# CHAPTER II.

## PROPORTION.

18. Definition. When two ratios are equal, the four quantities composing them are said to be **proportionals.** Thus if $\frac{a}{b} = \frac{c}{d}$, then $a$, $b$, $c$, $d$ are proportionals. This is expressed by saying that $a$ is to $b$ as $c$ is to $d$, and the proportion is written

$$a : b :: c : d;$$

or

$$a : b = c : d.$$

The terms $a$ and $d$ are called the *extremes*, $b$ and $c$ the *means*.

19. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let $a$, $b$, $c$, $d$ be the proportionals.

Then by definition $$\frac{a}{b} = \frac{c}{d};$$

whence $$ad = bc.$$

Hence if any three terms of a proportion are given, the fourth may be found. Thus if $a$, $c$, $d$ are given, then $b = \frac{ad}{c}$.

Conversely, if there are any four quantities, $a$, $b$, $c$, $d$, such that $ad = bc$, then $a$, $b$, $c$, $d$ are proportionals; $a$ and $d$ being the extremes, $b$ and $c$ the means; or vice versâ.

20. Definition. Quantities are said to be in **continued proportion** when the first is to the second, as the second is to the third, as the third to the fourth; and so on. Thus $a$, $b$, $c$, $d$,...... are in continued proportion when

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \dots\dots\dots$$

If three quantities $a$, $b$, $c$ are in continued proportion, then

$$a : b = b : c;$$

$$\therefore \quad ac = b^2. \qquad \text{[Art. 18.]}$$

In this case $b$ is said to be a **mean proportional** between $a$ and $c$; and $c$ is said to be a **third proportional** to $a$ and $b$.

21. *If three quantities are proportionals the first is to the third in the duplicate ratio of the first to the second.*

Let the three quantities be $a$, $b$, $c$; then $\dfrac{a}{b} = \dfrac{b}{c}$.

Now $$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$$

$$= \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2};$$

that is, $$a : c = a^2 : b^2.$$

It will be seen that this proposition is the same as the *definition* of duplicate ratio given in Euclid, Book V.

22. If $a : b = c : d$ and $e : f = g : h$, then will $ae : bf = cg : dh$.

For $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h};$$

$$\therefore \frac{ae}{bf} = \frac{cg}{dh},$$

or $$ae : bf = cg : dh.$$

COR. If $$a : b = c : d,$$

and $$b : x = d : y,$$

then $$a : x = c : y.$$

This is the theorem known as *ex æquali* in Geometry.

23. If four quantities $a$, $b$, $c$, $d$ form a proportion, many other proportions may be deduced by the properties of fractions. The results of these operations are very useful, and some of them are often quoted by the annexed names borrowed from Geometry.

(1) If $a : b = c : d$, then $b : a = d : c$. [*Invertendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is $$\frac{b}{a} = \frac{d}{c};$$

or $$b : a = d : c.$$

(2) If $a : b = c : d$, then $a : c = b : d$. [*Alternando.*]

For $ad = bc$; therefore $\frac{ad}{cd} = \frac{bc}{cd}$;

that is, $$\frac{a}{c} = \frac{b}{d};$$

or $$a : c = b : d.$$

(3) If $a : b = c : d$, then $a + b : b = c + d : d$. [*Componendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is $$\frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b = c + d : d.$$

(4) If $a : b = c : d$, then $a - b : b = c - d : d$. [*Dividendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $$\frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b = c - d : d.$$

(5) If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

For by (3) $$\frac{a+b}{b} = \frac{c+d}{d};$$

and by (4) $$\frac{a-b}{b} = \frac{c-d}{d};$$

$\therefore$ by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

or $$a + b : a - b = c + d : c - d.$$

This proposition is usually quoted as *Componendo and Dividendo.*

Several other proportions may be proved in a similar way.

24. The results of the preceding article are the algebraical equivalents of some of the propositions in the fifth book of Euclid, and the student is advised to make himself familiar with them in their verbal form. For example, *dividendo* may be quoted as follows:

*When there are four proportionals, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.*

25. We shall now compare the algebraical definition of proportion with that given in Euclid.

Euclid's definition is as follows:

Four quantities are said to be proportionals when if *any equimultiples whatever* be taken of the first and third, and also *any equimultiples whatever* of the second and fourth, the multiple of the third is greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

In algebraical symbols the definition may be thus stated:

Four quantities $a$, $b$, $c$, $d$ are in proportion when $pc \overset{>}{\underset{<}{=}} qd$ according as $pa \overset{>}{\underset{<}{=}} qb$, $p$ and $q$ being *any positive integers whatever.*

I. To deduce the geometrical definition of proportion from the algebraical definition.

Since $\frac{a}{b} = \frac{c}{d}$, by multiplying both sides by $\frac{p}{q}$, we obtain

$$\frac{pa}{qb} = \frac{pc}{qd};$$

hence, from the properties of fractions,

$$pc \overset{>}{\underset{<}{=}} qd \text{ according as } pa \overset{>}{\underset{<}{=}} qb,$$

which proves the proposition.

II. To deduce the algebraical definition of proportion from the geometrical definition.

Given that $pc \overset{>}{\underset{<}{=}} qd$ according as $pa \overset{>}{\underset{<}{=}} qb$, to prove

$$\frac{a}{b} = \frac{c}{d}.$$

If $\frac{a}{b}$ is not equal to $\frac{c}{d}$, one of them must be the greater. Suppose $\frac{a}{b} > \frac{c}{d}$; then it will be possible to find some fraction $\frac{q}{p}$ which lies between them, $q$ and $p$ being positive integers.

Hence $$\frac{a}{b} > \frac{q}{p} \quad\text{.............................. (1)},$$

and $$\frac{c}{d} < \frac{q}{p} \quad\text{.............................. (2)}.$$

From (1) $\quad pa > qb$;

from (2) $\quad pc < qd$;

and these contradict the hypothesis.

Therefore $\frac{a}{b}$ and $\frac{c}{d}$ are not unequal; that is $\frac{a}{b} = \frac{c}{d}$; which proves the proposition.

26. It should be noticed that the geometrical definition of proportion deals with *concrete* magnitudes, such as lines or areas, represented geometrically but not referred to any common unit of measurement. So that Euclid's definition is applicable to incommensurable as well as to commensurable quantities; whereas the algebraical definition, strictly speaking, applies only to commensurable quantities, since it tacitly assumes that $a$ is the same determinate multiple, part, or parts, of $b$ that $c$ is of $d$. But the proofs which have been given for commensurable quantities will still be true for incommensurables, since the ratio of two incommensurables can always be made to differ from the ratio of two integers *by less than any assignable quantity.* This has been shewn in Art. 7; it may also be proved more generally as in the next article.

27. Suppose that $a$ and $b$ are incommensurable; divide $b$ into $m$ equal parts each equal to $\beta$, so that $b = m\beta$, where $m$ is a positive integer. Also suppose $\beta$ is contained in $a$ more than $n$ times and less than $n+1$ times;

then $$\frac{a}{b} > \frac{n\beta}{m\beta} \text{ and } < \frac{(n+1)\beta}{m\beta},$$

that is, $$\frac{a}{b} \text{ lies between } \frac{n}{m} \text{ and } \frac{n+1}{m};$$

so that $\frac{a}{b}$ differs from $\frac{n}{m}$ by a quantity less than $\frac{1}{m}$. And since we can choose $\beta$ (our unit of measurement) as small as we please, $m$ can be made as great as we please. Hence $\frac{1}{m}$ can be made as small as we please, and two integers $n$ and $m$ can be found whose ratio will express that of $a$ and $b$ to any required degree of accuracy.

28. The propositions proved in Art. 23 are often useful in solving problems. In particular, the solution of certain equations is greatly facilitated by a skilful use of the operations *componendo* and *dividendo*.

*Example* 1.

If $(2ma+6mb+3nc+9nd)(2ma-6mb-3nc+9nd)$
$= (2ma-6mb+3nc-9nd)(2ma+6mb-3nc-9nd)$,

prove that $a$, $b$, $c$, $d$ are proportionals.

We have $\dfrac{2ma+6mb+3nc+9nd}{2ma-6mb+3nc-9nd} = \dfrac{2ma+6mb-3nc-9nd}{2ma-6mb-3nc+9nd}$;

$\therefore$ componendo and dividendo,

$$\frac{2(2ma+3nc)}{2(6mb+9nd)} = \frac{2(2ma-3nc)}{2(6mb-9nd)}.$$

Alternando, $\dfrac{2ma+3nc}{2ma-3nc} = \dfrac{6mb+9nd}{6mb-9nd}$.

Again, componendo and dividendo,

$$\frac{4ma}{6nc} = \frac{12mb}{18nd};$$

whence $\dfrac{a}{c} = \dfrac{b}{d}$,

or $a : b = c : d$.

*Example* 2. Solve the equation

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = \frac{4x-1}{2}.$$

We have, componendo and dividendo,

$$\frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{4x+1}{4x-3};$$

$$\therefore \frac{x+1}{x-1} = \frac{16x^2+8x+1}{16x^2-24x+9}.$$

Again, componendo and dividendo,

$$\frac{2x}{2} = \frac{32x^2-16x+10}{32x-8},$$

$$\therefore x = \frac{16x^2-8x+5}{16x-4};$$

whence $16x^2-4x = 16x^2-8x+5$;

$$\therefore x = \frac{5}{4}.$$

## EXAMPLES. II.

1. Find the fourth proportional to 3, 5, 27.

2. Find the mean proportional between

   (1) 6 and 24, (2) $360a^4$ and $250a^2b^2$.

3. Find the third proportional to $\dfrac{x}{y}+\dfrac{y}{x}$ and $\dfrac{x}{y}$.

If $a : b = c : d$, prove that

4. $a^2c+ac^2 : b^2d+bd^2 = (a+c)^3 : (b+d)^3$.

5. $pa^2+qb^2 : pa^2-qb^2 = pc^2+qd^2 : pc^2-qd^2$.

6. $a-c : b-d = \sqrt{a^2+c^2} : \sqrt{b^2+d^2}$.

7. $\sqrt{a^2+c^2} : \sqrt{b^2+d^2} = \sqrt{ac+\dfrac{c^3}{a}} : \sqrt{bd+\dfrac{d^3}{b}}$.

If $a$, $b$, $c$, $d$ are in continued proportion, prove that

8. $a : b+d = c^3 : c^2d+d^3$.

9. $2a+3d : 3a-4d = 2a^3+3b^3 : 3a^3-4b^3$.

10. $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

11. If $b$ is a mean proportional between $a$ and $c$, prove that

$$\frac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}} = b^4.$$

12. If $a : b = c : d$, and $e : f = g : h$, prove that

$$ae+bf : ae-bf = cg+dh : cg-dh.$$

Solve the equations:

13. $\dfrac{2x^3-3x^2+x+1}{2x^3-3x^2-x-1} = \dfrac{3x^3-x^2+5x-13}{3x^3-x^2-5x+13}$.

14. $\dfrac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3} = \dfrac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$.

15. $\dfrac{(m+n)x-(a-b)}{(m-n)x-(a+b)} = \dfrac{(m+n)x+a+c}{(m-n)x+a-c}$.

16. If $a$, $b$, $c$, $d$ are proportionals, prove that

$$a+d = b+c+\frac{(a-b)(a-c)}{a}.$$

17. If $a$, $b$, $c$, $d$, $e$ are in continued proportion, prove that

$$(ab+bc+cd+de)^2 = (a^2+b^2+c^2+d^2)(b^2+c^2+d^2+e^2).$$

18. If the work done by $x-1$ men in $x+1$ days is to the work done by $x+2$ men in $x-1$ days in the ratio of 9 : 10, find $x$.

19. Find four proportionals such that the sum of the extremes is 21, the sum of the means 19, and the sum of the squares of all four numbers is 442.

20. Two casks $A$ and $B$ were filled with two kinds of sherry, mixed in the cask $A$ in the ratio of 2 : 7, and in the cask $B$ in the ratio of 1 : 5. What quantity must be taken from each to form a mixture which shall consist of 2 gallons of one kind and 9 gallons of the other?

21. Nine gallons are drawn from a cask full of wine; it is then filled with water, then nine gallons of the mixture are drawn, and the cask is again filled with water. If the quantity of wine now in the cask be to the quantity of water in it as 16 to 9, how much does the cask hold?

22. If four positive quantities are in continued proportion, shew that the difference between the first and last is at least three times as great as the difference between the other two.

23. In England the population increased 15·9 per cent. between 1871 and 1881; if the town population increased 18 per cent. and the country population 4 per cent., compare the town and country populations in 1871.

24. In a certain country the consumption of tea is five times the consumption of coffee. If $a$ per cent. more tea and $b$ per cent. more coffee were consumed, the aggregate amount consumed would be $7c$ per cent. more; but if $b$ per cent. more tea and $a$ per cent. more coffee were consumed, the aggregate amount consumed would be $3c$ per cent. more: compare $a$ and $b$.

25. Brass is an alloy of copper and zinc; bronze is an alloy containing 80 per cent. of copper, 4 of zinc, and 16 of tin. A fused mass of brass and bronze is found to contain 74 per cent. of copper, 16 of zinc, and 10 of tin: find the ratio of copper to zinc in the composition of brass.

26. A crew can row a certain course up stream in 84 minutes; they can row the same course down stream in 9 minutes less than they could row it in still water: how long would they take to row down with the stream?