Chapter 13

Limits and Derivatives

Exercise 13.1

Question 1: Evaluate the Given $\lim_{x\to 3} x+3$

Answer 1:

$\lim_{x\to 3} x+3 = 3 + 3 = 6$

Question 2:Evaluate the Given $\lim_{x\to \pi}\left(x-\frac{22}{7}\right)$

Answer 2:

$\lim_{x\to \pi}\left(x-\frac{22}{7}\right) = \left(\pi-\frac{22}{7}\right)$

Question 3: Evaluate the Given $\lim_{r\to 1} \pi r^2$

Answer 3:

$\lim_{r\to 1} \pi r^2 = \pi(1)^2 = \pi$

Question 4: Evaluate the Given $\lim_{x\to 4}\frac{4x+3}{x-2}$

Answer 4:

$\lim_{x\to 4}\frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$

Question 5: Evaluate the Given $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

Answer 5:

$\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = \frac{-1}{2}$

Question 6:Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Answer 6:

$\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Put x + 1 = y so that y → 1 as x → 0.

Accordingly, $\lim_{x\to 0}\frac{(x+1)^5-1}{x} = \lim_{y\to 1}\frac{y^5-1}{y-1}$

$$= \lim_{y\to 1}\frac{y^5-1^5}{y-1}$$

$$= 5.1^{5-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 5$$

$$\lim_{x\to 0}\frac{x^{10}+x^5-1}{x} = 5$$

Question 7: Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Answer 7:

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$

$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$$= \lim_{x\to 2}\frac{3x+5}{x+2}$$

$$= \frac{3(2)+5}{2+2}$$

$$= \frac{11}{4}$$

Question 8: Evaluate the Given limit: $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Answer 8:

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$

$$= \lim_{x \to 3} \frac{(x+3)(x^2+9)}{2x+1}$$

$$= \frac{(3+3)(x^2+9)}{2(3)+1}$$

$$= \frac{6 \times 18}{7}$$

$$= \frac{108}{7}$$

Question 9: Evaluate the Given limit: $\lim_{x \to 0} \frac{ax+b}{cx+1}$

Answer 9:

$$\lim_{x \to 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Question 10: Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

Answer 10:

At z = 1, the value of the given function takes the form $\frac{0}{0}$

Put so that z →1 as x → 1

Accordingly, $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$$\lim_{x \to 1} \frac{x^1 - 1^1}{x - 1}$$

$= 2.1^{2-1}\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

= 2

$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{3}}-1} = 2$

Question 11: Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Answer 11:

$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a} = \frac{a(1)^2+b91)+c}{c(1)^2+b(1)+a}$

$= \frac{a+b+c}{a+b+c}$

= 1 [a + b + c ≠ 0]

Question 12: Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Answer 12:

$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

At x = –2, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$

$= \lim_{x\to -2}\frac{1}{2x}$

$= \frac{1}{2(-2)} = \frac{-1}{4}$

Question 13: Evaluate the Given limit: $\lim_{x\to 0}\frac{si\,nax}{bx}$

Answer 13:

$\lim_{x \to 0} \frac{si\,nax}{bx}$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \to 0} \frac{si\,nax}{bx} = \lim_{x \to 0} \frac{si\,nax}{bx} \times \frac{ax}{bx}$

$= \lim_{x \to 0} \left(\frac{si\,nax}{ax}\right) \times \left(\frac{a}{b}\right)$

$= \frac{a}{b} \lim_{x \to 0} \left(\frac{si\,nax}{ax}\right) [x \to 0 \Rightarrow ax \to 0]$

$= \frac{a}{b} \times 1 \left[\lim_{y \to 0} \frac{si\,ny}{y} = 1\right]$

$= \frac{a}{b}$

Question 14: Evaluate the Given limit: $\lim_{x \to 0} \frac{si\,nax}{si\,bbx}, a, b \neq 0$

Answer 14:

$\lim_{x \to 0} \frac{si\,nax}{si\,nbx}, a, b \neq 0$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \to 0} \frac{si\,nax}{si\,nbx} = \lim_{x \to 0} \frac{\left(\frac{si\,nax}{ax}\right) \times ax}{\left(\frac{si\,nbx}{bx}\right) \times bx}$

$= \left(\frac{a}{b}\right) \times \frac{\lim_{ax \to 0}\left(\frac{si\,nax}{ax}\right)}{\lim_{bx \to 0}\left(\frac{si\,nbx}{bx}\right)}$ $[x \to 0 \Longrightarrow ax \to 0]$

$[and\ x \to 0 \Longrightarrow bx \to 0]$

$= \left(\frac{a}{b}\right) \times \frac{1}{1}$ $\left[\lim_{y \to 0} \frac{si\,ny}{y} = 1\right]$

$= \frac{a}{b}$

Question 15: Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Answer 16:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that x → π ⇒ (π – x) → 0

$$=\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{\pi-x}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{si\,ny}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question 16: Evaluate the given limit: $\lim_{x\to 0}\frac{cosx}{\pi-x}$

Answer 16:

$$\lim_{x\to 0}\frac{cosx}{\pi-x}-\frac{cos0}{\pi-0}=\frac{1}{\pi}$$

Question 17: Evaluate the Given limit: $\lim_{x\to 0}\frac{cos2x-1}{cosx-1}$

Answer 17:

$$\lim_{x\to 0}\frac{cos2x-1}{cosx-1}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now,

$$=\lim_{x\to 0}\frac{cos2x-1}{cosx-1}=\lim_{x\to 0}\frac{1-2si\,n^2x-1}{1-2si\,n^2\frac{x}{2}-1} \qquad \left[cosx=1-2si\,n^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{si\,n^2x}{si\,n^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{si\,n^2x}{x^2}\right)\times x^2}{\left(\frac{si\,n^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$= 4\,\frac{\lim\limits_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$= 4\,\frac{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim\limits_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x \to 0 \Rightarrow \frac{x}{2} \to 0\right]$$

$$= 4\,\frac{1^2}{1^2} \qquad \left[\lim\limits_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$= 4$$

Question 18: Evaluate the Given limit: $\lim\limits_{x\to 0}\frac{ax + x\cos x}{b\sin x}$

Answer 18:

$$\lim\limits_{x\to 0}\frac{ax + x\cos x}{b\sin x}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now,

$$= \lim\limits_{x\to 0}\frac{ax + x\cos x}{b\sin x} - \frac{1}{b}\lim\limits_{x\to 0}\frac{x(a + \cos x)}{\sin x}$$

$$= \frac{1}{b}\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right) \times \lim\limits_{x\to 0}(a + \cos x)$$

$$= \frac{1}{b} \times \frac{1}{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)} \times \lim\limits_{x\to 0}(a + \cos x)$$

$$= \frac{1}{b} \times (a + \cos x) \qquad \left[\lim\limits_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= \frac{a+1}{b}$$

Question 19: Evaluate the Given limit: $\lim\limits_{x\to 0} x\sec x$

Answer 19:

$$\lim_{x\to 0} x\, secx = \lim_{x\to 0} \frac{x}{cosx} = \frac{0}{cos0} = \frac{0}{1} = 0$$

Question 20: Evaluate the Given limit: $\lim_{x\to 0} \frac{si\,nax+bx}{ax+si\,nbx} a, b, a+b \neq 0$

Answer 20:

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now,

$$\lim_{x\to 0} \frac{si\,nax+bx}{ax+si\,nbx}$$

$$= \lim_{x\to 0} \frac{\left(\frac{si\,nax}{ax}\right)ax+bx}{ax+bx\left(\frac{si\,nbx}{bx}\right)}$$

$$= \frac{\left(\lim_{ax\to 0} \frac{si\,nax}{ax}\right)\times \lim_{x\to 0}(ax)+\lim_{x\to 0} bx}{\lim_{x\to 0} ax+\lim_{x\to 0} bx\left(\lim_{x\to 0} \frac{si\,nbx}{bx}\right)} \qquad [As\ x \to 0 \Rightarrow ax \to 0\ and\ bx \to 0]$$

$$= \frac{\lim_{x\to 0}(ax)+\lim_{x\to 0} bx}{\lim_{x\to 0} ax+\lim_{x\to 0} bx} \qquad \left[\lim_{x\to 0} \frac{si\,nx}{x} = 1\right]$$

$$= \frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

$$= \lim_{x\to 0}(1)$$

= 1

Question 21: Evaluate the Given limit: $\lim_{x\to 0}(cosecx - cotx)$

Answer 21:

At x = 0, the value of the given function takes the form $\infty - \infty$

Now,

$$\lim_{x\to 0}(cosecx - cotx)$$

$$= \lim_{x\to 0}\left(\frac{1}{si\,nx} - \frac{cosx}{si\,nx}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-cosx}{si\,nx}\right)$$

$$= \lim_{x\to 0}\frac{\left(\frac{1-cosx}{x}\right)}{\left(\frac{si\,nx}{x}\right)}$$

$$= \frac{\lim_{x\to 0}\frac{1-cosx}{x}}{\lim_{x\to 0}\frac{si\,nx}{x}}$$

$$= \frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-cosx}{x} = 0 and \lim_{x\to 0}\frac{si\,nx}{x} = 1\right]$$

$$= 0$$

Question 22: $\lim_{x\to\frac{\pi}{2}}\frac{tan2x}{x-\frac{\pi}{2}}$Answer 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{tan2x}{x-\frac{\pi}{2}}$$

At, x = $\frac{\pi}{2}$the value of the given function takes the form $\frac{0}{0}$

Now, put $x - \frac{\pi}{2} = y$ so that x → $\frac{\pi}{2}, y \to 0$

$$= \lim_{x\to\frac{\pi}{2}}\frac{tan2x}{x-\frac{\pi}{2}} = \lim_{y\to 0}\frac{tan2\left(y+\frac{\pi}{2}\right)}{y} =$$

$$= \lim_{y\to 0}\frac{\tan(x+2y)}{y}$$

$$= \lim_{y\to 0}\frac{tan2y}{y} \qquad [\tan (x + 2y) = \tan 2y]$$

$$= \left(\frac{si\,n2y}{ycosx}\right)$$

$= \lim_{y \to 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right)$

$= \left(\lim_{2y \to 0} \frac{\sin 2y}{2y} \right) \times \lim_{y \to 0} \left(\frac{2}{\cos 2y} \right)$ $[y \to 0 \Rightarrow 2y \to 0]$

$= 1 \times \frac{2}{\cos 0}$ $\left[\lim_{x \to 0} \frac{\sin x}{x} = 1 \right]$

$= 1 \times \frac{2}{1}$

$= 2$

Question 23: Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$, where f (x) =

$\begin{cases} 2x + 3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Answer 23:

The given function is f(x) = $f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} [2 + 3] = 2(0) + 3 = 3$

$= \lim_{x \to 0} f(x) = \lim_{x \to 0} 3(x+1) = 3\,(0+1) = 3$

$= \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = 3$

$= \lim_{x \to 1} f(x) = \lim_{x \to 1} 3(x+1) = 3(1+1) = 6$

$= \lim_{x \to 1} f(x) = \lim_{x \to 1} 3(x+1) = 3(1+1) = 6$

$= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 6$

Question 24: Find $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Answer 24:

The given function is $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}[x^2 - 1] = 1^2 - 1 = 1 - 1 = 0$

$= \lim_{x\to 1^+} f(x) = \lim_{x\to 1}[-x^2 - 1] = -1^2 - 1 = -1 - 1 = -2$

It is observed that $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25: Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer 25:

The given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$

$= \lim_{x\to 0}\left(\frac{-x}{x}\right)$ [when x is negative, |x| = - x]

$= \lim_{x\to 0}(-1)$

= - 1

$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$

$= \lim_{x\to 0}\left[\frac{x}{x}\right]$ [when x is positive, |x| = x]

$= \lim_{x\to 0}(1)$

= 1

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 26: Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer 26:

The given function is $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[\frac{x}{|x|}\right]$

$= \lim_{x \to 0} \left[\frac{x}{-x}\right]$ [when $x < 0, |x| = -x$]

$= \lim_{x \to 0} (-1)$

= - 1

$= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[\frac{x}{|x|}\right]$

$= \lim_{x \to 0} \left[\frac{x}{x}\right]$ [when $x > 0, |x| = x$]

$= \lim_{x \to 0} (1)$

= 1

It is observed that $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 27: Find $\lim_{x \to 5} f(x)$, where $f(x) = |x| - 5$

Answer 27:

The given function is $f(x) = |x| - 5$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} [|x| - 5]$

$= \lim_{x \to 5}(x - 5)$ [when $x > 0, |x| = x$]

= 5 – 5

= 0

$= \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+}(|x| - 5)$

$= \lim_{x \to 5}(x - 5)$ [when $x > 0, |x| = x$]

= 5 – 5

= 0

$= \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 0$

Hence, $\lim_{x \to 5} f(x) = 0$

Question 28: Suppose $f(x) = \begin{cases} a + bx, & i\ f x < 1 \\ 4, & i\ f x = 0 \\ b - ax, & i\ f x > 1 \end{cases}$

and $\lim_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

Answer 28:

The given function is $f(x) = \begin{cases} a + bx, & i\ f x < 1 \\ 4, & i\ f x = 0 \\ b - ax, & i\ f x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(a + bx) = a + b$

$= \lim_{x \to 1^+} f(x) = \lim_{x \to 1}(b - ax) = b - a$

$f(x) = 4$

It is given that $\lim_{x \to 1} f(x) = f(1)$

$= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = f(1)$

= a + b = 4 and b – a = 4

On solving these two equations, we obtain a = 0 and b = 4

Thus, the respective possible values of a and b are 0 and 4.

Question 29: Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

What is $\lim_{x \to a} f(x)$? For some $a \neq a_1, a_2 \ldots a_n$, compute $\lim_{x \to a} f(x)$.

Answer 29:

Question 30:

The given function is $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

$$\lim_{x \to a_1} f(x) = \lim_{x \to a_1} [(x - a_1)(x - a_2) \ldots (x - a_n)]$$

$$= \left[\lim_{x \to a_1} (x - a_1)\right] \left[\lim_{x \to a_2} (x - a_2)\right] \ldots \left[\lim_{x \to a_n} (x - a_n)\right]$$

$$= (a_1 - a_1)(a_1 - a_2) \ldots (a_1 - a_n) = 0$$

$$= \lim_{x \to a_1} f(x) = 0$$

Now, $\lim_{x \to a} f(x) = \lim_{x \to a} [(x - a_1)(x - a_2) \ldots (x - a_n)]$

$$= \left[\lim_{x \to a} (x - a_1)\right] \left[\lim_{x \to a} (x - a_2)\right] \ldots \left[\lim_{x \to a} (x - a_n)\right]$$

$$= (a - a_1)(a - a_2) \ldots (a - a_n)$$

$$= \lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$$

Question 30: If $f(x) = \begin{cases} |x| + 1, x < 0 \\ 0, \quad x = 0 \\ |x| - 1, x > 0 \end{cases}$

For what value(s) of a does $\lim_{x \to a} f(x)$ exists?

Answer 30:

The given function is $f(x) = \begin{cases} |x| + 1, x < 0 \\ 0, \quad\quad x = 0 \\ |x| - 1, x > 0 \end{cases}$

When a = 0

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(|x| + 1)$

$= \lim_{x \to 0}(-x + 1) \quad\quad [\text{if } x < 0, |x| = -x]$

= - 0 + 1

= 1

$= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(|x| - 1)$

$= \lim_{x \to 0}(x - 1) \quad [\text{if } x > 0, [x] = x]$

= 0 – 1

= - 1

Here, is observed the $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\lim_{x \to 0} f(x)$ does not exist.

When a < 0,

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-}(|x| + 1)$

$= \lim_{x \to a}(-x + 1) \quad\quad [x < a < 0 \Rightarrow |x| = -x]$

= -a + 1

$= \lim_{x \to a^+} f(x) = \lim_{x \to a^+}(|x| + 1)$

$= \lim_{x \to a}(-x + 1) \quad [a < x < 0 \Rightarrow |x| = -x]$

= -a + 1

= $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = -a+1$

Thus, limit of f (x) exist at x = a, where a < 0

When a > 0

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x|-1)$

= $\lim_{x\to a}(x-1)$ [0$< x < a \Rightarrow |x| = x$]

= $a-1$

= $\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x|-1)$

= $\lim_{x\to a}(x-1)$ [0$< a < x \Rightarrow |x| = x$]

= $a-1$

= $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$

Thus, limit of $f(x)$exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$

Question 31: If the function f (x) satisfies, $\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$evaluate $\lim_{x\to 1} f(x)$

Answer 31:

$\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$

= $\frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}(x^2-1)} = \pi$

= $\lim_{x\to 1}(f(x)-2) = \pi \lim_{x\to 1}(x^2-1)$

$= \lim_{x\to 1}(f(x) - 2) = \pi(1^2 - 1)$

$= \lim_{x\to 1}(f(x) - 2) = 0$

$= \lim_{x\to 1} f(x) - \lim_{x\to 1} 2 = 0$

$= \lim_{x\to 1} f(x) - 2 = 0$

$= \lim_{x\to 1} f(x) = 2$

Question 32: If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$ For what integers m and n does $\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x)$ exist?

Answer 32:

The given function is $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(mx^2 + n)$

$= m\ (0)^2 + n$

$= n$

$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0}(nx + m)$

$= n(0) + m$

$= m$

Thus, $\lim_{x\to 0} f(x)$exists if m = n .

$= \lim_{x\to 1^-} f(x) = \lim_{x\to 1}(nx + m)$

$= n(1) + m$

$= m + n$

$= \lim_{x \to 1^+} f(x) = \lim_{x \to 1} (nx^3 + m)$

$= n(1)^3 + m$

$= m + n$

$= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x)$

Thus $\lim_{x \to 1} f(x)$ exist for any integral value of m and n.

Exercise 13.2

Question 1: Find the derivative of $x^2 - 2$ at $x = 10$.

Answer 1:

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0} \frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0} \frac{[(10+h)^2-2]-(10^2-2)}{h}$$

$$= \lim_{h\to 0} \frac{10^h+2.10.h+h^h-2-10^h+2}{h}$$

$$= \lim_{h\to 0} \frac{20h+h^2}{h}$$

$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2: Find the derivative of 99x at x = 100.

Answer 2:

Let f(x) = 99x. Accordingly,

$$f'(100) = \lim_{h\to 0} \frac{f(100+h)-f(100)}{h}$$

$$= \lim_{h\to 0} \frac{99(100+h)-99(100)}{h}$$

$$= \lim_{h\to 0} \frac{99\times 100+99h-99\times 100}{h}$$

$$= \lim_{h\to 0} \frac{99h}{h}$$

$$= \lim_{h\to 0}(99) = 99$$

Thus, the derivative of 99x at x = 100 is 99.

Question 3: Find the derivative of x at x = 1.

Answer 3:

Let f(x) = x. Accordingly,

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)-1}{h}$$

$$= \lim_{h\to 0}\frac{h}{h}$$

$$= \lim_{h\to 0}(1)$$

$$= 1$$

Thus, the derivative of x at x = 1 is 1.

Question 4: Find the derivative of the following functions from first principle.

(I) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer 4:

(I) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-27]-(x^3-27)}{h}$$

$$= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$

$$= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}$$

$= \lim_{h\to 0}(h^2 + 3x^2 + 3xh)$

$= 0 + 3x^2 + 0 = 3x^2$

(ii) Let f(x) = (x – 1) (x – 2). Accordingly, from the first principle,

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$

$= \lim_{h\to 0}\frac{(x^2+hx-2x+hx+hx^2-2h-x-h+2)-(x^2-2x-x+2)}{h}$

$= \lim_{h\to 0}\frac{(hx+hx+h^2-2h-h)}{h}$

$= \lim_{h\to 0}\frac{2hx+h^2-3h}{h}$

$= \lim_{h\to 0}(2x+h-3)$

$= (2x+0-3)$

$= 2x-3$

(iii) Let $f(x) = \frac{1}{x^2}$

Accordingly, from the first principle,

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$

$= \lim_{h\to 0} \left[\frac{-h-2hx}{x^2(x+h)^2}\right]$

$= \frac{0-2x}{x^2(x+0)^2} = \frac{-2}{x^3}$

(iv) Let $f(x) = \frac{x+1}{x-1}$

Accordingly, from the first principle,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\left\{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right\}}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h+1)}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$

$= \lim_{h\to 0} \left[\frac{-2}{(x-1)(x+h-1)}\right]$

$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$

Question 5: For the function $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$

Prove that $f'(1) = 100f'(0)$

Answer 5:

The given function is

$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$

$= \frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1\right]$

$= \frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{x^{100}}{100}\right) + \frac{d}{dx}\left(\frac{x^{99}}{99}\right) + \cdots + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{d}(x) + \frac{d}{dx}(1)$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$= \frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \cdots + \frac{2x}{2} + 1 + 0$

$= x^{99} + x^{98} + \cdots + x + 1$

$= f'(x) = x^{99} + x^{98} + \cdots + x + 1$

At $x = 0$

= f' (0) =1

At $x = 1$

$= f'(1) = 1^{99} + 1^{98} + \cdots + 1 + 1 = (1 + 1 + \cdots + 1 + 1)_{100 terms} = 1 \times 100 = 100$

Thus, $f'(1) = 100 \times f^{1}(0)$

Question 6: Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n$ for some fixed real number a.

Answer 6:

Let

$f'(x) = \frac{d}{dx}(x^n + ax^{n-1} + a^n x^{n-2} + \cdots a^{n-1}x + a^n)$

$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \cdots + a^{n-1}\frac{d}{dx}(x) + a^n\frac{d}{dx}(1)$

On using $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$= f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1} + a^n(0)$

$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1}$

Question 7: For some constants an a and b, find the derivative of

(i) (x – a) (x – b) (ii) $(ax^2 + b)^2$ (iii) $\frac{x-a}{x-b}$

Answer 7:

(i) Let f (x) = (x – a) (x – b)

$f(x) = x^2 - (a+b)x + ab$

$f'(x) = \frac{d}{dx}(x^2 - (a+b) + ab)$

$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$

On using theorem $\frac{d}{dx}(x^n) = nx^{x-1}$, we obtain

$f'(x) = 2x - (a+b) + 0 = 2x - a - b$

(ii) Let $f(x) = (ax^2 + b)^2$

$f(x) = a^2x^4 + 2abx^{22} + b^2$

$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$

$= 4a^2x^3 + 4abx$

$= 4ax(ax^2 + b)$

(iii) $f(x) = \frac{x-a}{x-b}$

$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question 8: Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Answer 8:

Let, $f(x) = \frac{x^n-a^n}{x-a}$

$$f'(x) = \frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)\left(nx^{n-1}-0\right)-(x^n-a^n)}{(x-a)^2}$$

$$= \frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9: Find the derivative of

(i) $2x - \frac{3}{4}$

(ii) (5x^3 + 3x – 1) (x – 1)

(iii) x–3 (5 + 3x)

(iv) x^5 (3 – 6x–9)

(v) x–4 (3 – 4x–5)

(vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Answer 9:

(i) Let, $f(x) = 2x - \frac{3}{4}$

= $f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$

= $2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$

= 2 – 0

= 2

(ii) Let f (x) = (5x³ + 3x – 1) (x – 1)

By Leibnitz product rule,

$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(5x^3 + 3x - 1)$

= $(5x^3 + 3x - 1)(1) + (x - 1)(5.3x^2 + 3 - 0)$

= $(5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$

= $5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$

= $20x^3 15x^2 + 6x - 4$

(iii) Let f (x) = x– 3 (5 + 3x)

By Leibnitz product rule,

$f'(x) = x^{-3}\frac{d}{dx}(5 + 3x) + (5 + 3x)\frac{d}{dx}(x^{-3})$

=$x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1})$

= $x^{-3}(3) + (5 + 3x)(-3x^{-4})$

= $3x^{-3} - 15x^{-4} - 9x^{-3}$

$= -6x^{-3} - 15x^{-4}$

(iv) Let f (x) = x5 (3 – 6x–9)

By Leibnitz product rule,

$f'(x) = x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9})\frac{d}{dx}(x^5)$

$= x^5\{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4)$

$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$

$= 54x^{-5} + 15x^4 - 30x^{-5}$

$= 24x^{-5} + 15x^4$

$= 15x^4 + \frac{24}{x^5}$

(v) Let f (x) = x–4 (3 – 4x–5)

By Leibnitz product rule,

$f'(x) = x^{-4}\frac{d}{dx}(3 - 4x^{-5}) + (3 - 4x^{-5})\frac{d}{dx}(x^{-4})$

$= x^{-4}\{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1}$

$= x^{-4}(20x^{-6}) + (3 - 4x^{-5})(-4x^{-5})$

$= 20x^{-10} - 12x^{-5} + 16x^{-10}$

$= 36x^{-10} - 12x^{-5}$

$= -\frac{12}{x^5} + \frac{36}{x^{10}}$

(vi) Let f (x) = $\frac{2}{x+1} - \frac{x^2}{3x-1}$

$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$

By quotient rule,

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left(\frac{(x+1)(0)-2(1)}{(x+1)^2}\right) - \left(\frac{(3x-1)(2x)-(x^2)(3)}{(3x+1)^2}\right)$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10: Find the derivative of cos x from first principle.

Answer 10:

Let f (x) = cos x. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h)-cosx}{h}$$

$$= \lim_{h\to 0}\left[\frac{cosxcosh-sinxsinh-cosx}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-cosx(1-\cos h)-sinxsinh}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-cosx(1-\cos h)}{h} - \frac{sinxsinh}{h}\right]$$

$$= -\cos x\left(\lim_{h\to 0}\frac{1-cosh}{h}\right) - sin\,x\lim_{h\to 0}\left(\frac{sinh}{h}\right)$$

$$= -cosx(0) - sin\,x(1) \qquad \left[\lim_{h\to 0}\frac{1-cosh}{h} = 0 \text{ and } \lim_{h\to 0}\frac{sinh}{h} = 1\right]$$

= - sin x

= $f'(x)$ = - sin x

Question 11: Find the derivative of the following functions:

(i) sin x cos x

(ii) sec x

(iii) 5 sec x + 4 cos x

(iv) cosec x

Answer 11:

(i) Let f (x) = sin x cos x.

Accordingly, from the first principle,

$$f'(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{sin(x+h)cos(x+h)-2sinxcosx}{h}$$

$$= \lim_{h\to 0} \frac{1}{2h}[2sin(x+h)cos(x+h) - sin2x]$$

$$= \lim_{h\to 0} \frac{1}{2h}[sin2(x+h) - sin2x]$$

$$= \lim_{h\to 0} \frac{1}{2h}\left[2cos\frac{2x+2h+2x}{2}.sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[cos\frac{4x+2h}{2} sin\frac{2h}{2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}[cos(2x+h)sinh]$$

$$= \lim_{h\to 0} cos(2x+h). \lim_{h\to 0}\frac{sinh}{h}$$

$$= \cos(2x+0).1$$

$$= cos2x$$

(ii) Let f (x) = sec x.

Accordingly, from the first principle,

$$f"(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec(x+h) - sec\ x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{cosx}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{cosx - \cos(x+h)}{cosxcos(x+h)}\right]$$

$$= \frac{1}{cosx}.\lim_{h \to 0} \frac{1}{h}\left[\frac{-2si\,n\left(\frac{x+x+h}{2}\right)si\,n\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{cosx}.\lim_{h \to 0} \frac{1}{h}\left[\frac{-2si\,n\left(\frac{2s+h}{2}\right)si\,n\left(\frac{h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{cosx}.\lim_{h \to 0} \frac{\left[si\,n\left(\frac{2x+h}{2}\right)\frac{si\,n\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$= \frac{1}{cosx}.\lim_{\frac{h}{2} \to 0} \frac{si\,n\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h \to 0} \frac{si\,n\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$= \frac{1}{cosx}.1.\frac{si\,nx}{cosx}$$

$$= sec\ x\ tan\ x$$

(iii) Let f (x) = 5 sec x + 4 cos x.

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5sec(x+h) + 4cos(x+h) - [5secx + 4cosx]}{h}$$

$$= 5 \lim_{h\to 0} \frac{[\sec(x+h) - secx]}{h} + 4 \lim_{h\to 0} \frac{[\cos(x+h) - cosx]}{h}$$

$$= 5 \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{cosx}\right] + 4 \lim_{h\to 0} \frac{1}{h}[\cos(x+h) - cosx]$$

$$= 5 \lim_{h\to 0} \frac{1}{h}\left[\frac{cosx - \cos(x+h)}{cosx cos(x+h)}\right] + 4 \lim_{h\to 0} \frac{1}{h}[cosx cosh - si\ nx si\ h - cosx]$$

$$= \frac{5}{cosx} \lim_{h\to 0} \frac{1}{h}\left[\frac{-2si\ n\left(\frac{x+x+h}{2}\right) si\ n\left(\frac{x-x-h}{2}\right)}{cos(x+h)}\right] + 4 \lim_{h\to 0} \frac{1}{h}[-cosx(1 - cosh) - si\ nx si\ h]$$

$$= \frac{5}{cosx} . \lim_{h\to 0} \frac{1}{h}\left[\frac{-2si\ n\left(\frac{2x+h}{2}\right) si\ n\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] + 4\left[-cosx \lim_{h\to 0} \frac{(1 - cosh)}{h} - si\ nx \lim_{h\to 0} \frac{si\ nh}{h}\right]$$

$$= \frac{5}{cosx} \lim_{h\to 0} \left[\frac{si\ n\left(\frac{2x+h}{2}\right) . \frac{si\ n\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)}\right] + 4[(-cosx).(0) - (si\ nx).1]$$

$$= \frac{5}{cosx} . \left[\lim_{h\to 0} \frac{si\ n\left(\frac{2x+h}{2}\right)}{\cos(x+h)} . \lim_{h\to 0} \frac{si\ n\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] - 4si\ nx$$

$$= \frac{5}{cosx} . \frac{si\ nx}{cosx} . 1 - 4si\ nx$$

$$= 5sec\ x\ tan\ x - 4si\ nx$$

(iv) Let f (x) = cosec x.

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}[cosec(x+h) - cosecx]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{si\ nx}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{si\,nx - \sin(x+h)}{\sin(x+h)\,si\,nx} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2cos\left(\frac{x+x+h}{2}\right).si\,n\left(\frac{x-x-h}{2}\right)}{\sin(x+h)si\,nx} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2cos\left(\frac{2x+h}{2}\right)si\,n\left(-\frac{h}{2}\right)}{\sin(x+h)\,si\,nx} \right]$$

$$= \lim_{h \to 0} \frac{-cos\left(\frac{2x+h}{2}\right).\frac{si\,n\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)si\,nx}$$

$$= \lim_{h \to 0} \left(\frac{-cos\frac{2x+h}{2}}{\sin(x+h)si\,nx} \right) . \lim_{\frac{h}{2} \to 0} \frac{si\,n\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-cosx}{si\,nxsi\,nx} \right) . 1$$

= - $cosec\ x\ cot\ x$

Miscellaneous Exercise

Question 1: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + a)

Answer 1:

Let f(x) = x + a. Accordingly, $f(x+h) = x + h + a$

By first principle,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{x+h+a-x-a}{h}$

$= \lim_{h \to 0} \left(\frac{h}{h}\right)$

$= \lim_{h \to 0} (1)$

= 1

Question 2: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(px + q)\left(\frac{r}{x} + s\right)$

Answer 2:

Let, $f(x) = (px + q)\left(\frac{r}{x} + s\right)$

$f'(x) = (px + q)\left(\frac{r}{x} + s\right) + \left(\frac{r}{x} + s\right)(px + q)$

$= (px + q)(rx^{-1} + s) + \left(\frac{r}{x} + s\right)(p)$

$= (px + q)(-rx^{-2}) + \left(\frac{r}{x} + s\right)p$

$= (px + q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x} + s\right)p$

$= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps$

$= ps - \frac{qr}{x^2}$

Question 3: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers)

Answer 3:

Let, $f(x) = (ax + b)(cx + d)^2$

By product rule,

$f'(x) = (ax + b)\frac{dd}{x}(cx + d)^2 + (cx + d)^2(ax + b)$

$= (ax + b)\frac{d}{dx}(c^2x^2 + 2cdx + d^2) + (cx + d^2)\frac{d}{dx}(ax + b)$

$= (ax + b)\left[\frac{d}{dx}(c^2x^2) + \frac{d}{dx}(2cdx) + \frac{d}{dx}d^2\right] + (cx + d)^2\left[\frac{d}{dx}ax\frac{d}{dx}b\right]$

$= (ax + b)(2x^2x + 2cd) + (cx + d)^2a$

$= 2c(ax + b)(cx + d) + a(cx + d)^2$

Question 4: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Answer 4:

Let $f(x) = \frac{ax+b}{cx+d}$

$f'(x) = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$

$= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$

$= \frac{acx+ad-acx-bc}{(cx+d)^2}$

$= \frac{ad-bc}{(cx+d)^2}$

Question 5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Answer 5:

Let, $f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$, where $x \neq 0$

By quotient rule,

$f'(x) = \frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1$

$= \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, x \neq 0,1$

$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0,1$

$= \frac{-2}{(x-1)^2}, x \neq 0,1$

Question 6: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2+bx+c}$

Answer 6:

Let $f(x) = \frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x) = \frac{(ax^2+bx+c)\frac{d}{dx}(1)-(1)\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$= \frac{(ax^2+bx+c)(0)-(1)(2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 7: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Answer 7:

Let, $f(x) = \frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x) = \frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

$$= \frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

$$= \frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$= \frac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$$

Question 8: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Answer 8:

Let, $f(x) = \frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$= \frac{(ax+b)(-px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$= \frac{2ap\,x^2+aqx+2bpx+bq-ap\,x^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{ap\,x^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 9:Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Answer 9:

Let, $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + cosx$

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(cosx)$$

$$= a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(cosx)$$

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-si\ nx)$$

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} and \frac{d}{dx}(cosx) = -si\ nx\right]$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - si\ nx$$

Question 10: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $4\sqrt{x} - 2$

Answer 10:

Let, $f(x) = 4\sqrt{x} - 2$

$f'(x) = \frac{d}{dx}\left(4\sqrt{x} - 2\right) = \frac{d}{dx}\left(4\sqrt{x}\right) - \frac{d}{dx}(2)$

$= 4\,\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0 = 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$

$= \left(2x^{\frac{1}{2}}\right) = \frac{2}{\sqrt{x}}$

Question 11: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a+b\sin x}{c+d\cos x}$

Answer 11:

Let, $f(x) = \frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x) = \frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x) - (a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$= \frac{(c+d\cos x)(b\cos x) - (a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$= \frac{cb\cos x + bd\cos^2 x + ad\sin x - bd\sin^2 x}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd(\cos^2 x + \sin^2 x)}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c+d\cos x)^2}$$

Question 12: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): x4 (5 sin x – 3 cos x)

Answer 12:

Let $f(x)$ = x4 (5 sin x – 3 cos x)

By product rule,

$f'(x) = x^4 \frac{d}{dx}(5si\ nx - 3cosx) + (5si\ nx - 3cosx)\frac{d}{dx}(x^4)$

$= x^4 \left[5\frac{d}{dx}(si\ nx) - 3\frac{d}{dx}(cosx)\right] + [5si\ nx - 3cosx]\frac{d}{dx}(x^4)$

$= x^4[5cosx - 3(-si\ nx)] + (5si\ nx - 3cosx)(4x^3)$

$= x^3[5xcosx + 3xsi\ nx + 20si\ nx - 12cosx]$

Question 13: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x2 + 1) cos x

Answer 13:

Let $f(x) =$ (x2 + 1) cos x

By product rule,

$f'(x) = (x^2 + 1)\frac{d}{dx}(cosx)cosx\frac{d}{dx}(x^2 + 1)$

$= (x^2 + 1)(-si\ nx) + cosx(2x)$

$= -x^2 si\ nx - si\ nx + 2xcosx$

Question 14: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (ax2 + sin x) (p + q cos x)

Answer 14:

Let $f(x) =$ (ax2 + sin x) (p + q cos x)

By product rule,

$f'(x) = (ax^2 + si\ nx)\frac{d}{dx}(p + qcosx) + (p + qcosx)\frac{d}{dx}(ax^2 + si\ nx)$

$= (ax^2 + si\ nx)(-qsi\ nx) + (p + qcosx)(2ax + cosx)$

$= -qsi\ nx(ax^2 + si\ nx) + (p + qcosx)(2ax + cosx)$

Question 15: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer 15:

Let $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x) = \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$= \frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 16: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

Answer 16:

Let $f(x) = \frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x) = cos\frac{\pi}{4}.\left[\frac{sin x\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(sin x)}{sin^2 x}\right]$$

$$= \cos\frac{\pi}{4}.\left[\frac{sin x 2x - x^2 cosx}{sin^2 x}\right]$$

$$= \frac{xcos\frac{\pi}{4}[2sin x - xcosx]}{sin^2 x}$$

Question 17: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+tanx}$

Answer 17:

Let $f(x) = \frac{x}{1+tanx}$

$$f'(x) = \frac{(1+tanx)\frac{d}{dx}(x) - x\frac{d}{dx}(1+tanx)}{(1+tanx)^2}$$

$$f'(x) = \frac{(1+tanx) - x.\frac{d}{dx}(1+tanx)}{(1+tanx)^2} \ldots (1)$$

Let, g (x) = 1 + tan x, accordingly, g $(x + h)$ = 1 + tan $(x + h)$

By first principle,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{1+\tan(x+h) - 1 - tanx}{h}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{sin x}{cosx}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)cosx - sin x cos(x+h)}{\cos(x+h)cosx}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)cosx}\right]$$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$

$= \left\{\lim_{h\to 0} \frac{\sin h}{h}\right\}.\left\{\lim_{h\to 0} \frac{1}{\cos(x+h)\cos x}\right\}$

$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$

$= \frac{d}{dx}(1 + \tan x) = \sec^2 x \ldots (2)$

From (i) and (ii), we obtain

$= f'(x) = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$

Question 18: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Answer 18:

Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$f'(x) = \frac{\sin^n x \frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$

It can be easily shown that $\frac{d}{dx}\sin^n x = n\sin^{n-1} x\cos x$

Therefore,

$f'(x) = \frac{\sin^n x \frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$

$= \frac{\sin^n .1 - x\left(n\sin^{n-1} x\cos x\right)}{\sin^{2x} x}$

$= \frac{\sin^{n-1} x(\sin nx - nx\cos x)}{\sin^{2n} x}$

$$= \frac{\sin nx - nx\cos x}{\sin^{n+1} x}$$