CHAPTER XV.

MULTINOMIAL THEOREM.

194. WE have already seen in Art. 175, how we may apply the Binomial Theorem to obtain the expansion of a multinomial expression. In the present chapter our object is not so much to obtain the complete expansion of a multinomial as to find the coefficient of any assigned term.

Example. Find the coefficient of $a^4b^2c^3d^5$ in the expansion of

$$(a+b+c+d)^{14}.$$

The expansion is the product of 14 factors each equal to $a+b+c+d$, and every term in the expansion is of 14 dimensions, being a product formed by taking one letter out of each of these factors. Thus to form the term $a^4b^2c^3d^5$, we take a out of any *four* of the fourteen factors, b out of any *two* of the remaining ten, c out of any *three* of the remaining eight. But the number of ways in which this can be done is clearly equal to the number of ways of arranging 14 letters when four of them must be a, two b, three c, and five d; that is, equal to

$$\frac{\underline{|14}}{\underline{|4}\,\underline{|2}\,\underline{|3}\,\underline{|5}}. \qquad \text{[Art. 151.]}$$

This is therefore the number of times in which the term $a^4b^2c^3d^5$ appears in the final product, and consequently the coefficient required is 2522520.

195. *To find the coefficient of any assigned term in the expansion of* $(a+b+c+d+\ldots)^p$, *where* p *is a positive integer.*

The expansion is the product of p factors each equal to $a+b+c+d+\ldots$, and every term in the expansion is formed by taking one letter out of each of these p factors; and therefore the number of ways in which any term $a^\alpha b^\beta c^\gamma d^\delta \ldots$ will appear in the final product is equal to the number of ways of arranging p letters when α of them must be a, β must be b, γ must be c; and so on. That is,

the coefficient of $\quad a^\alpha b^\beta c^\gamma d^\delta \ldots$ is $\dfrac{\underline{|p}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\,\underline{|\delta}\ldots}$,

where $\quad \alpha+\beta+\gamma+\delta+\ldots=p.$

COR. In the expansion of

$$(a + bx + cx^2 + dx^3 + \dots)^p,$$

the term involving $a^\alpha b^\beta c^\gamma d^\delta \dots$ is

$$\frac{\lfloor p}{\lfloor \alpha \lfloor \beta \lfloor \gamma \lfloor \delta \dots} a^\alpha (bx)^\beta (cx^2)^\gamma (dx^3)^\delta \dots,$$

or
$$\frac{\lfloor p}{\lfloor \alpha \lfloor \beta \lfloor \gamma \lfloor \delta \dots} a^\alpha b^\beta c^\gamma d^\delta \dots x^{\beta + 2\gamma + 3\delta + \dots},$$

where $\alpha + \beta + \gamma + \delta + \dots = p$.

This may be called *the general term* of the expansion.

Example. Find the coefficient of x^5 in the expansion of $(a + bx + cx^2)^9$.

The general term of the expansion is

$$\frac{\lfloor 9}{\lfloor \alpha \lfloor \beta \lfloor \gamma} a^\alpha b^\beta c^\gamma x^{\beta + 2\gamma} \quad \dots\dots\dots\dots (1),$$

where $\alpha + \beta + \gamma = 9$.

We have to obtain by trial all the positive integral values of β and γ which satisfy the equation $\beta + 2\gamma = 5$; the values of α can then be found from the equation $\alpha + \beta + \gamma = 9$.

Putting $\gamma = 2$, we have $\beta = 1$, and $\alpha = 6$;
putting $\gamma = 1$, we have $\beta = 3$, and $\alpha = 5$;
putting $\gamma = 0$, we have $\beta = 5$, and $\alpha = 4$.

The required coefficient will be the sum of the corresponding values of the expression (1).

Therefore the coefficient required

$$= \frac{\lfloor 9}{\lfloor 6 \lfloor 2} a^6 bc^2 + \frac{\lfloor 9}{\lfloor 5 \lfloor 3} a^5 b^3 c + \frac{\lfloor 9}{\lfloor 4 \lfloor 5} a^4 b^5$$

$$= 252a^6bc^2 + 504a^5b^3c + 126a^4b^5.$$

196. *To find the general term in the expansion of*

$$(a + bx + cx^2 + dx^3 + \dots)^n,$$

where n *is any rational quantity.*

By the Binomial Theorem, the general term is

$$\frac{n(n-1)(n-2)\dots(n-p+1)}{\lfloor p} a^{n-p} (bx + cx^2 + dx^3 + \dots)^p,$$

where p is a positive integer.

And, by Art. 195, the general term of the expansion of

$$(bx + cx^2 + dx^3 + \dots)^p$$

is
$$\frac{\underline{|p}}{\underline{|\beta}\,\underline{|\gamma}\,\underline{|\delta}\dots} b^\beta c^\gamma d^\delta \dots x^{\beta+2\gamma+3\delta+\dots},$$

where β, γ, δ ... are positive integers whose sum is p.

Hence the general term in the expansion of the given expression is

$$\frac{n(n-1)(n-2)\dots(n-p+1)}{\underline{|\beta}\,\underline{|\gamma}\,\underline{|\delta}\dots} a^{n-p} b^\beta c^\gamma d^\delta \dots x^{\beta+2\gamma+3\delta+\dots},$$

where
$$\beta + \gamma + \delta + \dots = p.$$

197. Since $(a + bx + cx^2 + dx^3 + \dots)^n$ may be written in the form

$$a^n \left(1 + \frac{b}{a}x + \frac{c}{a}x^2 + \frac{d}{a}x^3 + \dots\right)^n,$$

it will be sufficient to consider the case in which the first term of the multinomial is unity.

Thus the general term of

$$(1 + bx + cx^2 + dx^3 + \dots)^n$$

is
$$\frac{n(n-1)(n-2)\dots(n-p+1)}{\underline{|\beta}\,\underline{|\gamma}\,\underline{|\delta}} b^\beta c^\gamma d^\delta \dots x^{\beta+2\gamma+3\delta+\dots},$$

where
$$\beta + \gamma + \delta + \dots = p.$$

Example. Find the coefficient of x^3 in the expansion of

$$(1 - 3x - 2x^2 + 6x^3)^{\frac{2}{3}}.$$

The general term is

$$\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\dots\left(\frac{2}{3}-p+1\right)}{\underline{|\beta}\,\underline{|\gamma}\,\underline{|\delta}} (-3)^\beta (-2)^\gamma (6)^\delta x^{\beta+2\gamma+3\delta} \quad\dots\dots (1).$$

We have to obtain by trial all the positive integral values of β, γ, δ which satisfy the equation $\beta + 2\gamma + 3\delta = 3$; and then p is found from the equation $p = \beta + \gamma + \delta$. The required coefficient will be the sum of the corresponding values of the expression (1).

In finding β, γ, δ, ... it will be best to commence by giving to δ successive integral values beginning with the greatest admissible. In the present case the values are found to be

$$\delta=1, \quad \gamma=0, \quad \beta=0, \quad p=1;$$
$$\delta=0, \quad \gamma=1, \quad \beta=1, \quad p=2;$$
$$\delta=0, \quad \gamma=0, \quad \beta=3, \quad p=3.$$

Substituting these values in (1) the required coefficient

$$=\left(\frac{2}{3}\right)(6)+\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)(-3)(-2)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{\underline{|3}}(-3)^3$$

$$=4-\frac{4}{3}-\frac{4}{3}=\frac{4}{3}.$$

198. Sometimes it is more expeditious to use the Binomial Theorem.

Example. Find the coefficient of x^4 in the expansion of $(1-2x+3x^2)^{-3}$.

The required coefficient is found by picking out the coefficient of x^4 from the first few terms of the expansion of $(1-\overline{2x-3x^2})^{-3}$ by the Binomial Theorem; that is, from

$$1+3(2x-3x^2)+6(2x-3x^2)^2+10(2x-3x^2)^3+15(2x-3x^2)^4;$$

we stop at this term for all the other terms involve powers of x higher than x^4.

The required coefficient $=6 \,.\, 9+10 \,.\, 3(2)^2(-3)+15(2)^4$

$$=-66.$$

EXAMPLES. XV.

Find the coefficient of

1. $a^2b^3c^4d$ in the expansion of $(a-b-c+d)^{10}$.
2. a^2b^5d in the expansion of $(a+b-c-d)^8$.
3. a^3b^3c in the expansion of $(2a+b+3c)^7$.
4. $x^2y^3z^4$ in the expansion of $(ax-by+cz)^9$.
5. x^3 in the expansion of $(1+3x-2x^2)^3$.
6. x^4 in the expansion of $(1+2x+3x^2)^{10}$.
7. x^6 in the expansion of $(1+2x-x^2)^5$.
8. x^8 in the expansion of $(1-2x+3x^2-4x^3)^4$.

Find the coefficient of

9. x^{23} in the expansion of $(1-2x+3x^2-x^4-x^5)^5$.

10. x^5 in the expansion of $(1-2x+3x^2)^{-\frac{1}{2}}$.

11. x^3 in the expansion of $(1-2x+3x^2-4x^3)^{\frac{1}{2}}$.

12. x^8 in the expansion of $\left(1-\frac{x^2}{3}+\frac{x^4}{9}\right)^{-2}$.

13. x^4 in the expansion of $(2-4x+3x^2)^{-2}$.

14. x^6 in the expansion of $(1+4x^2+10x^4+20x^6)^{-\frac{3}{4}}$.

15. x^{12} in the expansion of $(3-15x^3+18x^6)^{-1}$.

16. Expand $(1-2x-2x^2)^{\frac{1}{4}}$ as far as x^2.

17. Expand $(1+3x^2-6x^3)^{-\frac{2}{3}}$ as far as x^5.

18. Expand $(8-9x^3+18x^4)^{\frac{4}{3}}$ as far as x^8.

19. If $(1+x+x^2+\ldots\ldots+x^p)^n=a_0+a_1x+a_2x^2+\ldots\ldots a_{np}x^{np}$, prove that

(1) $a_0+a_1+a_2+\ldots\ldots+a_{np}=(p+1)^n$.

(2) $a_1+2a_2+3a_3+\ldots\ldots+np\,.\,a_{np}=\frac{1}{2}np\,(p+1)^n$.

20. If a_0, a_1, a_2, $a_3\ldots$ are the coefficients in order of the expansion of $(1+x+x^2)^n$, prove that

$$a_0^2-a_1^2+a_2^2-a_3^2+\ldots\ldots+(-1)^{n-1}a_{n-1}^2=\frac{1}{2}a_n\{1-(-1)^na_n\}.$$

21. If the expansion of $(1+x+x^2)^n$

be $$a_0+a_1x+a_2x^2+\ldots+a_rx^r+\ldots+a_{2n}x^{2n},$$
shew that

$$a_0+a_3+a_6+\ldots=a_1+a_4+a_7+\ldots=a_2+a_5+a_8+\ldots=3^{n-1}.$$