23. LOGARITHMS

IMPORTANT FACTS AND FORMULAE

I. Logarithm : If a is a positive real number, other than 1 and $a^m = x$, then we write : $m = \log_a x$ and we say that the value of log x to the base a is m.

Example :

(*i*) $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$ (*ii*) $3^4 = 81 \Rightarrow \log_3 81 = 4$

(*ii*) $2^{-3} = \frac{1}{8} \Rightarrow \log_2 \frac{1}{8} = -3$ (*iv*) $(.1)^2 = .01 \Rightarrow \log_{(.1)} .01 = 2.$

II. Properties of Logarithms :

1. $\log_a (xy) = \log_a x + \log_a y$
2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_x x = 1$
4. $\log_a 1 = 0$
5. $\log_a (x^p) = p (\log_a x)$
6. $\log_a x = \frac{1}{\log_x a}$
7. $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\log x}{\log a}$.

Remember : When base is not mentioned, it is taken as 10.

III. Common Logarithms : Logarithms to the base 10 are known as common logarithms.

IV. The logarithm of a number contains two parts, namely *characteristic* and *mantissa*.

Characteristic : The integral part of the logarithm of a number is called its ***characteristic***.

Case I : When the number is greater than 1.

In this case, the characteristic is one less than the number of digits in the left of the decimal point in the given number.

Case II : When the number is less than 1.

In this case, the characteristic is one more than the number of zeros between the decimal point and the first significant digit of the number and it is negative.

Instead of -1, -2, etc. we write, $\overline{1}$ (one bar), $\overline{2}$ (two bar), etc.

Example :

Number	*Characteristic*	*Number*	*Characteristic*
348.25	2	0.6173	$\overline{1}$
46.583	1	0.03125	$\overline{2}$
9.2193	0	0.00125	$\overline{3}$

Mantissa : The decimal part of the logarithm of a number is known is its ***mantissa.*** For mantissa, we look through log table.

SOLVED EXAMPLES

Ex. 1. ***Evaluate : (i)*** $\log_3 27$ ***(ii)*** $\log_7 \left(\frac{1}{343}\right)$ ***(iii)*** $\log_{100} (0.01)$

Sol. (i) Let $\log_3 27 = n$.

Then, $3^n = 27 = 3^3$ or $n = 3$. $\therefore \log_3 27 = 3$.

(ii) Let $\log_7 \left(\frac{1}{343}\right) = n$.

Then, $7^n = \frac{1}{343} = \frac{1}{7^3} = 7^{-3}$ or $n = -3$. $\therefore \log_7 \left(\frac{1}{343}\right) = -3$.

(iii) Let $\log_{100} (0.01) = n$.

Then, $(100)^n = 0.01 = \frac{1}{100} = (100)^{-1}$ or $n = -1$ $\therefore \log_{100} (0.01) = -1$.

Ex. 2. ***Evaluate : (i)*** $\log_7 1 = 0$ ***(ii)*** $\log_{34} 34$ ***(iii)*** $36^{\log_6 4}$

Sol. (i) We know that $\log_a 1 = 0$, so $\log_7 1 = 0$.

(ii) We know that $\log_a a = 1$, so $\log_{34} 34 = 0$.

(iii) We know that $a^{\log_a x} = x$.

Now, $36^{\log_6 4} = (6^2)^{\log_6 4} = 6^{2(\log_6 4)} = 6^{\log_6 (4^2)} = 6^{\log_6 16} = 16$.

Ex. 3. ***If*** $\log_{\sqrt{8}} x = 3\frac{1}{3}$, ***find the value of x.***

Sol. $\log_{\sqrt{8}} x = \frac{10}{3} \Leftrightarrow x = (\sqrt{8})^{10/3} = (2^{3/2})^{10/3} = 2^{\left(\frac{3}{2} \times \frac{10}{3}\right)} = 2^5 = 32$.

Ex. 4. ***Evaluate : (i)*** $\log_5 3 \times \log_{27} 25$ ***(ii)*** $\log_9 27 - \log_{27} 9$

Sol. (i) $\log_5 3 \times \log_{27} 25 = \frac{\log 3}{\log 5} \times \frac{\log 25}{\log 27} = \frac{\log 3}{\log 5} \times \frac{\log (5^2)}{\log (3^3)} = \frac{\log 3}{\log 5} \times \frac{2 \log 5}{3 \log 3} = \frac{2}{3}$.

(ii) Let $\log_9 27 = n$.

Then, $9^n = 27 \Leftrightarrow 3^{2n} = 3^3 \Leftrightarrow 2n = 3 \Leftrightarrow n = \frac{3}{2}$.

Again, let $\log_{27} 9 = m$.

Then, $27^m = 9 \Leftrightarrow 3^{3m} = 3^2 \Leftrightarrow 3m = 2 \Leftrightarrow m = \frac{2}{3}$.

$\therefore \log_9 27 - \log_{27} 9 = (n - m) = \left(\frac{3}{2} - \frac{2}{3}\right) = \frac{5}{6}$.

Ex. 5. ***Simplify :*** $\left(\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243}\right)$ **(S.S.C. 2000)**

Sol. $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243} = \log \frac{75}{16} - \log \left(\frac{5}{9}\right)^2 + \log \frac{32}{243} = \log \frac{75}{16} - \log \frac{25}{81} + \log \frac{32}{243}$

$= \log \left(\frac{75}{16} \times \frac{32}{243} \times \frac{81}{25}\right) = \log 2$.

Ex. 6. ***Find the value of x which satisfies the relation***

$\log_{10} 3 + \log_{10} (4x + 1) = \log_{10} (x + 1) + 1$ **(M.B.A. 2002)**

Sol. $\log_{10} 3 + \log_{10} (4x + 1) = \log_{10} (x + 1) + 1$

$\Leftrightarrow \log_{10} 3 + \log_{10} (4x + 1) = \log_{10} (x + 1) + \log_{10} 10$

$\Leftrightarrow \log_{10} [3 (4x + 1)] = \log_{10} [10 (x + 1)]$

$\Leftrightarrow 3 (4x + 1) = 10 (x + 1) \Leftrightarrow 12x + 3 = 10x + 10 \Leftrightarrow 2x = 7 \Leftrightarrow x = \frac{7}{2}.$

Ex. 7. ***Simplify :*** $\left[\frac{1}{\log_{xy} (xyz)} + \frac{1}{\log_{yz} (xyz)} + \frac{1}{\log_{zx} (xyz)}\right]$

Sol. Given expression $= \log_{xyz} (xy) + \log_{xyz} (yz) + \log_{xyz} (zx)$

$= \log_{xyz} (xy \times yz \times zx) = \log_{xyz} (xyz)^2$ $\left[\because \log_a x = \frac{1}{\log_x a}\right]$

$= 2 \log_{xyz} (xyz) = 2 \times 1 = 2.$

Ex. 8. ***If*** $\log_{10} 2 = 0.30103$, ***find the value of*** $\log_{10} 50$. **(C.B.I. 1997)**

Sol. $\log_{10} 50 = \log_{10} \left(\frac{100}{2}\right) = \log_{10} 100 - \log_{10} 2 = 2 - 0.30103 = 1.69897.$

Ex. 9. ***If*** $\log 2 = 0.3010$ ***and*** $\log 3 = 0.4771$, ***find the values of :***

(i) $\log 25$ ***(ii)*** $\log 4.5$

Sol. (i) $\log 25 = \log\left(\frac{100}{4}\right) = \log 100 - \log 4 = 2 - 2 \log 2 = (2 - 2 \times 0.3010) = 1.398.$

(ii) $\log 4.5 = \log\left(\frac{9}{2}\right) = \log 9 - \log 2 = 2 \log 3 - \log 2$

$= (2 \times 0.4771 - 0.3010) = 0.6532.$

Ex. 10. ***If*** $\log 2 = 0.30103$, ***find the number of digits in*** 2^{56}.

Sol. $\log (2^{56}) = 56 \log 2 = (56 \times 0.30103) = 16.85768.$

Its characteristic is 16. Hence, the number of digits in 2^{56} is 17.

EXERCISE 23

(OBJECTIVE TYPE QUESTIONS)

Directions : ***Mark (✓) against the correct answer :***

1. The value of $\log_2 16$ is : **(M.B.A. 2003)**

(a) $\frac{1}{8}$ (b) 4 (c) 8 (d) 16.

2. The value of $\log_{343} 7$ is :

(a) $\frac{1}{3}$ (b) -3 (c) $-\frac{1}{3}$ (d) 3

3. The value of $\log_5 \left(\frac{1}{125}\right)$ is :

(a) 3 (b) -3 (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

4. The value of $\log_{\sqrt{2}} 32$ is :

(a) $\frac{5}{2}$ (b) 5 (c) 10 (d) $\frac{1}{10}$

5. The value of $\log_{10} (.0001)$ is :

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) -4 (d) 4

6. The value of $\log_{(.01)} (1000)$ is :

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

7. The logarithm of 0.0625 to the base 2 is :

(a) -4 (b) -2 (c) 0.25 (d) 0.5

8. If $\log_3 x = -2$, then x is equal to :

(a) -9 (b) -6 (c) -8 (d) $\frac{1}{9}$

9. If $\log_8 x = \frac{2}{3}$, then the value of x is :

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) 3 (d) 4

10. If $\log_x \left(\frac{9}{16}\right) = -\frac{1}{2}$, then x is equal to :

(a) $-\frac{3}{4}$ (b) $\frac{3}{4}$ (c) $\frac{81}{256}$ (d) $\frac{256}{81}$

11. If $\log_x 4 = 0.4$, then the value of x is : **(Asstt. Grade, 1998)**

(a) 1 (b) 4 (c) 16 (d) 32

12. If $\log_{10000} x = -\frac{1}{4}$, then x is equal to :

(a) $\frac{1}{10}$ (b) $\frac{1}{100}$ (c) $\frac{1}{1000}$ (d) $\frac{1}{10000}$

13. If $\log_x 4 = \frac{1}{4}$, then x is equal to :

(a) 16 (b) 64 (c) 128 (d) 256

14. If $\log_x (0.1) = -\frac{1}{3}$, then the value of x is :

(a) 10 (b) 100 (c) 1000 (d) $\frac{1}{1000}$

15. If $\log_{32} x = 0.8$, then x is equal to :

(a) 25.6 (b) 16 (c) 10 (d) 12.8

16. If $\log_x y = 100$ and $\log_2 x = 10$, then the value of y is : **(S.S.C. 1999)**

(a) 2^{10} (b) 2^{100} (c) 2^{1000} (d) 2^{10000}

17. The value of $\log_{(-1/3)} 81$ is equal to :

(a) -27 (b) -4 (c) 4 (d) 27

18. The value of $\log_{2\sqrt{3}} (1728)$ is :

(a) 3 (b) 5 (c) 6 (d) 9

19. $\frac{\log \sqrt{8}}{\log 8}$ is equal to : **(I.A.F. 2002)**

(a) $\frac{1}{\sqrt{8}}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

20. Which of the following statements is not correct ? **(M.B.A. 2003)**

(a) $\log_{10} 10 = 1$ (b) $\log (2 + 3) = \log (2 \times 3)$

(c) $\log_{10} 1 = 0$ (d) $\log (1 + 2 + 3) = \log 1 + \log 2 + \log 3$

21. The value of $\log_2(\log_5 625)$ is :
(a) 2 (b) 5 (c) 10 (d) 15

22. If $\log_2[\log_3(\log_2 x)] = 1$, then x is equal to :
(a) 0 (b) 12 (c) 128 (d) 512

23. The value of $\log_2 \log_2 \log_3 \log_3 27^3$ is :
(a) 0 (b) 1 (c) 2 (d) 3

24. If $a^x = b^y$, then : **(Hotel Management, 2001)**
(a) $\log\frac{a}{b} = \frac{x}{y}$ (b) $\frac{\log a}{\log b} = \frac{x}{y}$ (c) $\frac{\log a}{\log b} = \frac{y}{x}$ (d) None of these

25. log 360 is equal to :
(a) $2\log 2 + 3\log 3$ (b) $3\log 2 + 2\log 3$
(c) $3\log 2 + 2\log 3 - \log 5$ (d) $3\log 2 + 2\log 3 + \log 5$

26. The value of $\left(\frac{1}{3}\log_{10} 125 - 2\log_{10} 4 + \log_{10} 32\right)$ is :
(a) 0 (b) $\frac{4}{5}$ (c) 1 (d) 2

27. $2\log_{10} 5 + \log_{10} 8 - \frac{1}{2}\log_{10} 4 = ?$ **(M.B.A. 2002)**
(a) 2 (b) 4 (c) $2 + 2\log_{10} 2$ (d) $4 - 4\log_{10} 2$

28. If $\log_a(ab) = x$, then $\log_b(ab)$ is : **(M.A.T. 2002)**
(a) $\frac{1}{x}$ (b) $\frac{x}{x+1}$ (c) $\frac{x}{1-x}$ (d) $\frac{x}{x-1}$

29. If $\log 2 = x$, $\log 3 = y$ and $\log 7 = z$, then the value of $\log(4\cdot\sqrt[3]{63})$ is :
(a) $2x + \frac{2}{3}y - \frac{1}{3}z$ (b) $2x + \frac{2}{3}y + \frac{1}{3}z$
(c) $2x - \frac{2}{3}y + \frac{1}{3}z$ (d) $-2x + \frac{2}{3}y + \frac{1}{3}z$ **(S.S.C. 1998)**

30. If $\log_4 x + \log_2 x = 6$, then x is equal to :
(a) 2 (b) 4 (c) 8 (d) 16

31. If $\log_8 x + \log_8 \frac{1}{6} = \frac{1}{3}$, then the value of x is :
(a) 12 (b) 16 (c) 18 (d) 24

32. If $\log_{10} 125 + \log_{10} 8 = x$, then x is equal to :
(a) $\frac{1}{3}$ (b) .064 (c) -3 (d) 3

33. The value of $(\log_9 27 + \log_8 32)$ is :
(a) $\frac{7}{2}$ (b) $\frac{19}{6}$ (c) 4 (d) 7

34. $(\log_5 3) \times (\log_3 625)$ equals :
(a) 1 (b) 2 (c) 3 (d) 4

35. $(\log_5 5)(\log_4 9)(\log_3 2)$ is equal to :
(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) 5

36. If $\log_{12} 27 = a$, then $\log_6 16$ is : **(Assistant Grade, 1998)**
(a) $\frac{3-a}{4(3+a)}$ (b) $\frac{3+a}{4(3-a)}$ (c) $\frac{4(3+a)}{(3-a)}$ (d) $\frac{4(3-a)}{(3+a)}$

37. If $\log_{10} 5 + \log_{10} (5x + 1) = \log_{10} (x + 5) + 1$, then x is equal to : **(C.D.S. 2003)**
(a) 1 (b) 3 (c) 5 (d) 10

38. If $\log_5 (x^2 + x) - \log_5 (x + 1) = 2$, then the value of x is :
(a) 5 (b) 10 (c) 25 (d) 32

39. The value of $\left(\frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}\right)$ is :
(a) 0 (b) 1 (c) 5 (d) 60

40. The value of $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9)$ is :
(a) 2 (b) 7 (c) 8 (d) 33

41. The value of $16^{\log_4 5}$ is :
(a) $\frac{5}{64}$ (b) 5 (c) 16 (d) 25

42. If $\log x + \log y = \log (x + y)$, then :
(a) $x = y$ (b) $xy = 1$ (c) $y = \frac{x-1}{x}$ (d) $y = \frac{x}{x-1}$

43. If $\log \frac{a}{b} + \log \frac{b}{a} = \log (a + b)$, then :
(a) $a + b = 1$ (b) $a - b = 1$ (c) $a = b$ (d) $a^2 - b^2 = 1$

44. $\left[\log\left(\frac{a^2}{bc}\right) + \log\left(\frac{b^2}{ac}\right) + \log\left(\frac{c^2}{ab}\right)\right]$ is equal to :
(a) 0 (b) 1 (c) 2 (d) abc

45. $(\log_b a \times \log_c b \times \log_a c)$ is equal to :
(a) 0 (b) 1 (c) abc (d) $a + b + c$

46. $\left[\frac{1}{(\log_a bc) + 1} + \frac{1}{(\log_b ca) + 1} + \frac{1}{(\log_c ab) + 1}\right]$ is equal to :
(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) 3

47. The value of $\left[\frac{1}{\log_{(p/q)} x} + \frac{1}{\log_{(q/r)} x} + \frac{1}{\log_{(r/p)} x}\right]$ is :
(a) 0 (b) 1 (c) 2 (d) 3

48. If $\log_{10} 7 = a$, then $\log_{10}\left(\frac{1}{70}\right)$ is equal to : **(C.D.S. 2003)**
(a) $-(1 + a)$ (b) $(1 + a)^{-1}$ (c) $\frac{a}{10}$ (d) $\frac{1}{10a}$

49. If $a = b^x$, $b = c^y$ and $c = a^z$, then the value of xyz is equal to :
(a) -1 (b) 0 (c) 1 (d) abc

50. If $\log 27 = 1.431$, then the value of $\log 9$ is : **(Section Officers', 2001)**
(a) 0.934 (b) 0.945 (c) 0.954 (d) 0.958

51. If $\log_{10} 2 = 0.3010$, then $\log_2 10$ is equal to : **(S.S.C. 2000)**
(a) $\frac{699}{301}$ (b) $\frac{1000}{301}$ (c) 0.3010 (d) 0.6990

52. If $\log_{10} 2 = 0.3010$, the value of $\log_{10} 5$ is : **(S.S.C. 2001)**
(a) 0.3241 (b) 0.6911 (c) 0.6990 (d) 0.7525

53. If $\log_{10} 2 = 0.3010$, the value of $\log_{10} 80$ is :
(a) 1.6020 (b) 1.9030 (c) 3.9030 (d) None of these

54. If $\log 3 = 0.477$ and $(1000)^x = 3$, then x equals : **(S.S.C. 2000)**
(a) 0.0159 (b) 0.0477 (c) 0.159 (d) 10

55. If $\log_{10} 2 = 0.3010$, the value of $\log_{10} 25$ is :
(a) 0.6020 (b) 1.2040 (c) 1.3980 (d) 1.5050

56. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, the value of $\log_5 512$ is : **(M.A.T. 2002)**
(a) 2.870 (b) 2.967 (c) 3.876 (d) 3.912

57. If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, then the value of $\log_{10} 1.5$ is :
(a) 0.1761 (b) 0.7116 (c) 0.7161 (d) 0.7611

58. If $\log_{10} 2 = 0.3010$ and $\log_{10} 7 = 0.8451$, then the value of $\log_{10} 2.8$ is :
(a) 0.4471 (b) 1.4471 (c) 2.4471 (d) None of these
(S.S.C. 1999)

59. If $\log (0.57) = \bar{1}.756$, then the value of $\log 57 + \log (0.57)^3 + \log \sqrt{0.57}$ is :
(a) 0.902 (b) $\bar{2}.146$ (c) 1.902 (d) $\bar{1}.146$
(Section Officers', 2003)

60. If $\log 2 = 0.30103$, the number of digits in 2^{64} is : **(C.B.I. 1997)**
(a) 18 (b) 19 (c) 20 (d) 21

61. If $\log 2 = 0.30103$, the number of digits in 4^{50} is :
(a) 30 (b) 31 (c) 100 (d) 200

62. If $\log 2 = 0.30103$, then the number of digits in 5^{20} is :
(a) 14 (b) 16 (c) 18 (d) 25

ANSWERS

1. (b)	2. (a)	3. (b)	4. (c)	5. (c)	6. (d)	7. (a)	8. (d)
9. (d)	10. (d)	11. (d)	12. (a)	13. (d)	14. (c)	15. (b)	16. (c)
17. (b)	18. (c)	19. (c)	20. (b)	21. (a)	22. (d)	23. (a)	24. (c)
25. (d)	26. (c)	27. (a)	28. (d)	29. (b)	30. (d)	31. (a)	32. (d)
33. (b)	34. (d)	35. (a)	36. (d)	37. (b)	38. (c)	39. (b)	40. (a)
41. (d)	42. (d)	43. (a)	44. (a)	45. (b)	46. (a)	47. (a)	48. (a)
49. (c)	50. (c)	51. (b)	52. (c)	53. (b)	54. (c)	55. (c)	56. (c)
57. (a)	58. (a)	59. (a)	60. (c)	61. (b)	62. (a)		

SOLUTIONS

1. Let $\log_2 16 = n$. Then, $2^n = 16 = 2^4 \Rightarrow n = 4$.
$\therefore \log_2 16 = n$.

2. Let $\log_{343} 7 = n$. Then, $(343)^n = 7 \Leftrightarrow (7^3)^n = 7 \Leftrightarrow 3n = 1 \Leftrightarrow n = \frac{1}{3}$.
$\therefore \log_{343} 7 = \frac{1}{3}$.

3. Let $\log_5 \left(\frac{1}{125}\right) = n$. Then, $5^n = \frac{1}{125} \Leftrightarrow 5^n = 5^{-3} \Leftrightarrow n = -3$.
$\therefore \log_5 \left(\frac{1}{125}\right) = -3$.

4. Let $\log_{\sqrt{2}} 32 = n$. Then, $(\sqrt{2})^n = 32 \Leftrightarrow (2)^{n/2} = 2^5 \Leftrightarrow \frac{n}{2} = 5 \Leftrightarrow n = 10$.

$\therefore \log_{\sqrt{2}} 32 = 10$.

5. Let $\log_{10}(.0001) = n$.

Then, $10^n = .0001 \Leftrightarrow 10^n = \frac{1}{10000} = \frac{1}{10^4} \Leftrightarrow 10^n = 10^{-4} \Leftrightarrow n = -4$.

$\therefore \log_{10}(.0001) = -4$.

6. Let $\log_{(.01)}(1000) = n$.

Then, $(.01)^n = 1000 \Leftrightarrow \left(\frac{1}{100}\right)^n = 10^3 \Leftrightarrow (10^{-2})^n = 10^3 \Leftrightarrow -2n = 3 \Leftrightarrow n = -\frac{3}{2}$.

7. Let $\log_2 0.0625 = n$.

Then, $2^n = 0.0625 = \frac{625}{10000} \Leftrightarrow 2^n = \frac{1}{16} \Leftrightarrow 2^n = 2^{-4} \Leftrightarrow n = -4$.

$\therefore \log_2 0.0625 = -4$.

8. $\log_3 x = -2 \Leftrightarrow x = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

9. $\log_8 x = \frac{2}{3} \Leftrightarrow x = 8^{2/3} = (2^3)^{2/3} = 2^{\left(3 \times \frac{2}{3}\right)} = 2^2 = 4$.

10. $\log_x\left(\frac{9}{16}\right) = -\frac{1}{2} \Leftrightarrow x^{-1/2} = \frac{9}{16} \Leftrightarrow \frac{1}{\sqrt{x}} = \frac{9}{16} \Leftrightarrow \sqrt{x} = \frac{16}{9}$

$\Leftrightarrow x = \left(\frac{16}{9}\right)^2 = \frac{256}{81}$.

11. $\log_x 4 = 0.4 \Leftrightarrow \log_x 4 = \frac{4}{10} = \frac{2}{5} \Leftrightarrow x^{2/5} = 4 \Leftrightarrow x = 4^{5/2} = (2^2)^{5/2}$

$\Leftrightarrow x = 2^{\left(2 \times \frac{5}{2}\right)} = 2^5 \Leftrightarrow x = 32$.

12. $\log_{10000} x = -\frac{1}{4} \Leftrightarrow x = (10000)^{-1/4} = (10^4)^{-1/4} = 10^{-1} = \frac{1}{10}$.

13. $\log_x 4 = \frac{1}{4} \Leftrightarrow x^{1/4} = 4 \Leftrightarrow x = 4^4 = 256$.

14. $\log_x (0.1) = -\frac{1}{3} \Leftrightarrow x^{-1/3} = 0.1 \Leftrightarrow \frac{1}{x^{1/3}} = 0.1 \Leftrightarrow x^{1/3} = \frac{1}{0.1} = 10$

$\Leftrightarrow x = (10)^3 = 1000$.

15. $\log_{32} x = 0.8 \Leftrightarrow x = (32)^{0.8} = (2^5)^{4/5} = 2^4 = 16$.

16. $\log_2 x = 10 \Rightarrow x = 2^{10}$.

$\therefore \log_x y = 100 \Rightarrow y = x^{100} = (2^{10})^{100} \Rightarrow y = 2^{1000}$.

17. Let $\log_{(-1/3)} 81 = x$. Then, $\left(-\frac{1}{3}\right)^x = 81 = 3^4 = (-3)^4 = \left(-\frac{1}{3}\right)^{-4}$

$\therefore x = -4$ *i.e.*, $\log_{(-1/3)} 81 = -4$.

18. Let $\log_{2\sqrt{3}}(1728) = x$.

Then, $(2\sqrt{3})^x = 1728 = (12)^3 = [(2\sqrt{3})^2]^3 = (2\sqrt{3})^6$.

$\therefore x = 6$, *i.e.*, $\log_{2\sqrt{3}}(1728) = 6$.

19. $\dfrac{\log\sqrt{8}}{\log 8} = \dfrac{\log (8)^{1/2}}{\log 8} = \dfrac{\frac{1}{2}\log 8}{\log 8} = \dfrac{1}{2}.$

20. (*a*) Since $\log_a a = 1$, so $\log_{10} 10 = 1$.

(*b*) $\log (2 + 3) = 5$ and $\log (2 \times 3) = \log 6 = \log 2 + \log 3$

$\therefore$ $\log (2 + 3) \neq \log (2 \times 3)$.

(*c*) Since $\log_a 1 = 0$, so $\log_{10} 1 = 0$.

(*d*) $\log (1 + 2 + 3) = \log 6 = \log (1 \times 2 \times 3) = \log 1 + \log 2 + \log 3$.

So, (*b*) is incorrect.

21. Let $\log_5 625 = x$. Then, $5^x = 625 = 5^4$ or $x = 4$.

Let $\log_2 (\log_5 625) = y$. Then, $\log_2 4 = y$ or $2^y = 4 = 2^2$ or $y = 2$.

$\therefore$ $\log_2 (\log_5 625) = 2$.

22. $\log_2 [\log_3 (\log_2 x)] = 1 = \log_2 2$

$\Leftrightarrow$ $\log_3 (\log_2 x) = 2$ $\Leftrightarrow$ $\log_2 x = 3^2 = 9$ $\Leftrightarrow$ $x = 2^9 = 512$.

23. $\log_2 \log_2 \log_3 (\log_3 27^3) = \log_2 \log_2 \log_3 [\log_3 (3^3)^3] = \log_2 \log_2 \log_3 [\log_3 (3)^9]$

$= \log_2 \log_2 \log_3 (9 \log_3 3) = \log_2 \log_2 \log_3 9$ $\quad [\because \log_3 3 = 1]$

$= \log_2 \log_2 [\log_3 (3)^2] = \log_2 \log_2 (2 \log_3 3)$

$= \log_2 \log_2 2 = \log_2 1 = 0$.

24. $a^x = b^y \Rightarrow \log a^x = \log b^y \Rightarrow x \log a = y \log b \Rightarrow \dfrac{\log a}{\log b} = \dfrac{y}{x}$.

25. $360 = (2 \times 2 \times 2) \times (3 \times 3) \times 5$.

So, $\log 360 = \log (2^3 \times 3^2 \times 5) = \log 2^3 + \log 3^2 + \log 5 = 3 \log 2 + 2 \log 3 + \log 5$.

26. $\frac{1}{3} \log_{10} 125 - 2 \log_{10} 4 + \log_{10} 32$

$= \log_{10} (125)^{1/3} - \log_{10} (4)^2 + \log_{10} 32 = \log_{10} 5 - \log_{10} 16 + \log_{10} 32$

$= \log_{10} \left(\dfrac{5 \times 32}{16}\right) = \log_{10} 10 = 1$.

27. $2 \log_{10} 5 + \log_{10} 8 - \frac{1}{2} \log_{10} 4 = \log_{10} (5^2) + \log_{10} 8 - \log_{10} (4^{1/2})$

$= \log_{10} 25 + \log_{10} 8 - \log_{10} 2 = \log_{10} \left(\dfrac{25 \times 8}{2}\right) = \log_{10} 100 = 2$.

28. $\log_a (ab) = x \Leftrightarrow \dfrac{\log ab}{\log a} = x \Leftrightarrow \dfrac{\log a + \log b}{\log a} = x$

$\Leftrightarrow 1 + \dfrac{\log b}{\log a} = x \Leftrightarrow \dfrac{\log b}{\log a} = x - 1$

$\Leftrightarrow \dfrac{\log a}{\log b} = \dfrac{1}{x-1} \Leftrightarrow 1 + \dfrac{\log a}{\log b} = 1 + \dfrac{1}{x-1}$

$\Leftrightarrow \dfrac{\log b}{\log b} + \dfrac{\log a}{\log b} = \dfrac{x}{x-1} \Leftrightarrow \dfrac{\log b + \log a}{\log b} = \dfrac{x}{x-1}$

$\Leftrightarrow \dfrac{\log (ab)}{\log b} = \dfrac{x}{x-1} \Leftrightarrow \log_b (ab) = \dfrac{x}{x-1}$.

29. $\log (4 \cdot \sqrt[3]{63}) = \log 4 + \log (\sqrt[3]{63}) = \log 4 + \log (63)^{1/3} = \log (2^2) + \log (7 \times 3^2)^{1/3}$

$= 2 \log 2 + \frac{1}{3} \log 7 + \frac{2}{3} \log 3 = 2x + \frac{1}{3} z + \frac{2}{3} y$.

30. $\log_4 x + \log_2 x = 6 \Leftrightarrow \frac{\log x}{\log 4} + \frac{\log x}{\log 2} = 6$

$\Leftrightarrow \frac{\log x}{2\log 2} + \frac{\log x}{\log 2} = 6 \Leftrightarrow 3\log x = 12\log 2$

$\Leftrightarrow \log x = 4\log 2 \Leftrightarrow \log x = \log(2^4) = \log 16 \Leftrightarrow x = 16.$

31. $\log_8 x + \log_8\left(\frac{1}{6}\right) = \frac{1}{3} \Leftrightarrow \frac{\log x}{\log 8} + \frac{\log \frac{1}{6}}{\log 8} = \frac{1}{3}$

$\Leftrightarrow \log x + \log\frac{1}{6} = \frac{1}{3}\log 8 \Leftrightarrow \log x + \log\frac{1}{6} = \log(8^{1/3}) = \log 2$

$\Leftrightarrow \log x = \log 2 - \log\frac{1}{6} = \log\left(2 \times \frac{6}{1}\right) = \log 12$

$\therefore \quad x = 12.$

32. $\log_{10} 125 + \log_{10} 8 = x \Rightarrow \log_{10}(125 \times 8) = x$

$\Rightarrow x = \log_{10}(1000) = \log_{10}(10)^3 = 3\log_{10} 10 = 3.$

33. Let $\log_9 27 = x$. Then, $9^x = 27 \Leftrightarrow (3^2)^x = 3^3 \Leftrightarrow 2x = 3 \Leftrightarrow x = \frac{3}{2}$.

Let $\log_8 32 = y$. Then, $8^y = 32 \Leftrightarrow (2^3)^y = 2^5 \Leftrightarrow 3y = 5 \Leftrightarrow y = \frac{5}{3}$.

$\therefore \quad \log_9 27 + \log_8 32 = \left(\frac{3}{2} + \frac{5}{3}\right) = \frac{19}{6}.$

34. Given expression $= \left(\frac{\log 3}{\log 5} \times \frac{\log 625}{\log 3}\right) = \frac{\log 625}{\log 5} = \frac{\log(5^4)}{\log 5} = \frac{4\log 5}{\log 5} = 4.$

35. Given expression $= \frac{\log 9}{\log 4} \times \frac{\log 2}{\log 3}$ $[\because \log_5 5 = 1]$

$= \frac{\log 3^2}{\log 2^2} \times \frac{\log 2}{\log 3} = \frac{2\log 3}{2\log 2} \times \frac{\log 2}{\log 3} = 1.$

36. $\log_{12} 27 = a \Rightarrow \frac{\log 27}{\log 12} = a \Rightarrow \frac{\log 3^3}{\log(3 \times 2^2)} = a$

$\Rightarrow \frac{3\log 3}{\log 3 + 2\log 2} = a \Rightarrow \frac{\log 3 + 2\log 2}{3\log 3} = \frac{1}{a}$

$\Rightarrow \frac{\log 3}{3\log 3} + \frac{2\log 2}{3\log 3} = \frac{1}{a} \Rightarrow \frac{2}{3}\frac{\log 2}{\log 3} = \frac{1}{a} - \frac{1}{3} = \left(\frac{3-a}{3a}\right)$

$\Rightarrow \frac{\log 2}{\log 3} = \left(\frac{3-a}{2a}\right) \Rightarrow \log 3 = \left(\frac{2a}{3-a}\right)\log 2.$

$$\log_6 16 = \frac{\log 16}{\log 6} = \frac{\log 2^4}{\log(2 \times 3)} = \frac{4\log 2}{\log 2 + \log 3} = \frac{4\log 2}{\log 2\left[1 + \left(\frac{2a}{3-a}\right)\right]}$$

$$= \frac{4}{\left(\frac{3+a}{3-a}\right)} = \frac{4(3-a)}{(3+a)}.$$

37. $\log_{10} 5 + \log_{10}(5x+1) = \log_{10}(x+5) + 1$

$\Rightarrow \log_{10} 5 + \log_{10}(5x+1) = \log_{10}(x+5) + \log_{10} 10$

$\Rightarrow \log_{10}[5(5x+1)] = \log_{10}[10(x+5)] \Rightarrow 5(5x+1) = 10(x+5)$

$\Rightarrow 5x + 1 = 2x + 10 \Rightarrow 3x = 9 \Rightarrow x = 3.$

38. $\log_5(x^2+x) - \log_5(x+1) = 2 \Rightarrow \log_5\left(\frac{x^2+x}{x+1}\right) = 2$

$\Rightarrow \log_5\left[\frac{x(x+1)}{x+1}\right] = 2 \Rightarrow \log_5 x = 2 \Rightarrow x = 5^2 = 25.$

39. Given expression $= \log_{60} 3 + \log_{60} 4 + \log_{60} 5 = \log_{60}(3 \times 4 \times 5) = \log_{60} 60 = 1.$

40. Given expression $= \left(\frac{\log 4}{\log 3} \times \frac{\log 5}{\log 4} \times \frac{\log 6}{\log 5} \times \frac{\log 7}{\log 6} \times \frac{\log 8}{\log 7} \times \frac{\log 9}{\log 8}\right)$

$$= \frac{\log 9}{\log 3} = \frac{\log 3^2}{\log 3} = \frac{2 \log 3}{\log 3} = 2.$$

41. We know that : $a^{\log_a x} = x.$

$\therefore \quad 16^{\log_4 5} = (4^2)^{\log_4 5} = 4^{2\log_4 5} = 4^{\log_4 (5^2)} = 4^{\log_4 25} = 25.$

42. $\log x + \log y = \log(x+y) \Rightarrow \log(x+y) = \log(xy)$

$\Rightarrow x + y = xy \Rightarrow y(x-1) = x \Rightarrow y = \frac{x}{x-1}.$

43. $\log \frac{a}{b} + \log \frac{b}{a} = \log(a+b) \Rightarrow \log(a+b) = \log\left(\frac{a}{b} \times \frac{b}{a}\right) = \log 1.$

So, $a + b = 1.$

44. Given expression $= \log\left(\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}\right) = \log 1 = 0.$

45. Given expression $= \left(\frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log a}\right) = 1.$

46. Given expression $= \frac{1}{\log_a bc + \log_a a} + \frac{1}{\log_b ca + \log_b b} + \frac{1}{\log_c ab + \log_c c}$

$$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc}(abc) = 1.$$

47. Given expression $= \log_x\left(\frac{p}{q}\right) + \log_x\left(\frac{q}{r}\right) + \log_x\left(\frac{r}{p}\right) = \log_x\left(\frac{p}{q} \times \frac{q}{r} \times \frac{r}{p}\right) = \log_x 1 = 0.$

48. $\log_{10}\left(\frac{1}{70}\right) = \log_{10} 1 - \log_{10} 70 = -\log_{10}(7 \times 10) = -(\log_{10} 7 + \log_{10} 10) = -(a+1).$

49. $a = b^x, b = c^y, c = a^z \Rightarrow x = \log_b a, y = \log_c b, z = \log_a c$

$\Rightarrow xyz = (\log_b a) \times (\log_c b) \times (\log_a c) \Rightarrow xyz = \left(\frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log a}\right) = 1.$

50. $\log 27 = 1.431 \Rightarrow \log(3^3) = 1.431 \Rightarrow 3 \log 3 = 1.431$

$\Rightarrow \log 3 = 0.477$

$\therefore \quad \log 9 = \log(3^2) = 2 \log 3 = (2 \times 0.477) = 0.954.$

51. $\log_2 10 = \frac{1}{\log_{10} 2} = \frac{1}{0.3010} = \frac{10000}{3010} = \frac{1000}{301}$.

52. $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2 = (1 - 0.3010) = 0.6990.$

53. $\log_{10} 80 = \log_{10} (8 \times 10) = \log_{10} 8 + \log_{10} 10 = \log_{10} (2^3) + 1 = 3 \log_{10} 2 + 1$

$= (3 \times 0.3010) + 1 = 1.9030.$

54. $(1000)^x = 3 \Rightarrow \log [(1000)^x] = \log 3 \Rightarrow x \log 1000 = \log 3$

$\Rightarrow x \log (10^3) = \log 3 \Rightarrow 3x \log 10 = \log 3$

$\Rightarrow 3x = \log 3 \Rightarrow x = \frac{0.477}{3} = 0.159.$

55. $\log_{10} 25 = \log_{10}\left(\frac{100}{4}\right) = \log_{10} 100 - \log_{10} 4 = 2 - 2 \log_{10} 2 = (2 - 2 \times 0.3010)$

$= (2 - 0.6020) = 1.3980.$

56. $\log_5 512 = \frac{\log 512}{\log 5} = \frac{\log 2^9}{\log\left(\frac{10}{2}\right)} = \frac{9 \log 2}{\log 10 - \log 2}$

$= \frac{(9 \times 0.3010)}{1 - 0.3010} = \frac{2.709}{0.699} = \frac{2709}{699} = 3.876.$

57. $\log_{10} (1.5) = \log_{10}\left(\frac{3}{2}\right) = \log_{10} 3 - \log_{10} 2 = (0.4771 - 0.3010) = 0.1761.$

58. $\log_{10} (2.8) = \log_{10}\left(\frac{28}{10}\right) = \log_{10} 28 - \log_{10} 10$

$= \log_{10} (7 \times 2^2) - 1 = \log_{10} 7 + 2 \log_{10} 2 - 1$

$= 0.8451 + 2 \times 0.3010 - 1 = 0.8451 + 0.602 - 1 = 0.4471.$

59. $\log (0.57) = \bar{1}.756 \Rightarrow \log 57 = 1.756$ [∵ mantissa will remain the same]

$\therefore \log 57 + \log (0.57)^3 + \log \sqrt{0.57}$

$= \log 57 + 3 \log\left(\frac{57}{100}\right) + \log\left(\frac{57}{100}\right)^{1/2}$

$= \log 57 + 3 \log 57 - 3 \log 100 + \frac{1}{2} \log 57 - \frac{1}{2} \log 100$

$= \frac{9}{2} \log 57 - \frac{7}{2} \log 100 = \frac{9}{2} \times 1.756 - \frac{7}{2} \times 2 = 7.902 - 7 = 0.902.$

60. $\log (2^{64}) = 64 \times \log 2 = (64 \times 0.30103) = 19.26592.$

Its characteristic is 19. Hence, the number of digits in 2^{64} is 20.

61. $\log 4^{50} = 50 \log 4 = 50 \log 2^2 = (50 \times 2) \log 2 = 100 \times \log 2 = (100 \times 0.30103) = 30.103.$

$\therefore$ Characteristic = 30. Hence, the number of digits in 4^{50} = 31.

62. $\log 5^{20} = 20 \log 5 = 20 \times \left[\log\left(\frac{10}{2}\right)\right] = 20 (\log 10 - \log 2)$

$= 20 (1 - 0.3010) = 20 \times 0.6990 = 13.9800.$

$\therefore$ Characteristic = 13. Hence, the number of digits in 5^{20} is 14.