2 Continuity and Differentiability

- Continuity of a function –

A function $f(x)$ is said to be continuous at $x=a$, if $\lim_{x\to a} (x) = f(a)$

Left Hand Limit, Right hand Limit

*) LHL = RHL = value of a function at $x=a$.

→ if a function $f(x)$ is not continuous at $x=a$, we say $f(x)$ is discontinuous at $x=a$

→ $f(x)$ will be discontinuous at $x=a$, if

LHS RHS

→ $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

→ $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$

→ $f(a)$ is not defined.

→ At least one of the limit doesn't exists.

ex

Q1) If $f(x) = \frac{|x|}{x}$ is it continuous at $x=0$?

→ LHL, at $x\to 0$

let, $x = 0-h$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{|0-h|}{(0-h)}$$

$$= \lim_{h\to 0} \frac{h}{-h}$$

$$= -1$$

RHL, at $x\to 0$

$x = 0+h$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{|0+h|}{0+h}$$

$$= 1$$

$\boxed{LHL \neq RHL}$ not continuous at $x=0$.

Q2 if $f(x) = \begin{cases} 2x+3 & \text{when } x<0 \\ 0 & \text{when } x=0 \\ x^2+3 & \text{when } x>0 \end{cases}$

is it continuous at $x=0$?

$\Rightarrow$ LHL $x=0$

let, $x = 0-h$

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} 2(0-h)+3$

$= \lim_{h \to 0} (-2h+3)$

$= 3$

RHL $x=0$

let, $x = 0+h$

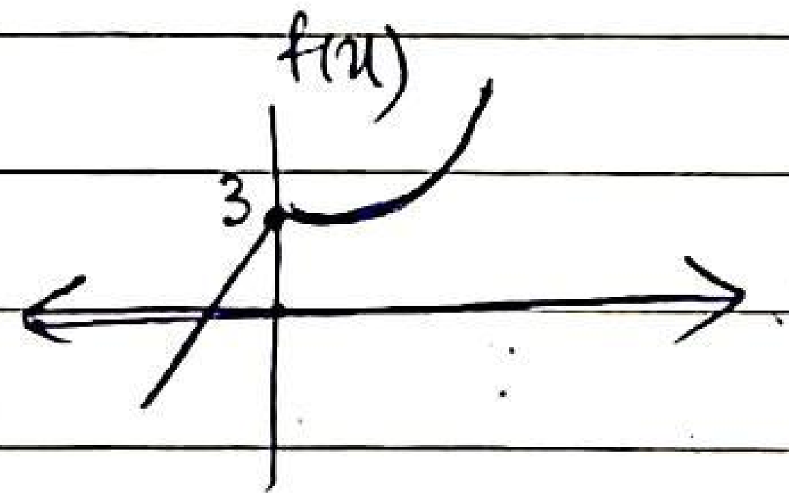
$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} [(0+h)^2+3]$

$= \lim_{h \to 0} (h^2+3)$

$= 3$

$\therefore$ LHL = RHL $\neq f(0)$ $\rightarrow$ not continuous



Q3 $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & x \neq 3 \\ 3 & x = 3 \end{cases}$ check continuity at $x=3$

$f(a) \overset{?}{=} \lim_{x \to a} f(x)$

$\Rightarrow$ $f(x) = f(3) = 3$

$\lim_{x \to 3} \dfrac{x^2-9}{x-3}$

$= \lim_{x \to 3} (x+3)$

$= 6$

$\lim_{x \to 3} f(x) \neq f(3)$

$\rightarrow f(x)$ discontinuous at $x=3$

Q4

if $f(x) = \begin{cases} (x+1)^{\cot x} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$

is continuous at $x=0$, then find 'K'?

→ Given, from defination of continuous.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

$$\Rightarrow \lim_{x \to 0} (x+1)^{\cot x} = f(0)$$

$$\Rightarrow e^{\lim_{x \to 0} \cot x (x+1-1)} = K$$

$$\Rightarrow e^{\lim_{x \to 0} \frac{x}{\tan x}} = K$$

$$\Rightarrow e^{\lim_{x \to 0} \frac{1}{\frac{\tan x}{x}=1}} = K$$

$$\Rightarrow e^{1} = K$$

$$\boxed{K = e}$$

Q5) if $f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 5x-4 & ,, \quad 0 < x \leq 1 \\ 4x^2-3x & ,, \quad 1 < x < 2 \\ 3x+4 & ,, \quad x \geq 2 \end{cases}$

then which of the following is true -

a) f(x) discontinuous at $x=0$ (true)

b) f(x) is continuous at $x=1$ ~~(false)~~ (true)

c) f(x) is left continuous at $x=2$ (true)

d) all of the above

=>

at $x=0$

LHL, at $x \to 0$	RHL,
$\lim_{x \to 0^-} f(x) = 0$	$\lim_{x \to 0^+} f(x)$
	$= \lim_{x \to 0} 5x-4$
	$= -4$

LHL ≠ RHL

at $x=1$

LHL, at $x \to 1$	RHL,
$= \lim_{x \to 1^-} 5x-4$	$= \lim_{x \to 1^+} (4x^2-3x)$
$= 1$	$= 1$

~~LHL ≠ RHL~~ LHL = RHL = f(1)

at $x=2$

LHL

$\lim_{x \to 2^-} 4x^2 - 3x$

$\Rightarrow 4 \times 4 - 6$

$= 10$

$\boxed{LHL = f(2)}$

$f(2) = 3x + 4$
$= 6 + 4$
$= 10$

C
D

- Differentiability: (Left Derivative = RD)

Note: Every differentiable function is continuous but a continuous function need not be differentiable.

ex-1

$f(x) = |x|$, Check differentiability.

Left

$\boxed{LD = RD}$

$\Rightarrow LD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \frac{|-h| - 0}{-h} = -1$

$RD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{+h} = \frac{|h| - 0}{h} = 1$

right

$\boxed{LD \neq RD}$ → not differentiable.

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(D) c

* ex-2

let $f(x) = x|x|$, $x \in R$ then $f(x)$ at $x=0$ is

(a) continuous, differentiable.
× (b) continuous, not differentiable.
× (c) differentiable but not continuous.
× (d) Neither differentiable nor continuous.

$$f(x) = \begin{cases} -x^2 & x<0 \\ x^2 & x>0 \\ 0 & x=0 \end{cases}$$

$\Rightarrow$ $L.H.L = \lim_{x\to 0^-} -x^2 = 0$

$R.H.L = \lim_{x\to 0^+} x^2 = 0$

L.H.L = R.H.L $\rightarrow$ continuous. $= f(0)$

$$L.D = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h(h)-0}{-h} = \lim_{h\to 0} \frac{-h^2}{-h} = 0$$

$$R.D = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

L.D = R.D $\rightarrow$ differentiable.

simple Ex-3

$f(x) = \frac{1}{1+|x|}$ check differentiability at $x=0$.

$$f(x) = \begin{cases} \frac{1}{1-x} & x<0 \\ \frac{1}{1+x} & x>0 \\ 1 & x=0 \end{cases}$$

$\Rightarrow$

$$\underline{LD} = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1-(-h)} - 1}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{-h} \Rightarrow \lim_{h\to 0} \frac{\cancel{1}-\cancel{1}-h}{(1+h)(-h)}$$

$$\Rightarrow \lim_{h\to 0} \frac{\cancel{(-h)}}{(1+h)\cancel{(-h)}}$$

$$\Rightarrow \frac{1}{1+0} = 1$$

$$RD = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{-\cancel{h}}{\cancel{h}(1+h)} \Rightarrow -1$$

$\boxed{LD \neq RD} \rightarrow$ not differentiability.

→ not differentiable.

ex-4

$f(x) = |x-1| + |x+1|$ Discuss ~~a~~ the continuity and ~~disconti~~ differentiability of the function.

$$\Rightarrow f(x) = \begin{cases} -(x-1)-(x+1) & x < -1 \\ -(x-1)+(x+1) & -1 \leq x \leq 1 \\ (x-1)+(x+1) & x > 1 \end{cases}$$

$-x+1+x+1$

$$LD = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{(-h)}$$

$$= \lim_{h\to 0} \frac{-((-h)-1)+((-h)+1) - 2}{-h}$$

$$= \lim_{h\to 0} \frac{h+1-h+1-2}{-h}$$

$$= \lim_{h\to 0} \frac{0}{-h} = \frac{0}{0}$$ Not Differentiable.

$$RD = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{2-2}{h} = \frac{0}{0}$$

	sin			
	0	$\frac{3\pi}{2}$	$\frac{\pi}{2}$	π
sin	0		1	

~~ex~~ 5

$f(x) = [\sin x]$ when $x \in [0, 2\pi]$, which of the following is true -

(a) $f(x)$ is discountinuous. at $x = \frac{\pi}{2}$.
(b) $f(x)$ is discontinuous. at $x = \pi$.
(c) $f(x)$ is continuous at $x = \frac{3\pi}{2}$.
(d) All of the above.

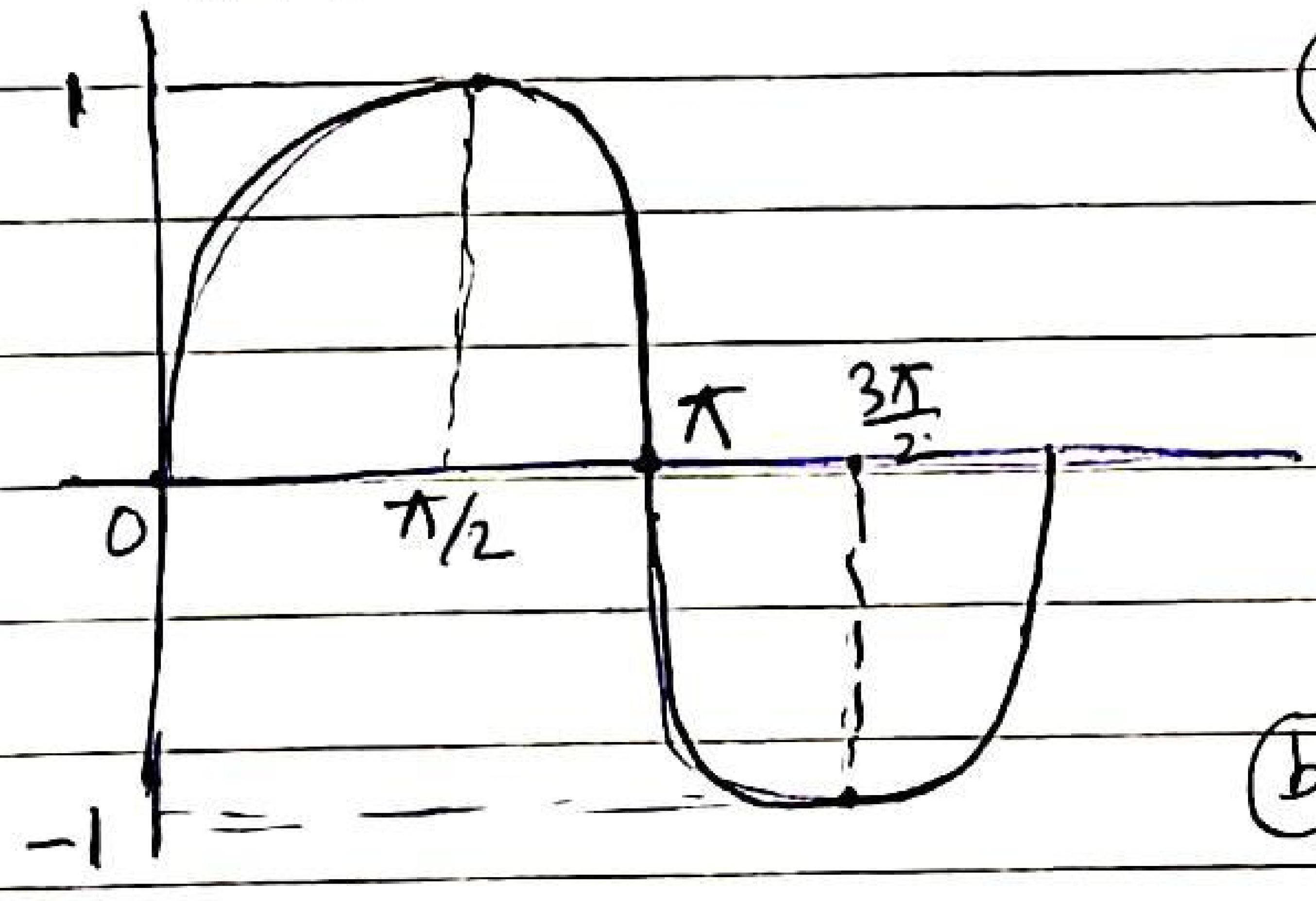
$-1 \leqslant \sin x \leqslant 1$

$\boxed{\sin x}$

-1 ; 0 ; 1

$\Rightarrow$ ~~$\text{LHL} = \lim_{h \to 0^-} f(0-h)$~~

~~$= \lim_{h \to 0} [\sin(-h)]$~~



(a) $f(\pi/2) = 1$

$f(\frac{\pi}{2}^-) = 0$ $\underline{\text{discont}}$

$f(\frac{\pi}{2}^+) = 0$

(b) $f(\pi^-) = 0$

$f(\pi^+) = -1$

(c) $f(\frac{3\pi}{2}^-) = -1$

$f(\frac{3\pi}{2}^+) = -1$