Exercise 14.2

Answer 1E.

In general, we use the notation $\lim_{(x,y)\to(a,b)} f(x,y) = L$ to indicate the values of $f(x,y)$ approach the number L as the point (x,y) approaches the point (a,b) along any path that stays within the domain of f.

It is given that

$$\lim_{(x,y)\to(3,1)} f(x,y) = 6$$

So it indicates that $f(x, y)$ approaches 6 as (x, y) approaches (3, 1) along any path that stays within the domain of f.

Since $\lim_{(x,y)\to(3,1)} f(x,y)$ is not necessarily equal to $f(3, 1)$, therefore nothing can be said about the values of $f(3, 1)$

A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Now if f is continuous then

$$\lim_{(x,y)\to(3,1)} f(x,y) = f(3,1)$$

Therefore $\boxed{f(3,1) = 6}$.

Answer 2E.

(a)

The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b)

Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation can jump from one value to another.

(c)

The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

Answer 3E.

$$f(x,y) = \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$$

x\y	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	-7	-3.5833	-2.0844	-2.5	-2.287	-2.05	-1.666
-0.5	-3.5833	-2.8928	-2.6352	-2.5	-2.382	-2.222	-1.95
-0.2	-2.8044	-2.6352	-2.5513	-2.5	-2.450	-2.379	-2.258
0	-2.5	-2.5	-2.5	-2.5	-2.5	-2.5	-2.5
0.2	-2.2872	-2.3828	-2.4509	-2.5	-2.550	-2.627	-2.751
0.5	-2.05	-2.2222	-2.3795	-2.5	-2.627	-2.821	-3.083
1.0	-1.6666	-1.95	-2.2581	-2.5	-2.751	-3.083	-3

From the above table showing values of f(x, y) for the above showing values of f(x, y) for points near the origin, it appears that as (x, y) approaches (0, 0), the values of f(x, y) approach -2.5

Then we can write

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{-5}{2}$$

Answer 4E.

$$f(x,y) = \frac{2xy}{x^2+2y^2}$$

We make a table of values of f(x, y) for a set of points

x\y	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0	-0.667	-0.444	-0.316
0	0	0	0	-	0	0	0
0.1	-0.316	-0.444	-0.667	0	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0	0.545	0.706	0.667

From the table above it appears that the values of f(x, y) do not approach a single value as (x, y) approach (0, 0), then we can say that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

Now let $(x,y)\to(0,0)$ along x – axis that is y = 0

Then $f(x,0)=0$ to $f(x,y)\to 0$

Let $(x,y)\to(0,0)$ along y = x, the $f(x,y) = \frac{2x^2}{x^2+2x}$

$$= \frac{2}{3} \neq 0$$

So $f(x,y)\to {}^2\!/\!_3$ along y = x as $(x,y)\to(0,0)$

Since f approaches different values along different paths so the limit $(x,y)\to 0$ does not exist. This proves above conclusion from the table

Answer 5E.

Given that

$$\lim_{(x,\,y)\to(1,\,2)}\left(5x^3 - x^2y^2\right)$$

Using direct substitution.

$$5(1)^3 - 1^2 2^2 = 5 - 4 = 1$$

Answer 6E.

Consider the limit,

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$$

To find the limit, first check the function $e^{-xy}\cos(x+y)$ is continuous or not.

The function $e^{-xy}\cos(x+y)$ is a composition of fundamental functions that are known to be continuous.

Since the function is continuous and defined at the point $(1,-1)$, direct substitution can be used to solve the limit.

$$\begin{aligned} e^{-1(-1)}\cos(1-1) &= e\cos(0) \\ &= e.1 \\ &= \boxed{e} \end{aligned}$$

Answer 7E.

Given that

$$\lim_{(x,\,y)\to(2,\,1)} 4 - xy / x^2 + 3y^2$$

Any rational function is continuous on its domain because it is a quotient of continuous functions.

So we can find the limit by direct substitution:

$$\lim_{(x,\,y)\to(2,\,1)} = 4 - 2\cdot 1 / 2^2 + 3\cdot 1^2 = 2/7$$

Answer 8E.

Consider the following limit:

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

Find the limit of the function at $x \to 1$ and $y \to 0$ as shown below:

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) &= \ln\left(\frac{1+(0)^2}{(1)^2+(1)(0)}\right) \\ &= \ln(1) \\ &= 0\end{aligned}$$

Therefore, $\boxed{\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = 0}$.

Answer 9E.

To find $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ if exists:

Let $f(x,y) = \frac{x^4-4y^2}{x^2+2y^2}$

If substitute the limit directly then,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2} &= \frac{(0)^4-4(0)^2}{(0)^2+2(0)^2} \\ &= \frac{0}{0}\end{aligned}$$

This is an indeterminate form, so evaluate this limit in the following way.

To evaluate the limit along *x*-axis, substitute $y = 0$

Then,

$$f(x,0) = \frac{x^4 - 4(0)^2}{x^2 + 2(0)^2}$$
$$= x^2$$

Along *x*-axis, as $(x, y) \to (0,0)$, $f(x, y) \to 0$

To evaluate the limit along *y*-axis, substitute $x = 0$

Then,

$$f(0, y) = \frac{(0)^4 - 4y^2}{(0)^2 + 2y^2}$$
$$= -2$$

Along *y*-axis, as $(x, y) \to (0,0)$, $f(x, y) \to -2$

Therefore both the limits are different, and hence given limit do not exist.

Answer 10E.

To find $\lim_{(x,y)\to(0,0)} \frac{5y^4 \cos^2 x}{x^4 + y^4}$ if exists:

Let $f(x, y) = \frac{5y^4 \cos^2 x}{x^4 + y^4}$

If substitute the limit directly then,

$$\lim_{(x,y)\to(0,0)} \frac{5y^4 \cos^2 x}{x^4 + y^4} = \frac{5(0)^4 \cos^2 0}{(0)^4 + (0)^4}$$
$$= \frac{0}{0}$$

This is an indeterminate form, so evaluate this limit in the following way.

To evaluate the limit along x-axis, substitute $y = 0$

Then,

$$f(x,0) = \frac{5(0)^4 \cos^2 x}{x^4 + (0)^4}$$
$$= 0$$

Along x-axis, as $(x,y) \to (0,0)$, $f(x,y) \to 0$

To evaluate the limit along y-axis, substitute $x = 0$

Then,

$$f(0,y) = \frac{5y^4 \cos^2 0}{(0)^4 + y^4}$$
$$= \frac{5y^4}{y^4}$$
$$= 5$$

Along y-axis, as $(x,y) \to (0,0)$, $f(x,y) \to 5$

Therefore both the limits are different, and hence given limit do not exist.

Answer 11E.

Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$$

Find the limit if exists otherwise limit does not exist.

Let the function $f(x,y) = \frac{y^2 \sin^2 x}{x^4 + y^4}$.

If we substitute the limit $(x,y) \to (0,0)$ directly then,

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4} = \frac{(0)^2 \sin^2 (0)}{(0)^4 + (0)^4}$$
$$= \frac{0}{0}$$

This is an indeterminate form, so evaluate this limit in the following way.

To evaluate the limit along *x*-axis, substitute $y=0$ in $f(x,y)=\frac{y^2\sin^2 x}{x^4+y^4}$.

Then, $f(x,0)=\frac{(0)^2\sin^2 x}{x^4+(0)^4}$

$=0$

Along *x*-axis, as $(x,y)\to(0,0)$, $f(x,y)\to 0$

To evaluate the limit along *y*-axis, substitute $x=0$ in $f(x,y)=\frac{y^2\sin^2 x}{x^4+y^4}$.

Then, $f(0,y)=\frac{y^2\sin^2 0}{(0)^4+y^4}$

$=\frac{y^2(0)}{y^4}$

$=0$

Along *y*-axis, as $(x,y)\to(0,0)$, $f(x,y)\to 0$

Although obtained identical limits along the axes that do not show that the given limit is 0.

Now, approach $(0,0)$ along another line say $y=x$, $x \neq 0$

Then, $f(x,x)=\dfrac{(x)^2 \sin^2 x}{x^4+(x)^4}$

$$=\frac{x^2 \sin^2 x}{x^4+x^4}$$

$$=\frac{x^2 \sin^2 x}{2x^4}$$

$$=\frac{\sin^2 x}{2x^2}$$

$$=\frac{1}{2}\left(\frac{\sin x}{x}\right)^2$$

$$\lim_{(x,y)\to(0,0)} f(x,x)=\lim_{(x,y)\to(0,0)} \frac{1}{2}\left(\frac{\sin x}{x}\right)^2$$

$$=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2$$

$$=\frac{1}{2}(1)^2$$

$$=\frac{1}{2}$$

Limit of the function $f(x,y)$ along *x*-axis is not coincides with the limit of $f(x,y)$ along the line $y=x$.

Therefore, the limit $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{y^2 \sin^2 x}{x^4+y^4}$ does not exist.

Answer 12E.

Consider the following limit

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$$

Its need to find the limit, if it exists

Let $f(x,y)=\frac{xy-y}{(x-1)^2+y^2}$.

If substitute the limit directly then,

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} &= \frac{(1)(0)-(0)}{(1-1)^2+(0)^2} \\ &= \frac{0}{0}\end{aligned}$$

This is an indeterminate form, so evaluate this limit in the following way.

To evaluate the limit along x-axis, substitute $y=0$.

Then,

$$\begin{aligned}f(x,0) &= \frac{x(0)-0}{(x-1)^2+(0)^2} \\ &= 0\end{aligned}$$

Thus, $f(x,y)\to 0$ along the line $y=0(x-\text{axis})$.

To evaluate the limit along the line parallel to y-axis, substitute $x=1$.

Then,

$$\begin{aligned}f(1,y) &= \frac{(1)y-y}{(1-1)^2+y^2} \\ &= 0\end{aligned}$$

Thus, along the line $x=1$ parallel to y-axis $f(x,y)\to 0$.

From the above resultants, the limits along the axes are identical and that do not show that the given limit is 0.

Next, try approach $f(x)$ along the line $y=x-1$.

Notice that this line passes through the point $(1,0)$ as required.

$$\begin{aligned} f(x,x-1) &= \frac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2} \\ &= \frac{(x-1)(x-1)}{2(x-1)^2} \\ &= \frac{(x-1)^2}{2(x-1)^2} \\ &= \frac{1}{2} \end{aligned}$$

Thus, $f(x,y)\to\frac{1}{2}$ along the line $y=x-1$.

As f has two different limits along two different paths, it follows that the **limit does not exist**.

Answer 13E.

Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

Let $f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$.

First let's approach $(0,0)$ along $x-$ axis.

Then $y=0$ gives $f(x,0)=\frac{x(0)}{\sqrt{x^2+0^2}}=0$ for all $x\neq 0$.

So $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the $x-\text{axis}$.

Now approach $(0,0)$ along the $y-$ axis by putting $x=0$.

$$f(0,y)=\frac{0(y)}{\sqrt{0^2+y^2}}=0$$

The function $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the $y-\text{axis}$.

Approach (0, 0) along any non-vertical line through the origin

Then $y = mx$, where m is the slope.

$$f(x,y) = f(x,mx)$$

$$= \frac{x(mx)}{\sqrt{x^2 + m^2x^2}}$$

$$= \frac{mx^2}{x\sqrt{1+m^2}}$$

$$= \frac{mx}{\sqrt{1+m^2}}$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = mx$.

Although $f(x,y) \to 0$ along the axes and any non-vertical line, this does not show that the given limit is 0.

Approach $(0,0)$ along the parabola $x = y^2$.

Then,

$$f(x,y) = f(y^2,y)$$

$$= \frac{(y^2)y}{\sqrt{(y^2)^2 + y^2}}$$

$$= \frac{y^3}{\sqrt{y^4 + y^2}}$$

$$= \frac{y^3}{y\sqrt{y^2+1}}$$

$$= \frac{y^2}{\sqrt{y^2+1}}$$

Therefore, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the parabola $x = y^2$.

Let $(x,y)\to(0,0)$ along parabola $y=x^2$.

Then, $f(x,y)=f(x,x^2)$

$$=\frac{x^3}{\sqrt{x^2+x^4}}$$

$$=\frac{x^2}{\sqrt{1+x^2}}$$

Therefore, $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $x=y^2$.

Therefore the function $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along any line and the parabolas $x=y^2$ and $y=x^2$.

The limit along any line through the origin is 0, but the limits along the parabolas $y=x^2$ **and** $x=y^2$ also turn out to be 0, so it is possible that the lime does exist and is equal to 0.

Let $\varepsilon>0$. Find $\delta>0$ such that $|f(x,y)-0|<\varepsilon$, whenever $0<\sqrt{x^2+y^2}<\delta$.

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon, \text{ whenever } 0<\sqrt{x^2+y^2}<\delta$$

$$\frac{|x||y|}{\sqrt{x^2+y^2}}<\varepsilon, \text{ whenever } 0<\sqrt{x^2+y^2}<\delta$$

Use the inequality $|x|\le\sqrt{x^2+y^2}$.

Then, $\frac{|x|}{\sqrt{x^2+y^2}}\le 1$ and $|y|\ge 0$.

Therefore, $\frac{|x||y|}{\sqrt{x^2+y^2}}\le|y|\le\sqrt{y^2}\le\sqrt{x^2+y^2}$

Choose $\delta=\varepsilon$, then

$$\frac{|x||y|}{\sqrt{x^2+y^2}}\le|y|\le\sqrt{y^2}\le\sqrt{x^2+y^2}<\delta=\varepsilon$$

Thus $\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon$ whenever $0<\sqrt{x^2+y^2}<\delta$.

Hence, the required value of the given limit is $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=\boxed{0}$.

Answer 14E.

Consider the limit $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$.

To find the value of the limit, first simplify the function and then apply the limit to the resultant.

First, the simplification is as follows:

$$\frac{x^4 - y^4}{x^2 + y^2} = \frac{\cancel{(x^2 + y^2)}(x^2 - y^2)}{\cancel{x^2 + y^2}}$$
$$= x^2 - y^2 \qquad \text{where } (x, y) \neq (0,0)$$

This works perfectly for the limit calculation, because the limit examines the functions at points that are specifically not equal to (0, 0).

Therefore, the all of the limits of the function are equal. Specifically,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x^2 - y^2)$$

Since the simplified function (at all points) is a polynomial, it is continuous at all points.

The value of limit is,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} &= \lim_{(x,y)\to(0,0)} (x^2 - y^2) \\ &= (0)^2 - (0)^2 \\ &= 0\end{aligned}$$

Therefore, the value of the limit is $\boxed{\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = 0}$.

Answer 15E.

Given limit is $\lim_{(x,\,y)\to(0,\,0)} x^2ye^y / x^4 + 4y^2$

First we'll approach (0,0) along the x- axis. Then y = 0 gives

$f\left(x,\,0\right) = 0 / x^4 = 0$ for all $x \neq 0$, so

$f(x,\,y) \longrightarrow 0 \text{as} (x,\,y) \longrightarrow (0,\,0)$ along the x- axis.

Now, we'll approach along the y- axis by putting x=0.

Then $f\left(0,\,y\right) = 0 / 4y^2 = 0$ for all $y \neq 0$

This is not enough to show that the given limit is 0. Now we'll approach (0,0) along the parabola,

$y = x^2$ for all $x \neq 0$

$\lim_{\left(x,\,x^2\right)\to(0,\,0)} f\left(x,\,x^2\right) = x^4 e^{x^2} / 5x^4 = 1/5$

Since I have obtained different limits along different paths, the given limit does not exist.

Answer 16E.

Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

The objective is to evaluate the value of the limit if it exists.

Let $f(x,y) = \dfrac{x^2 \sin^2 y}{x^2 + 2y^2}$

If $y = 0$ then $f(x,0) = 0$

So, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along x – axis

If $x = 0$ then $f(0,y) = 0$

So, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along y – axis

Now verify the limit at (0, 0) along any non – vertical line through origin.

Substitute $y = x$.

The function $f(x,y) = \frac{x^2(\sin x)^2}{x^2 + 2(x)^2}$

$= \frac{\sin^2 x}{3}$

Hence, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = x$

Now let $(x,y) \to (0,0)$ along the parabola $x = y^2$

Then the function $f(x,y) = f(y^2, y)$

$= \frac{y^4 \sin^2 y}{y^4 + 2y^2}$

$= \frac{y^2 \sin^2 y}{y^2 + 2}$

Hence, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along parabola $x = y^2$

Let $f(x,y) \to (0,0)$ along parabola $y = x^2$

Then $f(x,y) = \frac{x^2(\sin x^2)^2}{x^2 + 2x^4}$

$= \frac{(\sin x^2)^2}{1 + 2x^2}$

Then, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along parabola $y = x^2$

From the above, it is observed that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any line and parabolas $x = y^2$ and $y = x^2$.

Since different paths leads to same limiting value, the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$ exists.

Now $x^2 \le x^2 + 2y^2$

Then $\frac{x^2}{x^2+2y^2} \le 1$ and $0 \le \sin^2 y \le 1$

Therefore, $f(x,y) = \frac{x^2 \sin^2 y}{x^2+2y^2} \le \sin^2 y$

Since $\sin y$ is a bounded function and

$$\lim_{(x,y)\to(0,0)} \sin^2 y = 0$$

Hence, by squeeze theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2} = 0$

Answer 17E.

Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}\ -1}$$

Find the limit.

First, let's manipulate the expression so it can be factored.

Add and subtract 1 in the numerator. So,

$$\begin{aligned} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \frac{x^2+y^2+(1-1)}{\sqrt{x^2+y^2+1}-1} \\ &= \frac{(x^2+y^2+1)-1}{\sqrt{x^2+y^2+1}-1} \\ &= \frac{\left(\sqrt{x^2+y^2+1}-1\right)\left(\sqrt{x^2+y^2+1}+1\right)}{\sqrt{x^2+y^2+1}-1} \end{aligned}$$

$$= \sqrt{x^2+y^2+1}+1 \quad \text{where } (x,y) \ne (0,0)$$

This works perfectly for the limit calculation, because the limit examines the functions at points that are specifically not equal to $(0,0)$.

So, the all of the limits of the function are equal. Specifically,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right)$$

Since the simplified function (at all points) is a power function, it is continuous at all points on its domain.

Find the limit of the simplified function by substitution.

$$\lim_{(x,y)\to(0,0)}\left(\sqrt{x^2+y^2+1}+1\right)=\sqrt{(0)^2+(0)^2+1}+1$$
$$=1+1$$
$$=2$$

Therefore, $\lim_{(x,y)\to(0,0)}\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}=\boxed{2}$.

Answer 18E.

$$f(x,y)=\frac{xy^4}{x^2+y^8}$$

If $x=0$ then $f(0,y)=0$, therefore $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along y – axis

If $y=0$ then $f(x,0)=0$, thus $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along x – axis

Let (x, y) approaches $(0, 0)$ along the curve $x=y^4$

Then $\quad f(x,y)=\dfrac{y^8}{y^8+y^8}=\dfrac{1}{2} \qquad (\text{for } y\neq 0)$

Thus $f(x,y)\to\dfrac{1}{2}$ along the curve $x=y^4$ as $(x,y)\to(0,0)$

Since f(x, y) approaches different value along different paths then we can say that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

Answer 19E.

Replace x with π, y with 0, and z with $\dfrac{1}{3}$ in $e^{y^2}\tan(xz)$.

$$\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz) = e^{(0)^2}\tan\left[(\pi)\left(\frac{1}{3}\right)\right]$$
$$= 1\left(\sqrt{3}\right)$$
$$= \sqrt{3}$$

Therefore, the limit evaluates to $\sqrt{3}$.

Answer 20E.

To find the following limit if exists:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{y(x+z)}{x^2+y^2+z^2}$$

Let $f(x,y,z)=\dfrac{y(x+z)}{x^2+y^2+z^2}$

If substitute the limit directly then,

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \frac{y(x+z)}{x^2+y^2+z^2} &= \frac{0(0+0)}{0^2+0^2+0^2} \\ &= \frac{0}{0}\end{aligned}$$

This is an indeterminate form, so evaluate this limit in the following way.

To evaluate the limit along *x*-axis, substitute $y=0, z=0$

$$\begin{aligned}f(x,0,0) &= \frac{0(x+0)}{x^2+0^2+0^2} \\ &= 0\end{aligned}$$

Along *x*-axis, as $(x,y,z)\to(0,0,0)$, $f(x,y,z)\to 0$

To evaluate the limit along *y*-axis, substitute $x=0, z=0$

$$\begin{aligned}f(0,y,0) &= \frac{y(0+0)}{0^2+y^2+0^2} \\ &= 0\end{aligned}$$

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Along *z*-axis, as $(x,y,z)\to(0,0,0)$, $f(x,y,z)\to 0$

Although obtained identical limits along the axes that do not show that the given limit is 0.

Now, approach $(0,0,0)$ along another line, say $x=y=z, x,y,z\neq 0$

$$f(z,z,z)=\frac{z(z+z)}{z^2+z^2+z^2}$$
$$=\frac{2z^2}{3z^2}$$
$$=\frac{2}{3}$$

Therefore, along the line $x=y=z$, as $(x,y,z)\to(0,0,0)$, $f(x,y,z)\to\frac{2}{3}$

This limit does not coincide with the limits obtained along the axes.

Hence, the given limit does not exist.

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This limit does not coincide with the limits obtained along the axes.

Hence, the given limit does not exist.

Answer 21E.

To find the following limit if it exist:

$$\lim_{(x,y,z)\to(0,0,0)}\frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$$

Consider $f(x,y,z)=\frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$

We can approach the point $(0,0,0)$ in space through the coordinate axes, or through the coordinate planes or through the symmetrical or unsymmetrical lines.

If $f(x,y,z)$ tends to the same value as we approach the point $(0,0,0)$ in all directions, then we say that the limit of this function exists at $(0,0,0)$. Otherwise the limit does not exist.

Approach the point $(0,0,0)$ along *x*-axis:

Along *x*-axis, $y=0, z=0$

Then $f(x,0,0)=\frac{0}{x^2}$

$=0$ for $x \neq 0$

Thus $f(x,y,z) \to 0$ as $(x,y,z) \to (0,0,0)$ along *x*-axis

Approach the point $(0,0,0)$ along the lines $y=mx, z=0$ $(\text{for } x \neq 0)$ (the set of lines in *xy*-plane) where m is a parameter.

Let $(x,y,z) \to (0,0,0)$ along the line $y=mx$, $z=0$ $(\text{for } x \neq 0)$

Then

$$f(x,mx,0)=\frac{x(mx)}{x^2+(mx)^2}$$
$$=\frac{mx^2}{x^2(1+m^2)}$$

$=\frac{m}{1+m^2}$ for $x \neq 0$

This will have different values for different values of *m*, for $x \neq 0$

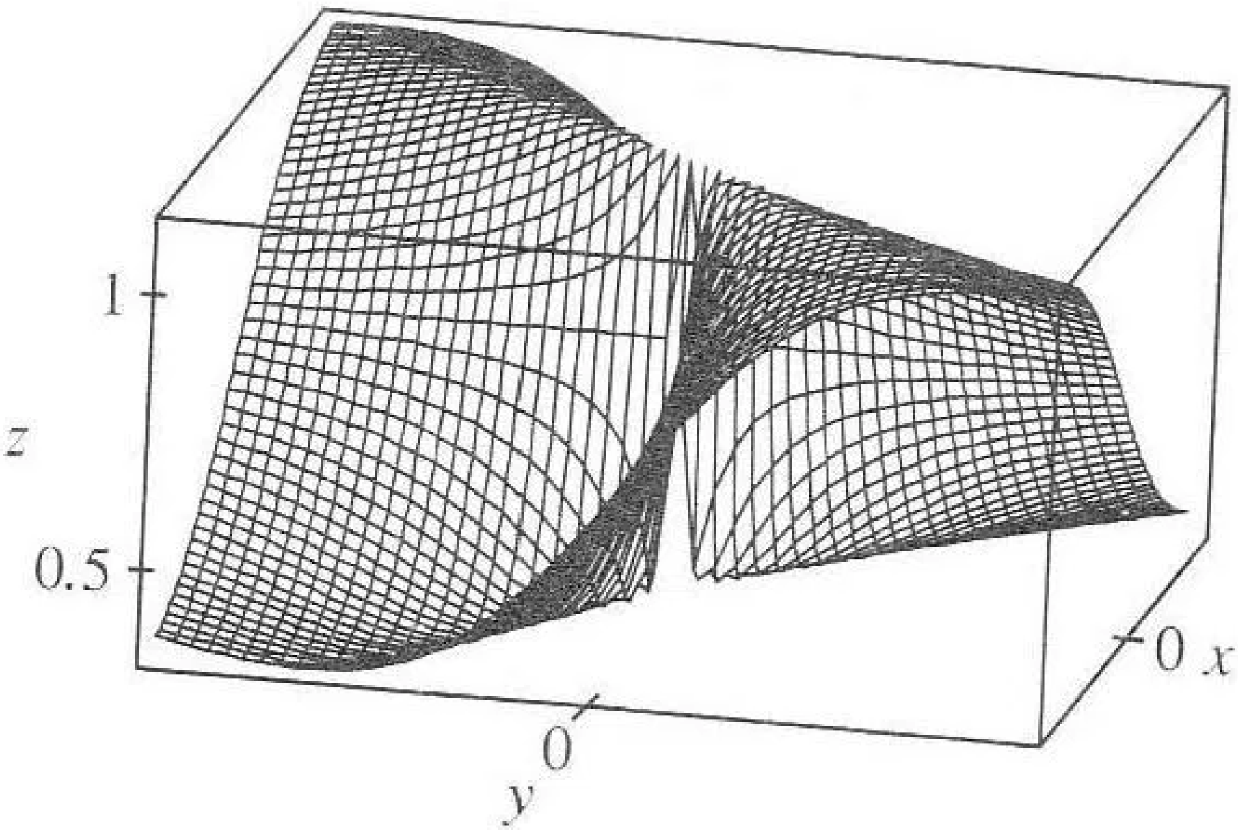
Thus $f(x,y,z)$ approaches different limits along different paths (lines).

Hence we say that limit of f does not exist.

That is, $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$ does not exist.

Answer 23E.

From the ridges on the graph, we see that as $(x,y) \to (0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.



Consider the limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}.$$

Let $f(x,y)=\dfrac{xy^3}{x^2+y^6}$.

Use Maple to graph the function.

Maple Input:

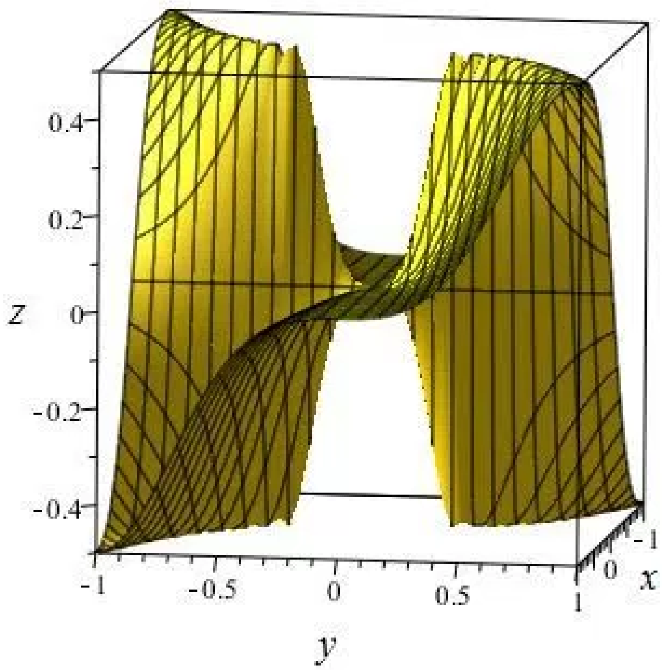
> with(plots):

> plot3d(x*y^3/x^2+y^6,x=-1..1,y=-1..1);

Maple Output:

> $with(plots):$

> $plot3d\left(\dfrac{x\cdot y^3}{x^2+y^6}, x=-1..1, y=-1..1\right);$



Answer 24E.

From the graph, it appears that approaching to the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas approaching the origin along a certain curve it has a constant value of about $\frac{1}{2}$.

Since the function approaches different values depending on the path of approach, the limit does not exist.

Therefore, the limit of the function $\frac{xy^3}{x^2 + y^6}$ as $(x, y) \to (0,0)$ $\boxed{\text{does not exists}}$.

Answer 25E.

$$g(t) = t^2 + \sqrt{t}$$

And $f(x,y) = 2x + 3y - 6$

Then
$$\begin{aligned} h(x,y) &= g(f(x,y)) \\ &= (2x+3y-6)^2 + \sqrt{2x+3y-6} \\ &= (2x)^2 + (3y)^2 + 36 + 2(2x)(3y) - 2(2x)(6) \\ &\quad -2(3y)(6) + \sqrt{2x+3y-6} \end{aligned}$$

i.e. $h(x,y) = 4x^2 + 9y^2 + 12xy - 24x - 36y + 36 + \sqrt{2x+3y-6}$

Now $h(x, y)$ is defined everywhere except for the values of x and y for which

$$2x + 3y - 6 < 0$$

Hence $h(x, y)$ is continuous on the set

$$\{(x,y): \ 2x + 3y - 6 \ge 0\}$$

i.e. $\boxed{\{(x,y): \ 2x + 3y \ge 6\}}$

Answer 26E.

$$g(t) = t + \ln t$$

$$f(x,y) = \frac{1 - xy}{1 + x^2y^2}$$

Then
$$\begin{aligned} h(x,y) &= g(f(x,y)) \\ &= \frac{1-xy}{1+x^2y^2} + \ln\left(\frac{1-xy}{1+x^2y^2}\right) \end{aligned}$$

i.e. $h(x,y) = \frac{1-xy}{1+x^2y^2} + \ln\left(\frac{1-xy}{1+x^2y^2}\right)$

Now h(x, y) is not defined for $1 - xy \le 0$

Hence h(x, y) is continuous everywhere except for (x, y) for which $xy \ge 1$

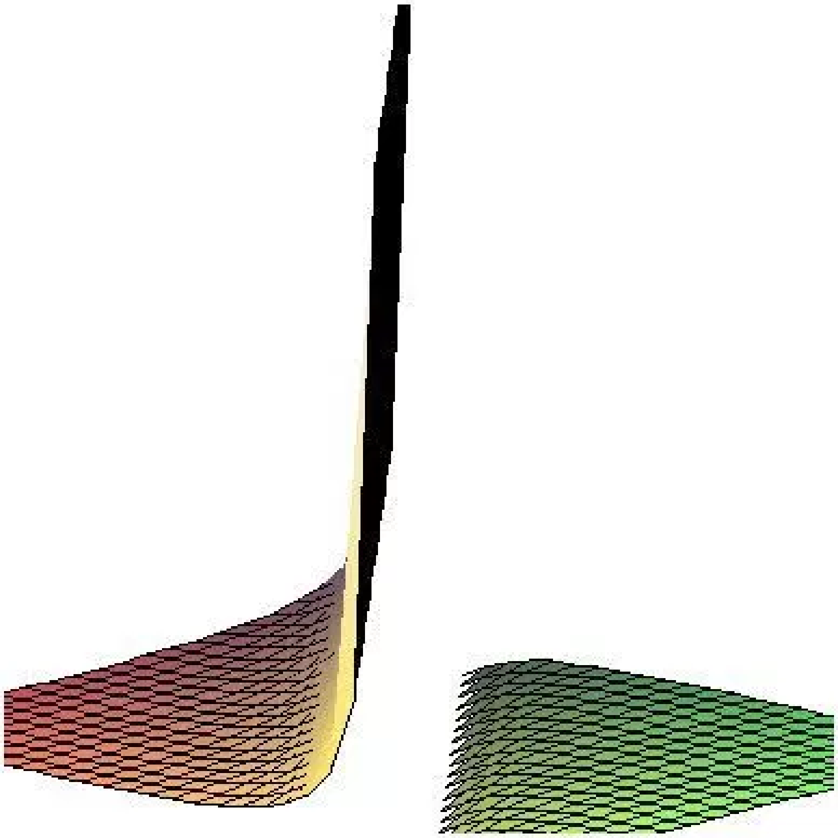
That is h(x, y) is continuous on the set $\boxed{\{(x, y): \quad xy < 1\}}$

Answer 27E.

Consider the function:

$$f(x, y) = e^{\frac{1}{(x-y)}}$$

Use maple to plot the graph of the function as shown below:



From the above graph, the given function is discontinuous along $y = x$.

Hence, the function $f(x, y) = e^{\frac{1}{(x-y)}}$ is **discontinuous** at all points which satisfy $y = x$.

Let,

$$x - y = 0$$
$$x = y$$

Thus, the function is not defined for $x = y$ or $y = x$ or discontinuous.

Answer 28E.

Consider the function:

$$f(x,y)=\frac{1}{1-x^2-y^2}.$$

Use Maple to graph the function.

Maple Input:

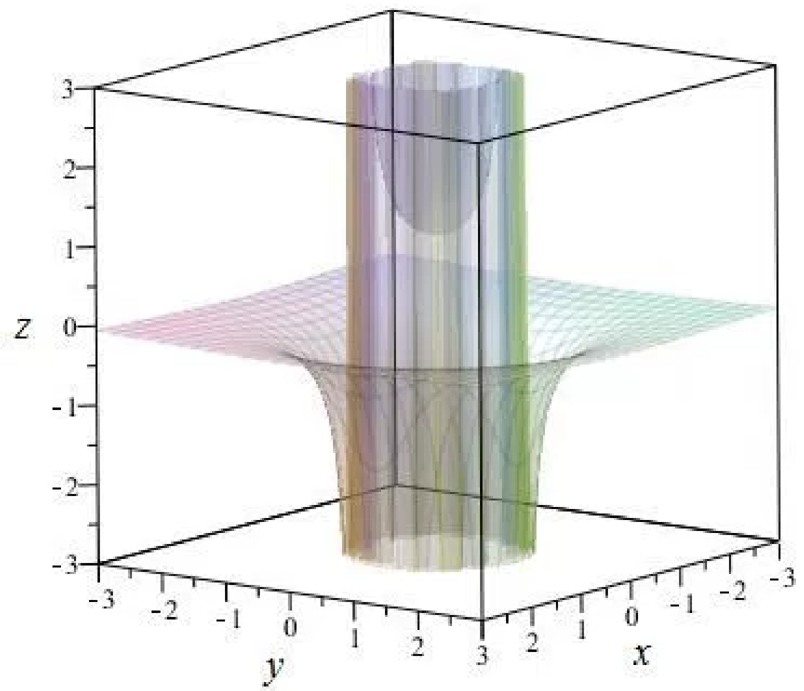
> with(plots):

> plot3d(exp(1/(1-x^2-y^2)),x=-2..2,y=-2..2);

Maple Output:

> $with(plots):$

> $plot3d\left(\frac{1}{1-x^2-y^2}, x=-3..3, y=-3..3\right);$



From the graph, observe that a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous.

As $f(x,y)=\frac{1}{1-x^2-y^2}$ is a rational function, it is continuous except where $1-x^2-y^2=0$, that is $x^2+y^2=1$.

Therefore, $f(x,y)$ is discontinuous on the circle $x^2+y^2=1$.

Answer 29E.

Consider the function $F(x, y) = \frac{xy}{1+e^{x-y}}$.

The given function is defined for all values of x and y except at $1+e^{x-y}=0$.

This implies $e^{x-y}=-1$.

But the value of e^{x-y} can never be negative it is always greater than zero. That is, $e^{x-y}>0$.

So the given function is defined for all values of x and y.

Therefore, $F(x, y) = \frac{xy}{1+e^{x-y}}$ is continuous on $\mathbb{R}^2$.

Answer 30E.

Consider the following function:

$$F(x, y) = \cos\sqrt{1+x-y}$$

The objective is to find the set of points at which the function is continuous.

The function is, $F(x, y) = \cos\sqrt{1+x-y}$

This function is continuous for all real values of x, y satisfying $1+x-y \ge 0$

$$1+x-y \ge 0$$
$$1+x \ge y$$

Because the square root of a number exists if the number is non-negative.

So, the set of points where F is continuous is, $\{x, y \mid y \le 1+x\}$

Therefore, the set of points at which the function is continuous is,

$$\boxed{\{(x, y) \mid y \le 1+x\}}.$$

Answer 31E.

Let us start by rewriting the given function.

$$F(x, y) = \frac{1+x^2+y^2}{1-(x^2+y^2)}$$

We note that when $x^2+y^2=1$ the denominator of the rational function becomes zero. We know that division by 0 is an indeterminate form. Therefore, we can say that the given function is continuous for all (x, y), except for $x^2+y^2=1$.

Answer 32E.

Consider the function:

$$H(x,y)=\frac{e^x+e^y}{e^{xy}-1}$$

The objective is to find the set of all points for which the function $H(x,y)=\frac{e^x+e^y}{e^{xy}-1}$ is continuous.

Generally the functions having the fractions are not continuous at points where the denominator becomes zero.

In similar way, check points for which the denominator of the function $H(x,y)=\frac{e^x+e^y}{e^{xy}-1}$ is zero.

The denominator of the function is $e^{xy}-1$ and obviously it is equal to zero if either of *x* and *y* are zero.

The numerator of the function is e^x+e^y and it is continuous for all values of *x* and *y*.

So the given function $H(x,y)=\frac{e^x+e^y}{e^{xy}-1}$ is continuous for all values of *x* and *y* such that $x\neq 0,$ and $y\neq 0$

Hence the set of points for which the function $H(x,y)=\frac{e^x+e^y}{e^{xy}-1}$ continuous $\forall(x,y)\in\mathbb{R}^2$ such that $x\neq 0,$ and $y\neq 0$

Answer 33E.

The given function is

$$G(x,y)=\ln\left(x^2+y^2-4\right)$$

This function is not defined when $x^2+y^2-4\leq 0$ that is when $x^2+y^2\leq 4$, because logarithms defined only for positive values.

Hence G(x, y) is continuous everywhere except when $x^2+y^2\leq 4$

That is G(x, y) is defined on the set

$$\boxed{\{(x,y):\ x^2+y^2>4\}}.$$

Answer 34E.

1) $\tan^{-1}\left((x+y)^{-2}\right) = g(x, y)$

2) the range of tanθ is (-∞, ∞) on its domain: that is arctan(x) is defined everywhere (x+y)^-2 is defined

3) so g(x,y) is continuous on its domain { (x,y) | $x \neq -y\}$

Answer 35E.

We know that the domain of the inverse sine function lies in the interval $[-1, 1]$. Since $x^2+y^2+z^2$ is the sum of squared terms, it cannot be negative. Therefore, the given functions is continuous for all values of x, y, z, such that $x^2+y^2+z^2 \le 1$. i.e.,

$\{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

Answer 36E.

We have $f(x, y, z) = \sqrt{y-x^2}\ln z$.

We know that $\ln z$ is well defined and continuous if $z > 0$.

Also, $\sqrt{y-x^2}$ is continuous and well defines if $y \ge x^2$ and the product of two continuous functions is continuous.

Therefore, $f(x, y, z) = \sqrt{y-x^2}\ln z$ is continuous, real, and well defined if $y \ge x^2$ and $z > 0$.

Answer 37E.

Consider the following function:

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 1 & \text{if} \quad (x,y)=(0,0)\end{cases}$$

The definition of continuity is as follows:

"A function *f* of two variables is called **continuous at** (*a*, *b*) if $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

and f is continuous on *D* if f is continuous at every point (*a*, *b*) in *D*".

Use this definition to find the continuity of $f(x,y)$.

Consider, $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2}$.

By direct substitution, the limit does not exist.

So, use another approach to find the limit.

For any real values x,y, $x^2,y^2\geq 0$

Then, $2x^2\geq x^2$.

$$2x^2+y^2\geq x^2$$

Thus, $\dfrac{x^2}{2x^2+y^2}\leq 1$

As the quantity $\dfrac{x^2}{2x^2+y^2}$ contains square terms, it must be greater than zero

$0\leq\left|\dfrac{x^2y^3}{2x^2+y^2}\right|\leq\left|y^3\right|$ For all *x* and *y*

The statement of squeeze theorem is as follows:

If $f(x) \le g(x) \le h(x)$ when *x* is near a and $\lim f = \lim h = L$ then $\lim g = L$

Here, $f(x,y) = 0, g(x,y) = \left|\frac{x^2y^3}{2x^2+y^2}\right|, h(x,y) = |y^3|$

However, $|y^3| \to 0$ as (x,y) approaches $(0,0)$.

By using the squeeze theorem, $\left|\frac{x^2y^3}{2x^2+y^2}\right| \to 0$ as (x,y) approaches $(0,0)$.

Thus, $\left|\frac{x^2y^3}{2x^2+y^2}\right| \to 0$

But, by the definition of the piece wise function $f(x,y)$, it is given that $f(0,0) = 1$.

This contradicts the above shown result.

Hence, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, $f(x,y)$ is discontinuous at $(0,0)$ and it is defined for all other points.

Therefore, the set of points where $f(x,y)$) is continuously determined by

$\boxed{\{(x,y) \mid (x,y) \ne (0,0)\}}$.

Answer 38E.

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+xy+y^2} & if \quad (x,y) \ne (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a,b) in D

Take $f(0,0)=0$

And $\lim_{(x,y)\to(0,0)} f(x,y)$

$$=\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$$

When $x=0$ then

$$\begin{aligned} f(x,y) &= f(0,y) \\ &= \frac{(0)y}{0+0+y^2} \\ &= 0 \end{aligned}$$

Then $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along y – axis ......(1)

When $y=0$ then

$$\begin{aligned} f(x,y) &= f(x,0) \\ &= \frac{x(0)}{x^2+0+0} \\ &= 0 \end{aligned}$$

Then $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along x – axis ......(2)

When $y=mx$, any non – vertical line through origin where m is its slope

Then

$$\begin{aligned} f(x,y) &= f(x,mx) \\ &= \frac{x(mx)}{x^2+x(mx)+(mx)^2} \\ &= \frac{mx^2}{x^2+mx^2+m^2x^2} \\ &= \frac{mx^2}{x^2(1+m+m^2)} \end{aligned}$$

$$=\frac{m}{1+m+m^2}$$

And then $f(x,y)\to \frac{m}{(1+m+m^2)}$ as $(x,y)\to(0,0)$ along any non – vertical line $y=mx$

......(3)

Thus f has the same limiting value along every non vertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x,y)\to(0,0)$ along the straight line

$y=x$

$$f(x,y)=f(x,x)$$

$$=\frac{x\cdot x}{x^2+x\cdot x+x^2}$$

$$=\frac{x^2}{x^2+x^2+x^2}$$

$$=\frac{x^2}{3x^2}$$

$$=\frac{1}{3}$$

So $f(x,y)\to\frac{1}{3}$ as $(x,y)\to(0,0)$ along $x=y$ ......(4)

Since $f(x,y)$ has different limits along different lines, then the above limits does not exist and therefore $f(x,y)$ is not continuous at zero

Also for $(x,y)\neq(0,0)$

$$f(x,y)=\frac{xy}{x^2+xy+y^2}$$

Now $f(x,y)$ is not defined for $x^2+xy+y^2=0$. $f(x,y)$ being a rational function is continuous on its domain

Hence from the results (1), (2), (3) and (4), $f(x,y)$ is not continuous on the set

$$\boxed{\{(x,y):\ (x,y)=(0,0),\ \ x^2+xy+y^2=0\}}$$

Answer 39E.

The given limit is

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}$$

Using polar co – ordinates

$x=r\cos\theta$, $y=r\sin\theta$ where $r\geq 0$

When $(x,y)\to(0,0)$, $r\to 0^+$.

Then the given limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0^+} \frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r^2\left(\cos^2\theta+\sin^2\theta\right)}$$

$$= \lim_{r\to 0^+} r\left(\cos^3\theta+\sin^3\theta\right) \qquad \left[\because \cos^2\theta+\sin^2\theta = 1\right]$$

$$= 0$$

Hence $\boxed{\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0}$.

Answer 40E.

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln\left(x^2+y^2\right)$$

Using polar co – ordinates $x = r\cos\theta,\ y = r\sin\theta,\ r \ge 0$

Thus when $(x,y)\to(0,0),\ r\to 0^+$

Then given limit becomes

$$\lim_{r\to 0^+} \left(r^2\cos^2\theta + r^2\sin^2\theta\right)\ln\left(r^2\cos^2\theta + r^2\sin^2\theta\right)$$

$$= \lim_{r\to 0^+} r^2 \ln r^2$$

$$= \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2}$$

$$= \lim_{r\to 0^+} \frac{\left(1/r^2\right)2r}{-2/r^3} \qquad \text{(Using L' Hospital's rule)}$$

$$= \lim_{r\to 0^+} \left(-r^2\right)$$

$$= 0$$

Hence $\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln\left(x^2+y^2\right) = 0$

Answer 41E.

Given $\lim_{(x,y)\to(0,0)} \frac{e^{x^2-y^2}-1}{x^2+y^2}$

The conversion equations from Cartesian to polar coordinates are

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

Rearrange the expression in the limit and substitute in using the third of these equations, noting that r approaches 0 as (x, y) approaches 0:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2}$$

$$\lim_{r\to 0} \frac{e^{-r^2}-1}{r^2}$$

Evaluate this limit by using L'Hôpital's Rule, which states that if the limit of the numerator and denominator of a fraction both approach 0, the whole limit is the same as the limit of the derivative of the numerator over the derivative of the denominator. Check that the numerator and denominator both approach 0:

$$\begin{aligned}\lim_{r\to 0} e^{-r^2}-1 &= e^0-1\\ &= 1-1\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{r\to 0} r^2 &= 0^2\\ &= 0\end{aligned}$$

So the limit as r approaches 0 of both the numerator and denominator approaches 0, and L'Hôpital's Rule applies. We can calculate the limit by taking the derivative of the numerator and the derivative of the denominator and then taking the limit:

$$\begin{aligned}\lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} &= \lim_{r\to 0} \frac{-2re^{-r^2}}{2r}\\ &= \lim_{r\to 0} \frac{-e^{-r^2}}{1}\\ &= \lim_{r\to 0} \left(-e^{-r^2}\right)\\ &= -e^{-0^2}\\ &= \boxed{-1}\end{aligned}$$

Answer 42E.

Consider the function:

$$f(x,y)=\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}.$$

Use polar coordinates to determine the value of the limit of $f(x,y)$ as $(x,y)\to(0,0)$.

The Cartesian coordinates x and y can be converted to the polar coordinates r and φ in the interval $(-\pi,\pi)$ by

$r^2=x^2+y^2$, For $x=r\cos\varphi$ and $y=r\sin\varphi$.

Plug in the value of $x^2+y^2=r^2$, to the function $f(x,y)=\dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$.

$$f(x,y)=\frac{\sin\left(r^2\right)}{r^2}$$

Apply the limit $(x,y)\to(0,0)$ as $r\to0^+$.

$$\lim_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}=\lim_{r\to0^+}\frac{\sin\left(r^2\right)}{r^2}$$

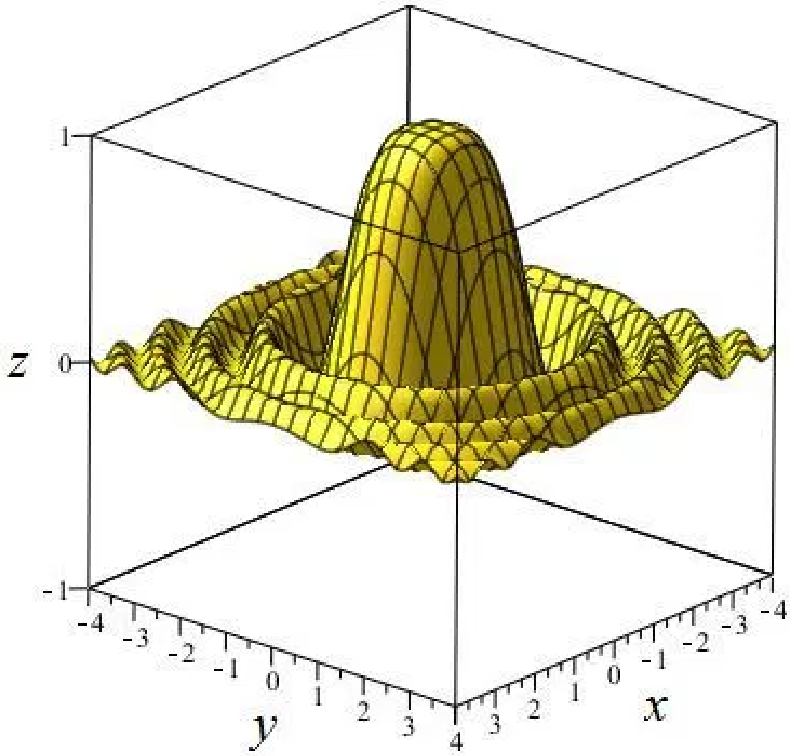
This is in the form indeterminate form $\dfrac{0}{0}$.

Use L'Hopital's Rule to find the limit of the function. First differentiate the numerator and denominator and then take the limit.

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}&=\lim_{r\to0^+}\frac{\cancel{2r}\cos\left(r^2\right)}{\cancel{2r}}\\&=\lim_{r\to0^+}\cos\left(r^2\right)\\&=1\end{aligned}$$

Therefore, the value of the limit at $(x,y)\to(0,0)$ is $\boxed{f(x,y)\to1}$.

Sketch the graph of $f(x,y)=\dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$.



Answer 43E.

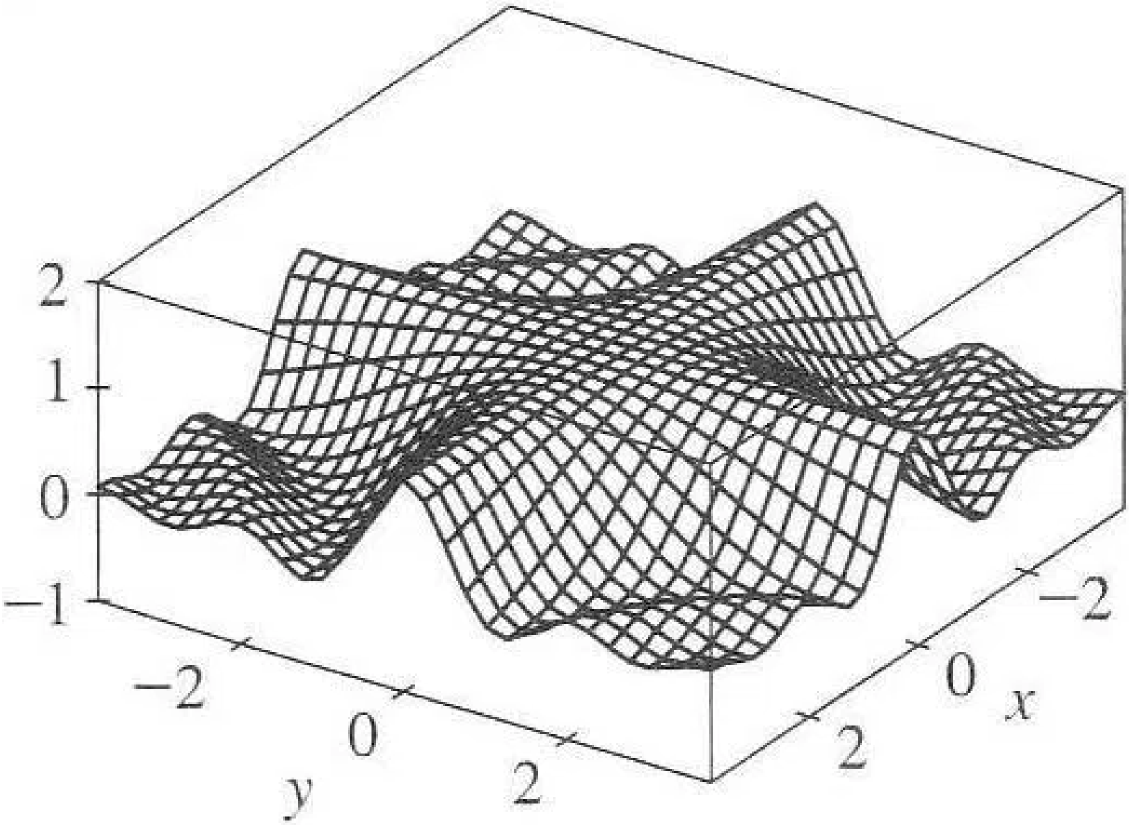
We are given the function $f\left(x, y\right) = \begin{cases} \frac{\sin(xy)}{xy} \text{ if } (x, y) \neq (0, 0) \\ 1 \text{ if } (x, y) = (0, 0) \end{cases}$. From the graph, it appears that f is continuous everywhere.

We know xy is continuous on $\mathbb{R}^2$ and sin t is continuous everywhere, so sin(xy) is continuous on $\mathbb{R}^2$ and $\frac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$ except possibly where xy = 0.

To show that f is continuous at those points, consider any point (a,b) in $\mathbb{R}^2$ where ab = 0. Because xy is continuous, xy → ab = 0 as (x,y) → (a,b).

If we let t = xy, then t → 0 as (x,y) → (a,b) and so then we also find that

$\lim_{(x, y) \to (a, b)} \frac{\sin(xy)}{xy} = \lim_{t \to 0} \frac{\sin(t)}{t} = 1$ by Equation 3.4.2. Therefore, we determine that $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$ and f is continuous on $\mathbb{R}^2$.



Answer 44E.

(a) Examine all of the possible approaches along paths of the form $y = mx^a$ through the origin and verify that the limit as $(x,y) \to (0,0)$ is 0 along each of them.
Examine some of the easiest paths first in the most straightforward directions.

First of all, note that we must have $a > 0$, as if $a \le 0$, the curve $y = mx^a$ does not pass through the origin (unless $m = a = 0$, which we will deal with next).

Second, note that if $m = 0$, the function is just $y = 0$, and by the definition of f, $f(x,y) = 0$ when $y \le 0$, so it equals 0 all along the line $y = 0$ and therefore approaches 0 along this path.

Third, note that if $m < 0$, then for positive $x, mx^a < 0$ which again means the function value is 0, since $f(x,y) = 0$ for $y \le 0$.

Finally, notice that it is impossible to approach (0, 0) along the vertical line $x = 0$, as $y = mx^a$ is a function and cannot include a vertical line as part of the curve. Therefore, we can confine our lines of approach to positive and negative x.

We have now verified that for a good number of approaches to the origin along curves of the form $y = mx^a$, the function $f(x,y)$ is always 0 so has limit 0 as the function approaches the origin. The only approaches remaining are when $m > 0$ and $x > 0$ or when $x < 0$ and $m \ne 0$.

Examine the case of $m > 0$ and $x > 0$. We wish to set up an inequality that will prove that sufficiently close to $x = 0$, we always have $mx^a \geq x^4$, as that will guarantee that the y value of the curve is greater than the function $y = x^4$ and therefore that the function value $f(x, y)$ equals 0 on close approach along the curve. Proposition: Provided $x \leq \sqrt[4-a]{m}$, it is true that $mx^a \geq x^4$, and therefore once we are sufficiently close to $x = 0$ the function value $f(x, y)$ is always 0, and thus approaches 0 as (x, y) approaches (0, 0) along curves of this type.

First we reveal how we discovered this condition. Start with the condition we want,
$mx^a \geq x^4$
Since all quantities are positive, we can multiply and divide across this inequality with impunity.
$m \geq x^{4-a}$
$x \leq \sqrt[4-a]{m}$

This tells us that if the mx^a curve is greater than the x^4 curve, then x is less than $\sqrt[4-a]{m}$.
We check that this arithmetic, and therefore our conclusion, is reversible:
$x \leq \sqrt[4-a]{m}$
$m \geq x^{4-a}$
$mx^a \geq x^4$

Note that this inequality only holds over our arithmetic provided $4 - a > 0$, or $a < 4$, as otherwise we are raising both sides of an inequality to a negative power which does not maintain the inequality (or raising both sides to 0, which does maintain inequality but does not give the same final result).

All shows that as long as x is positive and is within $\sqrt[4-a]{m}$ of 0, as long as $a < 4$, which is a given condition on the curve, the curve $y = mx^a$ is greater than $y = x^4$, and therefore, according to the definition of the function, $f(x, y) = 0$ and the function does indeed approach 0 along this path.

Finally, we look at approaching the origin from the negative x direction and either a positive or negative m.

Examine the graph of $y = mx^a$. If a is not an integer power, then y does not exist for negative x, so we are done. If a is even, then y is symmetric over the y-axis; since $y = x^4$ is also symmetric over the y-axis, $y = mx^a$ is still greater than $y = x^4$ on closest approach from the negative side and therefore $f(x, y)$ is 0 close to the origin, making the limit of the function 0.

If a is odd, y is symmetric over the origin, and if m is positive, is $mx^a < 0$ for $x < 0$. Since $y \le 0$ also makes $f(x, y)$ equal 0, again we have a curve of approach that has function value 0 all the way in and therefore has a limit of 0 for the function. Finally, if a is odd and m is negative, then for negative x the expression mx^a will have two negatives that will cancel each other out and reduce to the case of $m > 0$, $x > 0$, which we proved above also works as a path of approach with $f(x, y) \to 0$.

We have examined all possible paths of approach along curves of the form $y = mx^a$ with $a < 4$, and the function value $\boxed{f(x, y) \text{ always approaches } 0}$ along these curves.

(b) Approach the origin along the curve $y = x^5$. For $0 < x < 1$, we know $x^5 < x^4$, so as (x, y) approaches the origin, $f(x, y) = 1$. For $x < 0$, $x^5 < 0$ so by the function definition we have $f(x, y) = 0$. Therefore, as (x, y) approaches the origin along $y = x^5$ from the right, we have $f(x, y) \to 1$; as (x, y) approaches the origin along $y = x^5$ from the left, we have $f(x, y) \to 0$. Since the limits are different as we approach along this curve from opposite directions, they cannot both equal the function value, and $f(x, y)$ is $\boxed{\text{discontinuous}}$ here.

(c) The function $f(x, y)$ is discontinuous along the entire lengths of the two curves $\boxed{y = x^4}$ and $\boxed{y = 0}$. For both of these curves, as (x, y) approaches the curve from one side, the function value is 1, and from the other side, the function value is 0, so $f(x, y)$ is discontinuous at every point along both of these curves.

Answer 45E.

Consider the following function:

$$f(\mathbf{x}) = |\mathbf{x}|$$

Show that the function is continuous on $\mathbb{R}^n$.

In order to show that the function is continuous, show that the following equality holds for all **a** in $\mathbb{R}^n$.

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

Find the limit by using the following definition:

For any $\varepsilon > 0$, let $\delta = \varepsilon$. If $0 < |\mathbf{x} - \mathbf{a}| < \delta = \varepsilon$, then by the triangle inequality, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}|$, which implies that, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}| < \delta = \varepsilon$ or equivalently that $\big||\mathbf{x}| - |\mathbf{a}|\big| < \varepsilon$.

From the definition of a limit, $\lim_{\mathbf{x} \to \mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$.

Since $\lim_{\mathbf{x} \to \mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$, we know that $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$ for all **a** in $\mathbb{R}^n$, which means that *f* is continuous on $\mathbb{R}^n$.

Answer 46E.

Let us assume that $\varepsilon > 0$

We need to find $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ or $|c.x - c.a| < \varepsilon$

whenever $|x - a| < \delta$.

But $|c.x - c.a| = |c.(x - a)|$ and

$|c - (x - a)| \leq |c||x - a|$.

Let $\varepsilon > 0$ be given and set $\delta = \varepsilon / |c|$.

Now, whenever $0 < |ax - a| < \delta$,

$$
\begin{aligned}
|f(x) - f(a)| = |c.x - c.a| \leq |c||x - a| < |c|\delta \\
= |c|(\varepsilon / |c|) \\
= \varepsilon
\end{aligned}
$$

So f is continuous on R^n.