}}$

13.

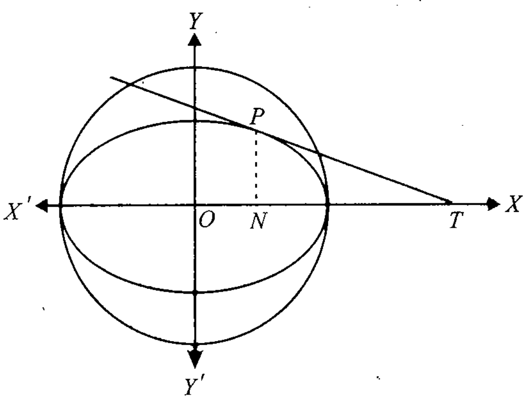


Fig. 4.47

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $(a\cos\theta, b\sin\theta)$ be a point on the ellipse.

The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

It meets the major axis at $T(a\sec\theta, 0)$.

The coordinates of N are $(a\cos\theta, 0)$.

The equation of the circle with NT as its diameter is

$$(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$$

$\Rightarrow$ It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally as $2g\cdot 0 + 2f\cdot 0 = a^2 - a^2 = 0$, which is true.

14. Let $P \equiv (h, k)$. The combined equation of tangents from (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2=0$$

Any curve that can be drawn through the points of intersection of these tangents with coordinate axes is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2+\lambda xy=0$$

or $x^2\left(\frac{k^2}{a^2b^2}-\frac{1}{a^2}\right)+y^2\left(\frac{h^2}{a^2b^2}-\frac{1}{b^2}\right)+xy\left(\lambda-\frac{2hk}{a^2b^2}\right)+\frac{2xh}{a^2}+\frac{2yk}{b^2}-\frac{h^2}{a^2}-\frac{k^2}{b^2}=0$

It should represent a circle

$$\Rightarrow \quad \lambda=\frac{2hk}{a^2b^2}, \frac{k^2}{a^2b^2}-\frac{1}{a^2}=\frac{h^2}{a^2b^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \lambda=\frac{2hk}{a^2b^2}, h^2-k^2=a^2-b^2$$

Hence, locus of P is

$$x^2 - y^2 = a^2 - b^2.$$

15. Equation of the ellipse is $\frac{x^2}{3}+\frac{y^2}{4}=1$.

Let point P be $(\sqrt{3}\cos\theta, 2\sin\theta)$, $\theta \in (0, \frac{\pi}{2})$

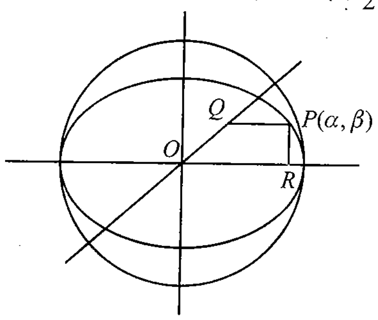


Fig. 4.48

Clearly line PQ is $y = 2\sin\theta$, line PR is $x = \sqrt{3}\cos\theta$ and OQ is $y = x$ and Q is $(2\sin\theta, 2\sin\theta)$.

$Z =$ area of the region PQORP (trapezium)

$$= \frac{1}{2}(OR + PQ)PR$$

$$= \frac{1}{2}(\sqrt{3}\cos\theta + (\sqrt{3}\cos\theta - 2\sin\theta))2\sin\theta$$

$$= \frac{1}{2}(2\sqrt{3}\cos\theta\sin\theta - 2\sin^2\theta)$$

$$= \frac{1}{2}(\sqrt{3}\sin 2\theta + \cos 2\theta - 1)$$

$$= \cos\left(2\theta - \frac{\pi}{3}\right) - \frac{1}{2}$$

which is maximum when $\cos\left(\theta - \frac{\pi}{3}\right)$ is maximum

or $\quad 2\theta - \frac{\pi}{3} = 0$

or $\quad \theta = \frac{\pi}{6}$

Hence, point P is $(\frac{3}{2}, 1)$.

Objective Type

1. c. We have $PQ = BP$

$$\Rightarrow \quad 2ae = \sqrt{a^2e^2 + b^2} = \sqrt{a^2} = a$$

$$\Rightarrow \quad e = \frac{1}{2}$$

2. c. Ellipse passing through $O(0, 0)$ and having foci $P(3, 3)$ and $Q(-4, 4)$,

then $\quad e = \frac{PQ}{OP + OQ}$

$$= \frac{\sqrt{50}}{3\sqrt{2} + 4\sqrt{2}}$$

$$= \frac{5}{7}$$

3. b. Here $a^2 + 2 > a^2 + 1$

$$\Rightarrow \quad a^2 + 1 = (a^2 + 2)(1 - e^2)$$

$$\Rightarrow \quad a^2 + 1 = (a^2 + 2)\frac{5}{6}$$

$$\Rightarrow \quad 6a^2 + 6 = 5a^2 + 10$$

$$\Rightarrow \quad a^2 = 10 - 6 = 4$$

$$\Rightarrow \quad a = \pm 2$$

Latus rectum $\quad = \frac{2(a^2+1)}{\sqrt{a^2+2}} = \frac{2\times 5}{\sqrt{6}} = \frac{10}{\sqrt{6}}$

4. a. Let $P(\theta)$, $Q\left(\theta + \frac{2\pi}{3}\right)$, $R\left(\theta + \frac{4\pi}{3}\right)$

then $\quad P' \equiv (a\cos\theta, b\sin\theta)$,

$$Q' \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$R' \equiv \left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let centroid of $\Delta P'Q'R' \equiv (x', y')$

$$x' = a\left[\frac{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3}\right]$$

$$= \frac{a}{3}\left[\cos\theta + 2\cos(\theta + \pi)\cos\frac{\pi}{3}\right] = 0$$

$$y' = \frac{a}{3}\left[\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right]$$

$$= \frac{a}{3}\left[\sin\theta + 2\sin(\theta + \pi)\sin\frac{\pi}{3}\right] = 0$$

$$= 0$$

5. a. $2a(1 + e) = 15$

$$1 + e = \frac{3}{2}$$

$$e = 0.5$$

6. d. For the two ellipses to intersect at four distinct points, $a > 1$

$\Rightarrow$ $b^2 - 5b + 7 > 1$

$\Rightarrow$ $b^2 - 5b + 6 > 0$

$\Rightarrow$ $b \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ b does not lie in $[2, 3]$

7. b. Let S be the given focus and ZM be the given line.

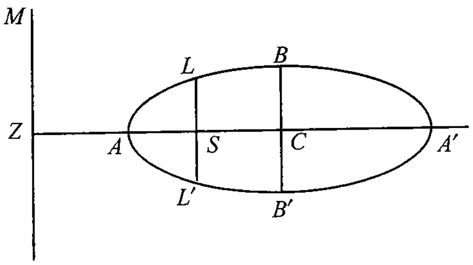


Fig. 4.49

Then $SZ = \frac{a}{e} - ae$

$$= \frac{a}{e}(1 - e^2)$$

$$= \frac{b^2}{ae} = k \text{ (say)}$$

as $b^2 = a^2(1 - e^2)$

Now take SC as x-axis and LSL' as y-axis. Let (x, y) be the coordinates of B with respect to these axes, then $x = SC = ae$, $y = CB = b$

Hence, $\frac{y^2}{x} = \frac{b^2}{ae} = SZ$, which is constant.

$\therefore$ $y^2 = kx$ is the required locus which is a parabola.

8. a. Let AB be the line

Let $AP = a$, $PB = b$,

so that $AB = a + b$

If AB makes an angle θ with x-axis and coordinates of P are (x, y),

then in ΔAPL, $x = a\cos\theta$

in ΔPBQ, $y = b\sin\theta$

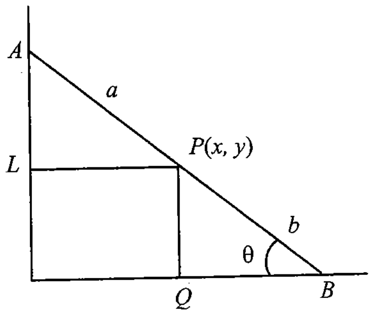


Fig. 4.50

$\therefore$ Locus of $P(x, y)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse

9. b. $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x - 4y + 7)^2}{4}$

$\Rightarrow$ $(x - 2)^2 + (y + 3)^2 = \left(\frac{1}{2}\,\frac{3x - 4y - 7}{5}\right)^2$

$\Rightarrow$ $\sqrt{(x - 2)^2 + (y + 3)^2} = \frac{1}{2}\,\frac{|3x - 4y - 7|}{5}$ is an ellipse, whose focus is $(2, -3)$, directrix $3x - 4y + 7 = 0$ and eccentricity is $\frac{1}{2}$.

Length of $\perp$ from focus to directrix is

$$\frac{|3 \times 2 - 4(-3) + 7|}{5} = 5$$

$\Rightarrow$ $\frac{a}{e} - ae = 5$

$\Rightarrow$ $2a - \frac{a}{2} = 5$

$\Rightarrow$ $a = \frac{10}{3}$

So length of major axis is $\frac{20}{3}$.

10. d.

$$\frac{(x - 4)^2}{25} + \frac{y^2}{4} = 1 \text{ and } (x - 1)^2 + \frac{y^2}{4} = 1$$

Clearly $m_{OP} \cdot m_{OQ} = -1$

$\Rightarrow$ OP and OQ are perpendicular to each other.

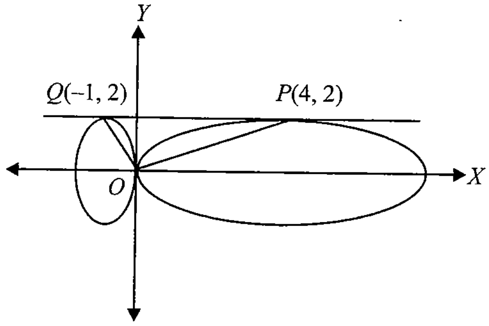


Fig. 4.51

11. a. Let ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=a^2e^2$

Radius of circle = ae

Point of intersection of circle and ellipse is $\left[\frac{a}{e}\sqrt{2e^2-1}, \frac{a}{e}(1-e^2)\right]$.

Now area of ΔPF_1F_2

$$=\frac{1}{2}\begin{vmatrix} \frac{a}{e}\sqrt{2e^2-1} & \frac{a}{e}(1-e^2) & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}=\frac{1}{2}\left|\frac{a}{e}(1-e^2)(2ae)\right|=30$$

$\Rightarrow \quad a^2(1-e^2)=30$ (given)

$\Rightarrow \quad a^2e^2=a^2-30=\left(\frac{17}{2}\right)^2-30=\frac{169}{4}$

$\Rightarrow \quad 2ae=13$

12. b. Solving the given line and the ellipse, we get

$$t^2+\frac{y^2}{9}=1$$

$\Rightarrow \quad y^2=9(1-t^2),$

which gives real and distinct values of y, if $1-t^2>0$

$\Rightarrow \quad t\in(-1, 1).$

13. a. Any point on the ellipse is $(2\cos\theta, \sqrt{3}\sin\theta)$.

The focus on the positive x-axis is $(1, 0)$.

Given that

$$(2\cos\theta-1)^2+3\sin^2\theta=\frac{25}{16}$$

$\Rightarrow \quad \cos\theta=\frac{3}{4}.$

14. c. Let $P(5\cos\theta, 4\sin\theta)$ be any point on the ellipse

Then $\quad SP=5+5e\cos\theta$

$S'P=5-5e\cos\theta$

$SP\cdot S'P=25-25e^2\cos^2\theta$

$=25\sin^2\theta+16\cos^2\theta$

$=16+9\sin^2\theta=f(\theta)$ (say)

$\Rightarrow \quad 16\le f(\theta)\le 25.$

15. a. Tangent at P($a\cos\alpha$, $b\sin\alpha$) is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1 \qquad \text{(i)}$$

Distance of focus $S(ae, 0)$ from this tangent is

$$d_1=\frac{|e\cos\alpha-1|}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

$$=\frac{1-e\cos\alpha}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

Distance of focus $S'(-ae, 0)$ from this line

$$d_2=\frac{1+e\cos\alpha}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

$\Rightarrow \quad \frac{d_1}{d_2}=\frac{1-e\cos\alpha}{1+e\cos\alpha}$

Now $SP=a-ae\cos\alpha$ and $S'P=a+ae\cos\alpha$

$\Rightarrow \quad \frac{SP}{S'P}=\frac{1-e\cos\alpha}{1+e\cos\alpha}$

$\Rightarrow \quad \frac{SP}{S'P}=\frac{d_1}{d_2}$

16. b.

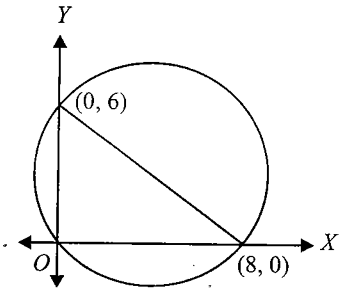


Fig. 4.52

Centre of family of ellipse is (4, 3) and distance of focus from centre = $ae=\frac{5}{2}$.

Hence, locus $(x-4)^2+(y-3)^2=\frac{25}{4}$

17. a. Let $P(x, y)$ be the position of the man at any time.

Let S (4, 0) and S' (− 4, 0) be the fixed flag, post, with C as the origin.

Since $SP+S'P=10$ m i.e., a constant, the locus of P is an ellipse with S and S' as foci

$\Rightarrow \quad ae=4$, and $2a=10$

$\Rightarrow \quad e=\frac{4}{5}$

Now $\quad b^2=a^2(1-e^2)$

$\Rightarrow \quad b^2 = 25\left(1 - \frac{16}{25}\right) = 9$

$\Rightarrow \quad b = 3$

Hence, the area of the ellipse $= \pi ab = \pi \times 5 \times 3 = 15\pi$

18. b. Since there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from centre is same, the points would be either end points of the major axis or of the minor axis.

But $\sqrt{\frac{a^2 + 2b^2}{2}} > b$, so the points are the vertices of major axis.

Hence, $\quad a = \sqrt{\frac{a^2 + 2b^2}{2}}$

$\Rightarrow \quad a^2 = 2b^2$

$\Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$

19. c. Let $p = 3h + 2$ and $q = k$

$\Rightarrow \quad h = \frac{p-2}{3}$ and $k = q$

Since (h, k) lies on $x^2 + y^2 = 1$

$\Rightarrow \quad h^2 + k^2 = 1$

$\Rightarrow \quad \left(\frac{p-2}{3}\right)^2 + q^2 = 1$

Locus is $\quad \left(\frac{x-2}{3}\right)^2 + y^2 = 1$

which has eccentricity $e = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$

20. b.

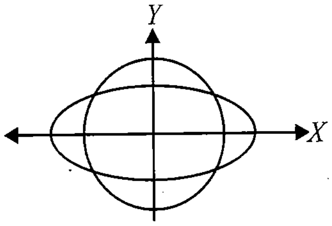


Fig. 4.53

Radius of the circle having SS' as diameter is $r = ae$

If it cuts an ellipse, then $r > b$

$\Rightarrow \quad ae > b$

$\Rightarrow \quad e^2 > \frac{b^2}{a^2}$

$\Rightarrow \quad e^2 > 1 - e^2$

$\Rightarrow \quad e^2 > \frac{1}{2}$

$\Rightarrow \quad e > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad e \in \left(\frac{1}{\sqrt{2}}, 1\right)$

21. b. Equation of conic through point of intersection of given two ellipse is

$$\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$$

$\Rightarrow \quad x^2\left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1 + \lambda) = 1 + \lambda$

$\Rightarrow \quad x^2\left(\frac{a^2 + 4\lambda}{4a^2(1 + \lambda)}\right) + y^2 = 1$

This equation is a circle if $\frac{a^2 + 4\lambda}{4a^2(1 + \lambda)} = 1$

$\Rightarrow$ Circle is $x^2 + y^2 = 1$

22. b.

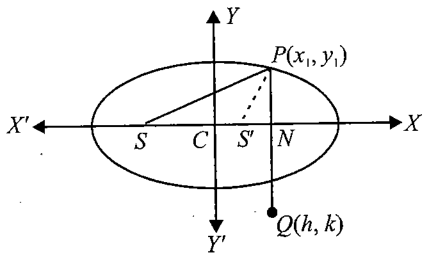


Fig. 4.54

$a^2 = 25$

and $\quad b^2 = 16$

$\Rightarrow \quad e = \sqrt{1 - \frac{16}{25}}$

$= \frac{3}{5}$

Let point Q be (h, k), where $k < 0$

Given that $k = SP = a + ex_1$, where $P(x_1, y_1)$ lies on the ellipse

$\Rightarrow \quad |k| = a + eh$ (as $x_1 = h$)

$\Rightarrow \quad -y = a + ex$

$\Rightarrow \quad 3x + 5y + 25 = 0$

23. a.

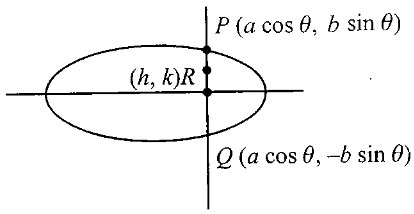


Fig. 4.55

Let $P(a\cos\theta, b\sin\theta)$, $Q(a\cos\theta, -b\sin\theta)$

$PR : RQ = 1:2$

Therefore, $\quad h = a\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{h}{a}$ (i)

and $\quad k = \frac{b}{3}\sin\theta$

$\Rightarrow \quad \sin\theta = \frac{3k}{b}$ (ii)

On squaring and adding Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$$

24. b. Let m be the slope of the common tangent, then

$$\pm\sqrt{3}\sqrt{1+m^2} = \pm\sqrt{4m^2+1}$$

$\Rightarrow \quad 3 + 3m^2 = 4m^2 + 1$

$\Rightarrow \quad m^2 = 2$

$\Rightarrow \quad m = \pm\sqrt{2}$

25. a.

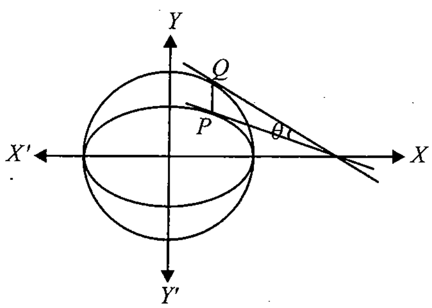


Fig. 4.56

Tangent to the ellipse at $P(a\cos\alpha, b\sin\alpha)$ is

$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ (i)

Tangent to the circle at $Q(a\cos\alpha, a\sin\alpha)$ is

$\cos\alpha x + \sin\alpha y = a$ (ii)

Now angle between tangents is θ,

then $\quad \tan\theta = \left|\dfrac{-\frac{b}{a}\cot\alpha - (-\cot\alpha)}{1 + \left(-\frac{b}{a}\cot\alpha\right)(-\cot\alpha)}\right|$

$$= \left|\frac{\cot\alpha\left(1 - \frac{b}{a}\right)}{1 + \frac{b}{a}\cot^2\alpha}\right|$$

$$= \left|\frac{a-b}{a\tan\alpha + b\cot\alpha}\right|$$

$$= \left|\frac{a-b}{(\sqrt{a\tan\alpha} - \sqrt{b\cot\alpha})^2 + 2\sqrt{ab}}\right|$$

Now the greatest value of the above expression is $\left|\frac{a-b}{2\sqrt{ab}}\right|$

when $\sqrt{a\tan\alpha} = \sqrt{b\cot\alpha}$

$\Rightarrow \quad \theta_{\text{maximum}} = \tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

26. c.

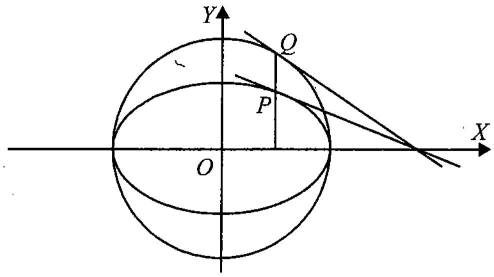


Fig. 4.57

Tangent to the ellipse at point $P(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Tangent to the circle at point $Q(a\cos\theta, a\sin\theta)$ is

$x\cos\theta + y\sin\theta = a$ (ii)

Equation (i) and (ii) intersect at $(\frac{a}{\cos\theta}, 0)$ which lies on $y = 0$

27. d. Tangent to the ellipse having slope m is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If it passes through the point $P(h, k)$, then

$$k = mh + \sqrt{a^2m^2 + b^2}$$

or $\quad (a^2 - h^2)m^2 + 2hkm + b^2 - k^2 = 0$

Now given $\tan\alpha + \tan\beta = \lambda$

$\Rightarrow \quad m_1 + m_2 = \lambda$

$\Rightarrow \quad \frac{-2hk}{a^2 - h^2} = \lambda$

$\Rightarrow \quad$ locus is $\lambda(x^2 - a^2) = 2xy$

28. c. Tangent to the ellipse at P and Q are

$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ (i)

and $\quad \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1$ (ii)

Solving Eqs. (i) and (ii), we get

$$\frac{x}{\begin{vmatrix}\frac{\sin\alpha}{b} & 1\\ \frac{\sin\beta}{b} & 1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}\frac{\cos\alpha}{a} & 1\\ \frac{\cos\beta}{a} & 1\end{vmatrix}} = \frac{1}{\begin{vmatrix}\frac{\cos\alpha}{a} & \frac{\sin\alpha}{b}\\ \frac{\cos\beta}{a} & \frac{\sin\beta}{b}\end{vmatrix}}$$

$\Rightarrow \quad x = \dfrac{a(\sin\alpha - \sin\beta)}{\sin(\beta - \alpha)}$,

$$y = \frac{-b(\cos\alpha - \cos\beta)}{\sin(\beta - \alpha)}$$

$$\Rightarrow \quad \frac{x\sin(\beta-\alpha)}{a} = \sin\alpha - \sin\beta,$$

$$\frac{y\sin(\beta-\alpha)}{b} = -(\cos\alpha - \cos\beta)$$

Squaring and adding, we get

$$\sin^2(\beta-\alpha)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = 2$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{2}{\sin^2 c}$$

(where $\beta - \alpha = c$ (constant) given)

which is an ellipse.

29. d. As in above question point of intersection is

$$(h, k) \equiv \left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

It is given that $\alpha + \beta = c$ = constant.

$$\Rightarrow \quad h = \frac{a\cos\frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)} \text{ and } k = \frac{b\sin\frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{h}{k} = \frac{a}{b}\cot\left(\frac{c}{2}\right)$$

$$\Rightarrow \quad k = \frac{b}{a}\tan\left(\frac{c}{2}\right)h$$

$\Rightarrow$ (h, k) lies on the straight line

30. a. Any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ having slope m is

$$y = mx + \sqrt{a^2m^2+b^2}$$

Points on the minor axis are $(0, ae)$, $(0, -ae)$.

$\therefore$ Sum of the squares of the perpendicular on the tangent from $(0, ae)$ and $(0, -ae)$

$$= \left[\frac{\sqrt{a^2m^2+b^2} - ae}{\sqrt{m^2+1}}\right]^2 + \left[\frac{\sqrt{a^2m^2+b^2} + ae}{\sqrt{m^2+1}}\right]^2$$

$$= \frac{2(a^2m^2+b^2+a^2e^2)}{m^2+1}$$

$$= \frac{2(a^2m^2+a^2-a^2e^2+a^2e^2)}{m^2+1}$$

$$= \frac{2a^2(m^2+1)}{m^2+1} = 2a^2$$

31. c. Equation of the tangent to the ellipse at $P(5\cos\theta, 4\sin\theta)$ is

$$\frac{x\cos\theta}{5} + \frac{y\sin\theta}{4} = 1$$

It meets the line $x = 0$ at $Q(0, 4\operatorname{cosec}\theta)$

Image of Q in the line $y = x$ is $R(4\operatorname{cosec}\theta, 0)$

$\therefore$ Equation of the circle is

$$x(x - 4\operatorname{cosec}\theta) + y(y - \operatorname{cosec}\theta) = 0$$

i.e., $\quad x^2 + y^2 - 4(x + y)\operatorname{cosec}\theta = 0$

$\therefore$ Each member of the family passes through the intersection of $x^2 + y^2 = 0$ and $x + y = 0$, i.e., the point $(0, 0)$.

32. c.

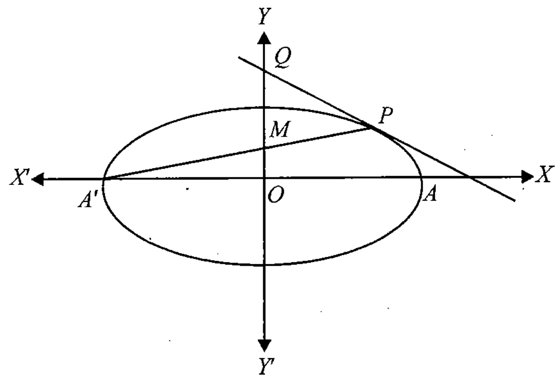


Fig. 4.58

Let point P be $(a\cos\theta, b\sin\theta)$.

Equation of the tangent at point P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

Then point Q is $(b\operatorname{cosec}\theta, 0)$

Equation of chord $A'P$ is

$$y - 0 = \frac{b\sin\theta}{a\cos\theta + a}(x + a)$$

Putting $x = 0$, we have $y = \frac{b\sin\theta}{\cos\theta + 1}$

Then

$$OQ^2 - MQ^2 = b^2\operatorname{cosec}^2\theta - \left(b\operatorname{cosec}\theta - \frac{b\sin\theta}{\cos\theta+1}\right)^2$$

$$= \frac{2b^2}{\cos\theta+1} - \frac{b^2\sin^2\theta}{(\cos\theta+1)^2}$$

$$= \frac{b^2}{\cos\theta+1}\left(\frac{2\cos\theta + 2 - \sin^2\theta}{(\cos\theta+1)}\right)$$

$$= \frac{b^2}{\cos\theta+1}\left(\frac{2\cos\theta + 1 + \cos^2\theta}{(\cos\theta+1)}\right)$$

$$= b^2 = 4$$

33. b. One of the tangents of slope m to the given ellipse is

$$y = mx + \sqrt{18m^2 + 32}$$

For $\quad m = -\frac{4}{3}$,

we have $\quad y = -\frac{4}{3}x + 8.$

Then points on the axis where tangents meet are $A(6, 0)$ and $B(0, 8)$.

Then area of triangle ABC is $\frac{1}{2}(6)(8) = 24$ units.

34. b.

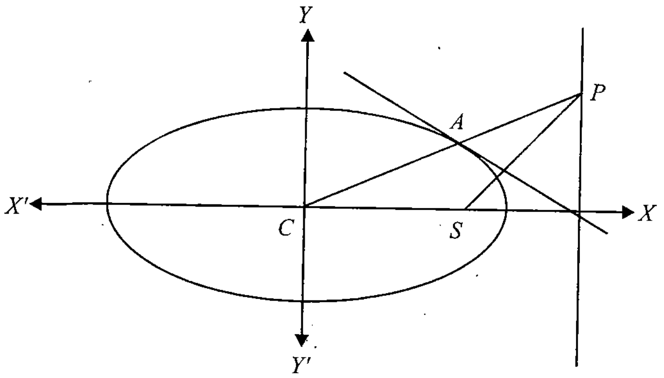


Fig. 4.59

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and let $\quad A \equiv (a\cos\theta, b\sin\theta)$

Equation of AC will be $y = \frac{b}{a}\tan\theta\, x$

Solving with $x = \frac{a}{e}$, we get

$$P \equiv \left(\frac{a}{e}, \frac{b}{e}\tan\theta\right)$$

Slope of tangent at A is $-\frac{b}{a\tan\theta}$

Slope of PS

$$= \frac{\frac{b}{e}\tan\theta}{\frac{a}{e} - ae} = \frac{b\tan\theta}{a(1-e^2)} = \frac{a}{b}\tan\theta$$

So $\quad \alpha = \frac{\pi}{2}$

35. a. A tangent of slope 2 to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = 2x \pm \sqrt{4a^2 + b^2} \qquad \text{(i)}$$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

$\Rightarrow$ Eq. (i), passes through $(-2, 0)$

$$\Rightarrow \quad 0 = -4 \pm \sqrt{4a^2 + b^2}$$

$$\Rightarrow \quad 4a^2 + b^2 = 16$$

Using A.M. ≥ G.M., we get

$$\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2 b^2}$$

$$\Rightarrow \quad ab \leq 4$$

36. a. Equation of tangent $\frac{x}{a}\frac{\sqrt{3}}{2} + \frac{y}{b}\frac{1}{2} = 1 \qquad \text{(i)}$

and equation of tangent at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \qquad \text{....(ii)}$$

Comparing (i) and (ii), we have $\cos\phi = \frac{\sqrt{3}}{2}$ and $\sin\phi = \frac{1}{2}$

Hence, $\quad \phi = \frac{\pi}{6}$

37. d.

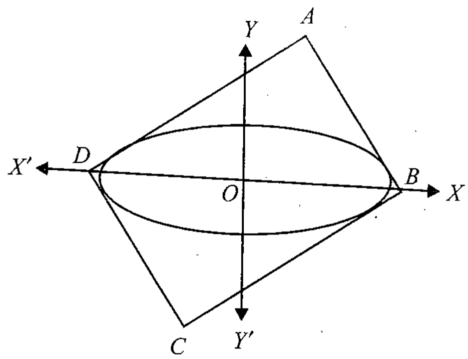


Fig. 4.60

Since sides of the square are tangent and perpendicular to each other, so the vertices lie on director circle

$$\Rightarrow \quad x^2 + y^2 = (a^2 - 7) + (13 - 5a)$$

$$= a^2 \quad (\sqrt{2}\,a \text{ is side of the square})$$

$$\Rightarrow \quad (a^2 - 7) + (13 - 5a) = a^2$$

$$\Rightarrow \quad a = \frac{6}{5}$$

But for an ellipse to exist $a^2 - 7 > 0$ and $13 - 5a > 0$

$$\Rightarrow \quad a \in (-\infty, -\sqrt{7})$$

Hence, $\quad a \neq \frac{6}{5}$

Hence, no such a exists.

38. d. Since locus of the point of intersection of the tangent at the end points of a focal chord is directrix

∴ Required locus is $x = \pm\frac{a}{e}$, which is pair of straight lines.

39. c. Normal at point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad \text{(i)}$$

It meets axes at $Q\left(\frac{(a^2 - b^2)\cos\theta}{a}, 0\right)$

and $R\left(0, -\frac{(a^2 - b^2)\sin\theta}{b}\right)$

Let $T(h, k)$ is a midpoint of QR.

Then $$2h = \frac{(a^2 - b^2)\cos\theta}{a}$$

and $$2k = -\frac{(a^2 - b^2)\sin\theta}{b}$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{4h^2a^2}{(a^2-b^2)^2} + \frac{4k^2b^2}{(a^2-b^2)^2} = 1$$

$$\Rightarrow \text{Locus is } \frac{x^2}{\frac{(a^2-b^2)^2}{4a^2}} + \frac{y^2}{\frac{(a^2-b^2)^2}{4b^2}} = 1 \quad \text{(ii)}$$

which is an ellipse, having eccentricity e', given by

$$e'^2 = 1 - \frac{\frac{(a^2-b^2)^2}{4a^2}}{\frac{(a^2-b^2)^2}{4b^2}} = 1 - \frac{b^2}{a^2} = e^2$$

$$\Rightarrow \quad e' = e$$

Note:
In Eq.* (ii),** $\frac{(a^2-b^2)}{4a^2} < \frac{(a^2-b^2)}{4b^2}$. ***Hence, x-axis is minor axis.

40. c. Equation of normal to the ellipse at P is

$5x \sec\theta - 3y \operatorname{cosec}\theta = 16$ (i)

Equation of normal to the circle $x^2 + y^2 = 25$ at point Q is

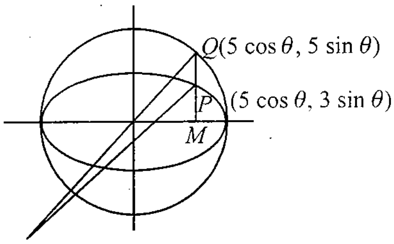


Fig. 4.61

$y = x\tan\theta$ (ii)

Eliminating θ from (i) and (ii), we get $x^2 + y^2 = 64$.

41. c. $\frac{x^2}{169} + \frac{y^2}{25} = 1$

Equation of normal at the point $(13\cos\theta, 5\sin\theta)$ is

$\frac{13x}{\cos\theta} - \frac{5y}{\sin\theta} = 144$, it passes through $(0, 6)$

$\Rightarrow \quad (15 + 72\sin\theta) = 0$

$$\Rightarrow \quad \sin\theta = -\frac{5}{24}$$

$$\Rightarrow \quad \theta = 2\pi - \sin^{-1}\left(\frac{5}{24}\right),$$

and $$\pi + \sin^{-1}\frac{5}{24}$$

Also y-axis is one of the normals.

42. c. Equation has three roots, hence three normal can be drawn.

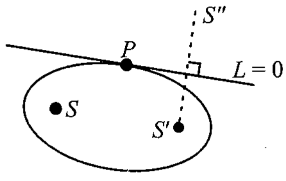


Fig. 4.62

Let image of S' be with respect to $x + y - 5 = 0$

$$\Rightarrow \quad \frac{h-2}{1} = \frac{k+1}{1} = \frac{-2(-4)}{2}$$

$$\Rightarrow \quad S'' = (6, 3)$$

Let P be the point of contact.

Because the line $L = 0$ is tangent to the ellipse, there exists a point P uniquely on the line such that $PS + PS' = 2a$.

Since $PS' = PS''$.

There exists one and only one point P on $L = 0$ such that $PS + PS'' = 2a$.

Hence, P should be the collinear with SS''.

Hence, P is a point of intersection of SS'' $(4x - 5y = 9)$, and $x + y - 5 = 0$, i.e. $P = \left(\frac{34}{9}, \frac{11}{9}\right)$.

43. c. A cyclic parallelogram will be a rectangle or square

So, $\angle QPR = 90°$

$\Rightarrow$ P lies on director circle of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $x^2 + y^2 = 25$ is director circle of $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad 16 + b^2 = 25$$

$$\Rightarrow \quad b^2 = 9$$

$$\Rightarrow \quad a^2(1 - e^2) = 9$$

$$\Rightarrow \quad 1 - e^2 = \frac{9}{16}$$

$$\Rightarrow \quad e^2 = \frac{7}{16}$$

$$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$$

44. d.

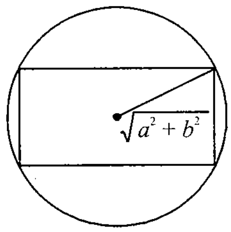


Fig. 4.63

Since mutually perpendicular tangents can be drawn from vertices of rectangle. So all the vertices of rectangle should lie on director circle $x^2 + y^2 = a^2 + b^2$.

Let breadth = $2l$ and length = $4l$, then

$$l^2 + (2l)^2 = a^2 + b^2$$

$$\Rightarrow \quad l^2 = \frac{a^2 + b^2}{5}$$

$$\Rightarrow \quad \text{Area} = 4l \times 2l = 8\,\frac{a^2 + b^2}{5}$$

45. b. Normal at $P(\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ (i)

Normal at $P\left(\frac{\pi}{2} + \theta\right)$ is $\frac{ax}{\cos\left(\frac{\pi}{2}+\theta\right)} - \frac{by}{\sin\left(\frac{\pi}{2}+\theta\right)} = a^2 - b^2$

or $\quad -\frac{ax}{\sin\theta} - \frac{by}{\cos\theta} = a^2 - b^2$ (ii)

Equations (i) and (ii) meet major axis at $G\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$ and $g\left(-\frac{(a^2-b^2)\sin\theta}{a}, 0\right)$

Now $PG^2 + Qg^2$

$$= \left(\frac{(a^2-b^2)\cos\theta}{a} - a\cos\theta\right)^2 + (0 - b\sin\theta)^2 + \left(-\frac{(a^2-b^2)\sin\theta}{a} - a\sin\theta\right)^2 + (0 - b\cos\theta)^2$$

$$= \frac{(a^2-b^2)^2}{a^2} + b^2 + a^2$$

$$= a^2\left(\frac{(a^2-b^2)^2}{a^4} + \frac{b^2}{a^2} + 1\right)$$

$$= a^2\left(\left(1 - \frac{b^2}{a^2}\right)^2 + \frac{b^2}{a^2} + 1\right)$$

$$= a^2(e^4 + 2 - e^2)$$

46. c. The equation of the normal to the given ellipse at the point $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$.

$$\Rightarrow \quad y = \left(\frac{a}{b}\tan\theta\right)x - \frac{(a^2-b^2)}{b}\sin\theta \quad \text{(i)}$$

Let $\quad \frac{a}{b}\tan\theta = m$, so that

$$\sin\theta = \frac{bm}{\sqrt{a^2 + b^2m^2}}$$

Hence, the equation of the normal Eq. (i) becomes

$$y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2 + b^2m^2}}$$

$\therefore \quad m \in R$, as $m = \frac{a}{b}\tan\theta \in R$

47. d.

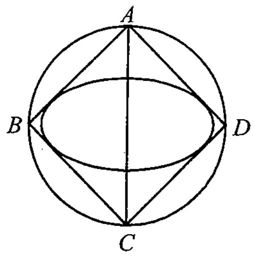


Fig. 4.64

Clearly, vertices of the square lie on the director circle of the ellipse $\frac{x^2}{7} + \frac{2y^2}{11} = 1$

which is $\quad x^2 + y^2 = 7 + \frac{11}{2}$ or $x^2 + y^2 = \frac{25}{2}$

Clearly, $\quad AC = 2\sqrt{\frac{25}{2}}$

Now $AB = BC = CD = AD$

and in ΔACD, $AC^2 = CD^2 + AD^2$

$$\Rightarrow \quad 2AD^2 = \left(2\sqrt{\frac{25}{2}}\right)^2$$

$$\Rightarrow \quad AD^2 = 25$$

$$\Rightarrow \quad AD = 5 \text{ units}$$

48. d. Equation of QR is $T = 0$ (chord of contact)

$$\frac{8x}{4} + \frac{27y}{9} = 1$$

$$\Rightarrow \quad 2x + 3y = 1 \quad \text{(i)}$$

Now, equation of the pair of lines passing through origin and points Q, R is given by

$\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = (2x + 3y)^2$ (Making equation of ellipse homogeneous using Eq. (i))

$$\Rightarrow \quad 9x^2 + 4y^2 = 36(4x^2 + 12xy + 9y^2)$$

$$\Rightarrow 135x^2 + 432xy + 320y^2 = 0$$

$\therefore$ Required angle is $\tan^{-1}\frac{2\sqrt{216^2 - 135 \times 320}}{455}$

$$= \tan^{-1}\frac{8\sqrt{2916 - 2700}}{455}$$

$$= \tan^{-1}\frac{8\sqrt{216}}{455}$$

$$= \tan^{-1}\frac{48\sqrt{6}}{455}$$

49. c. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Any point on the directrix is $P\left(\frac{a}{e}, k\right)$.

Chord of contact of P with respect to the ellipse is

$$\frac{a}{e}\frac{x}{a^2}+\frac{ky}{b^2}=1 \quad \text{(i)}$$

Chord of contact of P with respect to the auxiliary circle is

$$\frac{a}{e}x+ky=a^2 \quad \text{(ii)}$$

Equation (i) and (ii) intersect at $(ae, 0)$

50. a. Let (h, k) be the midpoint of the chord $7x+y-1=0$

$$\Rightarrow \quad \frac{hx}{1}+\frac{ky}{7}=\frac{h^2}{1}+\frac{k^2}{7} \quad \text{(i)}$$

and $$7x+y=1 \quad \text{(ii)}$$

represents same straight line

$$\Rightarrow \quad \frac{h}{7}=\frac{k}{7}\Rightarrow h=k$$

$\Rightarrow$ Equation of the line joining (0, 0) and (h, k) is $y-x=0$.

51. c. Given $m(n-1)=n$.

n is divisible by $n-1$

$$\Rightarrow \quad n=2\Rightarrow m=2$$

Hence, chord of contact of tangents drawn from (2, 2) to

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \text{ is } \frac{2x}{9}+\frac{2y}{4}=1$$

$$\Rightarrow \quad 4x+9y=18$$

52. b. The chord of contact of tangents from (h, k) is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$.

It meets the axes at points $\left(\frac{a^2}{h}, 0\right)$ and $\left(0, \frac{b^2}{k}\right)$.

Area of the triangle $=\frac{1}{2}\times\frac{a^2}{h}\times\frac{b^2}{k}=c$ (constant)

$$\Rightarrow \quad hk=\frac{a^2b^2}{2c} \text{ (} c \text{ is constant)}$$

$xy=c^2$ is the required locus.

53. d. Since x-axis and y-axis are perpendicular tangents to the ellipse. (0, 0) lies on the director circle and midpoint of foci (2, 2) is centre of the circle.

Hence, radius $=2\sqrt{2}$

$\Rightarrow$ the area is 8π units.

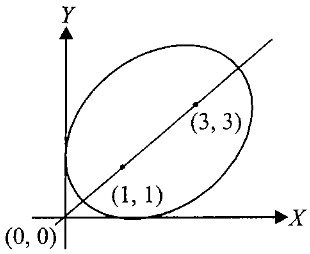


Fig. 4.65

54. a. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We know that the general equation of the tangent to the ellipse is

$$y=mx\pm\sqrt{a^2m^2+b^2} \quad \text{(i)}$$

since $3x-2y-20=0$ or $y=\frac{3}{2}x-10$ is tangent to the ellipse.

Comparing with Eq. (i), $m=\frac{3}{2}$ and $a^2m^2+b^2=100$

$$\Rightarrow \quad a^2\times\frac{9}{4}+b^2=100$$

$$\Rightarrow \quad 9a^2+4b^2=400 \quad \text{(ii)}$$

Similarly, since $x+6y-20=0$, i.e., $y=-\frac{1}{6}x+\frac{10}{3}$ is tangent to the ellipse, therefore comparing with Eq. (i),

$$m=\frac{1}{6} \text{ and } a^2m^2+b^2=\frac{100}{9}$$

$$\Rightarrow \quad \frac{a^2}{36}+b^2=\frac{100}{9}$$

$$\Rightarrow \quad a^2+36b^2=400 \quad \text{(iii)}$$

Solving Eqs. (ii) and (iii), we get $a^2=40$ and $b^2=10$

Therefore, the required equation of the ellipse is

$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

55. b. We know that product of length of perpendiculars from foci upon any tangents to ellipse is b^2.

Hence, from the diagram, x_1 and x_2 are length of perpendiculars from foci upon tangent y-axis of the given ellipse, hence $x_1x_2=b^2$.

Similarly $y_1y_2=b^2$

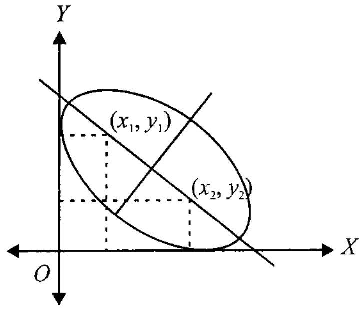


Fig. 4.66

56. c. $\sum_{i=1}^{10}(SP_i)(S'P'_i)=2560$

$$\Rightarrow \quad 10b^2=2560$$

$$\Rightarrow \quad b^2=256$$

$$\Rightarrow \quad b=16$$

$$\Rightarrow \quad 256=400(1-e^2)$$

$$\Rightarrow \quad 1 - e^2 = \frac{16}{25}$$

$$\Rightarrow \quad e = \frac{3}{5}$$

57. d. For the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, equation of director circle is $x^2 + y^2 = 25$. The director circle will cut the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ at 4 points

Hence, number of points = 4.

58. c. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of the parabola with focus $S(ae, 0)$ and directrix $x + ae = 0$ is $y^2 = 4aex$

Now length of latus rectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

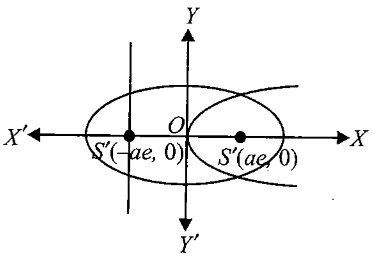


Fig. 4.67

For the two latus recta to be equal, we get

$$\frac{2b^2}{a} = 4ae$$

$$\Rightarrow \quad \frac{2a^2(1-e^2)}{a} = 4ae$$

$$\Rightarrow \quad 1 - e^2 = 2e$$

$$\Rightarrow \quad e^2 + 2e - 1 = 0$$

Therefore, $\quad e = -\frac{2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

Hence, $\quad e = \sqrt{2} - 1$

59. c. Here centre of the ellipse is (0, 0)

Let $P(r\cos\theta, r\sin\theta)$ be any point on the given ellipse then $r^2\cos^2\theta + 2r^2\sin^2\theta + 2r^2\sin\theta\cos\theta = 1$

$$\Rightarrow \quad r^2 = \frac{1}{\cos^2\theta + 2\sin^2\theta + \sin 2\theta}$$

$$= \frac{1}{\sin^2\theta + 1 + \sin 2\theta}$$

$$= \frac{2}{1 - \cos 2\theta + 2 + 2\sin 2\theta}$$

$$= \frac{2}{3 - \cos 2\theta + 2\sin 2\theta}$$

$$\Rightarrow \quad r_{\max} = \frac{\sqrt{2}}{\sqrt{3} - \sqrt{5}}$$

60. a. Combined equation of pair of lines through the origin joining the points of intersection of line $y = \sqrt{m}x + 1$ with the given curve is $x^2 + 2xy + (2 + \sin^2\alpha)y^2 - (y - \sqrt{m}x)^2 = 0$

for the chord to subtend a right angle at the origin $(1 - m) + (2 + \sin^2\alpha - 1) = 0$ (as sum of the coefficients of $x^2 + y^2 = 10$)

$$\Rightarrow \quad \sin^2\alpha = m - 2$$

$$\Rightarrow \quad 0 \le m - 2 \le 1$$

$$\Rightarrow \quad 2 \le m \le 3$$

Multiple Correct Answers Type

1. a., c. $r^2 - r - 6 > 0$ and $r^2 - 6r + 5 > 0$

$\Rightarrow (r-3)(r+2) > 0$ and $(r-1)(r-5) > 0$

$\Rightarrow (r < -2$ or $r > 3)$ and $(r < 1$ or $r > 5)$

$\Rightarrow r < -2$ or $r > 5$

Also $\quad r^2 - r - 6 \ne r^2 - 6r + 5$

$$\Rightarrow \quad r \ne \frac{11}{5}$$

2. a, c.

If both foci are fixed, then the ellipse is fixed, that is, both the directrices can be decided (eccentricity is given). Similar is the case for option (c). Thus (a) and (c) are the correct choices. In the remaining cases, size of the ellipse is fixed, but its position is not fixed.

3. a, b. Any point on this ellipse is $(\sqrt{6}\cos\phi, \sqrt{2}\sin\phi)$

Here centre is (0, 0), so $6\cos^2\phi + 2\sin^2\phi = 4$

$$\Rightarrow \quad 2\cos^2\phi = 1$$

$$\Rightarrow \quad \cos^2\phi = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\frac{\pi}{4}$$

$$\Rightarrow \quad \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

4. a, b, c. $3x^2 + 2y^2 + 6x - 8y + 5 = 0$

$$\Rightarrow \quad \frac{(x+1)^2}{2} + \frac{(y-2)^2}{3} = 1$$

Therefore, centre is (–1, 2) and ellipse is vertical ($\because b > a$)

$$a^2 = 2, b^2 = 3$$

Now $\quad 2 = 3(1 - e^2)$

$$\Rightarrow \quad e = \frac{1}{\sqrt{3}}$$

Foci are $(-1, 2 \pm be)$ and $(-1, 2 \pm 1) \equiv (-1, 3)$ and $(-1, 1)$ and directrix are $y = 2 \pm \frac{b}{e} \Rightarrow y = 5$ and $y = -1$

5. c, d. Ellipse is $16x^2 + 11y^2 = 256$

Equation of tangent at $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ is $16x(4\cos\theta) + 11y\left(\frac{16}{\sqrt{11}}\sin\theta\right) = 256$

It touches $(x-1)^2 + y^2 = 4^2$

if $\left|\dfrac{4\cos\theta - 16}{\sqrt{16\cos^2\theta + 11\sin^2\theta}}\right| = 4$

$\Rightarrow \quad (\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

$\Rightarrow 4\cos^2\theta + 8\cos\theta - 5 = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

6. a, c. Tangent and normal are bisectors of $\angle SPS'$

Now equation of SP is $y = 3x/2$ and that of $S'P$ is $y = 5x$

Then their equations are $\dfrac{3x - 2y}{\sqrt{13}} = \pm\dfrac{5x - y}{\sqrt{26}}$

or $\quad 3x - 2y = \pm\dfrac{5x - y}{\sqrt{2}}$

$\Rightarrow$ lines are $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$ and $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$

Now (2, 3) and (1, 5) lie on the same side of $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$, which is equation of tangent.

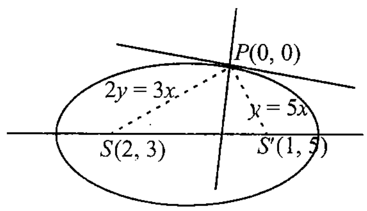


Fig. 4.68

Points (2, 3) and (1, 5) lie on the different sides of $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$, which is equation of normal.

7. c, d. The equation of the line joining θ and ϕ is

$$\frac{x}{5}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{3}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

If it passes through the point (4, 0), then

$$\frac{4}{5}\cos\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \quad \frac{4}{5} = \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}$$

$$\Rightarrow \quad \frac{4+5}{4-5} = \frac{\cos\left(\frac{\theta-\phi}{2}\right) + \cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right) - \cos\left(\frac{\theta+\phi}{2}\right)}$$

$$= \frac{2\cos\frac{\theta}{2}\cos\frac{\phi}{2}}{2\sin\frac{\phi}{2}\sin\frac{\theta}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{1}{9}$$

If it passes through the point (−5, 0),

then $\tan\frac{\phi}{2}\tan\frac{\theta}{2} = 9$

8. a, b, c, d.

$(\sqrt{3}x - 3\sqrt{3})^2 + (2y + 4)^2 = k$

so no locus for $k < 0$

ellipse for $k > 0$ and point for $k = 0$

9. a, c.

Let the point of equation of intersection of tangents A and B be $P(h, k)$, then equation of AB is

$$\frac{xh}{4} + \frac{yk}{1} = 1 \qquad \text{(i)}$$

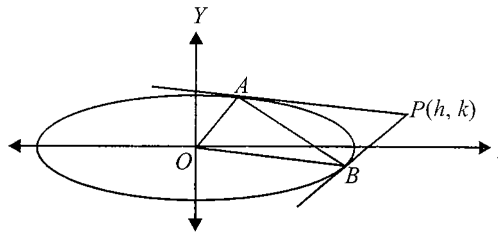


Fig. 4.69

Homogenizing the equation of ellipse using Eq. (i), we get

$$\frac{x^2}{4} + \frac{y^2}{1} = \left(\frac{xh}{4} + \frac{yk}{1}\right)^2$$

$$\Rightarrow x^2\left(\frac{h^2-4}{16}\right) + y^2(k^2 - 1) + \frac{2hk}{4}xy = 0 \qquad \text{(ii)}$$

Given equation of OA and OB is

$$x^2 + 4y^2 + \alpha\, xy = 0 \qquad \text{(iii)}$$

$\because$ Equation (ii) and (iii) represent same line,

Hence, $\quad \dfrac{h^2-4}{16} = \dfrac{k^2-1}{4} = \dfrac{hk}{2a}$

$\Rightarrow \quad h^2 - 4 = 4(k^2 - 1)$

$\Rightarrow \quad h^2 - 4k^2 = 0$

$\Rightarrow$ Locus $(x - 2y)(x + 2y) = 0$

10. a, b, c.

a. The given equation is $\left(x - \frac{1}{13}\right)^2 + \left(y - \frac{2}{13}\right)^2$

$$= \frac{1}{a^2}\left(\frac{5x + 12y - 1}{13}\right)^2$$

It represents ellipse if $\frac{1}{a^2} < 1 \Rightarrow a^2 > 1 \Rightarrow a > 1$

b. $4x^2 + 8x + 9y^2 - 36y = -4$

$\Rightarrow 4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = 36$

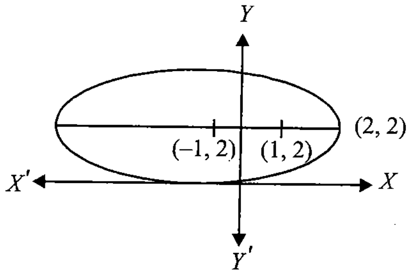


Fig. 4.70

$$\Rightarrow \quad \frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

Hence, (– 1, 2) is focus and (1, 2) lies on the major axis. Then required minimum distance is 1.

c. Equation of normal at $P(\theta)$ is $5 \sec\theta \, x - 4 \operatorname{cosec}\theta \, y = 25 - 16$, and it passes through $P(0, \alpha)$

$$\therefore \quad \alpha = \frac{-9}{4 \operatorname{cosec}\theta}$$

$$\Rightarrow \quad \alpha = \frac{-9}{4}\sin\theta$$

$$\Rightarrow \quad |\alpha| < \frac{9}{4}$$

d. $\frac{2b^2}{a} = \frac{2a}{3} \Rightarrow 3b^2 = a^2$

$\Rightarrow$ From $b^2 = a^2(1 - e^2)$, $1 = 3(1 - e^2) \Rightarrow e = \sqrt{2/3}$

11. a, c, d.

$x^2 + 4y^2 - 2x - 16y + 13 = 0$

$\Rightarrow (x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 4$

$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$$

$\therefore$ Length of latus rectum $= \frac{2 \times 1}{2} = 1$

Also $\quad e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

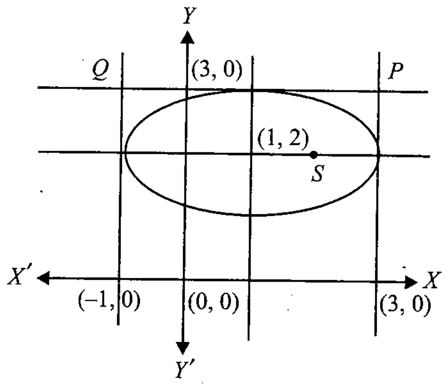


Fig. 4.71

$$\Rightarrow \quad 2ae = 2 \times 2 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Sum of the focal distance $= 2a = 4$

Tangents at the vertices are $x - 1 = \pm 2$

or $\quad x = 3, -1$

Therefore, the line $y = 3$ intersect these at points $P(3, 3)$ and $Q(-1, 3)$.

Coordinate of focus are $S(\sqrt{3} + 1, 2)$

Slope of PS is $\frac{1}{2 - \sqrt{3}}$, slope of QS is $\frac{1}{-2 - \sqrt{3}}$

$$\Rightarrow \quad \text{product of slopes} = \frac{1}{2 - \sqrt{3}} \times \frac{1}{-2 - \sqrt{3}} = -1$$

12. b, d.

Differentiating the equation of ellipse $x^2 + 3y^2 = 37$ w.r.t. x,

$$\frac{dy}{dx} = -\frac{x}{3y}$$

Slope of the given line is $\frac{6}{5}$, which is normal to the ellipse.

Hence, $\frac{3y}{5} = \frac{6}{5}$ or $2x = 5y$.

Points in the option (b) and (d) are satisfying the above relation.

13. a, c.

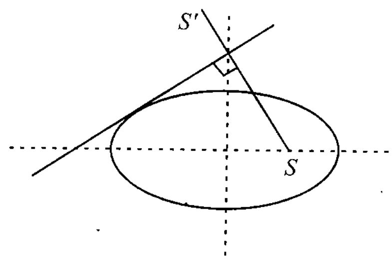


Fig. 4.72

Let $S'(h, k)$ be the image.

SS' cuts a tangent at a point which lies on the auxiliary circle of the ellipse

$$\Rightarrow \quad \left(\frac{h \pm 4}{2}\right)^2 + \frac{k^2}{4} = 25$$

$\Rightarrow$ locus is $(x \pm 4)^2 + y^2 = 100$

14. a, c.

Let $P(h, k)$ be the point of intersection of E_1 and E_2

$$\Rightarrow \quad \frac{h^2}{a^2} + k^2 = 1$$

$$\Rightarrow \quad h^2 = a^2(1 - k^2) \qquad \text{(i)}$$

and $$\frac{h^2}{1} + \frac{k^2}{a^2} = 1$$

$$\Rightarrow \quad k^2 = a^2(1 - h^2) \qquad \text{(ii)}$$

Eliminating a from Eqs. (i) and (ii), we get

$$\frac{h^2}{1 - k^2} = \frac{k^2}{1 - h^2}$$

$$\Rightarrow \quad h^2(1 - h^2) = k^2(1 - k^2)$$

$$\Rightarrow (h - k)(h + k)(h^2 + k^2 - 1) = 0$$

Hence, the locus is a set of curves consisting of the straight lines

$y = x, y = -x$ and circle $x^2 + y^2 = 1$.

15. a, c.

$f(x)$ is a decreasing function and for major axis to be x-axis

$$f(k^2 + 2k + 5) > f(k + 11)$$

$$\Rightarrow \quad k^2 + 2k + 5 < k + 11$$

$$\Rightarrow \quad k \in (-3, 2)$$

Then for the remaining values of k, i.e., $k \in (-\infty, -3) \cup (2, \infty)$, major axis is y-axis.

16. a, b, c.

Clearly O is the midpoint of SS' and HH'

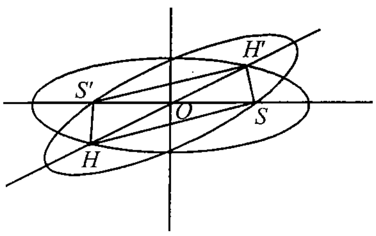


Fig. 4.73

$\Rightarrow$ Diagonals of quadrilateral $HSH'S'$ bisect each other, so it is a parallelogram.

Let $H'OH = 2r \Rightarrow OH = r = ae'$

H lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (suppose)

$$\therefore \quad \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$e'^2\cos^2\theta + \frac{e'^2\sin^2\theta}{1 - e^2} = 1 \qquad [\because b^2 = a^2(1 - e'^2)]$$

$$\Rightarrow e'^2\cos^2\theta - \frac{e'^2\cos^2\theta}{1 - e^2} = 1 - \frac{e'^2}{1 - e^2}$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2e'^2}$$

$$\Rightarrow \quad \theta = \cos^{-1}\sqrt{\frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2e'^2}}$$

For $$\theta = 90°, \frac{e^2 + e'^2}{e^2e'^2} = \frac{1}{e^2e'^2}$$

$$\Rightarrow \quad e^2 + e'^2 = 1$$

17. a, d. The equation of the tangent at $(t^2, 2t)$ to the parabola

$y^2 = 4x$ is

$$2ty = 2(x + t^2)$$

$$\Rightarrow \quad ty = x + t^2$$

$$\Rightarrow \quad x - ty + t^2 = 0 \qquad \text{(i)}$$

The equation of the normal at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $5x^2 + 5y^2 = 20$ is

$\Rightarrow (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y = 5 - 4$

$\Rightarrow (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y = 1 \qquad$ (ii)

Given that Eqs. (i) and (ii) represent the same line.

$$\Rightarrow \quad \frac{\sqrt{5}\sec\theta}{1} = \frac{-2\,\text{coses}\,\theta}{-t} = \frac{-1}{t^2}$$

$$\Rightarrow \quad t = \frac{2}{\sqrt{5}}\cot\theta \text{ and } t = -\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad \frac{2}{\sqrt{5}}\cot\theta = -\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad 4\cos\theta = -\sqrt{5}\sin^2\theta$$

$$\Rightarrow \quad 4\cos\theta = -\sqrt{5}(1 - \cos^2\theta)$$

$$\Rightarrow \sqrt{5}\cos^2\theta - 4\cos\theta - \sqrt{5} = 0$$

$$\Rightarrow (\cos\theta - \sqrt{5})(\sqrt{5}\cos\theta + 1) = 0$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{\sqrt{5}} \qquad [\because \cos\theta \neq -\sqrt{5}]$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$$

Putting $\cos\theta = -\frac{1}{\sqrt{5}}$ in $t = -\frac{1}{2}\sin\theta$, we get

$$t = -\frac{1}{2}\sqrt{1 - \frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

Hence, $$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \text{ and } t = -\frac{1}{\sqrt{5}}.$$

Reasoning Type

1. **d.** Statement 1 is false as locus of (x, y) is a line segment joining points $(2, 0)$ and $(-2, 0)$.
2. **c.** Statement 2 is false (locus of P may be a line segment also). statement 1 is true.
3. **a.** It is fundamental property of an ellipse.
4. **c.** $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$

   Ends of the major axis are $(0, 6)$ and $(0, 0)$.

   Equation of tangent at $(0, 6)$ and $(0, 0)$ is $y = 6$ and $y = 0$.

   Hence, statement 1 is true.

   But statement 2 is false, as tangents at the ends of major axis may be lines parallel to y-axis when $a < b$.
5. **a.** Locus of point of intersection of perpendicular tangents is director circle, which is $x^2 + y^2 = a^2 + b^2$.

   Now line $px + qy + r = 0$ may intersect this circle maximum at two points.

   Thus there can be maximum two points on the line from which perpendicular tangents can be drawn to the ellipse.
6. **a.** Statement 2 is true as it is one of the properties of ellipse.

   Ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$

   Focus $\equiv (\sqrt{5}, 0)$, $e = \frac{\sqrt{5}}{3}$

   One of the points on the ellipse $\equiv \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

   Equation of the circle as the diameter joining the points $(3/\sqrt{2}, 2/\sqrt{2})$ and focus $(\sqrt{5}, 0)$ is

   $(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$

   Hence, statement 1 is true and statement 2 is correct explanation of statement 1.
7. **d.** Let C_1, C_2 the centres and r_1, r_2 be the radii of the two circles. Let $S_1 = 0$ lies completely inside the circle. $S_2 = 0$. Let C and r be centre and radius of the variable circle, respectively.

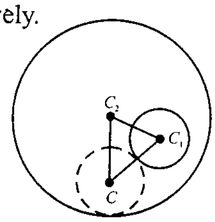


Fig. 4.74

then $CC_2 = r_2 - r$ and $C_1C = r_1 + r$

$\Rightarrow C_1C + C_2C = r_1 + r_2$ (constant)

$\Rightarrow$ Locus of C is an ellipse

$\Rightarrow S_2$ is true

Statement 1 is false (two circles are intersecting).

8. **b.** By formula $p_1p_2 = b^2 = 3$.

   Also foot of perpendicular lies on auxiliary circle of the ellipse.

   Thus both the statements are true. But statement 2 is not correct explanation of statement 1.
9. **d.** Locus of point of intersection of perpendicular tangents is director circle.

   If there exists exactly one such point on the line $3x + 4y + 5\sqrt{5} = 0$, then it must touch the director circle

   $$x^2 + y^2 = a^2 + 1$$
   $$\Rightarrow \quad 5 = a^2 + 1$$
   $$\Rightarrow \quad a^2 = 4$$
   $$\Rightarrow \quad a = 2$$

   Hence, eccentricity $= \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$.
10. **a.** Let $y = mx$ be any chord through $(0, 0)$. This will meet conic at points whose x-coordinates are given by $x^2 + m^2x^2 + mx^2 = 1$

    $$\Rightarrow \quad (1 + m + m^2)x^2 - 1 = 0$$
    $$\Rightarrow \quad x_1 + x_2 = 0$$
    $$\Rightarrow \quad \frac{x_1 + x_2}{2} = 0$$
    Also $\quad y_1 = mx_1, y_2 = mx_2$
    $$\Rightarrow \quad y_1 + y_2 = m(x_1 + x_2) = 0$$
    $$\Rightarrow \quad \frac{y_1 + y_2}{2} = 0$$

    $\Rightarrow$ Midpoint of chord is $(0, 0)$ $\forall m$.

    Hence, statement 1 is true as $(0, 0)$ is also centre of the ellipse.

    Statement 2 is fundamental property of the ellipse, hence statement 2 is correct explanation of statement 1.
11. **b.** Given ellipse is $\frac{x^2}{3} + \frac{y^2}{2} = 1$, whose area is $= \pi\sqrt{3}\sqrt{2} = \pi\sqrt{6}$.

    Circle is $x^2 + y^2 - 2x + 4y + 4 = 0$ or $(x-1)^2 + (y-2)^2 = 1$.

    Its area is π. Hence, statement 1 is true.

    Also statement 2 is true but it is not the correct explanation of statement 1.

    Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{1} = 1$, whose area is 5π and circle $x^2 + y^2 = 16$ whose area is 16π.

    Also here semi-major axis of ellipse (= 5) is more than the radius of the circle (= 4).
12. **d.** Statement 2 is true, see theory.

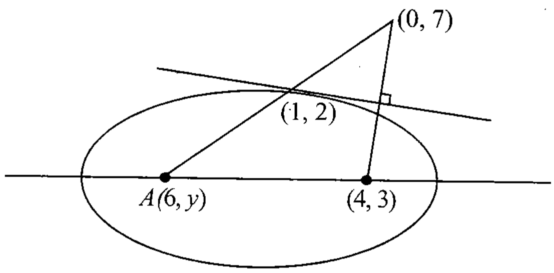


Fig. 4.75

Image of (4, 3) in the line $x + y - 3 = 0$ is

$$\frac{x-4}{1} = \frac{y-3}{1} = -2\frac{(4+3-3)}{2} = -4$$

$\Rightarrow$ $x = 0$ and $y = 7$

Now points (0, 7), (1, 2) and (6, y) are collinear.

$\Rightarrow$ $\frac{7-2}{0-1} = \frac{y-2}{6-1}$

$\Rightarrow$ $y = 27$

13. a. Statement 2 is correct as ellipse is a central conic and it also explains statement 1.

14. d. Statement 1: Locus of point of intersection of only perpendicular lines is a circle, and other vertices B and C do not form a circle

Statement 2 is obvious (standard definition).

15. a. Chord of contact of the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ with respect to point (8, 6) is

$$\frac{8x}{4} + \frac{6x}{2} = 1$$

or $2x + 3y = 1.$

Hence, statement 1 is correct, also statement 2 is correct and explains the statement 1.

16. a. Statement 2 is true as it is one of the properties of the ellipse.

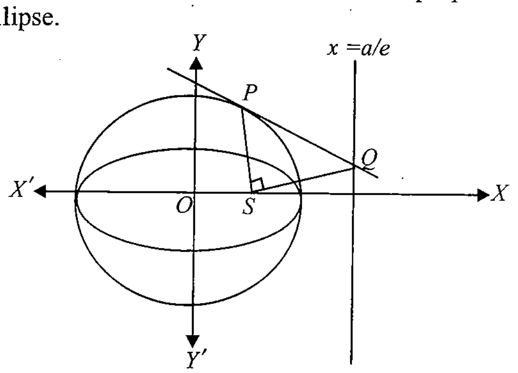


Fig. 4.76

Circle with minimum radius having PQ as chord when PQ is diameter of the circle, hence as shown in the figure it passes through the focus.

17. d. $|ay - bx| = c\sqrt{(x-a)^2 + (y-b)^2}$

$$\Rightarrow \quad \frac{|ay - bx|}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}\sqrt{(x-a)^2 + (y-b)^2}$$

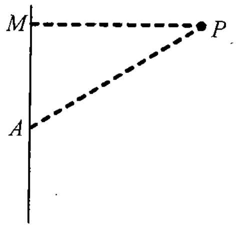


Fig. 4.77

or $PM = kPA$, where m is the length of perpendicular from P on the line $ay - bx = 0$ and PA is the length of line segment joining P to the point A (a, b) and A lies on the line, so the locus of P is a straight line through A inclined at an angle $\sin^{-1}\frac{c}{\sqrt{a^2+b^2}}$ to the given line (provided $c < \sqrt{a^2+b^2}$).

Linked Comprehension Type

For Problems 1–3

1. c., 2. a., 3. b.

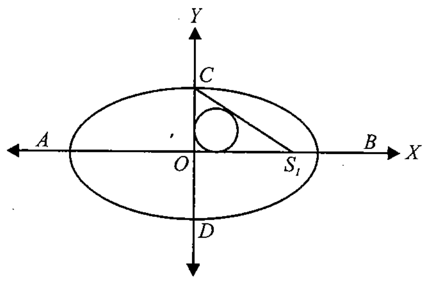


Fig. 4.78

1. a. $\because OS_1 = ae = 6, OC = b$

Sol.

Also $CS_1 = a$

$\Rightarrow$ Area of $\Delta OCS_1 = \frac{1}{2}(OS_1) \times (OC) = 3b$

$\Rightarrow$ Semi-perimeter of $\Delta OCS_1 = \frac{1}{2}(OS_1 + OC + CS_1)$

$$= \frac{1}{2}(6 + a + b) \quad \text{(i)}$$

$\Rightarrow$ In radius of $\Delta OCS_1 = 1$

$\Rightarrow$ $\frac{3b}{\frac{1}{2}(6+a+b)} = 1$

$\Rightarrow$ $5b = 6 + a$ (ii)

also $b^2 = a^2 - a^2e^2$

$= a^2 - 36$ (iii)

$\Rightarrow$ From (ii), we get

$25(a^2 - 36) = 36 + a^2 + 12a$

$\Rightarrow$ $2a^2 - a - 78 = 0$

$\Rightarrow$ $a = \frac{13}{2}, -6$

$\Rightarrow$ $a = \frac{13}{2}$

and $b = \frac{5}{2}$

Area of ellipse $= \pi ab = \frac{65\pi}{4}$ sq. unit

Perimeter of $\Delta\, OCS_1 = 6 + a + b = 6 + \frac{13}{2} + \frac{5}{2} = 15$ units

Equation director circle is $x^2 + y^2 = a^2 + b^2$

or $$x^2 + y^2 = \frac{97}{2} = r^2$$

For Problems 4–6

4. c., 5. b., 6. a.

Sol.

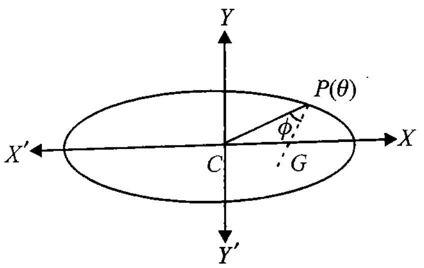


Fig. 4.79

Any point P on the ellipse at $(a\cos\theta, b\sin\theta)$

$\therefore$ Equation of CP is $y = \left(\frac{b}{a}\tan\theta\right)x$

The normal to the ellipse at P is $\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2$

Slope of the lines CP and the normal GP are $\frac{b}{a}\tan\theta$ and $\frac{a}{b}\tan\theta$, respectively

$$\therefore \quad \tan\phi = \frac{\frac{a}{b}\tan\theta - \frac{b}{a}\tan\theta}{1 + \frac{a}{b}\tan\theta\,\frac{b}{a}\tan 0}$$

$$= \frac{a^2 - b^2}{ab}\,\frac{\tan\theta}{\sec^2\theta}$$

$$= \frac{a^2 - b^2}{ab}\sin\theta\cos\theta = \frac{a^2 - b^2}{2ab}\sin 2\theta$$

Therefore, the greatest value of $\tan\phi = \frac{a^2 - b^2}{2ab} \times 1 = \frac{a^2 - b^2}{2ab}$

Given that $\frac{a^2 - b^2}{2ab} = \frac{3}{2}$. Let $\frac{a}{b} = t$

$$\Rightarrow \quad t - \frac{1}{t} = \frac{3}{2}$$

$$\Rightarrow \quad 2t^2 - 3t - 2 = 0$$

$$\Rightarrow \quad 2t^2 - 4t + t - 2 = 0$$

$$\Rightarrow \quad (2t + 1)(t - 2) \Rightarrow \frac{a}{b} = 2$$

$$\Rightarrow \quad e^2 = 1 - \frac{1}{4}$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

Rectangle inscribed in the ellipse whose one vertex is $(a\cos\theta, b\sin\theta)$ is $(2a\cos\theta)(2b\sin\theta) = 2ab\sin(2\theta)$ which has maximum value $2ab$. Given that $a = 10$, then $b = 5 \Rightarrow$ maximum area is 100.

Locus of intersection point of perpendicular tangents is $x^2 + y^2 = 10^2 + 5^2$ or $x^2 + y^2 = 125$ (director circle).

For Problems 7–9

7. b., 8. a., 9. c.

$$21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

$$3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$$

$$\Rightarrow \frac{(x - 3y + 3)^2}{60} + \frac{(3x + y - 1)^2}{90} = 1$$

$$\Rightarrow \left(\frac{x - 3y + 3}{\sqrt{1 + 3^2}\sqrt{6}}\right)^2 + \left(\frac{3x + y - 1}{\sqrt{1 + 3^2}3}\right)^2 = 1$$

Thus C is an ellipse whose lengths of axes are 6, $2\sqrt{6}$.

The minor and the major axes are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.

Their point of intersection gives the centre of the conic.

$\therefore$ Centre $\equiv (0, 1)$

For Problems 10–12

10. b., 11. d., 12. a.

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The line $y = mx \pm \sqrt{a^2m^2 + b^2}$ touches the ellipse for all m.

Hence, it is identical with
$$y = -\frac{2px}{\sqrt{1 - p^2}} + \frac{1}{\sqrt{1 - p^2}}$$

Hence, $$m = -\frac{2p}{\sqrt{1 - p^2}}$$

and $$a^2m^2 + b^2 = \frac{1}{1 - p^2}$$

$$\Rightarrow \quad a^2\frac{4p^2}{1 - p^2} + b^2 = \frac{1}{1 - p^2}$$

$$\Rightarrow \quad p^2(4a^2 - b^2) + b^2 - 1 = 0$$

This equation is true for all real p if $b^2 = 1$ and $4a^2 = b^2$

$$\Rightarrow \quad b^2 = 1 \text{ and } a^2 = \frac{1}{4}$$

Therefore, the equation of the ellipse is
$$\frac{x^2}{1/4} + \frac{y^2}{1} = 1$$

If e is its eccentricity, then
$$\frac{1}{4} = 1 - e^2 \Rightarrow e^2 = \Rightarrow e = \frac{\sqrt{3}}{2}$$

$be = \frac{\sqrt{3}}{2}$, hence foci are $\left(0, \pm\frac{\sqrt{3}}{2}\right)$

Equation of director circle is $x^2 + y^2 = \frac{5}{4}$.

For Problems 13–15

13. a., 14. b., 15. a.

Sol.

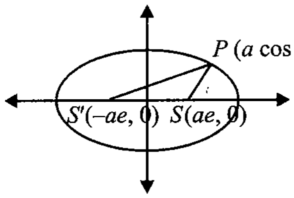


Fig. 4.80

Let the coordinates of P be $(a\cos\theta, b\sin\theta)$.

Here SP = focal distance of the point $P = a - ae\cos\theta$

$$S'P = a + ae\cos\theta$$

$$SS' = 2ae$$

If (h, k) be the coordinates of the incentre of $\Delta PSS'$, then

$$h = \frac{2ae(a\cos\theta) + a(1-e\cos\theta)(-ae) + a(1+e\cos\theta)ae}{2ae + a(1-e\cos\theta) + a(1+e\cos\theta)}$$

$$\Rightarrow \quad h = ae\cos\theta \quad \text{(i)}$$

and

$$k = \frac{2ae(b\sin\theta) + a(1-e\cos\theta)\times 0 + a(1+e\cos\theta)\times 0}{2ae + a(1-e\cos\theta) + a(1+e\cos\theta)}$$

$$\Rightarrow \quad k = \frac{eb\sin\theta}{(e+1)} \quad \text{(ii)}$$

Eliminating θ from (i) and (ii), we get

$$\frac{x^2}{a^2e^2} + \frac{y^2}{\left(\frac{be}{e+1}\right)^2} = 1$$

which clearly represents an ellipse. Let e_1 be its eccentricity. Then

$$\frac{b^2e^2}{(e+1)^2} = a^2e^2(1-e_1^2)$$

$$\Rightarrow \quad e_1^2 = 1 - \frac{b^2}{a^2(e+1)^2}$$

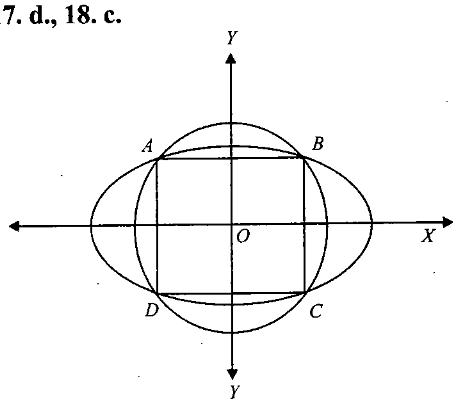
$$\Rightarrow \quad e_1^2 = 1 - \frac{1-e^2}{(e+1)^2} = 1 - \frac{1-e}{1+e}$$

$$\Rightarrow \quad e_1^2 = \frac{2e}{e+1} \Rightarrow e_1 = \sqrt{\frac{2e}{e+1}}$$

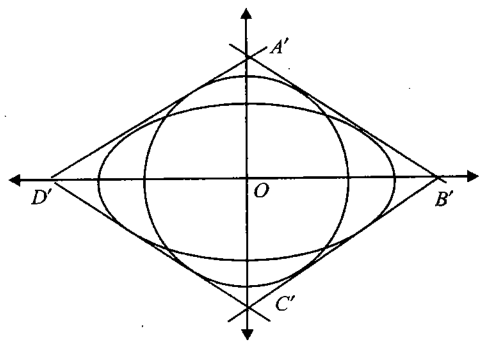
Maximum area of rectangle is $2(ae)\left(\frac{be}{e+1}\right) = \frac{2abe^2}{e+1}$

For Problems 16–18

16. a., 17. d., 18. c.



(a)



(b)

Fig. 4.81

16. a. Solving the curves given (eliminating x^2), we have

$$\frac{r^2 - y^2}{16} + \frac{y^2}{9} = 1$$

$$\Rightarrow \quad y^2 = \frac{144 - 9r^2}{7}$$

Solving the curves given (eliminating y^2), we have

$$\frac{x^2}{16} + \frac{r^2 - x^2}{9} = 1$$

$$\Rightarrow \quad x^2 = \frac{16r^2 - 144}{7}$$

If $ABCD$ is a square, then

$$x^2 = y^2$$

or $$\frac{144 - 9r^2}{7} = \frac{16r^2 - 144}{7}$$

or $$25r^2 = 288$$

or $$r = \frac{12}{5}\sqrt{2}$$

17. d. Tangent of slope m to circle and ellipse is

$$y = mx \pm \sqrt{r^2m^2 + r^2}$$

and $$y = mx \pm \sqrt{16m^2 + 9}, \text{ respectively.}$$

For common tangent

$$r^2m^2 + r^2 = 16m^2 + 9$$

Also if A', B', C', D' is a square, then

$$m = \pm 1$$

$$\Rightarrow \quad r^2 + r^2 = 25$$

$$\Rightarrow \quad r = 5/\sqrt{2}$$

18. c. If $A'B'C'D'$ is a square, then tangents

$$y = \pm x \pm 5$$

for which diagonal length $A'C' = 10$.

Then area of $\Delta A'B'C'$ is 25π.

Also area of circle C_1 is $\frac{25\pi}{2}$. Hence, the required ratio is $\frac{1}{2}$.

For Problems 19–21

19. c., 20. b., 21. d.

19. c. $\lambda x - y + 2(1 + \lambda) = 0$

$$\Rightarrow \quad \lambda(x + 2) - (y - 2) = 0$$

This line passes through (–2, 2).

$$\mu x - y + 2(1 - \mu) = 0$$

$$\Rightarrow \quad \mu(x - 2) - (y - 2) = 0$$

This line passes through (2, 2).

Clearly these represent the foci of the ellipse. So $2ae = 4$.

The circle $x^2 + y^2 - 4y - 5 = 0 \Rightarrow x^2 + (y - 2)^2 = 9$ represents auxiliary circle. Thus $a^2 = 9 \Rightarrow e = \frac{2}{3}$ and $b^2 = 5$.

20. c. Required area $= \frac{1}{2}(2ae)(b) = 3\frac{2}{3}\sqrt{5} = 2\sqrt{5}$

For Problems 22–24

22. b., 23. d., 24. c.

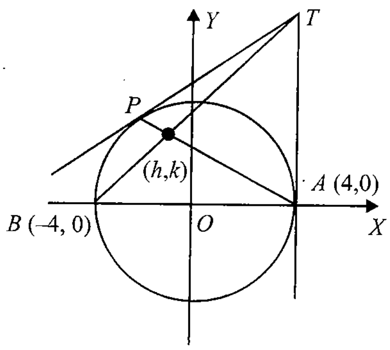


Fig. 4.82

Tangents at $P(4\cos\theta, 4\sin\theta)$ to $x^2 + y^2 = 16$ is

$$x\cos\theta + y\sin\theta = 4 \quad \text{(i)}$$

Equation of AP is

$$y = \frac{\sin\theta}{\cos\theta - 1}(x - 4) \quad \text{(ii)}$$

From (i), coordinates of the point T are given by

$$\left(4, \frac{4(1 - \cos\theta)}{\sin\theta}\right)$$

Equation of BT is

$$y = \frac{1 - \cos\theta}{2\sin\theta}(x + 4) \quad \text{(iii)}$$

Let (h, k) be the point of intersection of the lines (ii) and (iii). Then we have

$$k^2 = -\frac{1}{2}(h^2 - 16)$$

$$\Rightarrow \quad \frac{h^2}{16} + \frac{k^2}{8} = 1$$

Therefore, locus of (h, k) is

$$\frac{x^2}{16} + \frac{y^2}{8} = 1$$

which is an ellipse with eccentrically $e = \frac{1}{\sqrt{2}}$.

Sum of focal distances of any point $= 2a = 8$.

Considering circle $x^2 + y^2 = a^2$, we find that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$ which is constant and does not change by changing the radius of the circle.

For Problems 25–27

25. a, 26. c., 27. b

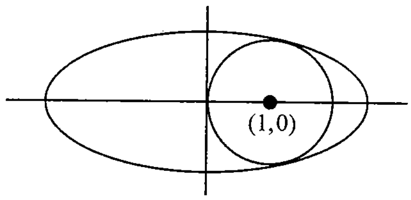


Fig. 4.83

Solving both equations, we have

$$\frac{x^2}{a^2} + \frac{1 - (x - 1)^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 + a^2[1 - (x - 1)^2] = a^2b^2$$

$$\Rightarrow \quad (b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0 \quad \text{(i)}$$

for least area circle must touch the ellipse

$\Rightarrow$ Discriminant of (1) is zero

$$\Rightarrow \quad 4a^4 + 4a^2b^2(b^2 - a^2) = 0$$

$$\Rightarrow \quad a^2 + b^2(b^2 - a^2) = 0$$

$$\Rightarrow \quad a^2 + b^2(-a^2e^2) = 0$$

$$\Rightarrow \quad 1 - b^2e^2 = 0 \Rightarrow b = \frac{1}{e}$$

Also $$a^2 = \frac{b^2}{1 - e^2} = \frac{1}{e^2(1 - e^2)}$$

$$\Rightarrow \quad a = \frac{1}{e\sqrt{1 - e^2}}$$

Let S be the area of the ellipse.

$$\Rightarrow \quad S = \pi ab = \frac{\pi}{e^2\sqrt{1 - e^2}}$$

$$= \frac{\pi}{\sqrt{e^4 - e^6}}$$

Area is minimum if $f(e) = e^4 - e^6$ is maximum

when $f'(e) = 4e^3 - 6e^5 = 0$

or $e = \sqrt{\frac{2}{3}}$ (which is point of maxima for $f(e)$)

$\Rightarrow$ S is least when $e = \sqrt{\frac{2}{3}}$

$\Rightarrow$ Ellipse is $2x^2 + 6y^2 = 9$

Equation of auxiliary circle of ellipse is $x^2 + y^2 = 4.5$

Length of latus rectum of ellipse is $\frac{2b^2}{a} = \frac{2\frac{9}{4}}{\frac{9}{2}} = 1$

Matrix-Match Type

1. a. → p, q, r, s; b → p, q, r, s; c → q, r, s; d → p.

a. Equation of tangent at $\left(\frac{\cos\theta}{2}, \frac{\sin\theta}{3}\right)$ is

$2x\cos\theta + 3y\sin\theta = 1$

which is parallel to the given line $8x = 9y$

$\therefore\ \cos\theta = \pm\frac{4}{5}, \sin\theta = \mp\frac{3}{5}$

Hence, points are $\left(\frac{2}{5}, -\frac{1}{5}\right)$ and $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

Distance between the points is

$$\sqrt{\frac{16}{25} + \frac{4}{25}} = \frac{2}{\sqrt{5}}$$

which is less than 1.

b. The given equation is

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

$\Rightarrow$ $e^2 = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

Hence, the foci are $S, S' \equiv (-1, -2 \pm 4) \equiv S(-1, 2)$ and $S'(-1, -6)$

The required sum of distances = 2 + 6 = 8.

c. Equation of normal at $(3\cos\theta, 2\sin\theta)$ is

$3x\sec\theta - 2y\,\text{cosec}\,\theta = 5$

which is parallel to the given line $2x + y = 1$. Therefore,

$\cos\theta = \mp\frac{3}{5}, \sin\theta = \pm\frac{4}{5}$

Hence, points are $\left(\frac{-9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

The required sum of distances $= \frac{16}{5}$.

d. Consider any point $(t, t+2)$, $t \in R$, on the line $x - y + 2 = 0$.

The chord of contact of ellipse w.r.t. this point is

$xt + 2y(t+2) = 2$

$\Rightarrow (4y - 2) + t(x + 2y) = 0$

This line passes through point of intersection of lines

$4y - 2 = 0$ and $x + 2y = 0$

$\therefore$ $x = -1$

Hence, the point is (–1, 1/2), whose distance from (2, 1/2) is 3.

2. a → r, s; b → p, q, r, s; c → r, s; d → r, s.

Equation of any tangent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(i)}$$

is given by

$$y = mx + \sqrt{a^2m^2 + b^2}$$

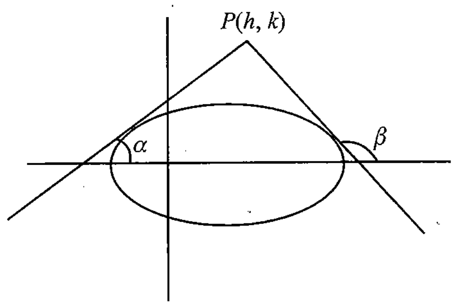


Fig. 4.84

Since it passes through $P(h, k)$

$$k = mh + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow m^2(h^2 - a^2) - 2kmh + (k^2 - b^2) = 0 \quad \text{(ii)}$$

As (ii) is quadratic in m, having two roots m_1 and m_2 (say), Therefore,

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2},\ m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2} \quad \text{(iii)}$$

$\Rightarrow$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{m_1 + m_2}{1 - m_1m_2}$$

$$= \frac{\frac{2hk}{h^2 - a^2}}{1 - \frac{k^2 - b^2}{h^2 - a^2}}$$

$$= \frac{2hk}{h^2 - k^2}$$

a. $\alpha + \beta = \frac{c\pi}{2}$

When c is even,

$$m_1 + m_2 = 0$$

$$\frac{2kh}{h^2 - a^2} = 0 \Rightarrow 2kh = 0$$

$\Rightarrow xy = 0$, which is the equation of a pair of straight lines.

When c is odd,

$$1 - m_1m_2 = 0$$

$$\Rightarrow \quad \frac{k^2 - b^2}{h^2 - b^2} = 1$$

Therefore, the locus of (h, k) is

$$y^2 - b^2 = x^2 - a^2$$

which is a hyperbola.

b. $m_1m_2 = c$

$$\Rightarrow \quad \frac{k^2 - b^2}{h^2 - a^2} = c$$

When $c = 0$, $k = \pm b$, the locus is pair of straight lines.

When $c = 1$, $h^2 - k^2 = a^2 - b^2$, the locus is hyperbola.

When $c = -1$, $h^2 + k^2 = a^2 + b^2$, the locus is circle.

When $c = -2$, $2h^2 + k^2 = 2a^2 + b^2$, the locus is ellipse.

c. $\tan\alpha + \tan\beta = c$

$$\Rightarrow \quad m_1 + m_2 = c$$

$$\Rightarrow \quad \frac{2kh}{h^2 - a^2} = c$$

When $c = 0$, $kh = 0$, the locus is pair of straight lines.

When $c \neq 0$

$$c(h^2 - a^2) - 2kh = 0$$

Locus of (h, k) is

$$cx^2 - 2xy - ca^2 = 0$$

$$\Delta = -ca^2 \neq 0$$

Also,

$$h^2 - ab = 1 > 0$$

Therefore, the locus is a hyperbola for $c \neq 0$.

d. $\cot\alpha + \cot\beta = c$

$$\Rightarrow \quad \frac{1}{m_1} + \frac{1}{m_2} = c$$

$$\Rightarrow \quad \frac{m_1 + m_2}{m_1m_2} = c$$

$$\Rightarrow \quad \frac{2kh}{k^2 - b^2} = c$$

$$\Rightarrow \quad c(k^2 - b^2) - 2kh = 0$$

When $c = 0$, locus is a pair of straight lines.

When $c \neq 0$, locus is a hyperbola (as in previous case c).

3. a. → p, r; b → r, s; c → p; d → q.

Tangent to ellipse at $P(\phi)$ is $\frac{x}{4}\cos\phi + \frac{y}{2}\sin\phi = 1$.

It must pass through the centre of the circle. Hence,

$$\frac{4}{4}\cos\phi + \frac{2}{2}\sin\phi = 1$$

$$\Rightarrow \quad \cos\phi + \sin\phi = 1$$

$$\Rightarrow \quad 1 + \sin 2\phi = 1$$

$$\text{or} \quad \sin 2\phi = 0$$

$$\Rightarrow \quad 2\phi = 0 \text{ or } \pi$$

$$\Rightarrow \quad \frac{\phi}{2} = 0 \text{ or } \frac{\pi}{4}$$

b. Consider any point $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$.

Given that $\quad OP = 2$

$$\Rightarrow \quad 6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow \quad 4\cos^2\theta = 2$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

c. Solving the equation of ellipse and parabola (eliminating x^2), we have

$$y - 1 + 4y^2 = 4$$

$$\Rightarrow \quad 4y^2 + y - 5 = 0$$

$$\Rightarrow \quad (4y + 5)(y - 1) = 0$$

$$\Rightarrow \quad y = 1, x = 0$$

The curves touch at (0, 1). So, the angle of intersection is 0.

d. The normal at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2$$

where $\quad a^2 = 14, b^2 = 5.$

It meets the curve again at $Q(2\theta)$, i.e. $(a\cos 2\theta, b\sin 2\theta)$. Hence,

$$\frac{a}{\cos\theta}a\cos 2\theta - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$$

$$\Rightarrow \quad \frac{14}{\cos\theta}\cos 2\theta - \frac{5}{\sin\theta}(\sin 2\theta) = 14 - 5$$

$$\Rightarrow \quad 28\cos^2\theta - 14 - 10\cos^2\theta = 9\cos\theta$$

$$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta - 2) = 0$$

$$\Rightarrow \quad \cos\theta = -\frac{2}{3}$$

4. a → s; b → p; c → q; d → r.

a. The locus is

$$\Rightarrow \quad \frac{x^2}{16}+\frac{y^2}{36}=1$$

$$\Rightarrow \quad e=\sqrt{1-\frac{16}{36}}=\sqrt{\frac{20}{36}}=\frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad 3e=\sqrt{5}$$

b. $3(x^2+2x+1)+2(y^2-2y+1)=3+2+1$

$$\Rightarrow \frac{(x+1)^2}{2}+\frac{(y-1)^2}{3}=1$$

$$\Rightarrow \quad e=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$$

$$\therefore \quad a=\sqrt{3},\ b=\sqrt{2}$$

$$\therefore \quad \text{Area}=\frac{1}{2}\times 2\sqrt{3}\times\sqrt{2}=\sqrt{6}$$

c. Eliminating θ from $x=1+4\cos\theta, y=2+3\sin\theta$, we have

$$\frac{(x-1)^2}{16}+\frac{(y-2)^2}{9}=1$$

Hence, $a=4$ and $e=\frac{\sqrt{7}}{4}$.

$$\Rightarrow \quad ae=\sqrt{7}$$

$\Rightarrow$ Distance between the foci $=2\sqrt{7}$

d. $\frac{x^2}{16}+\frac{y^2}{7}=1$

$$e=\sqrt{1-\frac{7}{16}}=\frac{3}{4}$$

One end of latus rectum is

$$\left(ae,\frac{b^2}{a}\right)=\left(3,\frac{7}{4}\right)$$

Therefore, equation of tangent is

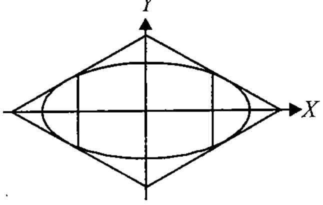


Fig. 4.85

$$\frac{3x}{16}+\frac{7}{4}\frac{y}{7}=1$$

or $$\frac{3x}{16}+\frac{y}{4}=1$$

It meets x-axis at $\left(\frac{16}{3},0\right)$ and y-axis at (0, 4).

Hence, the area of quadrilateral $=2\times\frac{16}{3}\times 4=\frac{128}{3}$.

5. a → q; b → r; c → r; d → p.

a. Points are $O(0, 0)$, $P(3, 4)$ and $Q(6, 8)$

$$2a=OP+OQ$$
$$=5+10=15$$

$$\Rightarrow \quad a=\frac{15}{2}$$

Also distance between foci,

$$2ae=\sqrt{(6-3)^2+(8-4)^2}=5$$

$$\Rightarrow \quad e=\frac{1}{3}$$

$$\Rightarrow \quad b^2=\frac{225}{4}\left(1-\frac{1}{9}\right)=50$$

$$\Rightarrow \quad b=5\sqrt{2}$$

$$\Rightarrow \quad 2b=10\sqrt{2}$$

b. We know that $\frac{1}{SP}+\frac{1}{SQ}=\frac{2a}{b^2}$

$$\Rightarrow \quad \frac{1}{2}+\frac{1}{SQ}=\frac{10}{16}$$

$$\Rightarrow \quad SQ=8$$

$$\Rightarrow \quad PQ=10$$

c. If the line $y=x+k$ touches the ellipse $9x^2+16y^2=144$, then

$$k^2=16(1)^2+9$$

$$\Rightarrow \quad k=\pm 5$$

d. Sum of the distances of a point on the ellipse from the foci $=2a=8$.

6. a → q; b → q; c → s; d → p.

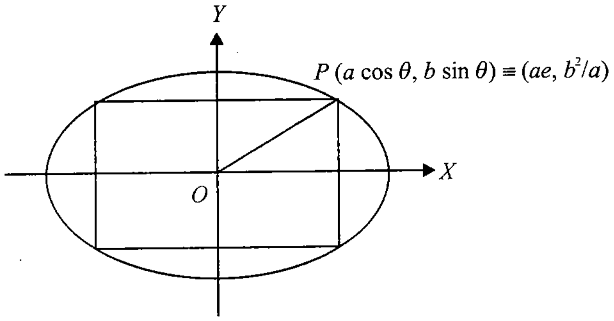


Fig. 4.86

a. Let one of the vertices of the rectangle be $P(a\cos\theta, b\sin\theta)$.

Then its area $A=(2a\cos\theta)(2b\sin\theta)=2ab\sin 2\theta$.

Hence, $A_{\max}=2ab$

Now area of rectangle formed by extremities of LR $=(2ae)(2b^2/a)=4eb^2$.

Given that

$$2ab=4eb^2 \Rightarrow \frac{2b}{a}e=1$$

$$\Rightarrow \quad \frac{4b^2}{a^2}e^2=1$$

$$\Rightarrow \quad 4(1-e^2)e^2=1$$

$$\Rightarrow \quad 4e^4-4e^2+1=0$$

$$\Rightarrow \quad (2e^2-1)^2=0$$

$$\Rightarrow \quad e=\frac{1}{\sqrt{2}}$$

b.

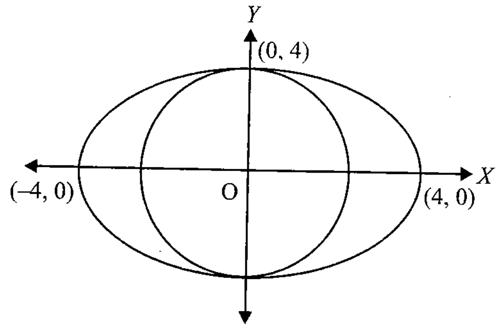


Fig. 4.87

For ellipse, distance c between the foci, $2ae = 8$ and length of semi-minor axis, is $b = 4$.

Now,

$$b^2 = a^2 - a^2e^2$$

$\Rightarrow$ $16 = a^2 - 16$

$\Rightarrow$ $a^2 = 32$

$\Rightarrow$ $e = \sqrt{1 - \frac{16}{32}} = \frac{1}{\sqrt{2}}$

c. Normal at point P (6, 2) to the ellipse passes through its focus Q (5, 2). Then P must be extremity of the major axis. Now $ae = QR = 1$ (where R is centre) and $a - ae = 1$.

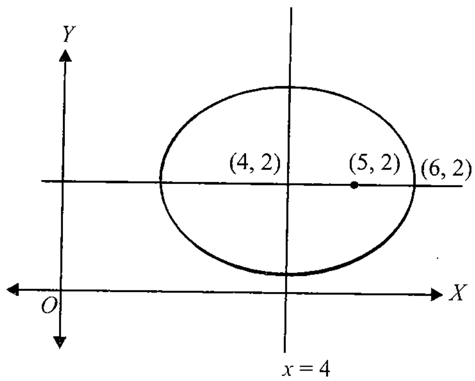


Fig. 4.88

$\therefore$ $a = 2$

$$b^2 = a^2 - a^2e^2 = 4 - 1 = 3$$

$\Rightarrow$ $e = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

d.

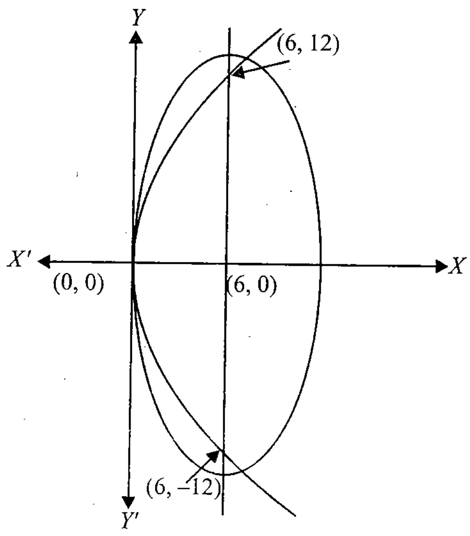


Fig. 4.89

Extremities of LR of parabola $y^2 = 24x$ are $(6, \pm 12)$.

For ellipse, $2be = 24$ and extremity of minor axis is (0, 0). Hence, $a = 6$.

Now, $a^2 = b^2 - b^2e^2$

$\Rightarrow$ $b^2 = 36 + 144 = 180$

$\Rightarrow$ $e^2 = \sqrt{1 - \frac{36}{180}} = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$

Integer type

1. (8)

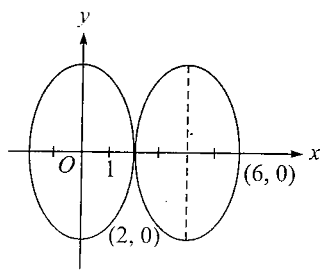


Fig. 4.90

Let $x - 4 = 2\cos\theta \Rightarrow x = 2\cos\theta + 4$

and $y = 3\sin\theta$

Now $E = \frac{x^2}{4} + \frac{y^2}{9}$

$$= \frac{(2\cos\theta + 4)^2}{4} + \sin^2\theta$$

$$= \frac{4\cos^2\theta + 16 + 16\cos\theta + 4\sin^2\theta}{4}$$

$= \dfrac{20+16\cos\theta}{4}$

$= 5 + 4\cos\theta$

Hence $E_{max} - E_{min} = (9 - 1) = 8$

2. **(2)** $\left(\pm ae, \dfrac{b^2}{a}\right)$ are extremities of the latus-rectum having positive ordinates.

$\Rightarrow a^2e^2 = -2\left(\dfrac{b^2}{a} - 2\right)$ (1)

But $b^2 = a^2(1 - e^2)$ (2)

$\therefore$ From (1) and (2), we get $a^2e^2 - 2ae^2 + 2a - 4 = 0$

$\Rightarrow ae^2(a - 2) + 2(a - 2) = 0$

$\therefore (ae^2 + 2)(a - 2) = 0$

Hence $a = 2$.

3. **(7)** By using condition of tangency, we get $4h^2 = 3k^2 + 2$

$\therefore$ Locus of $P(h, k)$ is $4x^2 - 3y^2 = 2$ (which is hyperbola)

Hence $e^2 = 1 + \dfrac{4}{3} \Rightarrow e = \sqrt{\dfrac{7}{3}}$

4. **(9)** Center of the given circle is $O(4, -3)$.

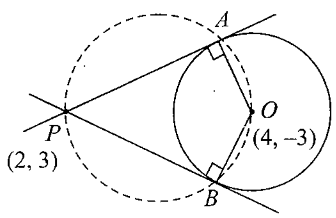


Fig. 4.91

The circumcircle of ΔPAB will circumscribe the quadrilateral $PBOA$ also, hence one of the diameters must be OP.

$\therefore$ Equation of circumcircle of ΔPAB will be

$(x - 2)(x - 4) + (y - 3)(y + 3) = 0$

$\Rightarrow x^2 + y^2 - 6x - 1 = 0$ (1)

Director circle of given ellipse will be

$(x + 5)^2 + (y - 3)^2 = 9 + b^2$

$\Rightarrow x^2 + y^2 + 10x - 6y + 25 - b^2 = 0$ (2)

$\therefore$ From (1) and (2), by applying condition of orthogonality, we get

$2[-3(5) + 0(-3)] = -1 + 25 - b^2 \Rightarrow -30 = 24 - b^2$

Hence $b^2 = 54$

5. **(9)** Equation of normal at $P(\theta)$ is $5\sec\theta x - 4\operatorname{cosec}\theta y = 25 - 16$ and it passes through $P(0, \alpha)$

$\alpha = \dfrac{-9}{4\operatorname{cosec}\theta}$ i.e. $\alpha = \dfrac{-9}{4}\sin\theta \Rightarrow |\alpha| < \dfrac{9}{4}$

6. **(5)** Points are $A(3, 4)$, $B(6, 8)$ and $O(0, 0)$.

$OA + OB = 2a$ (where a is semi-major axis.)

$2a = 5 + 10 = 15$

$\therefore \quad a = \dfrac{15}{2}$

Now $2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$

$e = \dfrac{1}{3}$

$\therefore \quad b^2 = \dfrac{225}{4}\left(1 - \dfrac{1}{9}\right) = 50$

7. **(8)** Equation of the chord whose mid point is (0, 3) is

$\dfrac{3y}{25} - 1 = \dfrac{9}{25} - 1$, i.e. $y = 3$

intersects the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

at $\dfrac{x^2}{16} = 1 - \dfrac{9}{25} = \dfrac{16}{25} \Rightarrow x = \pm\dfrac{16}{5}$

$\therefore$ length of the chord $= \dfrac{32}{5}$

thus $\dfrac{4k}{5} = \dfrac{32}{5} \therefore k = 8$

8. **(8)** $\dfrac{a}{e} - ae = 8 \Rightarrow 2a - \dfrac{a}{2} = 8$, i.e. $a = \dfrac{16}{3}$

$b^2 = a^2(1 - e^2) = \dfrac{256}{9}\left(1 - \dfrac{1}{4}\right) = \dfrac{64}{3}$

$\therefore$ length of minor axis $= 2b = \dfrac{16}{\sqrt{3}}$

$\therefore k = 8$

9. **(4)** $\because OS_1 = ae = 6$, $OC = b$ (let)

also $CS_1 = a$

$\because$ Area of $\Delta OCS_1 = \dfrac{1}{2}(OS_1) \times (OC) = 3b$

$\because$ semi-perimeter of $\Delta OCS_1 = ½ (OS_1 + OC + CS_1)$

$= ½ (6 + a + b)$ (1)

$\because$ Inradius of $\Delta OCS_1 = 1$

$\Rightarrow \dfrac{3b}{\frac{1}{2}(6 + a + b)} = 1 \Rightarrow 5b = 6 + a$ (2)

also $b^2 = a^2 - a^2e^2 = a^2 - 36$ (3)

$\Rightarrow$ from (2)

$25b^2 = 36 + 12a + a^2$

$\therefore 25(a^2 - 36) = 36 + a^2 + 12a$

$2a^2 - a - 78 = 0$

$\therefore a = \dfrac{13}{2}, -6$

$a = \dfrac{13}{2} \therefore b = \dfrac{5}{2}$

10. (4) Equation of tangent is $y = 2x \pm \sqrt{4a^2 + b^2}$

$\Longrightarrow$ this is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

$\Longrightarrow$ this tangent passes through (–2, 0)

$\Longrightarrow 0 = -4 \pm \sqrt{4a^2 \cdot b^2} \quad \Rightarrow \quad 4a^2 + b^2 = 16$

$\Longrightarrow$ Using A.M. ≥ G.M, we get

$$\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2 + b^2} \quad \Rightarrow \quad ab \leq 4$$

11. (8) $x^2 + 9y^2 - 4x + 6y + 4 = 0$

$\Longrightarrow x^2 - 4x + 9y^2 + 6y + 4 = 0$

$\Longrightarrow \quad (x-2)^2 + (3y+1)^2 = 1$

$$\Longrightarrow \quad (x-2)^2 + \frac{\left(y + \frac{1}{3}\right)^1}{\frac{1}{9}} = 1$$

which is an equation of ellipse having centre at $\left(2, -\frac{1}{3}\right)$

General point on ellipse is.

$P(x, y) = (2 + a \cos \theta, -1/3 + b \sin \theta)$

$= (2 + \cos \theta, -1/3 + 1/3 \sin \theta)$

$x = 2 + \cos \theta$ and $y = -1/3 + 1/3 \sin \theta$

$\therefore\ 4x - 9y = 4(2 + \cos \theta) - 9\left(-\frac{1}{3} + \frac{1}{3}\sin\theta\right)$

$\Rightarrow f(\theta) = 8 + 4 \cos \theta + 3 - 3 \sin \theta$

$= 11 + 4 \cos \theta - 3 \sin \theta$

$\therefore f(\theta)_{max.} = 11 + 5 = 16$

12. (4) Let sides of rectangle be p and q

Area of rectangle $= pq = 200$ (1)

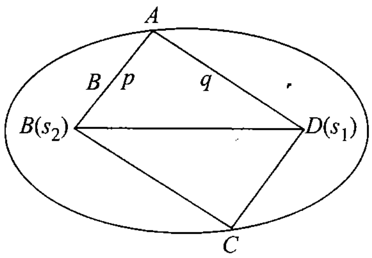


Fig. 4.92

Area of ellipse $= \pi ab = 200\pi$

$\therefore\ ab = 200$ (2)

we have to find the perimeter of rectangle $= 2(p + q)$

From triangle ABD

Distance $BD = \sqrt{p^2 + q^2}$ = distance between foci

or $p^2 + q^2 = 4a^2e^2$

or $(p + q)^2 - 2pq = 4(a^2 - b^2)$ (3)

Also from the definition of ellipse sum of focal lengths is $2a$,

Then $AB + AD = p + q = 2a$ (4)

putting value of $(p + q)$ in equation (3) from (4)

we have $(2a)^2 - 2pq = 4a^2 - 4b^2$ (using equation (1))

$\Rightarrow 4a^2 - 2 \times 200 = 4(a^2 - b^2)$

$\Rightarrow a^2 - 100 = a^2 - b^2$

$\Rightarrow b = 10$

from equation (2), $ab = 200 \Rightarrow a = 20$

since $p + q = 2a$ (from equation (4))

therefore perimeter $= 2(p + q) = 4a = 4 \times 20 = 80$

Archives

Subjective Type

1. Equation to the tangent at the point P ($a \cos \theta$, $b \sin \theta$) on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \text{(i)}$$

Perpendicular distance of (i) from the centre (0, 0) of the ellipse is given by

$$d = \frac{1}{\sqrt{\frac{1}{a^2}\cos^2\theta + \frac{1}{b^2}\cos^2\theta}}$$

$$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore\ 4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2\left\{1 - \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2}\right\}$$

$$= 4(a^2 - b^2)\cos^2\theta$$

$$= 4a^2e^2\cos^2\theta \quad \text{(ii)}$$

The coordinates of foci F_1 and F_2 are $F_1 \equiv (ae, 0)$ and $F_2 \equiv (-ae, 0)$.

$\therefore \quad PF_1 = e(1 - e\cos\theta)$

and $\quad PF_2 = a(1 + e\cos\theta)$

$$\therefore\ (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta \quad \text{(iii)}$$

Hence, from (ii) and (iii), we have

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

2.

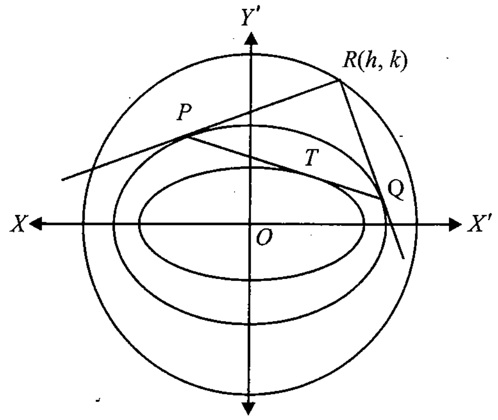


Fig. 4.93

The given ellipses are

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \qquad \text{(i)}$$

and $$\frac{x^2}{6}+\frac{y^2}{3}=1 \qquad \text{(ii)}$$

Then the equation of tangent to (i) at any point $T(2\cos\theta, \sin\theta)$ is given by

$$\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1 \qquad \text{(iii)}$$

Let this tangent meet the ellipse (ii) at P and Q. Let the tangents drawn to ellipse (ii) at P and Q meet each other at $R(h, k)$.

Then PQ is chord of contact of ellipse (ii) with respect to the point $R(h, k)$ and is given by

$$\frac{xh}{6}+\frac{yk}{3}=1 \qquad \text{(iv)}$$

Clearly, equations (iii) and (iv) represent the same line and hence should be identical.

Therefore, comparing the ratio of coefficients, we get

$$\frac{\cos\theta/2}{\frac{h}{6}}=\frac{\sin\theta}{k/3}=\frac{1}{1}$$

$\Rightarrow$ $$h=3\cos\theta,\ k=3\sin\theta$$

$\Rightarrow$ $$h^2+k^2=9$$

Therefore, the locus of (h, k) is

$$x^2+y^2=9$$

which is the director circle of the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ and we know that the director circle is the locus of point of intersection of the tangents which are at right angle.

Thus tangents at P and Q are perpendicular.

3. Let the midpoint of AB is (h, k), then coordinates of A and B are $(2h, 0)$ and $(0, 2k)$.

Then equation of line AB is $\frac{x}{2h}+\frac{y}{2k}=1$ or $y=-\frac{k}{h}+2k$

It touches the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$, if $4k^2=25\left(-\frac{k}{h}\right)^2+4$

or $\frac{25}{h^2}+\frac{4}{k^2}=4$

Therefore, locus of (h, k) is $\frac{25}{x^2}+\frac{4}{y^2}=4$.

(For the given tangent to the ellipse, radius of the circle is automatically fixed.)

4.

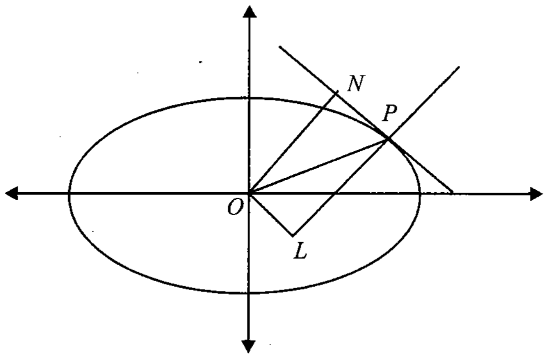


Fig. 4.94

The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since this ellipse is symmetrical in all four quadrants, either there exists no such point P or there are four points, one in each quadrant. Without loss of generality, we can assume that $a > b$ and P lies in the first quadrant.

Let P be $(a\cos\theta; b\sin\theta)$. Then equation of tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$\therefore$ $$ON=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Equation of ON is

$$\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0$$

Equation of normal at P is

$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$

$\therefore$ $$OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}}$$

$$=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Now, $NP = OL$

$\therefore \quad NP = \dfrac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

$\therefore \quad \Delta =$ Area of triangle OPN

$= \dfrac{1}{2} \times ON \times NP$

$$= \frac{1}{2}ab(a^2-b^2)\frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{1}{2}ab(a^2-b^2)\frac{1}{a^2\tan\theta + b^2\cot\theta}$$

$$= \frac{1}{2}ab(a^2-b^2)\frac{1}{(a\sqrt{\tan\theta} - b\sqrt{\cot\theta})^2 + 2ab}$$

Now maximum Δ occurs when

$a\sqrt{\tan\theta} - b\sqrt{\cot\theta} = 0$ or $\tan\theta = \frac{b}{a}$

Therefore, P has coordinates $\left(\dfrac{a^2}{\sqrt{a^2+b^2}}, \dfrac{b^2}{\sqrt{a^2+b^2}}\right)$.

By symmetry, we have four such points, i.e.

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right).$$

5. Let A, B, C be the points on circle whose coordinates are $A\ (a\cos\theta, a\sin\theta)$

$$B\left(a\cos\left(\theta+\frac{2\pi}{3}\right), a\sin\left(\theta+\frac{2\pi}{3}\right)\right)$$

$$C\left(a\cos\left(\theta+\frac{4\pi}{3}\right), a\sin\left(\theta+\frac{4\pi}{3}\right)\right)$$

Further, $P\ (a\cos\theta, b\sin\theta)$ (given)

Hence,

$$Q\left(a\cos\left(\theta+\frac{2\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right)$$

$$R\left(a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{4\pi}{3}\right)\right)$$

It is given that P, Q, R are on the same side of x-axis as A, B, C.

So required normals to the ellipse are

$$ax\sec\theta - by\ \mathrm{cosec}\ \theta = a^2 - b^2 \quad \text{(i)}$$

$$ax\sec\left(\theta+\frac{2\pi}{3}\right) - by\ \mathrm{cosec}\left(\theta+\frac{2\pi}{3}\right) = a^2 - b^2 \quad \text{(ii)}$$

$$ax\sec\left(\theta+\frac{4\pi}{3}\right) - by\ \mathrm{cosec}\left(\theta+\frac{4\pi}{3}\right) = a^2 - b^2 \quad \text{(iii)}$$

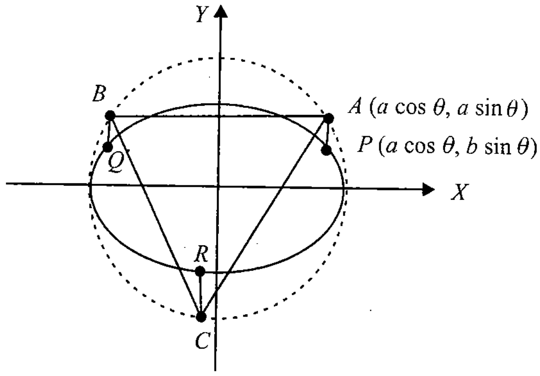


Fig. 4.95

Now, above three normals are concurrent. Hence, $\Delta = 0$.

$$\Rightarrow \Delta = \begin{vmatrix} \sec\theta & \mathrm{cosec}\ \theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \mathrm{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta+\frac{4\pi}{3}\right) & \mathrm{cosec}\left(\theta+\frac{4\pi}{3}\right) & 1 \end{vmatrix}$$

Multiplying and dividing the different rows by $\sin\theta\cos\theta$, we get

$\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right)$ and $\sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)$

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\pi-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

where $k = \sin\theta\cos\theta\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)$

[Operating $R_2 \to R_2 + R_3$ and simplifying R_2 we get $R_2 \equiv R_1$.] Hence, $\Delta = 0$.

6. Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Then coordinates of Q are $(a\cos\theta, a\sin\theta)$.

Let $R\ (h, k)$ divides PQ in the ratio $r{:}s$. Then

$$h = \frac{s\ (a\cos\theta) + r\ (a\cos\theta)}{(r+s)}$$

$$= a\cos\theta$$

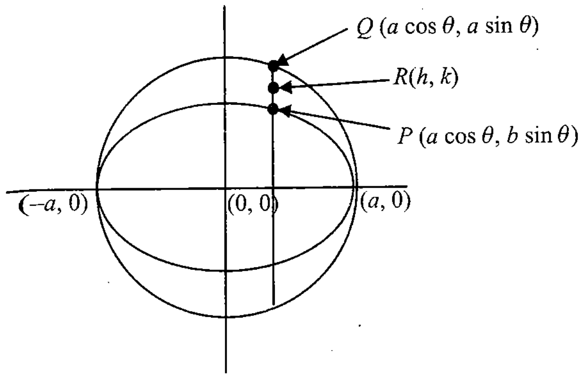


Fig. 4.96

$\Rightarrow \quad \cos\theta = \frac{h}{a}$

$$k = \frac{s(b\sin\theta) + r(a\sin\theta)}{(r+s)}$$

$$= \frac{\sin\theta\,(bs + ar)}{(r+s)}$$

$\Rightarrow \quad \sin\theta = \frac{k(r+s)}{(bs+ar)}$

We know that $\cos^2\theta + \sin^2\theta = 1$. Therefore,

$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(bs+ar)^2} = 1$

Hence, locus of R is

$$\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(bs+ar)^2} = 1$$

which is an ellipse.

7.

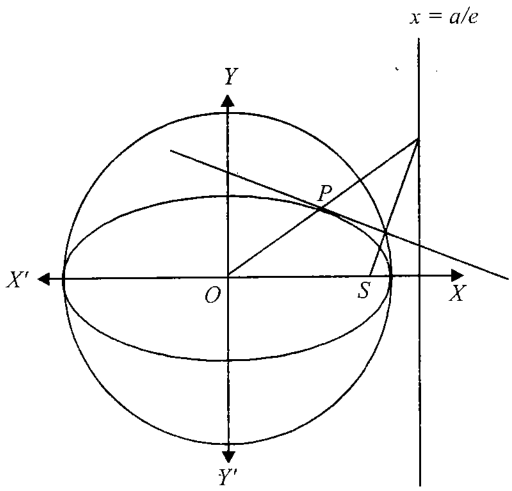


Fig. 4.97

Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and O be the centre.

Tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

whose slope is $-\frac{b^2x_1}{a^2y_1}$

Focus of the ellipse is $S(ae, 0)$.

Equation of the line through $S(ae, 0)$ perpendicular to tangent at P is

$$y = \frac{a^2y_1}{b^2x_1}(x - ae) \qquad \text{(i)}$$

Equation of OP is

$$y = \frac{y_1}{x_1}x \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$\Rightarrow \quad \frac{y_1}{x_1}x = \frac{a^2y_1}{b^2x_1}(x - ae)$

$\Rightarrow \quad x(a^2 - b^2) = a^3e$

$\Rightarrow \quad x\,a^2e^2 = a^3e$

$\Rightarrow \quad x = a/e$

This is the corresponding directrix.

8. Any tangent on ellipse is

$$y = mx \pm \sqrt{25m^2 + 4}$$

If this is also the tangent on the circle, then

$$\left|\frac{0 - m\times 0 \pm \sqrt{25m^2+4}}{\sqrt{1+m^2}}\right| = 4$$

$\Rightarrow \quad m = \pm\frac{2}{\sqrt{3}}$

Since common tangent is in first quadrant so

$$m = -\frac{2}{\sqrt{3}}$$

Hence, the common tangent in first quadrant is given by

$$\sqrt{3}\,y + 2x = 4\sqrt{7} \qquad \text{(i)}$$

The points of intersection of this tangent with x-axis and y-axis are $(2\sqrt{7}, 0)$ and $\left(0, \frac{4\sqrt{7}}{\sqrt{3}}\right)$.

Therefore, the length of intercept

$$= \sqrt{(2\sqrt{7} - 0)^2 + \left(0 - \frac{4\sqrt{7}}{\sqrt{3}}\right)^2}$$

$$= \frac{14}{\sqrt{3}}.$$

Objective Type

Fill in the blanks

1.

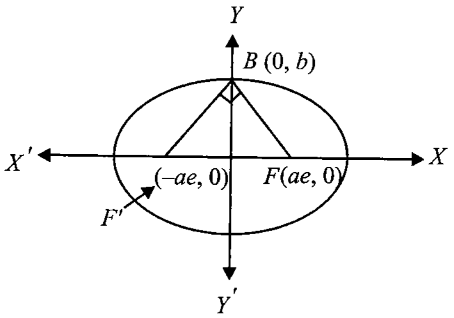


Fig. 4.98

$$m_{BF} \cdot m_{BF'} = -1$$

$$\Rightarrow \quad \frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\Rightarrow \quad \frac{b^2}{a^2e^2} = 1$$

$$\Rightarrow \quad e^2 = \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow \quad e^2 = \frac{1}{2}$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

Multiple choice questions with one correct answer

1. a. The given ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Here $\quad a^2 = 16$ and $b^2 = 9$

$\therefore \quad b^2 = a^2(1 - e^2) \Rightarrow 9 = 16(1 - e^2)$

$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$

Hence, the foci are $(\pm \sqrt{7}, 0)$.

Radius of the circle = distance between $(\pm \sqrt{7}, 0)$ and $(0, 3) = \sqrt{7+9} = 4$.

2. c. For given slope there exists two parallel tangents to ellipse. Hence, there are two values of c.

3. c. The ellipse can be written as

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Here $\quad a^2 = 25, b^2 = 16$

Now, $\quad b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad \frac{16}{25} = 1 - e^2$$

$$\Rightarrow \quad e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\Rightarrow \quad e = \frac{3}{5}$$

Foci of the ellipse are $(\pm ae, 0) \equiv (\pm 3, 0)$, i.e. F_1 and F_2 are foci of the ellipse.

Therefore, we have $PF_1 + PF_2 = 2a = 10$ for every point P on the ellipse.

4. d. The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Then $\quad a^2 = 9, b^2 = 5$

$$\Rightarrow \quad e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

Hence, end point of latus rectum in first quadrant is $L(2, 5/3)$.

Equation of tangent at L is

$$\frac{2x}{9} + \frac{y}{3} = 1$$

The tangent meets x-axis at $A(9/2, 0)$ and y-axis at $B(0, 3)$.

Therefore, area of $\Delta OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$

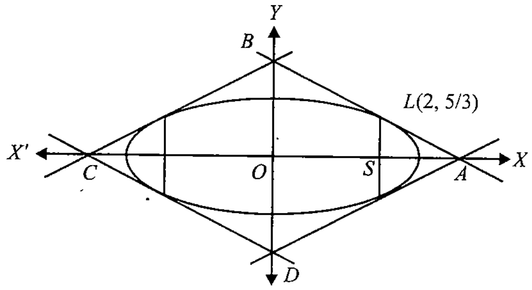


Fig. 4.99

By symmetry, area of quadrilateral

$$= 4 \times (\text{Area of } \Delta OAB)$$

$$= 4 \times \frac{27}{4} = 27 \text{ sq. units}$$

5. a. Any tangent to ellipse

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

is given by $\quad \frac{x \cos\theta}{\sqrt{2}} + y \sin\theta = 1$

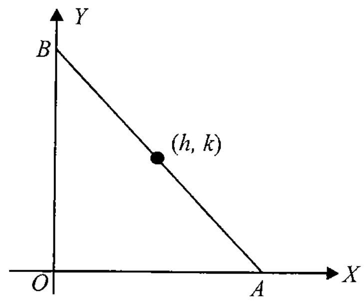


Fig. 4.100

Using midpoint formula, we have

$$A(\sqrt{2}\sec\theta, 0) \text{ and } B(0, \operatorname{cosec}\theta).$$

Hence, $2h = \sqrt{2}\sec\theta$ and $2k = \operatorname{cosec}\theta$

$$\Rightarrow \left(\frac{1}{\sqrt{2}h}\right)^2 + \left(\frac{1}{2k}\right)^2 = 1$$

$$\Rightarrow \frac{1}{2h^2} + \frac{1}{4k^2} = 1$$

$$\Rightarrow \frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

6. a. Any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$ is given by

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

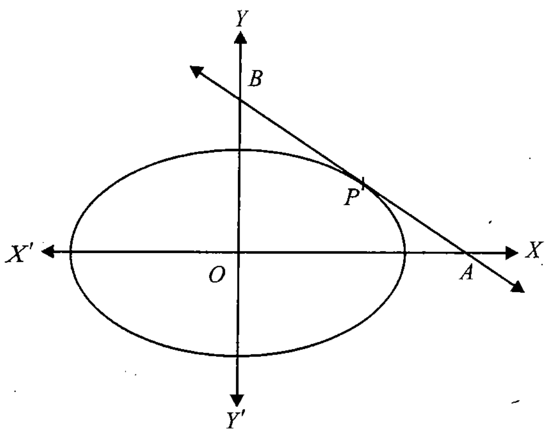


Fig. 4.101

It meets coordinate axes at $A(a\sec\theta, 0)$ and $B(0, b\operatorname{cosec}\theta)$

$\therefore$ Area of $\Delta OAB = \frac{1}{2} \times a\sec\theta \times b\operatorname{cosec}\theta$

$$\Rightarrow \Delta = \frac{ab}{\sin 2\theta}$$

For area to be minimum $\sin 2\theta$ should be maximum and we know maximum value of $\sin 2\theta = 1$.

$$\therefore \Delta_{max} = ab$$

7. d.

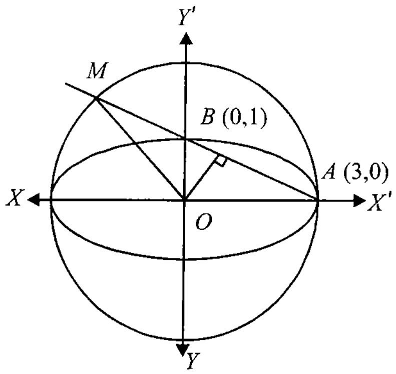


Fig. 4.102

Equation of line AM is

$$x + 3y - 3 = 0$$

Perpendicular distance of line from the origin $= \frac{3}{\sqrt{10}}$

Length of $AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

$$\Rightarrow \text{Area} = \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}}$$

$$= \frac{27}{10} \Rightarrow \text{q.units}$$

8. c. Normal is given by $4x\sec\phi - 2y\operatorname{cosec}\phi = 12$

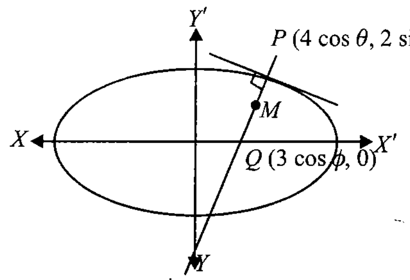


Fig. 4.103

$$Q \equiv (3\cos\phi, 0)$$

$$M \equiv (\alpha, \beta)$$

$$\Rightarrow \alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$$

$$\Rightarrow \cos\phi = \frac{2}{7}\alpha$$

$$\beta = \sin\phi$$

Using $\cos^2\phi + \sin^2\phi = 1$, we have

$$\frac{4}{49}\alpha^2 + \beta^2 = 1$$

$$\Rightarrow \frac{4}{9}x^2 + y^2 = 1 \quad \text{(i)}$$

Now latus rectum,

$$x = \pm 2\sqrt{3} \quad \text{(ii)}$$

Solving (i) and (ii), we have $\frac{48}{49} + y^2 = 1$

$$\Rightarrow y = \pm\frac{1}{7}$$

Points of intersection are $(\pm 2\sqrt{3}, \pm 1/7)$.

9. d. Since $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$ both P and Q lie inside C. Also $\frac{1^2}{9} + \frac{2^2}{4} = \frac{1}{9} + 1 > 1$ and $\frac{2^2}{9} + \frac{1}{4} = \frac{25}{36} < 1$. Hence, P lies outside E and Q lies inside E. Thus P lies inside C but outside E.

Multiple choice questions with one or more than one correct answer

1. b., d. Let (x_1, y_1) be the point at which tangents to ellipse $4x^2 + 9y^2 = 1$ are parallel to $8x = 9y$.

Then slope of the tangent $= \frac{8}{9}$.

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{8}{9} \qquad \text{(i)}$$

Differentiating equation of ellipse w.r.t. x, we get

$$8x + 18y\frac{dx}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-8x_1}{18y_1} = \frac{-4x_1}{9y_1}$$

Substituting in equation (i), we get

$$\frac{-4x_1}{9y_1} = \frac{8}{9} \Rightarrow -x_1 = 2y_1 \qquad \text{(ii)}$$

Also (x_1, y_1) is the point of contact which must be on curve. Hence,

$$4x_1^2 + 9y_1^2 = 1$$

$$\Rightarrow \quad 4x \times 4x_1^2 + 9y_1^2 = 1 \text{ [using (2)]}$$

$$\Rightarrow \quad y_1^2 = \frac{1}{25}$$

$$\Rightarrow \quad y_1 = \pm\frac{1}{5}$$

$$\Rightarrow \quad x_1 = \mp\frac{2}{5}$$

Thus the required points are $\left(\frac{-2}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, -\frac{1}{5}\right)$.

Alternative Method:

Let $y = \frac{8}{9}x + c$ be the tangent to $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

where $c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{\frac{1}{4} \times \frac{64}{81} + \frac{1}{9}} = \pm\frac{5}{9}$

So, points of contact are $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right) = \left(\frac{2}{5}, \frac{-1}{5}\right)$ or $\left(\frac{-2}{5}, \frac{1}{5}\right)$.

2. b., c.

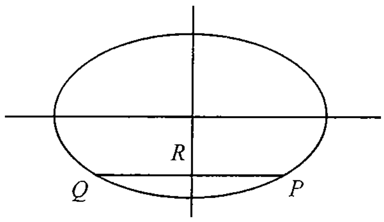


Fig. 4.104

The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

Hence, the end points P and Q of the latus rectum are given by

$$P \equiv \left(\sqrt{3}, -\frac{1}{2}\right)$$

and $\quad Q \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$ (given $y_1, y_2 < 0$)

Coordinates of midpoint of PQ are

$$R \equiv \left(0, -\frac{1}{2}\right)$$

Length of latus rectum, $PQ = 2\sqrt{3}$

Hence, two parabolas are possible whose vertices are

$\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$.

The equations of the parabolas are

$$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$$

and $\quad x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. b., c.

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$

$$\Rightarrow 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$$

$$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) = 2\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} = 2$$

$$\Rightarrow \quad \frac{\sin B + \sin C}{\sin A} = 2$$

$$\Rightarrow \quad b + c = 2a \text{ (constant)}$$

Hence, locus of vertex A is ellipse with B and C as foci.

Comprehension type

1. d. Equation of tangent having slope m is

$y = mx + \sqrt{9m^2 + 4}$

Tangent passes through the point (3, 4) then

$4 - 3m = \sqrt{9m^2 + 4}$

Squaring, we have

$16 + 9m^2 - 24m = 9m^2 + 4 \Rightarrow m = \frac{12}{24} = \frac{1}{2}$

$\therefore$ Equation of tangent is $y - 4 = \frac{1}{2}(x - 3)$ or $x - 2y + 5 = 0$

Let point of contact on the curve is $B(\alpha, \beta)$

$$\Rightarrow \frac{x\alpha}{9}+\frac{y\beta}{4}-1=0 \Rightarrow \frac{\alpha/9}{1}=\frac{\beta/4}{-2}=-\frac{1}{5}$$

$$\Rightarrow \alpha=-\frac{9}{5}, \beta=\frac{8}{5}$$

$$B \equiv \left(-\frac{9}{5}, \frac{8}{5}\right)$$

Another slope of tangent is ∞, then equation of tangent is $x = 3$ and corresponding point of contact is $A(3, 0)$

2. c. Since slope of PA is ∞,

Slope of altitude through B must be 0, for which orthocenter is $\left(\frac{11}{5}, \frac{8}{5}\right)$

3. a. Locus is parabola

Equation of AB is $\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow \frac{x}{3}+y=1 \Rightarrow x+3y-3=0$

$\therefore$ Equation of locus is

$$(x-3)^2+(y-4)^2=\frac{(x+3y-3)^2}{10}$$

or $9x^2+y^2-6xy-54x-62y+241=0$