Chapter 8

Binomial Theorem

EXERCISE 8.1

Q. 1 Expand each of the expressions in $(1 – 2x)^5$

Answer:

By using binominal theorem, the expression $(1 – 2x)^5$ can be expanded as $(1 – 2x)^5$

$= {}^5C_0(1)^5 – {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 – {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 – {}^5C_5(2x)^5$

$= 1 – 5(2x) + 10(4x)^2 – 10(8x)^3 + 5\,(16x)^4 – (32x)^5$

$= 1 – 10x + 40x^2 – 80x^3 + 80x^4 – 32x^5$

Q. 2 Expand each of the expressions in $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Answer:

By using binomial theorem, the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ can be expanded as

$$\left\{\frac{2}{X} - \frac{X}{2}\right\}^5 = 5C0\left(\frac{2}{X}\right)^5 - 5C1\left(\frac{2}{X}\right)^4\left(\frac{X}{2}\right) + 5C2\left(\frac{2}{X}\right)^3\left(\frac{X}{2}\right)^2 - 5C3\left(\frac{2}{X}\right)^2\left(\frac{X}{2}\right)^3 + 5C4\left(\frac{2}{X}\right)\left(\frac{X}{2}\right)^4 - 5C5\left(\frac{X}{2}\right)^5$$

$$= \frac{32}{X^5} - 5\left(\frac{16}{X^4}\right)\left(\frac{X}{2}\right) + 10\left(\frac{8}{X^3}\right)\left(\frac{X^2}{4}\right) - 10\left(\frac{4}{X^2}\right)\left(\frac{X^2}{8}\right) + 5\left(\frac{2}{X}\right)\left(\frac{X^4}{16}\right) - \frac{X^5}{32}$$

$$= \frac{32}{X^5} - \frac{40}{X^3} + \frac{20}{X} - 5x + \frac{5}{8}X^2 - \frac{X^5}{32}$$

Q. 3Expand each of the expressions in $(2x – 3)^6$

Answer:

By using binomial theorem, the expression $(2x - 3)^6$ can be expended as

$(2x - 3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6$

$= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Q. 4 Expand each of the expressions in $\left\{\frac{X}{3} + \frac{1}{X}\right\}^5$

Answer:

By using binomial theorem, the expression $\left\{\frac{X}{3} + \frac{1}{X}\right\}^5$ can be expanded as

$$\left\{\frac{X}{3} + \frac{1}{X}\right\}^5 = 5C0\left(\frac{X}{3}\right)^5 + 5C1\left(\frac{X}{3}\right)^4\left(\frac{1}{X}\right) + 5C2\left(\frac{X}{3}\right)^3\left(\frac{1}{X}\right)^2 + 5C3\left(\frac{X}{3}\right)^2\left(\frac{1}{X}\right)^3 + 5C4\left(\left(\frac{X}{3}\right)\right)\left(\frac{1}{X}\right)^4 + 5C5\left(\frac{1}{X}\right)^5$$

$$= \frac{X^5}{243} + 5\left(\frac{X^4}{81}\right)\left(\frac{1}{X}\right) + 10\left(\frac{X^3}{27}\right)\left(\frac{1}{X^2}\right) + 10\left(\frac{X^2}{9}\right)\left(\frac{1}{X^3}\right) + 5\left(\frac{X}{3}\right)\left(\frac{1}{X^4}\right) + \frac{1}{X^5}$$

$$= \frac{X^5}{243} + \frac{5X^3}{81} + \frac{10X}{9X} + \frac{5}{3X^3} + \frac{1}{X^5}$$

Q. 5Expand each of the expressions in $\left\{X + \frac{1}{X}\right\}^6$

Answer:

By using binomial theorem, the expression $\left\{X + \frac{1}{X}\right\}^6$ can be expanded as

$$\left\{x + \frac{1}{X}\right\}^6 = {}^5C_0(x)^6 + {}^5C_1(x)^5\left(\frac{1}{X}\right) + {}^6C_2(x)^4\left(\frac{1}{X}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{X}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{X}\right)^4 + {}^6C_5(x)\left(\frac{1}{X}\right)^4 + {}^6C^6\left(\frac{1}{X}\right)^4$$

$= x^6 + 6(x)^5\left(\frac{1}{X}\right) + 15(x)^4\left(\frac{1}{X^2}\right) + 20(x)^3\left(\frac{1}{X^3}\right) + 15(x)^2\left(\frac{1}{X^4}\right) + 6(x)\left(\frac{1}{X^5}\right) + \frac{1}{X^6}$

$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{X^2} + \frac{6}{X^4} + \frac{1}{X^6}$

Q. 6 Using binomial theorem, evaluate each of the following: $(96)^3$

Answer:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100 – 4

$(96)^3 = (100 - 4)^3$

$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)\,(4)^2 - {}^3C_2(4)^3$

$= 100)^3 - 3(100)^2(4) + 3(100)\,(4)^2 - (4\,)^3$

= 1000000 – 120000 + 4800 – 64

= 884736

Q. 7 Using binomial theorem, evaluate each of the following: $(102)^5$

Answer:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 102 = 100 + 2

$(102)^5 = (100 + 2)^5$

$= {}^5C_0\,(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C^3(100)^4(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$= 1000000000 + 100000000 + 40000000 + 80000 + 8000\ 32$

$= 11040808032$

Q. 8 Using binomial theorem, evaluate each of the following: $(101)^4$

Answer:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $101 = 100 + 1$

$(101)^4 = (100 + 1)^4$

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)\,(1)^3 + {}^4C_4(1)^4$

$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$= 10000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

Q. 9 Using binomial theorem, evaluate each of the following: $(99)^5$

Answer:

99 can be written as the sum of difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $99 = 100 – 1$

$(99)^5 = (100 – 1)^5$

$= {}^5C_0(100)^5 – {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 – {}^5C_3(100)^2(1)^3 + {}^5C_4(100)$
$(1)^4 – {}^5C_5(1)^5$

$= (100)^5 – 5(100)^4 + 10(100)^3 – 10(100)^2 + 5(100) – 1$

$= 10000000000 - 500000000 + 1000000 – 100000 + 500 – 1$

$= 10010000500 – 500100001$

= 9509900499

Q. 10 Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Answer:

By splitting 1.1 and then applying binomial theorem, the first few term of $(1.1)^{10000}$ or 1000.

as

$(1.1)^{10000} = (1 + 0.1)1000$

$= {}^{10000}C_0 + {}^{10000}C_2(1.1) +$ other positive terms

$= 1 + 10000 \times 1.1 +$ other positive terms

$= 1 + 10000 +$ other positive terms

> 1000

Hence, (1.1)10000 > 1000.

Q. 11 Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$

Answer:

Using binomial theorem, the expressions, (a + b)4 and (a – b)4, can be expanded as

$(a + b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$

$(a - b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$

$(a + b)^4 - (a - b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 - [{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4]$

$= 2\ ({}^4C_2a^3b + {}^4C_3ab^3) = 2(4a^3b + 4ab^3)$

$= 8ab\ (a^2 + b^2)$

By putting a = $\sqrt{3}$ and b = $\sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^2 - \left(\sqrt{3}-\sqrt{2}\right)^2 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$

$$= 8\left(\sqrt{6}\right)[3+2] = 40\sqrt{6}$$

Q. 12 Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

Answer:

Using binomial theorem, the expression, $(x + 1)^6$ and $(x - 1)^6$, can be expanded as

$(x + 1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$

$(x - 1)^6 = {}^6C_0x^5 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 - {}^6C_2x^2 + {}^6C_4x^2 - {}^6C_5x^6 + {}^6C_6$

$(x + 1)^6 + (x - 1)^6 = 2[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6]$

$= 2\,[x^6 + 15x^4 + 15x^2 + 1]$

By putting x = $\sqrt{2}$, we obtain

$$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$

$= 2(8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2(8 + 60 + 30 + 1)$

$= 2\,(99) = 198$

Q. 13 Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Answer:

In order to show that 9n-1 – 8n – 9 is divisible by 64, it has to be prove that, 9n-1 – 8n – 9 = 64k

Where k is some natural number

By binomial theorem,

$(1 + a)^{n} = {}^{n}C_0 + {}^{n}C_1 a + {}^{n}C_2 a2 + \ldots + {}^{n}Cn\ a^{n}$

For a = 8 and m = n + 1, we obtain

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$

$= 9^{n+1} = 9 + 8n + 64\ \{{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}\ (8)^{n+1}\}$

$= 9n+1 - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}\ (8)^{n+1}$ is a natural number.

Thus, 9n+1 – 8n -9 is divisible by 64, whenever n is a positive integer.
Q. 14 Prove that

$\sum_{r=0}^{n} 3^r n_{c_r} = 4^n$

Answer:

By binomial theorem,

$\sum_{r=0}^{n} n_{c_r}$, $a^{n-r}\ b^{r} = (a + b)^{n}$

By putting b = 3 and a = 1 in the above equation, we obtain

$\sum_{r=0}^{n} n_{c_r}\ (1)^{n-r}\ (3)^{r} = (1 + 3)^{n}$

$= \sum_{r=0}^{n} 3^r\ {}^{n}C\ r = 4^{n}$

Hence, proved

Exercise 8.2

Q. 1 Find the coefficient of

x^5 in $(x + 3)^8$

Answer:

It is known that $(r + 1)^n$ term, (T_{r+1}), in the binomial expression of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x5 occurs in the $(r + 1)^n$ term of the expression $(x + 3)^8$, we obtain

$T_{r+1} = {}^8C_r(x)^{8-r} (3)^r$

Comparing the indices of x in x^5 in T_{r+1}

We, obtain r = 3

Thus, the coefficient of x5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8.7.6.5!}{3.2.5!} . 3^3 = 1512$

Q. 2 Find the coefficient of

a^5b^7 in $(a - 2b)^{12}$

Answer:

It is known that $(r + 1)^n$ term (T_{r+1}), in the binomial expression of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that a^5b^7 occurs in the $(r + 1)^{12}$ term of the expression $(a - 2b)^{12}$, we obtain

$T_{r+1} = {}^{12}C_r(a)^{12-r} (-2b)^r = {}^{12}C_r (a)^{12-r} (b)^r$

Comparing the indices of a and b in a^5b^7 in T_{r+1}

We, obtain r = 7

Thus, the coefficient of a^5b^7 is

$^{12}C_r\,(-2)^7 = \frac{12!}{7!5!}.2^7 = \frac{12.11.10.9.8.7!}{5.4.3.2.7!}.(-2)^7 = -(792)(128) = -101376$

Q. 3 Write the general term in the expansion of$(x^2 - y^6)^6$

Answer:

It is known that the general term T_{r+1} {which is the $(r + 1)^n$ term} in the binomial expression of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$T^{r+1} = {}^6C_r\,(x^2)^{6-r}\,(-y)^r = (-1)^r\,{}^6C_r\,.x^{12-2r}.\,y^r$

Q. 4 Write the general term in the expansion of$(x^2 - y\,x)^{12}$, $x \neq 0$.

Answer:

It is known that the general term T_{r+1}{which is the $(r + 1)^n$ term} in the binomial expansion of $(a + b)^n$is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Thus, the general term in the expansion of $(x^2 - y\,x)^{12}$ is

$T_{r+1} = {}^{12}C_r\,(x^2)^{12-r}\,(-y\,x)^r = (-1)^r\,{}^{12}C_r.\,x^{24-2r}.\,y^r = (-1)^r\,{}^{12}C_r.\,x^{24-r}.\,y^r$

Q. 5 Find the 4th term in the expansion of $(x - 2y)^{12}$.

Answer:

It is known $(r + 1)^n$ term, T_{r+1} in the binomial expansion of (a + b) n is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Thus, the 4^{th} term is the expansion of $(x^2 - 2y)^{12}$ is

$T_4 = T_{3+1} = {}^{12}C_3\,(x)^{12-3}\,(-2y)^3 = (-1)^3.\frac{12!}{3!9!}.x^9.\,(2)^3.\,y^3 = \frac{12.11.10}{3.2}.(2)^3\,x^9\,y^3$
$= -1760x^9\,y^3$

Q. 6 Find the 13^{th} term in the expansion of $\left\{9x - \frac{1}{3\sqrt{x}}\right\}^{18}$

Answer:

It is known $(r + 1)^{n}$ term, T_{r+1} in the binomial expansion of $(a + b)$ n is given by $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$

Thus, the 13^{th} term in the expansion of $\left\{9X - \frac{1}{3\sqrt{X}}\right\}^{18}$ is

$T_{13} = T_{12-1} = {}^{18}C_{12}\,(9x)^{18-12}\left\{-\frac{1}{3\sqrt{x}}\right\}^{12}$

$= (-1)^{12}\frac{18!}{12!6!}(9)^{6}(x)^{6}\left(\frac{1}{3}\right)^{12} x\left(\frac{1}{\sqrt{x}}\right)^{12}$

$= \frac{18.17.16.15.14.13.12!}{12!6.5.4.3.2.}\,.\,x^{6}\,\frac{1}{x^{6}}\,.3^{12}\,\frac{1}{3^{12}}$

$= 18564$

Q. 7 Find the middle terms in the expansions of $\left(3 - \frac{x^{3}}{6}\right)^{7}$

Answer:

It is known that in the expansion of $(a + b)^{n}$ in n is odd, then there are two middle terms

Namely $\left(\frac{n+1}{2}\right)^{n}$ term and $\left(\frac{n+1}{2} + 1\right)^{n}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^{3}}{6}\right)^{7}$ are $\left(\frac{7+1}{2}\right)^{n} = 4^{th}$ and $\left(\frac{7+1}{2} + 1\right)^{n} = 5^{th}$ term.

$T_{4} = T_{3+1} = {}^{7}C_{3}(3)^{7-3} - \frac{x^{3}}{6} = (-1)^{3}\,\frac{7!}{3!4!}.\,3^{4}.\,\frac{x^{9}}{6^{3}}$

$= \frac{7.6.5.4!}{3.2.4!}.\,3^{4}.\,\frac{1}{2^{3}.3^{3}}.\,x^{9} = -\,\frac{105}{8}\,x^{4}$

$T_{5} = T_{4+1} = {}^{7}C_{4}(3)^{7-4}\left(-\frac{x^{3}}{6}\right)^{4} = (-1)^{4}\,\frac{7!}{4!3!}.\,3^{3}.\,\frac{x^{12}}{6^{4}}$

$= \frac{7.6.5.4!}{4!3.2}.\,\frac{3^{3}}{2^{4}.3^{4}}\,.\,x^{12} = \frac{35}{48}x^{12}$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$

Q. 8Find the middle terms in the expansions of $\left(\frac{x}{3}+9y\right)^{10}$

Answer:

It is known that in the expansion of (a + b)n, $i\ nn$ is even the middle term is $\left(\frac{n}{2}+1\right)^n$ term.

Therefore, the middle term in the expansion of $\left\{\frac{x}{3}+9y\right\}^{10}$ is $\left(\frac{10}{2}+1\right)^n=6^{th}$

$T_4 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5}(9y)^5 = \frac{10!}{5!5!}.\frac{x^5}{3^6}.\ 9^5.\ Y^5$

$= \frac{10.9.8.7.6.5!}{5.4.3.2.5!}.\frac{1}{3^6}.3^{10}.x^5y^5$ $[\ 9^5 = (3^2)^5 = 3^{10}]$

$= 252 \times 3^6.x^5.\ y^5 = 6123x^5y_5$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $6123x^5y^5$

Q. 9 In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Answer:

It is known that $(r + 1)^n$ term, (T_{r+1}) in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Assuming that a n occurs in the $(r+1)^n$ term of the expansion (1 + a) m + n, we obtain

$T_{r+1} = {}^{m+n}C_r\,(1)^{m+n-r}\,(a)^r = {}^{m+n}C_r\, a^r$

Comparing the indices of a in a^n in T_{r+1}

We, obtain r = m

Therefore, the coefficient of an is

$$^{M+n}C_r = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m!+n!)}{m!n!} \ldots (1)$$

Assuming that an occurs in the $(k + 1)^n$ term of the expansion (1 + a) m + n, we obtain

$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$

Comparing the indices of a in a^n and T_{k+1}

We, obtain

K = n

Therefore, the coefficient of an is

$$^{M+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \ldots (2)$$

Thus, from (1) and (2), it can be observed that the coefficient of a^n in the expansion of $(1 + a)^{m+n}$ is equal.

Q. 10 The coefficients of the $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1: 3: 5. Find n and r.

Answer:

It is known that $(k + 1)^n$ term (T_{k+1}) in the binomial expansion of (a + b) n is given by $T_{k+1} = {}^nC_k\, a^{n-k}\, b_k$

Therefore, $(r – 1)^n$ term in the expansion of $(x + 1)^1$ is

$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2}\, x^{n-r-2}$

(r + 1) term in the expansion of $(x + 1)^n$ is

$T_{r-1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_1\, x^{n-r}$

r th term in the expansion of $(x + 1)^n$ is

$T_r = {}^nC_{r-1}\,(x)^{n-(r-1)} = {}^nC_{r-1}\,x^{n-r+1}$

Therefore, the coefficient of the (r – 1) th, r th and (r + 1) th term in the expansion of $(x + 1)^n$

${}^nc_{r}-2$, ${}^nc_{r}-1$, and n c r are respectively. Since these - coefficient are in the ratio 1: 3: 5, we obtain

$= \frac{n_{c_{r-2}}}{n_{c_{r-1}}} = \frac{1}{3}$ and $\frac{n_{c_{r-1}}}{n_{c_r}} = \frac{3}{5}$

$\frac{n_{c_{r-2}}}{n_{c_{r-1}}} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{(r-1)!(r-2)!9n-r+1)}{(r-2)!(n-r+1)!(n-r+2)!}$

$= \frac{r}{n-r+2}$

$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$

= 3r – 3 = n – r + 2

= n – 4r + 5 = 0 …. (1)

$\frac{n_{c_{r-1}}}{n_{c_r}} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)(n-r)!}{(r-1)!(n-r+1)(n-r)!}$

$= \frac{r}{n-r+1}$

$\therefore \frac{r}{n-r+1} = \frac{3}{5}$

= 5r = 3n – 3r + 3

= 3n – 8r + 3 = 0 … (2)

Multiplying (1) by 3 and subtracting it from (2), we obtain

4r – 12 = 0

= r = 3

Putting the value of r in (1), we obtain n

- 12 + 5 = 0

- $n = 7$

thus, $n = 7$ and $r = 3$

Q. 11 Prove that the coefficient of x^n in the expansion of $(1 + x)2n$ is twice the coefficient of x^n in the expansion of $(1 + x)2n - 1$.

Answer:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)\ n$ is given by

$T_{r+1} = {}^n c_r\, a^{n-r}\, b^r$

Assuming that $x\ n$ occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$T_{r+1} = {}^{2n} c_r\, (1)^{2n-r}\, (x)^r = {}^{2n} c_r\, (x)^r$

Comparing the indices of x in x^n and in T_{r+2}, we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

$${}^{2n} c_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{n!^2} \ldots (1)$$

Assuming that $x\ n$ occurs in the $(k + 1)^{th}$ term of the expansion of $(1 + x)^{2n-2}$, we obtain

$T_{k+1} = {}^{2n} c_k\, (1)^{2n-r-k}\, (x)^k = {}^{2n} c_k\, (x)^k$

Comparing the indices of x in x^n and in T_{k+1}, we obtain $k = n$

Therefore, the coefficient of $x\ n$ in the expansion of $(1 + x)^{2n-1}$ is

$$2n - 1\ c\ n = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{2n(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \ldots (2)$$ From (1) and (2), it is observed that

$\frac{1}{2}({}^{2n} c_r) = {}^{2n-1} c_n$

$= {}^{2n} c_n = 2\,({}^{2n-1} c_n)$

Therefore, the coefficient of x^n expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$

Hence, proved.

Q. 12 Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer:

It is known that $(r + 1)^{th}$ term, (T_{r+1}) in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nc_r \, a^{n-r} \, b^r$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^n$, we obtain

$T_{r+1} = {}^nC_r \, (1)^{n-r} \, (x)^r = {}^nc_r \, (x)^r$

Comparing, the coefficient of x in x^2 and in T_{r+1}, we obtain r = 2

Therefore, the coefficient of x^2 is ${}^nc_2$

It is given that the coefficient of x2 in the expansion (1 + x) n is 6.

$= {}^nC_2 = 6$

$= \frac{m!}{2!(m-2)!} = 6$

$= \frac{m(m+1)(m-2)!}{2\times(m-2)!} = 6$

$= m(m - 1) = 12$

$= m^2 - m - 12 = 0$

$= m^2 - 4m + 3m - 12 = 0$

$= m(m - 4) + 3(m - 4) = 0$

$= (m - 4)(m + 3) = 0$

= (m – 4) = 0 or (m + 3) = 0

= m = 4 or m = - 3

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1 + x)_n$ is 6, is 4.

Miscellaneous Exercise

Q. 1 Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer:

It is known that (r + 1)th term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is $T_{r+1} = {}^nc_r a^{n-r} b^r$

The first three terms of the expansion are given as 729, 7290 and 30375 respect

Therefore, we obtain

$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \dots (1)$

$T_2 = {}^nC_1 a^{n-2} b^2 = {}^n a^{n-1} b = 7290 \dots (2)$

$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b_2 = 30375 \dots (3)$

Dividing (2) by (1), we obtain

$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$

$= \frac{nb}{a} = 10 \dots (4)$

Dividing (3) by 92), we obtain

$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$

$= \frac{(n-1)b}{2a} = \frac{30375}{7290}$

$= \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{2}{3}$

$= \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$

$= 10 - \frac{b}{a} = \frac{25}{3}$ [using (4)]

$= \frac{b}{a} - 10 - \frac{25}{3} = \frac{5}{3}$ … (5)

From (4) and (5), we obtain

n, $\frac{5}{3}$ = 10

= n = 6

Substituting n = 6 in equation (1), we obtain a 6

= 729

= a = $\sqrt[6]{729}$ = 3

From (5), we obtain

$\frac{b}{3} = \frac{5}{3}$ b = 5

Thus, a = 3, b = 5, and n = 6

Q. 2 Find a if the coefficients of x_2 and x_3 in the expansion of $(3 + ax)^9$ are equal.

Answer:

It is known that (r + 1) th term, (T_{r+1}), in the binomial expansion of 9a + b) ^{n} is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Assuming that x2 occurs in the (r + 1) th term in the expansion of 93 + ax) ^{9}, we obtain

$T_{r+1} = {}^9C_r\,(3)^{9-r}\,(ax)^r = {}^9C_r\,(3)^{9-r}\,a^r\,x^r$

Comparing the indices of x in x^2 and in T_{r+2}, we obtain

r = 2

thus, the coefficient of x^2 is

${}^9C_2\,(3)^{9-2}\,a^2 = \frac{9!}{2!7!}(3)^7\,a^2 = 36\,(3)^7\,a^2$

Assuming that x^2 occurs in the (k + 1) th term in the expansion of (3 + ax)^9, we obtain

$T_{k+1} = {}^9C_k (3)^{9-k} (ax)^k = {}^9C_k (3)^{9-k} a^k x^k$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain k = 3

Thus, the coefficient of x^3 is

${}^9C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!} (3)^6 a^3 = 84(3)^6 a^3$

It is given that the coefficient of x^2 and x^3 are the same.

$84(3)^6 a^3 = 36 (3)^7 a^2$

$= a = \frac{36\times3}{84} = \frac{104}{84}$

$= a = \frac{9}{7}$

Thus, the required value of is $\frac{9}{7}$.

Q. 3 Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer:

Using binomial theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$

$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$

$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

$(1 - x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2 (x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

$\therefore (1 + 2x)^6 (1 - x)^7$

$= \{1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\} \{1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7\}$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$1 (-21x^5) + (12x) (32x^4) + (60x^2) (-35x^3) + (160x^3) (21x^3) + (240x^4) (-7x) + (192x^5) (1) = 171x^5$

Thus, the coefficient of x5 in the given product is 171.

Q. 4 If a and b are distinct integers, prove that a – b is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint write an = $(a - b + b)^n$ and expand]

Answer:

In order to prove that (a – b) is a factor of $(a^n - b^n)$, it has to be prove that

$a^n - b^n = k (a - b)$, where k is some natural formula

It can be written that, a = a – b + b

$\therefore$ an $= (a - b + b)^n = [(a - b) + b]^n$

$= {}^nC_0 (a - b)^n + {}^nC_2 (a - b)^{n-1} b + \ldots + {}^nC_{n-1} (a - b) b^{n-1} + {}^nC_n b_n$

$= (a - b)^n + {}^nC_2 (a - b)^{n-1} b + \ldots + {}^nC^{n-1} (a - b) b^{n-1} + b^n$

$= a^n - b^n = (a - b) [(a - b)^{n-1} + {}^nC_2 (a - b)^{n-2} b + \ldots + {}^nC^{n-1} b^{n-1}]$

$= a^n - b^n = k (a - b)$

Where, $k = [(a - b)^{n-1} + {}^nC_2 (a - b)^{n-2} b + \ldots {}^nC_{n-1} b^{n-1}$ is a natural number.

This, shows that (a – b) is a factor of $(a^n – b^n)$, where n is a positive integer.

Q. 5 Evaluate $\left(\sqrt{3}-\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

Answer:

Firstly, the expression $(a + b)^6 – (a – b)^6$ is simplified by using binomial theorem. This, can be done as

$(a + b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$

$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

$(a – b)^6 = {}^6C_0a^6 – {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 – {}^6C_5ab^5 + {}^6C^6b^6$

$= a^6 -6a^5b + 15a^4b^2 – 20a^3b^3 + 15a^2b^4 – 6ab^5 + b^6$

$\therefore (a + b)^6 – (a – b)^6 = 2[6a^5b + 20a^3b^3 + 6ab^5]$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$\left(\sqrt{3}-\sqrt{2}\right)^6$ - $\left(\sqrt{3}-\sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$

$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$

$= 2 \times 198\sqrt{6}$

$= 396\sqrt{6}$

Q. 6 Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2+1}\right)^4$

Answer:

Firstly, the expression $(x + y)^4 + (x – y)^4$ is simplified by using binomial theorem

This can be done as

$(x + y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 + 4x_3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$\therefore (x + y)^4 + (x - y)^4 = 2 (x^4 + 6x^2y^2 + y^4)$

Putting x = a^2 and y = $\sqrt{a^2 + 1}$, we obtain

$$\left(a^2 + \sqrt{a^2} + 1\right)^4 + \left(a^2 - \sqrt{a^2} - 1\right)^4 = 2\left[(a)^{2^4} + 6(a)^{2^2}\left(\sqrt{a^2+1}\right)^2\left(\sqrt{a^2-1}\right)^4\right]$$

$= 2[a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2]$

$= 2 [a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$

$= 2 [a^8 + 6a^6 - 5a^4 - 2a^2 + 1]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

Q. 7 Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer:

$0.99 = 1 - 0.01$

$\therefore (0.99)^5 = (1 - 0.01)^5$

$= {}^5C_0(1)^5 - {}^5C_2(1)^4 (0.01) + {}^5C_2(1)^3 (0.01)^2$ [Approximately]

$= 1 - 5 (0.01) + 10 (0.01)^2$

$= 1 - 0.05 + 0.001$

$= 1.001 - 0.05$

$= 0.951$

Thus, the value of (0.99)5 is approximately 0.951.

Q. 8 Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left\{\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right\}^{n}$ is $\sqrt{6}$: 1

Answer:

In the expansion, (a + b) ^{n} = $^{n}C_0$ a $^{n-2}$ b ^{2} + … + ^{n} C_1 ab $^{n-2}$ + ^{n} C_n b ^{n}

Fifth term from the beginning = ^{n} C $_4$ a $^{n-4}$ b^4

Fifth term from the end = ^{n} C $_4$ a^4 b $^{n-4}$

Therefore, it is evident that in the expansion of $\left\{\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right\}^{n}$ are fifth term from the beginning is

$^{n}C_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^{4}$ and the fifth term from the end is $^{n}C_{n-4}$ $\left(\sqrt[4]{2}\right)^{4}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$^{n}C_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^{4} = {}^{n}C_4\frac{\left(\sqrt[4]{2}\right)^{n}}{\left(\sqrt[4]{2}\right)^{4}}.\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^{n}$ … (1)

$^{n}C_{n-4}\left(\sqrt[4]{2}\right)^{4}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^{n}C_{n-4}.2.\frac{3}{\left(\sqrt[4]{3}\right)^{n}} = \frac{6n!}{(n-4)!4!}.\frac{1}{\left(\sqrt[4]{3}\right)^{n}}$ … (2)

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}$: 1 therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^{n} : \frac{6n!}{(n-4)!4!}.\frac{1}{\left(\sqrt[4]{3}\right)^{n}} = \sqrt{6} : 1$$

$$= \frac{\left(\sqrt[4]{2}\right)^{n}}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^{n}} = \sqrt{6}: 1$$

$$= \frac{\left(\sqrt[4]{2}\right)^{n}}{6} \times \frac{\left(\sqrt[4]{3}\right)^{n}}{6} = \sqrt{6}$$

$$= \left(\sqrt[4]{6}\right)^{n} = 36\sqrt{6}$$

$= 6^{n/4} = 6^{5/2}$

$= \frac{n}{4} = \frac{5}{2}$

$= n = 4 \times \frac{5}{2} = 10$

Thus, the value of n is 10.

Q. 9Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^n$

Answer:

$= {}^nC_0\left(1 + \frac{x}{2}\right)^4 - {}^nC_1\left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + {}^nC_2\left(1 + \frac{x}{2}\right)^2 \left(\frac{2}{x}\right)^2 - {}^nC_3\left(1 + \frac{x}{2}\right) \left(\frac{2}{x}\right)^3 +$
${}^nC_4\left(\frac{2}{x}\right)^4$

$= \left(1 + \frac{x}{2}\right)^4 - 4\left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + 6\left(1 + x + \frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1 + \frac{x}{2}\right)\left(\frac{8}{x^3}\right) +$
$\frac{16}{x^4}$

$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$

$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$ … (1)

Again by using binomial theorem, we obtain

$\left(1 + \frac{x}{2}\right)^4 = 4C0(1)4 + 4C1(1)3\left(\frac{x}{2}\right) + 4C2(1)2\frac{x^2}{2} + 4C3(1)\left(\frac{x}{2}\right)^3 + 4C4\frac{x^4}{2}$

$= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^4}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16}$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}$ … (2)

$= \left(1 + \frac{x}{2}\right)^3 = 3C0(1)3 + 3C1(1)2\left(\frac{x}{2}\right) + 3C2(1)\left(\frac{x}{2}\right)^2 + 3C3\left(\frac{x}{2}\right)^3$

$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \ldots (3)$

From (1), (2) and (3), we obtain

$= \left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$

Q. 10 Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Answer:

Using binomial theorem, the given expression $(3x^2 - 2ax + 3a^2)^3$ can be expanded as $[(3x^2 - 2ax) + 3a^2]^3$

$= {}^3C_0(3x^2 - 2ax^2)^3 + {}^3C_1(3x^2 - 3ax)^2(3a^2) + {}^3C_2(3x^2 - 2ax)(3a^2)^2 + {}^3C_3(3a^2)^3$

$= (3x^2 - 2ax)^3 + 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^4$

$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^4 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$

$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \ldots (1)$

Again by using binomial theorem, we obtain

$(3x^2 - 2ax)^3$

$= {}^3C_0(3x^2)^3 - {}^3C_1(3x^2)^2(2ax) + {}^3C_2(3x^2)(2ax)^2 - {}^3C_3(2ax)^3$

$= 27x^5 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3$

$= 27x^5 - 54ax^5 + 36a^2x^4 - 5a^3x^3 \ldots (2)$

From (1) and (2), we obtain

$(3x^2 - 2ax + 3a^2)^3$

$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$