Chapter 04

Arches

4.1 Introduction

Arches can be used to reduce the bending moments in long span structures. Essentially, an arch acts as an inverted cable, so it receives its load mainly in compression although because of its rigidity, it also resist some bending and shear depending upon how it is loaded and shaped. When an arch is subjected to uniformly distributed vertical load then only compression forces will be resisted by the arch. Under these conditions the arch shape is called a funicular arch because no bending or shear forces occurs within the arch.

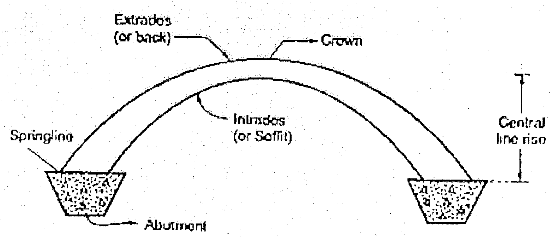


Fig. 4.1

4.2 Comparison between Cable and Arch

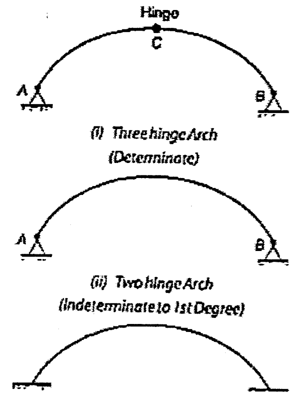
CABLE	ARCH
1. Cable is a tension member.	1. Generally arch is a compression member.
2. A cable is flexible member and it changes its shape with different type and position of loads.	2. A arch is a rigid structure and it does not changes its shape with different type and position of loads.
3. It cannot resist any bending moment hence bending moment is zero every where.	3. Basically, it is a compression member but it is often subjected to bending moment and shear force of small magnitude.

Types of Arches: There is three type of Arches :

1. **Three hinge arch:** A three hinge arch is statically determinate structure. There are four unknown reactions and three equations of equilibrium and one extra condition of equilibrium ($M_c = 0$) at internal hinge.

2. **Two hinge arch:** The two-hinge arch is statically indeterminate structure of first degree. There are four unknown reaction and only three equations of equilibrium are available.

3. **Fixed arch (Hingeless arch):** The fixed arch is statically indeterminate structure to third degree. There are six unknown reactions and only three equations of equilibrium are available.



(i) Three hinge Arch (Determinate)

(ii) Two hinge Arch (Indeterminate to 1st Degree)

(iii) Fixed Arch (Indeterminate to 3rd Degree)

Fig. 4.2

4.3 Comparison between the Behaviour of Arch and Beam

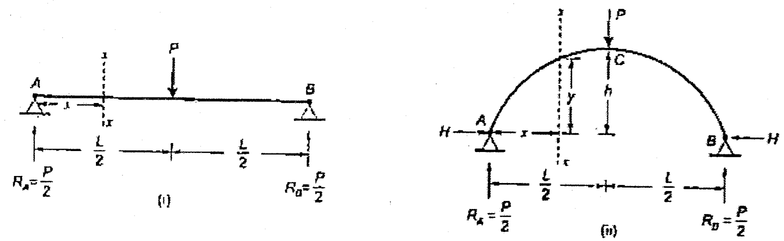


Fig. 4.3

(i)
$$M_x(\text{beam}) = R_A x$$
$$M_x = \frac{P}{2}x = M(\text{say})$$

(ii)
$$M_x(\text{arch}) = R_A x - Hy$$
$$M_x = \frac{P}{2}x - Hy$$
$$\therefore \quad M_x = M - Hy$$

From the above analysis it is clear that the bending moment at any section in arch is smaller than that of beam by an amount of Hy.

4.3.1 Advantages of Arch over Beam

1. For same loading and same span, the bending moment in arch section is smaller than bending moment in beam section at same location.
2. For long span, arch section are cheaper due to thinner cross-section needed for less bending moment. However for small span beam may be advantageous.

4.3.2 Disadvantages of Arch over Beam

1. Arches are very difficult to construct because of curved shape.
2. In multi-storey buildings arches provide constraint to height. So aesthetically arches less preferred.

4.4 Analysis of Three-Hinged Arches

4.4.1 Three Hinge Parabolic Arch Subjected to UDL over Entire Span

$\Sigma F_x = 0$; $\quad H_A = H_B = H$ (say)

$\Sigma F_y = 0$; $\quad R_A + R_B = wl \qquad$ ...(i)

Also, $\quad \Sigma M_A = 0$

$$\Rightarrow \quad R_B \times l - wl \times \frac{l}{2} = 0$$

$$\therefore \quad R_B = \frac{wl}{2}$$

From (i), $\quad R_A = \frac{wl}{2}$

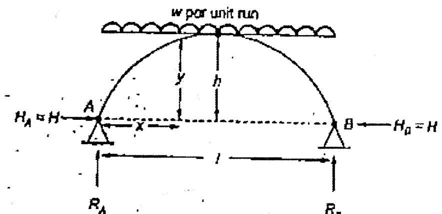


Fig. 4.4

Since there is a hinge at C, hence $M_c = 0$

$$\therefore \quad \Sigma M_C = 0 \text{ (From left)}$$

$$\Rightarrow \quad R_A \times \frac{l}{2} - H \times h - w \times \frac{l}{2} \times \frac{l}{4} = 0$$

$$\Rightarrow \quad \frac{wl}{2} \times \frac{l}{2} - H \times h - \frac{wl^2}{8} = 0$$

$$\therefore \quad H = \frac{wl^2}{8h}$$

Equation of Parabolic Arch

$$y = \frac{4h}{l^2} x(l - x)$$

Bending Moment at any section

$$M_x = \text{Beam moment} - \text{'}H\text{' Moment}$$

$$\Rightarrow \quad M_x = R_A x - w x \cdot \frac{x^2}{2} - H_A y$$

$$\Rightarrow \quad M_x = -\frac{wx^2}{2} \times \frac{wl}{2} \times x - \frac{wl}{8h} \times \frac{4h}{l^2} x(l - x)$$

$$\Rightarrow \quad M_x = 0$$

Hence 3-hinge parabolic arch subjected to UDL on entire span is free from bending moment.

Radial Shear and Normal Thrust

$$\Sigma F_x = 0$$

$$H_A - H_x = 0$$

$$\therefore \quad H_x = H_A = \frac{wl^2}{8h} \qquad \text{... (i)}$$

$$\Sigma F_y = 0$$

$$R_A - wx = V_x$$

$$\therefore \quad V_x = R_A - wx$$

$$V_x = \frac{wl}{2} - wx \qquad \text{... (ii)}$$

Radial shear at $x - x$,

$$S_x = V_x \cos\theta - H_x \sin\theta \quad \text{... (iii)}$$

We know,

$$y = \frac{4h}{l^2} x(l - x)$$

$$\tan\theta = \frac{dy}{dx} = \frac{4h}{l^2}(l - 2x)$$

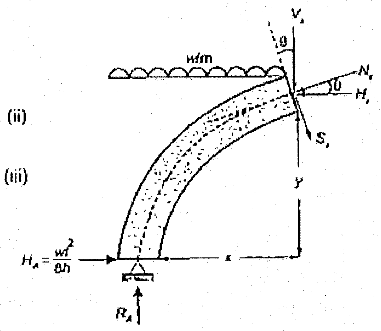


Fig. 4.5

From (iii),

$$S_x = \left(\frac{wl}{2} - wx\right)\cos\theta - \frac{wl^2}{8h}\sin\theta$$

$$S_x = \cos\theta\left[\frac{wl}{2} - wx - \frac{wl^2}{8h}\tan\theta\right]$$

$$S_x = \cos\theta\left(\frac{wl}{2} - wx - \frac{wl^2}{8h} \times \frac{4h}{l^2}(l - 2x)\right)$$

$$S_x = \cos\theta\left[\frac{wl}{2} - wx - \frac{wl}{2} + wx\right]$$

$$S_x = 0$$

Hence 3-hinge parabolic arch subjected to UDL on entire span is free from radial shear and bending moment.

Normal thrust at $x - x$,

$$N_x = V_x \sin\theta + H_x \cos\theta \qquad \text{...(iv)}$$

4.4.2 Three Hinge Semicircular Arch Subjected to UDL over Entire Span

$\Sigma F_x = 0$, $\quad H_A = H_B = H$(say)

$\Sigma F_y = 0$, $\quad R_A + R_B = w \times 2R \qquad$ ... (i)

Also, $\quad \Sigma M_B = 0$

$$R_A \times 2R - w \times 2R \times R = 0$$

$$\therefore \quad R_A = wR$$

and $\quad R_B = wR$

Since there is a hinge at C, hence $M_C = 0$,

$$\Sigma M_C = 0 \text{ (From left)}$$

$$\Rightarrow \quad R_A R - H \times R - wR \cdot \frac{R}{2} = 0$$

$$\therefore \quad H = \frac{wR}{2}$$

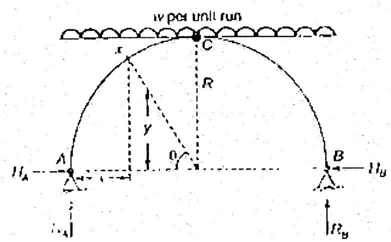


Fig. 4.6

Bending moment at any section $x - x$,

$$M_x = \text{Beam moment} - \text{'}H\text{' moment}$$

$\Rightarrow \quad M_x = R_A x - H.y - w.x.\frac{x}{2}$

$\Rightarrow \quad M_x = wR.R(1-\cos\theta) - \frac{w}{2}[R(1-\cos\theta)]^2 - \frac{wR}{2} \times R\sin\theta$

$\Rightarrow \quad M_x = wR^2(1-\cos\theta) - \frac{wR^2}{2}(1-\cos\theta)^2 - \frac{wR^2}{2}\sin\theta$

$\Rightarrow \quad M_x = wR^2(1-\cos\theta)\left[1 - \frac{(1-\cos\theta)}{2}\right] - \frac{wR^2}{2}\sin\theta$

$\Rightarrow \quad M_x = wR^2\left[(1-\cos\theta)\frac{(1+\cos\theta)}{2} - \frac{\sin\theta}{2}\right]$

$\Rightarrow \quad M_x = \frac{wR^2}{2}[\sin^2\theta - \sin\theta]$

$\therefore \quad M_x = -\frac{wR^2}{2}(\sin\theta - \sin^2\theta) \leq 0$ [since $\sin\theta > \sin^2\theta$]

For maximum bending moment,

$$\frac{dM_x}{d\theta} = 0$$

$\therefore \quad -\frac{wR^2}{2}(\cos\theta - 2\sin\theta\cos\theta) = 0$

$\cos\theta(1 - 2\sin\theta) = 0$

If $\cos\theta = 0$ $\tan\theta = 90°$

at $\theta = 90°$ minima occurs at C i.e. $M = 0$

if $(1 - 2\sin\theta) = 0$ then $\theta = 30°$

$\therefore$ M_{max} occurs at $\theta = 30°$

$$M_{max} = -\frac{wR^2}{2}(\sin 30° - \sin^2 30°) = -\frac{wR^2}{8}$$

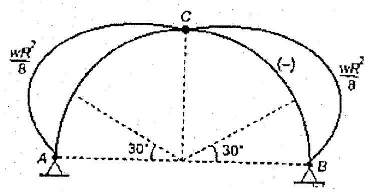


Fig. 4.7

4.4.3 Three Hinge Semicircular Arch Subjected to Concentrated Load at Crown

$$R_A = R_B = \frac{W}{2}$$

Also, $\Sigma M_C = 0$ (From left)

$R_A \times R - H \times R = 0$

$\therefore \quad H = R_A = \frac{W}{2}$

Bending moment at any section $x - x$,

$M_x = R_A . x - H.y$

$= \frac{W}{2}x - \frac{W}{2}y$

$= \frac{W}{2}(x - y)$

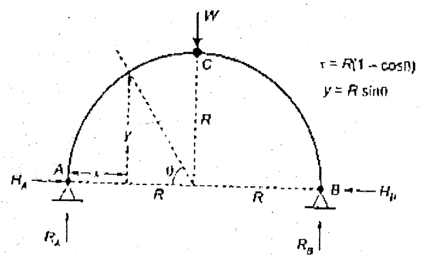


Fig. 4.8

$= \frac{W}{2}[R(1-\cos\theta) - R\sin\theta]$

$= \frac{WR}{2}[1 - \cos\theta - \sin\theta]$

For bending moment to be maximum,

$$\frac{dM_x}{d\theta} = 0$$

$\frac{WR}{2}[0 + \sin\theta - \cos\theta] = 0$

$\sin\theta = \cos\theta$

$\tan\theta = 1$

$\therefore \quad \theta = 45°$

$\therefore \quad M_{max} = \frac{WR}{2}[1 - \cos 45° - \sin 45°]$

$= \frac{WR}{2}\left[1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$

$= \frac{WR}{2}\left[1 - \sqrt{2}\right]$

$= -\frac{WR}{2}\left[\sqrt{2} - 1\right]$

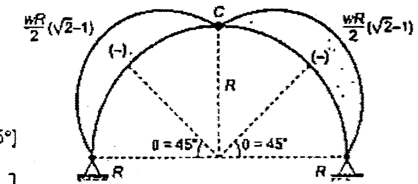


Fig. 4.9

Radial shear and normal thrust:

$\Sigma F_x = 0$

$H - H_x = 0$

$\therefore \quad H_x = H = \frac{W}{2}$ (←)

$\Sigma F_y = 0$

$R_A - V_x = 0$

$\therefore \quad V_x = R_A = \frac{W}{2}$ (↓)

$\therefore$ Radial shear, $\quad S_x = V_x \sin\theta - H_x\cos\theta$

$\therefore \quad S_x = \frac{W}{2}\sin\theta - \frac{W}{2}\cos\theta$

$S_x = \frac{W}{2}(\sin\theta - \cos\theta)$

At $\theta = 0$, $\quad S_A = \frac{W}{2}(\sin\theta - \cos\theta) = -\frac{W}{2}$

At $\theta = \frac{\pi}{2}$, $\quad S_C = \frac{W}{2}\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) = \frac{W}{2}$

For S_x to be zero,

$\frac{W}{2}(\sin\theta - \cos\theta) = 0$

$\therefore \quad \theta = 45°$

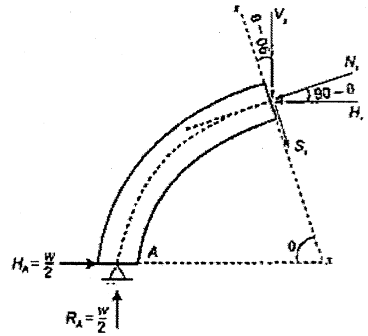


Fig. 4.10

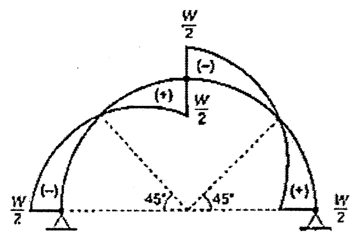
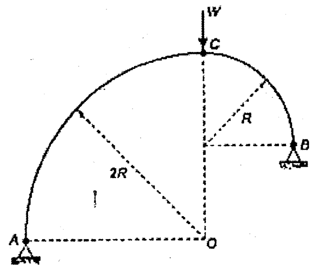


Fig. 4.11

Example 4.1 A three-hinged circular arch *ACB* is formed by two quadrants of circles *AC* and *BC* of radii $2R$ and R respectively with *C* as crown as shown in figure. Consider the following in respect of the horizontal reactions developed at supports *A* and *B* due to concentrated load at crown is



(a) $\frac{W}{2}, W$ (b) $W, \frac{W}{2}$

(c) $\frac{W}{2}, \frac{W}{2}$ (d) W, W

Ans. (c)

$$\Sigma F_y = 0$$

$\Rightarrow$ $R_A + R_B = W$ ... (i)

$$\Sigma F_x = 0$$

$\Rightarrow$ $H_A = H_B = \text{say } H$

$$\Sigma M_B = 0$$

$\Rightarrow$ $R_A \times 3R - H_A \times R - W \times R = 0$

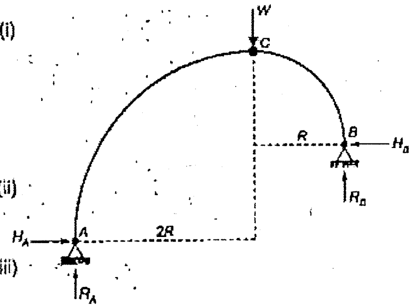
$\Rightarrow$ $3R_A - H - W = 0$

$\Rightarrow$ $3R_A = H + W$ ... (ii)

$\Rightarrow$ $\Sigma M_C = 0$ (From left)

$\Rightarrow$ $R_A \times 2R - H_A \times 2R = 0$

$\Rightarrow$ $R_A = H_A = H$ ... (iii)



From (ii) and (iii), we get

$$3H = H + W$$

$\therefore$ $H = \frac{W}{2}$

Hence option (c) is correct.

Example 4.2 A parabolic arch, symmetrical with hinges at centers and ends, carries a point load *P* at distance *x* from left support. The arch has a span of 20 m and rise of 5 m. What is the value of *x* if the left hinge reaction incline with a slope of two vertical on one horizontal?

(a) 8 m (b) 5 m

(c) 4 m (d) 2.5 m

Ans. (c)

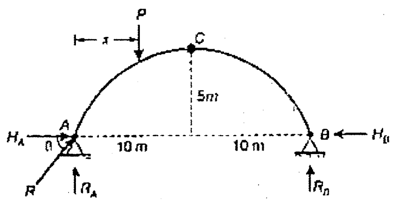
$$\Sigma M_B = 0$$

$\Rightarrow$ $R_A \times 20 - P(20 - x) = 0$

$\therefore$ $R_A = \frac{P(20-x)}{20}$

and $\Sigma M_c = 0$ (from left)

$\Rightarrow$ $-H_A \times 5 + R_A \times 10 - P \times (10 - x) = 0$



$\Rightarrow$ $-5H_A + \frac{P(20-x) \times 10}{20} - P(10-x) = 0$

$\Rightarrow$ $-5H_A + \frac{P(20-x)}{2} - P(10-x) = 0$

$\therefore$ $H_A = \frac{2P.x}{20} = \frac{P.x}{10}$

given that left hand reaction is inclined with a slope of two vertical on one horizontal

$\therefore$ $\frac{R_A}{H_A} = \frac{2}{1}$

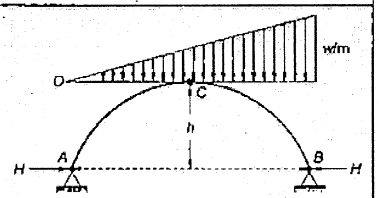
$\Rightarrow$ $R_A = 2H_A$

$\Rightarrow$ $\frac{P(20-x)}{20} = 2 \times \frac{Px}{10}$

$\Rightarrow$ $P(20-x) = 4Px$

$\therefore$ $x = 4$ m from left support

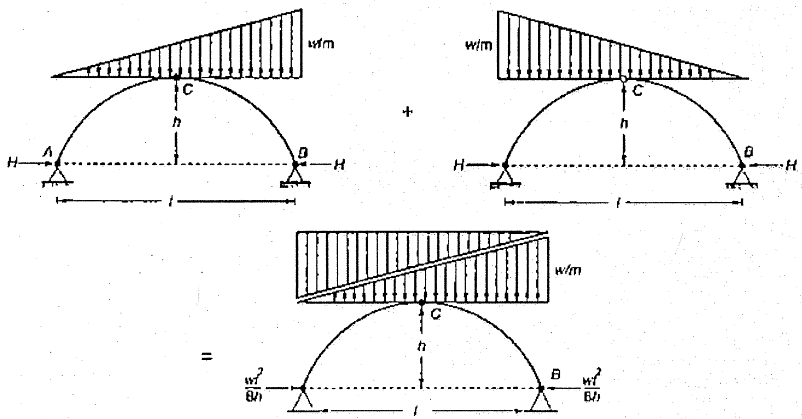
Example 4.3 Horizontal thrust for 3-hinged parabolic arch subjected to UVL as shown in Figure is



(a) $\frac{wl^2}{8h}$ (b) $\frac{wl^2}{16h}$

(c) $\frac{wl^2}{24h}$ (d) zero

Ans. (b)



By the principle of superposition,

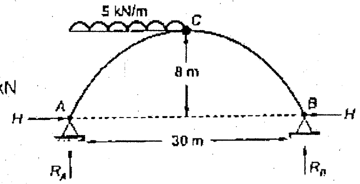
$$H + H = \frac{wl^2}{8h} \quad \therefore H = \frac{wl^2}{16h}$$

Hence option (b) is correct.

Example 4.4 A uniformly distributed load of 5 kN/m covers left half of the span of three-hinged parabolic arch, span 30 m and rise 8 m. The normal thrust and radial shear at 5 m from the left support are

(a) 46.10 kN, 5.25 kN (b) 5 kN, 40 kN
(c) 45 kN, 10 kN (d) 46.75 kN, 5.10 kN

Ans. (d)



$$H_A = H_B = H$$

$$= \frac{wl^2}{16h} = \frac{5\times 30^2}{16\times 8} = 35.15 \text{ kN}$$

$\Sigma M_B = 0;\ R_A \times 30 - 5 \times 15 \times 22.5 = 0$

$\therefore$ $R_A = 56.25$

and $R_B = 18.75$ kN

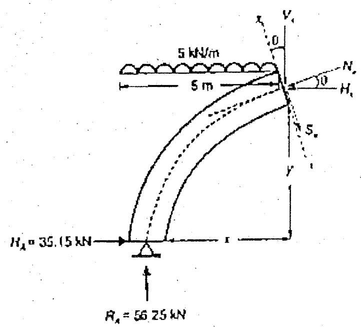
$\Sigma F_x = 0$

$\therefore$ $H_x = 35.15$ kN

$\Sigma F_y = 0$

$\Rightarrow$ $56.25 - 5 \times 5 - V_x = 0$

$\therefore$ $V_x = 31.25$ kN



$$\frac{dy}{dx} = \tan\theta = \frac{4h}{l^2}(l-2x)$$

$$= \frac{4\times 8}{30^2}\times(30-2\times 5)$$

$\therefore$ $\theta = \tan^{-1}(0.711) = 35.41°$

Normal thrust, $N_x = V_x \sin\theta + H_x \cos\theta$

$= 31.25 \times \sin 35.41 + 35.15 \times \cos 35.41°$

$= 46.75$ kN

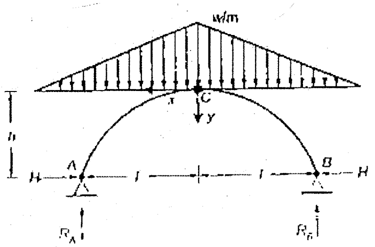
Radial shear, $S_x = V_x \cos\theta - H_x \cos\theta$

$= 31.25 \cos 35.41 - 35.15 \sin 35.41$

$= 5.10$ kN

Hence option (d) is correct.

Example 4.5 The equation of the three hinged symmetrical arch for the given coordinate is, when the arch perform as perfectly compression member.



(a) $y = \frac{hx^2}{2l^2}\left(3-\frac{x}{L}\right)$ (b) $y = \frac{hx^3}{2l^3}\left(3-\frac{x^2}{L^2}\right)$

(c) $y = \frac{hx}{2l}\left(3-\frac{x}{L}\right)$ (d) $y = \frac{hx^2}{l^2}\left(1.5-\frac{x^2}{L}\right)$

Ans. (a)

Reactions, $R_A = R_B = \frac{1}{2}w\times l = \frac{wl}{2}$

$\Sigma M_c = 0$ (From left)

$$\Rightarrow \quad R_A\times l - H_A\times h - \frac{1}{2}\times w\times l\times\frac{1}{3}\times l = 0$$

$$\Rightarrow \quad \frac{wl}{2}\times l - H_A h - \frac{wl^2}{6} = 0$$

$$\Rightarrow \quad H_A\times h = \frac{wl^2}{2} - \frac{wl^2}{6} = \frac{wl^2}{3}$$

$$\therefore \quad H_A = H = \frac{wl^2}{3h}$$

For perfectly compression member, arch should be free from bending moment.

$\therefore$ $M_x = 0$

M_x (x from c),

$$R_A.(l-x) - H\times(h-y) - \frac{1}{2}\times\frac{w}{l}(l-x)^2.\frac{1}{3}(l-x) = 0$$

$$\Rightarrow \quad y = \frac{3h}{wl^2}\left[\frac{w(l-x)^2}{6l} - \frac{wl^2}{6} + \frac{wlx}{2}\right]$$

$$\Rightarrow \quad y = \frac{3h}{wl^2}\left[\frac{wx^2}{2} - \frac{wx^3}{6l}\right]$$

$$y = \frac{hx^2}{2l^2}\left(3-\frac{x}{L}\right)$$

Hence option (a) is correct.

4.4.4 Three Hinged Parabolic Arch with Abutments at Different Heights Subjected to UDL over Entire Span

$\Sigma F_x = 0$

$\therefore$ $H_A = H_B =$ say H ...(i)

$\Sigma F_y = 0$

$R_A + R_B = wL$ ...(ii)

$\Sigma M_B = 0$

$$R_A\times l + H\times(h_2 - h_1) - \frac{wl^2}{2} = 0 \quad ...(iii)$$

$\Sigma M_c = 0$ (From left)

$$R_A\times l_1 - H\times h_1 - \frac{wl_1^2}{2} = 0 \quad ...(iv)$$

we know that,

$y \propto x^2$

$\therefore$ $\sqrt{y} \propto x$

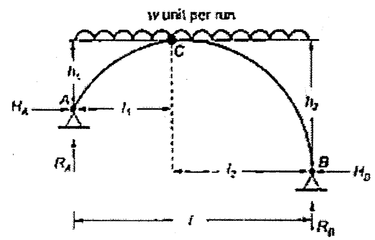


Fig. 4.12

$$\Rightarrow \quad \frac{x}{\sqrt{y}} = \text{constant}$$

$$\Rightarrow \quad \frac{l_1}{l_2} = \frac{\sqrt{h_1}}{\sqrt{h_2}}$$

$$\text{or} \quad \frac{l_1 + l_2}{l_2} = \frac{\sqrt{h_1} + \sqrt{h_2}}{\sqrt{h_2}}$$

$$\therefore \quad \frac{l_1}{\sqrt{h_1}} = \frac{l_2}{\sqrt{h_2}}$$

$$\text{or} \quad \frac{l_1}{l_2} + 1 = \frac{\sqrt{h_1}}{\sqrt{h_2}} + 1$$

$$\therefore \quad l_2 = \frac{l\sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}}$$

Similarly, $l_1 = \dfrac{l\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}}$

From (iii) and (iv), we get $H = \dfrac{wl^2}{2\left(\sqrt{h_1} + \sqrt{h_2}\right)^2}$

Example 4.6 A parabolic arch, hinged at springings and crown of span l and central rise y_C carries a uniformly distributed load of w per unit length over the left half of the span. Calculate the position and magnitude of maximum positive bending moment.

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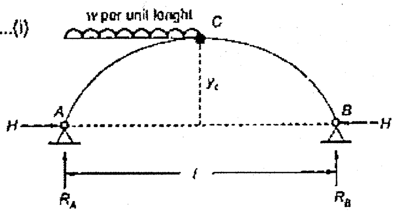
Solution:

Support reactions:



$$\Sigma f_y = 0; \quad R_A + R_B = \frac{wl}{2} \quad \text{...(i)}$$

$$\Sigma M_B = 0; \quad R_A \times l - \frac{wl}{2} \times \frac{3l}{4} = 0$$

$$R_A = \frac{3wl}{8}(\uparrow)$$

From eq. (i), we get

$$R_B = \frac{wl}{2} - R_A$$

$$R_B = \frac{wl}{2} - \frac{3wl}{8} = \frac{wl}{8}(\uparrow)$$

$$\Sigma M_C = 0 \quad \text{(From right)}$$

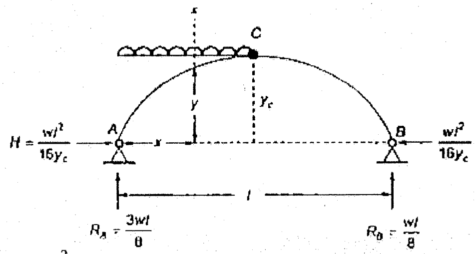
$$\Rightarrow \quad H \times y_C - R_B \times \frac{l}{2} = 0$$

$$\Rightarrow \quad H \times y_C = \frac{wl}{8} \times \frac{l}{2} = \frac{wl^2}{16}$$

$$H = \frac{wl^2}{16y_C}$$

Maximum positive BM: Maximum positive BM will occur in portion AC.

Bending moment at a section x-x at a distance x from the end A.



$$M_x = R_A x - \frac{wx^2}{2} - Hy$$

Here, $y = \dfrac{4h}{l^2} x(l - x)$

$$\therefore \quad M_x = \frac{3wl}{8}x - \frac{wl^2}{16y_c} \times \frac{4y_c}{l^2}x(l-x) - \frac{wx^2}{2}$$

$$= \frac{3wl}{8}x - \frac{w}{4}x(l-x) - \frac{wx^2}{2}$$

$$= \frac{3wlx}{8} - \frac{wlx}{4} + \frac{wx^2}{4} - \frac{wx^2}{2}$$

$$= \frac{wlx}{8} - \frac{wx^2}{4} \quad \text{(Parabolic)}$$

For M_x to be maximum,

$$\frac{dM_x}{dx} = 0$$

$$\therefore \quad \frac{wl}{8} - \frac{2wx}{4} = 0$$

$$x = \frac{l}{4} \quad \text{(From A)}$$

Hence maximum positive bending moment is,

$$M_{max} = \frac{wl}{8}\left(\frac{l}{4}\right) - \frac{w}{4}\left(\frac{l}{4}\right)^2$$

$$= \frac{wl^2}{32} - \frac{wl^2}{64} = \frac{wl^2}{64} \quad \text{(sagging)}$$

Example 4.7 A symmetrical three hinged circular arch has a span of 16 m and a rise to the central hinge of 4 m. It carries a vertical load of 16 kN at 4 m from the left end. Find

(a) the vertical reaction at the supports

(b) the magnitude of the horizontal thrust at the springing

(c) bending moment at 6m from the left hand hinge

(d) the maximum positive and negative bending moment.

Solution:

(a) The vertical reactions at the supports:

$$\Sigma f_y = 0; \quad R_A + R_B - 16 = 0$$

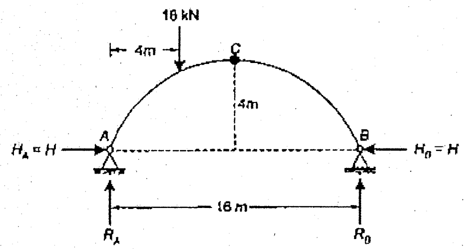
$$R_A + R_B = 16 \quad \text{...(i)}$$

$\Sigma M_B = 0$:

$R_A \times 16 - 16 \times 12 = 0$

$16R_A = 16 \times 12$

$R_A = \dfrac{16 \times 12}{16} = 12$ kN



From eq. (i), we get

$R_B = 16 - R_A$

$= 16 - 12 = 4$ kN

(b) Horizontal thrust at the springing:

$\Sigma M_C = 0$ (From left)

$-H \times 4 + R_A \times 8 - 16 \times 4 = 0$

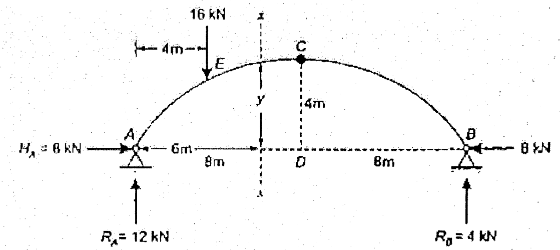
$H \times 4 = R_A \times 8 - 16 \times 4$

$H = 8$ kN

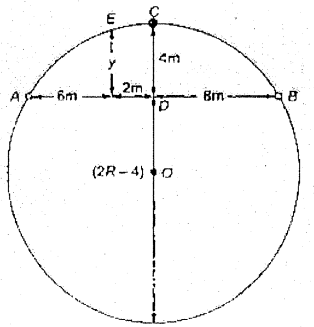
Thus, $H_A = 8$ kN $(\rightarrow)$

$H_B = 8$ kN $(\leftarrow)$

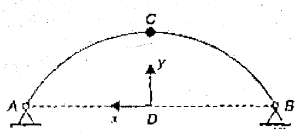
(c) BM at a section 6m from left:



Here, y can be obtain by the property of circle.



(i)



(ii)

$4 \times (2R - 4) = 8 \times 8$

$(2R - 4) = 2 \times 8$

$2R = 20$

$R = 10$ m

The equation of circle with respect to origin at O

$$x^2 + y^2 = R^2 = 10^2 \quad \text{...(i)}$$

The equation of circle with respect to origin at D.

Shifted coordinate are, $x = X$, $y = Y + 6$

$\therefore \quad X^2 + (Y + 6)^2 = 10^2$

Here, $x = 2$ m,

$\therefore \quad 2^2 + (y + 6)^2 = 10^2$

$y = 3.8$ m

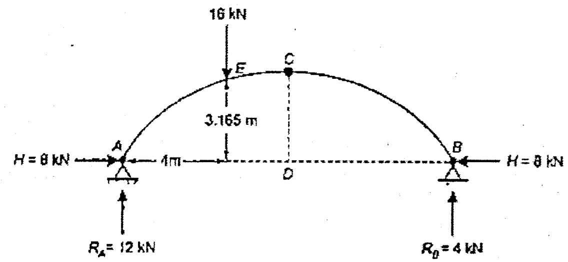
BM at a section 6 m from left,

$M_x = R_A \times 6 - H_A \times y - 16 \times 2$

$= 12 \times 6 - 8 \times 3.8 - 16 \times 2 = 9.6$ kNm

(d) Maximum positive and negative BM

(i) **Maximum positive BM:** Maximum positive BM will occur in portion AC under the point load

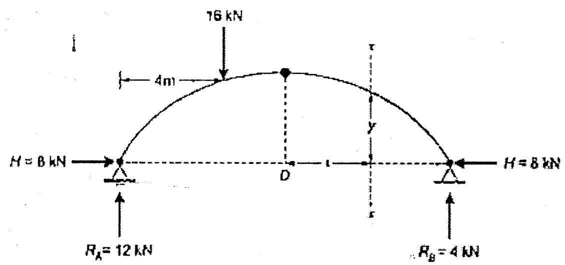


y at 4 m from left $= \sqrt{10^2 - 4^2} - 6 = 3.165$ m

Max. positive BM, $M_D = R_A \times 4 - H \times 3.165$

$= 12 \times 6 - 8 \times 3.165 = 22.68$ kN-m

(ii) **Maximum negative BM:** Maximum negative BM will occur in portion CB, consider a section x-x at a distance x from D.



$$y = \sqrt{R^2 - x^2} - \sqrt{R^2 - \left(\frac{l}{2}\right)^2}$$

$$= \sqrt{10^2 - x^2} - \sqrt{10^2 - 8^2} = \sqrt{100 - x^2} - 6$$

BM at section x-x,

$$M_x = R_A(8 + x) - H \times y - 16 \times (4 + x)$$

$$= 12(8 + x) - 8 \times \left[\sqrt{100 - x^2} - 6\right] - 16 \times (4 + x)$$

$$= 12 \times 8 + 12x - 8\sqrt{100 - x^2} + 8 \times 6 - 16 \times 4 - 16x$$

$$= 80 - 4x - 8\sqrt{100 - x^2} \qquad \text{...(i)}$$

For maximum negative BM,

$$\frac{dM_x}{dx} = 0$$

$$\Rightarrow \quad 0 - 4 \times 1 - 8 \times \frac{1}{2} \times \frac{1}{\sqrt{100 - x^2}} \times -2x = 0$$

$$\Rightarrow \quad -4 + \frac{8x}{\sqrt{100 - x^2}} = 0$$

$$\Rightarrow \quad \frac{8x}{\sqrt{100 - x^2}} = 4$$

$$\Rightarrow \quad (2x)^2 = \left(\sqrt{100 - x^2}\right)^2$$

$$x = 4.47 \text{ m}$$

Substituting value of x in (i), we get

$\therefore$ Maximum negative BM $= 80 - 4 \times 4.47 - 8\sqrt{100 - 20}$

$$= -9.434 \text{ kNm}$$

4.4.5 Temperature Effect on Three Hinged Arches

Three hinged arch is a determinate structure. If arch is initially unloaded (neglecting self weight) then due to rise of temperature there will be no horizontal or vertical reaction included.

The rise of temperature increases the length of the arch. Since the ends A and B do not move hence the crown hinge will rise to D from C. $CD = \delta$ is the increase in rise of the arch and is given by

$$\delta h = \delta = \left(\frac{l^2 + 4h^2}{4h}\right)\alpha T$$

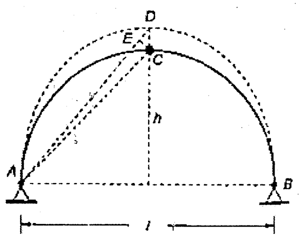


Fig. 4.13

If temperature change is applied on loaded arch then horizontal thrust will changed. The change in horizontal thrust due to change in the rise of arch is given by

$$\frac{\delta H}{H} = -\frac{dh}{h}$$

Thus there is decrease in horizontal thrust due to rise in temperature.

$\therefore$ decrease in horizontal thrust $= \frac{\delta}{h} \times H$ $\quad [\because dh = \delta]$

4.5 Two Hinges Arches

Two hinge arch is an indeterminate structure to first degree. Hence to analyses it one additional compatibility condition is required. R_A and R_B can be determined by taking moment about either end. The horizontal thrust at each support may be determined from the condition that the horizontal displacement of the either end with respect to other is zero.

$$\therefore \quad \frac{\partial U}{\partial H} = 0 \qquad \text{...(i)}$$

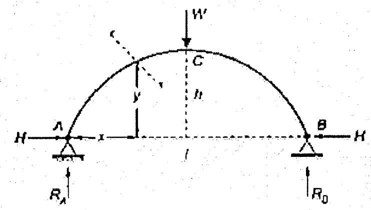


Fig. 4.14

Where, U = the total strain energy stored due to bending

$$U = \int \frac{M_x^2 ds}{2EI}$$

$$M_x = R_A x - H.y$$

Note that $R_A . x$ is beam moment due to vertical forces

$$\therefore \quad R_A x = M \text{ (say)}$$

$$\Rightarrow \quad M_x = M - Hy$$

Hence,

$$U = \int \frac{(M - Hy)^2 ds}{2EI}$$

$$\Rightarrow \quad \frac{\partial U}{\partial H} = \int 2 \frac{(M - Hy)(-y) ds}{2EI} = 0$$

$$\Rightarrow \quad -\int \frac{My\,ds}{EI} + \int \frac{H y^2\,ds}{EI} = 0$$

$$\therefore \quad H = \frac{\int \frac{My\,ds}{EI}}{\int \frac{y^2\,ds}{EI}}$$

if EI is constant then

$$H = \frac{\int My\,ds}{\int y^2\,ds}$$

4.5.1 Two Hinged Semicircular Arch Subjected to Concentrated Load at Crown

$\Sigma F_y = 0$; $\quad R_A + R_B - W = 0$

$$\Rightarrow \quad R_A + R_B = W \qquad \text{...(i)}$$

$\Sigma M_A = 0$; $\quad R_A \times 2R - W \times R = 0$

$$\Rightarrow \quad R_B = \frac{W}{2}$$

From (ii),

$$R_A = \frac{W}{2}$$

$$H = \frac{\int M y \, ds}{\int y^2 ds}$$

Where, M = beam moment

$$M = R_A x = \frac{W}{2} . x$$

$$\therefore \quad H = \frac{2\int_0^{\pi/2} \frac{W}{2} R(1-\cos\theta) R\sin\theta . Rd\theta}{2\int_0^{\pi/2} R^2 \sin^2\theta . Rd\theta}$$

$$= \frac{WR^3}{2} \frac{\int_0^{\pi/2} (1-\cos\theta)\sin\theta d\theta}{R^3 \int_0^{\pi/2} \sin^2\theta d\theta}$$

$$= \frac{\frac{WR^3}{2}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta}{\frac{\pi}{4}R^3} = \frac{2W}{\pi}\left[-\cos\theta - \frac{\cos 2\theta}{4}\right]_0^{\pi/2}$$

$$H = \frac{W}{\pi}$$

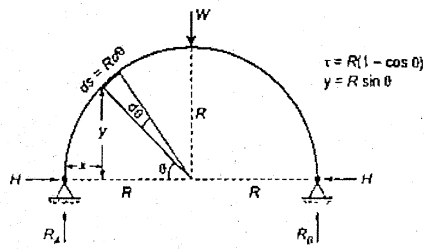


Fig. 4.15

Special Case

If concentrated load W acts at any point which makes an angle α from horizontal on semicircular 2-hinge arch.

The horizontal thrust H is given by

$$H = \frac{W}{\pi}\sin^2\alpha$$

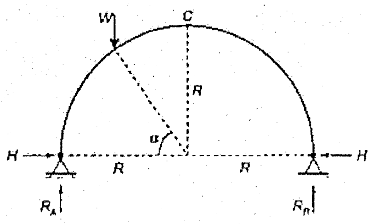


Fig. 4.16

4.5.2 Two Hinged Semicircular Arch Subjected to UDL over Entire Span

$$M = R_A . x - \frac{wx^2}{2}$$

$$= w.R.R(1-\cos\theta) - wR^2\frac{(1-\cos\theta)^2}{2}$$

$$= wR^2\left[(1-\cos\theta) - \frac{(1-\cos\theta)^2}{2}\right]$$

$$= wR^2\left[(1-\cos\theta)\frac{(1+\cos\theta)}{2}\right]$$

$$= \frac{wR^2}{2}(1-\cos^2\theta) = \frac{wR^2}{2}\sin^2\theta$$

$$H = \frac{2\int_0^{\pi/2}\frac{Myds}{EI}}{2\int_0^{\pi/2}\frac{y^2ds}{EI}}$$

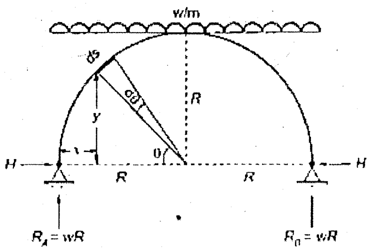


Fig. 4.17

$$\Rightarrow \quad H = \frac{\int_0^{\pi/2}\frac{wR^2}{2}\sin^2\theta . R\sin\theta Rd\theta}{\int_0^{\pi/2} R^2\sin^2\theta . Rd\theta} = \frac{\frac{wR}{2}\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta d\theta}{\int_0^{\pi/2}\sin^2\theta d\theta} \qquad ...(ii)$$

Let,

$$\cos\theta = t$$

$$-\sin\theta \, d\theta = dt$$

$$\therefore \quad \int_0^{\pi/2}(1-\cos^2\theta)\sin\theta d\theta = -\int_1^0 (1-t^2)dt$$

$$= -\left[t - \frac{t^3}{3}\right]_1^0 = -\left[-1+\frac{1}{3}\right] = \frac{2}{3}$$

From (i), we get

$$H = \frac{\frac{wR}{2}\times\frac{2}{3}}{\frac{\pi}{4}} = \frac{4}{3}\frac{wR}{\pi}$$

4.5.3 Two Hinge Parabolic Arch Subjected to UDL over Entire Span

The moment of inertia of parabolic arch varies as $I = I_0 \sec\theta$, where I_0 is moment of inertia about NA at crown and θ is angle of tangent with the horizontal i.e. $\frac{dy}{dx} = 0$

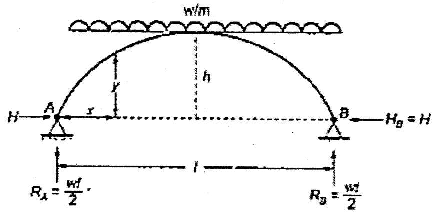


Fig. 4.18

$$y = \frac{4h}{l^2}x(l-x)$$

$$I = I_0 \sec\theta$$

$$\frac{dy}{ds} = \sin\theta$$

and

$$\frac{dx}{ds} = \cos\theta$$

$$\therefore \quad ds = dx\sec\theta$$

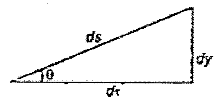


Fig. 4.19

Beam moment,

$$M = R_A . x - \frac{wx^2}{2} = \frac{wl}{2}x - \frac{wx^2}{2}$$

we know that,

$$H = \frac{\int\frac{Myds}{EI}}{\int\frac{y^2ds}{EI}} = \frac{2\int_0^{l/2}\frac{\left(\frac{wl}{2}x - \frac{wx^2}{2}\right)\frac{4h}{l^2}x(l-x).dx\sec\theta}{EI_0\sec\theta}}{\left(\frac{4h}{l^2}\right)^2\int_0^{l/2}\frac{x^2(l-x)^2dx\sec\theta}{EI_0\sec\theta}}$$

$$\Rightarrow \quad H = \frac{\frac{w}{2}\frac{4h}{l^2}\int_0^{l/2}x^2(l-x)^2dx}{\left(\frac{4h}{l^2}\right)^2\int_0^{l/2}x^2(l-x)^2dx} = \frac{wl^2}{8h}$$

Remember: The radial shear and bending moment at any section in a 2-hinge parabolic arch subjected to UDL over entire span are always zero.

4.5.4 Two Hinged Parabolic Arch Subjected to Concentrated Load at Crown

The moment of inertia of parabolic arch varies as $I = I_0 \sec\theta$, where I_0 is moment of inertia about NA at crown.

$$y = \frac{4h}{l^2}x(l-x)$$

$$I = I_0 \sec\theta$$

Beam moment, $M = R_A . x = \frac{W}{2}.x$

We know that,

$$H = \frac{\int \frac{Myds}{EI}}{\int \frac{y^2ds}{EI}}$$

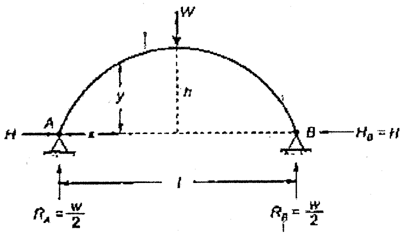


Fig. 4.20

$$\therefore \quad H = \frac{2\int_0^{l/2} \frac{\frac{W}{2}x\,\frac{4h}{l^2}x(l-x)dx\sec\theta}{EI_0\sec\theta}}{2\int_0^{l/2} \frac{\frac{16h^2}{l^4}x^2(l-x)^2dx\sec\theta}{EI_0\sec\theta}}$$

$$H = \frac{Wl^2}{8h}\frac{\int_0^{l/2} x^2(l-x)dx}{\int_0^{l/2} x^2(l-x)^2dx} \quad ...(i)$$

On solving (i), we get

$$H = \frac{125}{128}\frac{Wl}{h}$$

4.5.5 Temperature Effect on Two Hinged Arches

Two hinged arch is an indeterminate structure. If two hinged arch is unloaded then there will be no horizontal thrust. But on heating, horizontal thrust will produced.

Let before heating arch is unloaded and there is no horizontal thrust and vertical reaction. On heating arch try to expand freely in the direction of *AB*. If the free expansion is permitted then $l\alpha T$ will be the free expansion but both ends are hinged hence free expansion is not permitted. Therefore horizontal thrust will introduce.

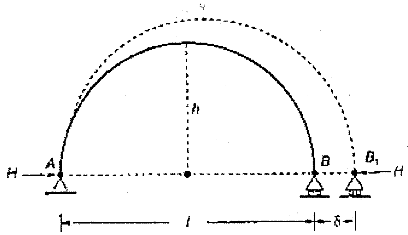


Fig. 4.21

Net expansion in the direction of *AB* is zero,

$$\therefore \quad l\alpha T - \frac{\partial U}{\partial H} = 0$$

$$\therefore \quad \frac{\partial U}{\partial H} = l\alpha T \quad ...(i)$$

Where, $U = \int \frac{(-Hy)^2ds}{2EI}$ $\quad [\because M = 0 - Hy]$

$$\therefore \quad \frac{\partial U}{\partial H} = \int \frac{2(Hy).y\,ds}{2EI}$$

$$\frac{\partial U}{\partial H} = H\int \frac{y^2ds}{EI} \quad ...(ii)$$

From (i) and (ii),

$$H\int \frac{y^2ds}{EI} = l\alpha T$$

$$\therefore \quad H = \frac{l\alpha T}{\int \frac{y^2ds}{EI}}$$

Special Case - 1 :
For a two hinged semicircular arch

$$H = \frac{4EI\alpha T}{\pi R^2}$$

Special Case - 2 :
For a two hinged parabolic arch

$$H = \frac{15}{8}\frac{EI_0\alpha T}{h^2}$$

4.5.6 Reaction Locus for a Two Hinged Arch

The reaction locus for a two hinged arch is the locus of the point of intersection of the two resultant reactions at the supports as a point load moves on the span of the arch.

1. **For two hinged Semicircular Arch:** The reaction locus is a straight line parallel to the line joining abutments and at a height of $\frac{\pi R}{2}$ above the base.

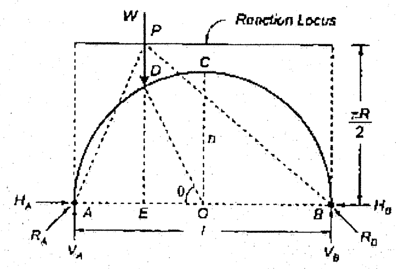


Fig. 4.22

2. **For two hinged Parabolic Arch:** The reaction locus for a parabolic arch is a curve whose equation is

$$y = PE = \frac{1.6hl^2}{l^2 + lx - x^2}$$

at $x = 0$, $PE = 1.6$ h

$x = \frac{l}{2}$, $y = PE = 1.28$ h

$x = l$, $y = PE = 1.6$ h

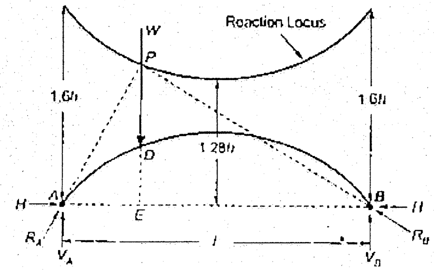


Fig. 4.23

4.6 The Linear Arch or Theoretical Arch

Linear arch is an imaginary structure which is obtained by a funicular polygon for given arch which has all joint like a truss and loaded at joints only. All the member of theoretical arch subjected to axial compressive forces only and there is no shear force and bending moment in any member of theoretical arch.

Imagine a structure *ACDEB* consisting of member *AC, CD, DE* and *EB*, which are pin jointed like a truss, having shape of funicular polygon and loaded with W_1, W_2 and W_3 at joints, such a structure is called **theoretical arch.**

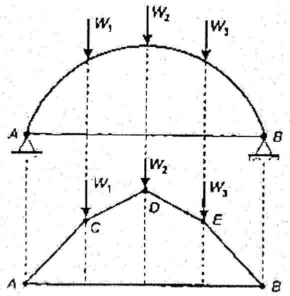


Fig. 4.24 Theoretical or linear arch

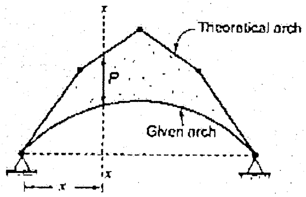


Fig. 4.25 Eddy theorem

Eddy Theorem: If a linear arch is superimposed on a given arch then bending moment at any section on given arch is proportional to the ordinate of intersect between given arch and theoretical arch.

$$M_x \propto p$$

where p = intersect between given arch and theoretical arch.

4.7 Influence Line Diagrams for Three Hinged Arches

4.7.1 ILD for Horizontal Thrust (*H*)

Consider a section x - x at a distance '*a*' from left support at point *D*, the rise of point *D* is y_D.

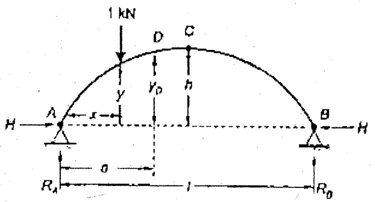


Fig. 4.26

Case - 1 :

When unit load is in portion *AC*

$$\Sigma M_B = 0$$

$$\Rightarrow \quad R_A \times L - 1\,(L - x) = 0$$

$$\Rightarrow \quad R_A = \frac{(L - x)}{L}$$

Also, $\quad \Sigma M_A = 0$

$$\therefore \quad -R_B \times L + 1 \times x = 0$$

$$\Rightarrow \quad R_B = \frac{x}{L}$$

Taking, $\quad \Sigma M_c = 0$ (From right)

$$\Rightarrow \quad -R_B \times \frac{L}{2} + H \times h = 0$$

$$\Rightarrow \quad H \times h = \frac{x}{L} \times \frac{L}{2}$$

$$\therefore \quad H = \frac{x}{2h} \text{ (linear)}$$

when $x = 0$, $H = 0$

when $x = \frac{L}{2}$, $H = \frac{L}{4h}$

Case - 2 :

When unit load is in portion *CB*.

Taking $\quad \Sigma M_C = 0$ (from left)

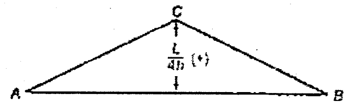
$$\therefore \quad R_A \times \frac{L}{2} - H \times h = 0$$

$$\Rightarrow \quad H \times h = \frac{(L - x)}{L} \times \frac{L}{2}$$

$$\Rightarrow \quad H = \frac{(L - x)}{2h} \text{ (linear)}$$

when $x = \frac{L}{2}$, $H = \frac{L}{4h}$

when $x = L$, $H = 0$



ILD for Term H_D

Fig. 4.27

4.7.2 ILD for Bending Moment at *D* (M_D)

Bending moment at any section = beam moment − *Hy*

Hence *ILD* can be drawn by super imposing beam *ILD* and (*Hy*) *ILD*.

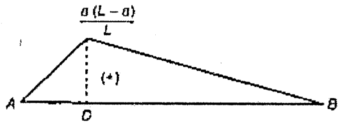


Fig. 4.28 (i) Beam ILD for M_D

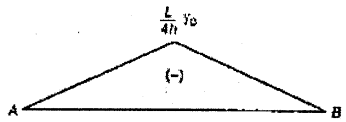


Fig. 4.28 (ii) ILD for term Hy

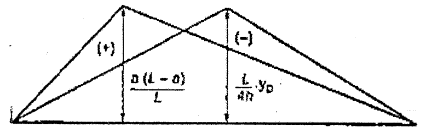


Fig. 4.28 (iii) ILD for M_D (for given arch)

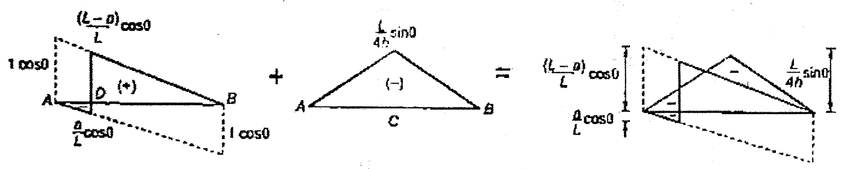
4.7.3 ILD for Radial Shear at D(S_D)

Radial shear at any section is given by

$$S_t = V_t \cos\theta - H \sin\theta$$

$$S_t = \text{(Beam shear)} \cos\theta - H \sin\theta$$

$$\therefore \quad ILD \text{ of } S_t = (\text{Beam } ILD \text{ for shear}) \times (H - ILD) \times \sin\theta$$



(i) ILD for (Beam shear) × cosθ (ii) ILD for H × sinθ (iii) ILD for S_D

Fig. 4.29

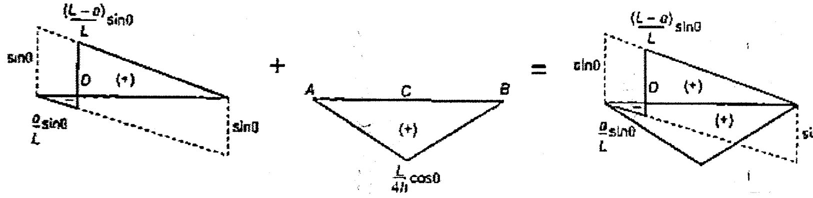
4.7.4 ILD for Normal Thrust at D (N_D)

The normal thrust at any section is given by.

$$N_x = V_x \sin\theta + H\cos\theta$$

$$N_x = (\text{Beam shear})\sin\theta - H\cos\theta$$

$\therefore$ ILD of N_D = (ILD for beam shear at D) × sin θ (ILD for H_D) × cos θ



(i) Beam shear × sinθ (ii) ILD for H_D × cosθ (iii) ILD for N_D

Fig. 4.30

4.8 Influence for Two Hinged Arches

4.8.1 ILD for Horizontal Thrust (H)

The horizontal thrust due to a concentrated load W at a distance x from end A is given by,

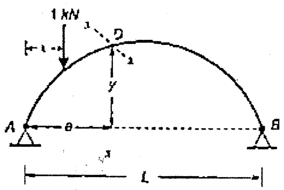


Fig. 4.31

$$H = \frac{5}{8}\frac{W}{L^3 h}x(L-x)(L^2+xL-x^2)$$

If $W = 1\text{ kN}$

Then $H = \frac{5}{8hL^3}x(L-x)(L^2+xL-x^2)$

$\Rightarrow$ $H = \frac{5}{8h}x\left(1-\frac{2x^2}{L^2}+\frac{x^3}{L^3}\right)$ (4°-curve)

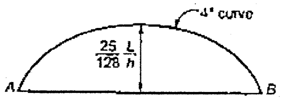


Fig. 4.32

4.8.2 ILD for Bending Moment at D (M_D)

Bending moment at any section is given by

$$M_x = \text{beam moment} - Hy$$

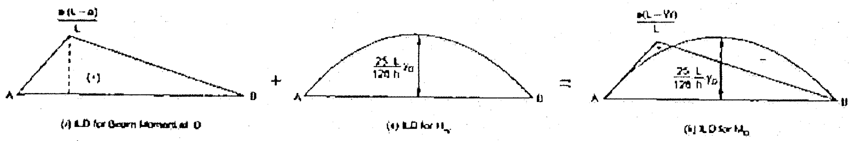


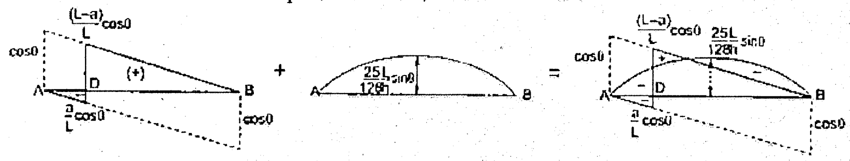
Fig. 4.33

4.8.3 ILD for Radial Shear at D (S_D)

Radial shear at any section is given by

$$S_x = V_x\cos\theta - H_x\sin\theta$$

$$S_x = (\text{Beam shear}) \times \cos\theta - H \times \cos\theta$$



(i) ILD for (Beam Shear) X cosθ (ii) ILD for HX sinθ (ii) ILD for radial shear (S_D)

Fig. 4.34

4.8.4 ILD for Normal Thrust at D (N_D)

Normal thrust at any section is given by

$$N_x = V_x\sin\theta + H_x\cos\theta$$

$$N_x = (\text{Beam shear}) \times \sin\theta + H_x \times \cos\theta$$

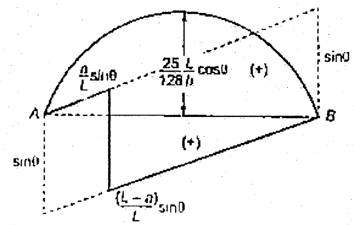


Fig. 4.35 ILD for Normal thrust N_D

Summary

- A three hinge arch is statically determinate structure. For a three hinge arch bending moment, radial shear and normal thrust can be computed by following formulae

$$M_x = R_A \cdot x \quad H_y$$

$$S_x = R_A\cos\theta - H\sin\theta$$

$$N_x = R_A\sin\theta + H\cos\theta$$

where, M_x = Bending moment, S_x = Radial shear, N_x = Normal thrust

H = Horizontal thrust at ends, R_A = Vertical reaction at end A

- A parabolic arch, either two hinge or three hinge carrying a UDL over entire span is free from bending moment and radial shear.
- The horizontal reaction for 2-hinge semicircular arch subjected to concentrated load at crown is

$$H = \frac{W}{\pi}$$

Objective Brain Teasers

Q.1 In a two hinged arch an increase in temperature induces
(a) no bending moment in the arch rib
(b) uniform bending moment in the arch rib
(c) maximum bending at the crown
(d) minimum bending moment at the crown

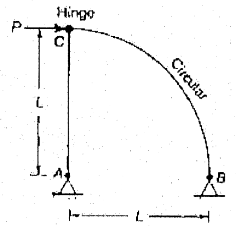
Q.2 A symmetrical parabolic arch of span 20 meters and rise 5 m is hinged at the springings. It supports a uniformly distributed load of 2 tonnes per meter run of the span. The horizontal thrust in tonnes at each of the springing is
(a) 8 t (b) 16 t
(c) 20 t (d) zero

Q.3 A symmetrical three-hinged parabolic arch of span L and rise h is hinged at springing and crown. It is subjected to a UDL of w throughout the span. What is the bending moment at a section $L/4$ from the left support

(a) $\frac{wL^2}{8}$ (b) $\frac{wL^2}{16}$

(c) $\frac{wL^2}{8h}$ (d) zero

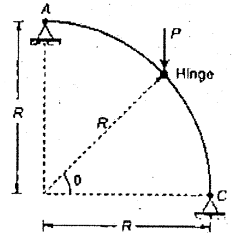
Q.4 Vertical reaction at support B of the structure is



(a) P (b) $P\sqrt{2}$
(c) $\frac{P}{\sqrt{2}}$ (d) $P/2$

Q.5 A parabolic arch, symmetrical with hinges at centre and ends, carries a point load P at distance x from left support. The arch has a span of 20 m and rise of 5 m. What is the value of x if the left hinge reaction inclined with a slope of three vertical to two horizontal?
(a) 4 m (b) 5 m
(c) 6 m (d) 8 m

Q.6 A three hinged arch shown in the figure is quarter of a circle. If the vertical and horizontal components of reaction at A are equal, the value of θ is



(a) 60° (b) 45°
(c) 30° (d) none

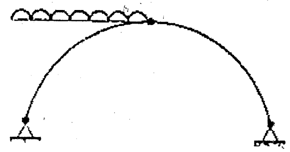
Q.7 A three hinged parabolic arch ABC has a span of 20 m and a central rise of 4 m. The arch has hinges at the ends and at the centre. A train of two point loads of 20 kN and 10 kN 5 m apart crosses the arch from left to right with 20 kN load leading. The maximum thrust induced at the support is
(a) 25.00 kN (b) 28.13 kN
(c) 31.25 kN (d) 32.81 kN

Q.8 A uniformly distributed load of 2 kN/m covers left half of the span of a three-hinged parabolic arch, span 40 m and central rise 10 m. Which of the following statements relating to different function at the loaded quarter point are correct?

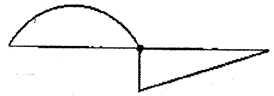
1. The slope is $\tan^{-1}\left(\frac{1}{2}\right)$.
2. The normal thrust is $6\sqrt{5}$ kN.
3. The shear force is not zero.
4. The bending moment is zero.

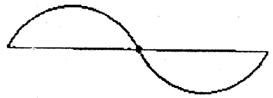
Select the correct answers using the codes given below:
(a) 1, 2 and 4 (b) 2 and 3
(c) 1 and 3 (d) 3 and 4

Q.9 For the three hinged parabolic arch shown below, which one among the following represents the bending moment diagram?



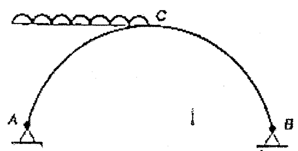
(a)


(b)


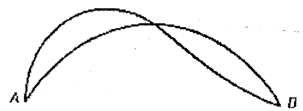
(c)


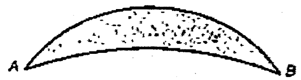
(d)

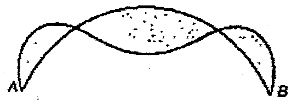

Q.10 For the two hinged parabolic arch as shown in figure above, which one of the following diagrams represents the shape of the bending variation?



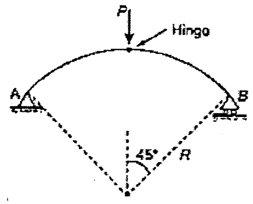
(a)


(b)


(c)


(d)


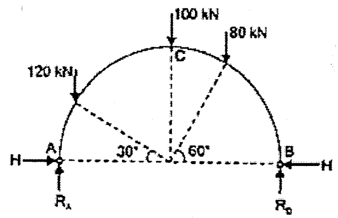
Q.11 A three hinged arch is subjected to point load at crown. The magnitude of horizontal reaction at A is



(a) $\frac{P}{2(\sqrt{2}-1)}$ (b) $\frac{P}{2}$

(c) $\frac{P}{(\sqrt{2}-1)}$ (d) $\frac{P}{2\sqrt{2}}$

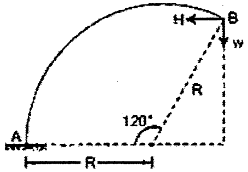
Q.12 A two hinged semicircular arch of radius 20 meters carries a load system shown in figure. The horizontal thrust at each support is (Assume uniform flexural rigidity)



(a) $\frac{180}{\pi}$ kN (b) $\frac{190}{\pi}$ kN

(c) $\frac{200}{\pi}$ kN (d) None of these

Q.13 A circular arch rib of constant flexural rigidity is fixed at *A* as shown in figure. The end '*B*' is tied horizontally with a force *H* such that it can only move vertically when a load '*W*' is hung at *B*. The ratio of *H/w* is



(a) 1.0 (b) 0.55
(c) 0.65 (d) 2.5

Answers

1. (c)	2. (c)	3. (d)	4. (a)	5. (b)
6. (d)	7. (c)	8. (c)	9. (c)	10. (b)
11. (a)	12. (b)	13. (c)		

Hints and Explanations:

1. **(c)**
Increase in temperature in a two hinged arch (degree of indeterminacy one) will cause horizontal thrust only.
Moment due to horizontal thrust is $-Hy$.
So, maximum bending moment will be at crown as crown has highest value of y.

3. **(d)**
The thrust of support,

$$H = \frac{wl^2}{8h}$$

∴ bending moment at any section,

$$M_x = R_A\, x - Hy$$

$$= \frac{wl}{2}x - \frac{wl^2}{8h} \times \frac{4h}{l^2}x(l-x) = 0$$

Thus at every section bending moment is zero if three hinged parabolic arch is subjected to UDL of intensity w through out the span.

5. **(b)**
Vertical reaction at *A*,

$$R_A = \frac{P(20-x)}{20}$$

Horizontal thrust,

$$H = \frac{R_B \times 10}{5} = 2R_B$$

$$\therefore \quad H = \frac{2 \times P \times x}{20} = \frac{Px}{10}$$

given that the left hinge reaction is inclined with a slop of two vertical to one horizontal

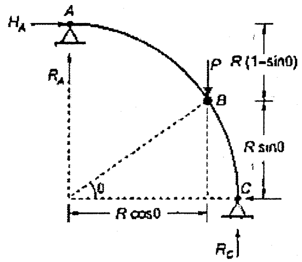
$$\therefore \quad 2R_A = 3H$$

$$2\frac{P(20-x)}{20} = 3\frac{Px}{10}$$

$$(20-x) = 3x$$

$$\therefore \quad x = 5\text{m}$$

6. **(d)**



Given, $H_A = R_A$

Now, $\Sigma M_B = 0$ (from top)

$$R_A.R\cos\theta + H_A.R(1-\sin\theta) = 0$$

$$H_A\, R\cos\theta + H_A\, R(1-\sin\theta) = 0$$

$$\cos\theta + 1 - \sin\theta = 0$$

$$\sin\theta - \cos\theta = 1$$

$$(\sin\theta - \cos\theta)^2 = 1$$

$$\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = 1$$

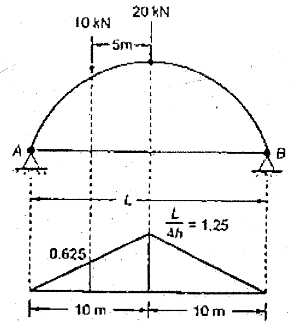
$$1 - \sin 2\theta = 1$$

$$\sin 2\theta = 0$$

$$2\theta = 0 \text{ or } 180$$

$$\theta = 0 \text{ or } 90°$$

7. **(c)**
In case of a three hinged parabolic arch, the influence line diagram for horizontal thrust is linear. Maximum thrust will be induced at the supports when 20 kN load is at the crown.



Ordinate of the *ILD* at a distance of 5 m from *A*

$$= \frac{5}{10} \times \frac{L}{4h} = \frac{1}{2} \times \left(\frac{20}{4 \times 4}\right)$$

$$= 0.625$$

Also, $\frac{L}{4h} = \frac{20}{4 \times 4} = 1.25$

Thus, the horizontal thrust,

$$H = 10 \times 0.625 + 20 \times 1.25$$

$$= 31.25 \text{ kN}$$

8. **(c)**

$$H = \frac{wL^2}{16h} = 20 \text{ kN}$$

$$R_A = \frac{3}{8}wL = 30 \text{ kN}$$

the equation of parabolic arch is

$$y = \frac{4h}{L^2}x(L-x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{4h}{L^2}(L-2x) = \tan\theta$$

at $x = \frac{L}{4}$, $\tan\theta = \frac{2h}{l}$

for given arch, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

So statement (i) is correct.
For BMD,

$$\sin\theta = \frac{1}{\sqrt{5}}; \quad \cos\theta = \frac{2}{\sqrt{5}}$$

$$\therefore \quad M = \frac{wx}{8}(L-2x) \text{ (for loaded half)}$$

at $x = \frac{L}{4}$, $M = \frac{wL}{8}\frac{L}{4} \times \frac{L}{2} = \frac{wL^2}{64}$

$$M = 50 \text{ kN-m}$$

So bending moment is not negative hence statement (4) is wrong.

Normal thrust, $N = R_A \sin\theta + H\cos\theta$

$$= 14\sqrt{5}\text{kN}$$

Radial shear, $S = R_A\cos\theta - H\sin\theta$

$$= 8\sqrt{5}\text{kN}$$

9. **(c)**
A point *c* moment will be zero,

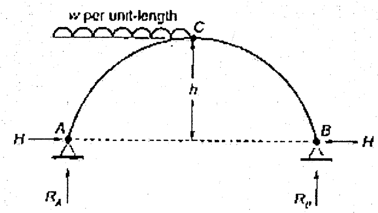
$$R_A = \frac{3}{8}wL$$

$$R_B = \frac{wL}{8}$$

and

$$H = \frac{wL^2}{16h}$$

$$y = \frac{4h}{L^2}x(L-x)$$



Portion AC: M_x (x from *A*) for *AC*

$$= R_A.x - \frac{wx^2}{2} - H_y \quad \left[x \le \frac{L}{2}\right]$$

$$= \frac{3}{8}wLx - \frac{wx^2}{2} - \frac{w}{4}x(L-x)$$

$$= \frac{3wLx}{8} - \frac{wx^2}{2} - \frac{wLx}{4} + \frac{wx^2}{4}$$

$$= \frac{wLx}{8} - \frac{wx^2}{4}$$

$$\Rightarrow \quad \frac{wx}{8}(L-2x) > 0$$

Portion BC:

$$M_x (x \text{ from } B) = \frac{wL}{8}x - \frac{wx}{4}(L-x)$$

$$= -\frac{wL}{8}x + \frac{wx^2}{4}$$

$$= -\frac{wx}{8}(L-2x) < 0$$

So bending moment is positive for AC and negative for CB and their shapes are parabolic.

11. (a)

Vertical reaction at both support = $P/2$ (by symmetry)

Now, taking moment about hinge at crown from LHS = 0

$$\Rightarrow \quad \frac{P}{2} \times \frac{R}{\sqrt{2}} - H \times \left(R - \frac{R}{\sqrt{2}}\right) = 0$$

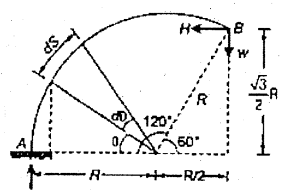
$$\Rightarrow \quad H = \frac{P}{2(\sqrt{2}-1)}$$

12. (b)

$$H = \frac{120}{\pi}\sin^2 30° + \frac{100}{\pi} + \frac{80}{\pi}\sin^2 60°$$

$$= \frac{190}{\pi} \text{kN}$$

13. (c)



$$\delta_{BH} = \int M \frac{\partial M}{\partial H} \frac{\partial S}{EI} = 0 \quad \text{...(i)}$$

Consider an element dS, the angular distance of its radius vector being at θ from A

$$M = H\left[\frac{\sqrt{3}}{2}R - R\sin\theta\right] - w\left(\frac{1}{2}R + R\cos\theta\right)$$

$$\frac{\partial M}{\partial H} = R\left(\frac{\sqrt{3}}{2} - \sin\theta\right) \text{ and } dS = Rd\theta$$

From eq. (i)

$$\int_0^{2\pi/3}\left[HR\left(\frac{\sqrt{3}}{2} - \sin\theta\right) - wR\left(\frac{1}{2} + \cos\theta\right)\right]$$

$$R\left(\frac{\sqrt{3}}{2} - \sin\theta\right)Rd\theta = 0$$

$$\therefore \frac{H}{w} = \frac{\int_0^{2\pi/3}\left(\frac{1}{2} + \cos\theta\right)\left(\frac{\sqrt{3}}{2} - \sin\theta\right)d\theta}{\int_0^{2\pi/3}\left(\frac{\sqrt{3}}{2} - \sin\theta\right)^2 d\theta}$$

$$= \frac{1.28}{1.97} = 0.65$$

Conventional Practice Questions

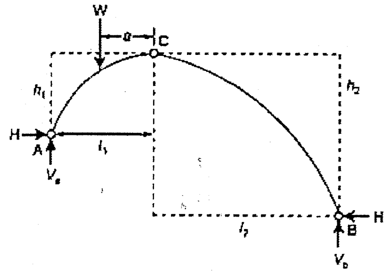
Q.1 A symmetrical three hinged parabolic arch of span 24 m and central rise 5 m carries a single point load of 10 t. Locate the position of load on the arch in order that the bending moment in the arch is zero at a section 8 m from the left hinge. For this position of the load, calculate the bending moment under the load.

Ans. x = 1.285 m, M = − 14.54 t

Q.2 A symmetrical two-hinged parabolic arch is of span 40 m and of central rise 6 m. It carries a single point load of 16 t at a horizontal distance of 15 m from the left hinge. If the moment of inertia of the cross-section of the arch varies as the secant of the slope of the arch axis, calculate from first principle, the horizontal thrust in the arch.

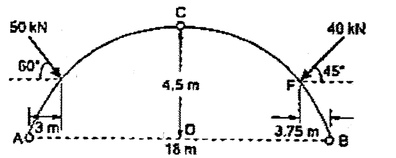
Ans. H = 19.287t

Q.3 Determine the horizontal thrust at each support for the three-hinged arch shown in figure.



Ans. $H = \frac{Wl_2(l_1 - a)}{h_1l_2 + h_2l_1}$

Q.4 A circular segmental three hinged arch hinged at the ends and at the crown has a span of 18 m and a rise of 4.50 m. The arch carries the loads as shown in figure. Find the reactions at the supports and the bending moment at the loaded points.



Ans. V_A = 43.163 kN, V_B = 28.422 kN

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