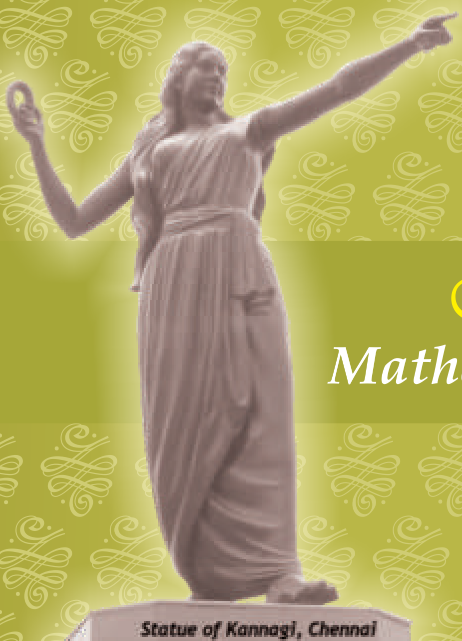
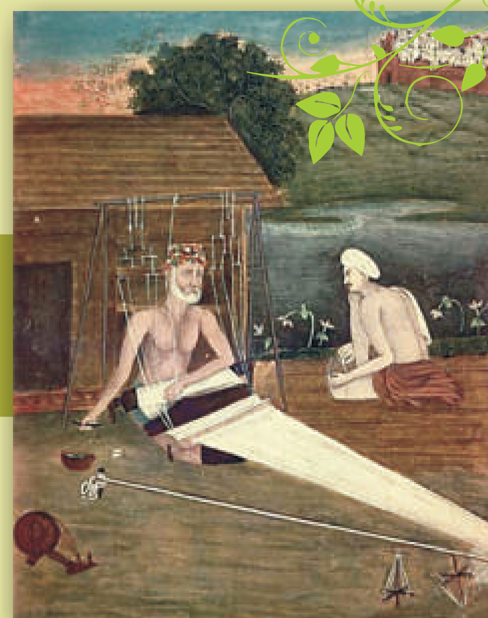
Knowledge
TRADITIONS & PRACTICES OF INDIA

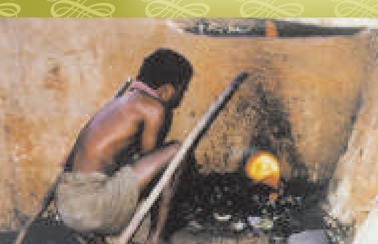
Textbook for Class XI

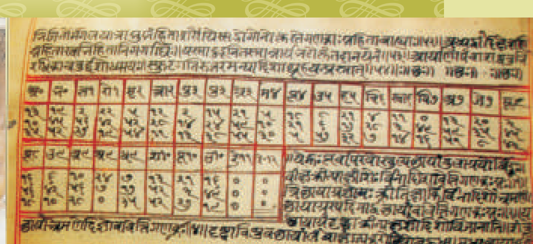


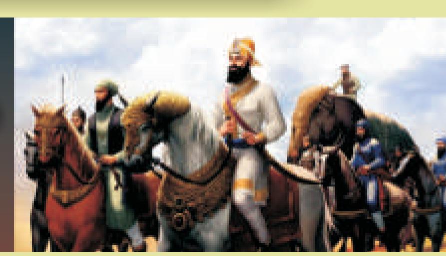
Statue of Kannagi, Chennai



Module 7
Mathematics in India







CENTRAL BOARD OF SECONDARY EDUCATION

Shiksha Kendra, 2, Community Centre, Preet Vihar,
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नया आगाज़

आज समय की माँग पर
आगाज़ नया इक होगा
निरंतर योग्यता के निर्णय से
परिणाम आकलन होगा।

परिवर्तन नियम जीवन का
नियम अब नया बनेगा
अब परिणामों के भय से
नहीं बालक कोई डरेगा

निरंतर योग्यता के निर्णय से
परिणाम आकलन होगा।

बदले शिक्षा का स्वरूप
नई खिले आशा की धूप
अब किसी कोमल-से मन पर
कोई बोझ न होगा

निरंतर योग्यता के निर्णय से
परिणाम आकलन होगा।
नई राह पर चलकर मंज़िल को हमें पाना है
इस नए प्रयास को हमने सफल बनाना है
बेहतर शिक्षा से बदले देश, ऐसे इसे अपनाए
शिक्षक, शिक्षा और शिक्षित
बस आगे बढते जाएँ
बस आगे बढते जाएँ
बस आगे बढते जाएँ.........

Knowledge TRADITIONS & PRACTICES OF INDIA

Textbook for Class XI

Module 7

Mathematics in India

केन्द्रीय माध्यमिक शिक्षा बोर्ड दिल्ली
असतो मा सद्गमय
CENTRAL BOARD OF SECONDARY EDUCATION
Shiksha Kendra, 2, Community Centre, Preet Vihar, Delhi-110 092 India

Preface

India has a rich tradition of intellectual inquiry and a textual heritage that goes back to several hundreds of years. India was magnificently advanced in knowledge traditions and practices during the ancient and medieval times. The intellectual achievements of Indian thought are found across several fields of study in ancient Indian texts ranging from the Vedas and the Upanishads to a whole range of scriptural, philosophical, scientific, technical and artistic sources.

As knowledge of India's traditions and practices has become restricted to a few erudite scholars who have worked in isolation, CBSE seeks to introduce a course in which an effort is made to make it common knowledge once again. Moreover, during its academic interactions and debates at key meetings with scholars and experts, it was decided that CBSE may introduce a course titled 'Knowledge Traditions and Practices of India' as a new Elective for classes XI - XII from the year 2012-13. It has been felt that there are many advantages of introducing such a course in our education system. As such in India, there is a wide variety and multiplicity of thoughts, languages, lifestyles and scientific, artistic and philosophical perceptions. The rich classical and regional languages of India, which are repositories of much of the ancient wisdom, emerge from the large stock of the shared wealth of a collective folklore imagination. A few advantages given below are self explanatory.

- India is a land of knowledge and traditions and through this course the students will become aware of our ancient land and culture.
- Learning about any culture particularly one's own culture - whatever it may be - builds immense pride and self-esteem. That builds a community and communities build harmony.
- The students will be learning from the rich knowledge and culture and will get an objective insight into the traditions and practices of India. They will delve deeply to ascertain how these teachings may inform and benefit them in future.
- The textbook has extracts and translations that will develop better appreciation and understanding of not only the knowledge, traditions and practices of India but also contemporary questions and issues that are a part of every discipline and field in some form or another.

This course once adopted in schools across India can become central to student learning: each student brings a unique culture, tradition and practice to the classroom. The content is devised in a way that the educator becomes knowledgeable about his/her students' distinctive cultural background. This can be translated into effective instruction and can enrich the curriculum thereby benefitting one and all. This insight has close approximation with the pedagogy of CCE.

The course is designed in a way that it embodies various disciplines and fields of study ranging from Language and Grammar, Literature, Fine Arts, Agriculture, Trade and Commerce, Philosophy and Yoga to Mathematics, Astronomy, Chemistry, Metallurgy, Medicine and Surgery, Life Sciences, Environment and Cosmology. This can serve as a good foundation for excellence in any discipline pursued by the student in her/his academic, personal and professional life.

This book aims at providing a broad overview of Indian thought in a multidisciplinary and interdisciplinary mode. It does not seek to impart masses of data, but highlights concepts and major achievements while engaging the student with a sense of exploration and discovery. There is an introduction of topics so that students who take this are prepared for a related field in higher studies in the universities.

The examination reforms brought in by CBSE have strengthened the Continuous and Comprehensive Evaluation System. It has to be ascertained that the teaching and learning methodology of CCE is adopted by the affiliated schools when they adopt this course. The contents have to cultivate critical appreciation of the thought and provide insights relevant for promoting cognitive ability, health and well-being, good governance, aesthetic appreciation, value education and appropriate worldview.

This document has been prepared by a special committee of convenors and material developers under the direction of Dr. Sadhana Parashar, Director (Academic & Training) and co-ordinated by Mrs. Neelima Sharma, Consultant, CBSE.

The Board owes a wealth of gratitude to Professor Jagbir Singh, Professor Kapil Kapoor, Professor Michel Danino, and all those who contributed to the extensive work of conceptualizing and developing the contents. I sincerely hope that our affiliated schools will adopt this new initiative of the Board and assist us in our endeavour to nurture our intellectual heritage.

Vineet Joshi
Chairman

Convenor's Note by Professor Jagbir Singh

In 2012, CBSE decided to introduce an Elective Course 'Knowledge Traditions and Practices of India' for classes XI and XII and an Advisory Committee was constituted to reflect on the themes and possible content of the proposed course. Subsequently Module-Preparation Committees were constituted to prepare ten modules for the first year of the programme to include the following Astronomy, Ayurveda (Medicine and Surgery), Chemistry, Drama, Environment, Literature, Mathematics, Metallurgy, Music and Philosophy.

Each module has;

I. A Survey article

ii. Extracts from primary texts

iii. Suitably interspersed activities to enable interactive study and class work

iv. Appropriate visuals to engender reading interest, and

v. Further e- and hard copy readings.

Each module in the course has kept in mind what would be a viable amount of reading and workload, given all that the class IX students have to do in the given amount of time, and controlled the word-length and also provided, where needed, choices in the reading materials.

Each Module consists of:

I. A Survey Essay (about 1500-2000 words) that introduces and shows the growth of ideas, texts and thinkers and gives examples of actual practice and production.

ii. A survey-related selection of extracts (in all about 2000 words) from primary sources (in English translation, though for first hand recognition, in some cases, where feasible, the extracts are also reproduced in the original language and script).

iii. Three kinds of interactive work are incorporated, both in the survey article and the extracts - comprehension questions, individual and collective activities and projects (that connect the reading material and the student to the actual practice and the environment).

iv. Visuals of thinkers, texts, concepts (as in Mathematics), practices.

v. Internet audiovisual resources in the form of URLs.

vi. List of further questions, and readings.

The objective of each module, as of the whole course, is to re-connect the young minds with the large body of intellectual activity that has always happened in India and, more importantly, to enable them (i) to relate the knowledge available to the contemporary life, theories and practices, (ii) to develop, wherever feasible, a comparative view on a level ground of the contemporary Western ideas and the Indian theories and practices, and (iii) to extend their horizons beyond what is presented or is available and contemplate on possible new meanings, extensions and uses of the ideas - in other words to make them think.

We have taken care to be objective and factual and have carefully eschewed any needless claims or comparisons with western thought. Such things are best left to the readers' judgement.

This pedagogical approach clearly approximates CBSE's now established activity-oriented interactive work inviting the students' critical responses.

It is proposed to upload the first year's modular programme to be downloaded and used by schools, teachers and students.

As a first exercise, we are aware that the content selection, a major difficult task, can be critically reviewed from several standpoints. We do not claim perfection and invite suggestions and concrete proposals to develop the content. We are eagerly looking forward to receiving the feedback from both teachers and students. That would help us refining the content choice, the length and the activities. We will also thankfully acknowledge any inadvertent errors that are pointed out by readers.

The finalisation of this course is thus envisaged as a collective exercise and only over a period of time, the Course will mature. We know that perfection belongs only to God.

If our students enjoy reading these materials, that would be our true reward.

Prof. Jagbir Singh
Convenor

Acknowledgement

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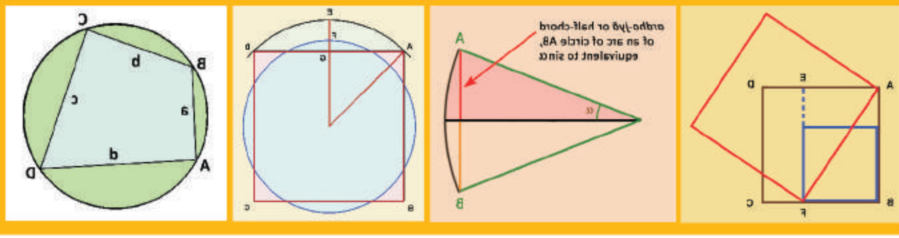
Content of Module 7



Mathematics in India: A Survey*

As early Indian astronomers tried to quantify the paths of the sun, the moon, the planets and the stars on the celestial sphere with ever more accuracy, or to predict the occurrence of eclipses, they were naturally led to develop mathematical tools. Astronomy and mathematics were thus initially regarded as inseparable, the latter being the maid-servant of the former. Indeed, about 1400 BCE, the *Vedāṅga Jyotiṣa,* the first extant Indian text of astronomy, states in two different versions:

> Like the crest on the head of a peacock, like the gem on the hood of a cobra, *jyotiṣa* (astronomy) / *gaṇita* (mathematics) is the crown of the *Vedāṅga śāstras* [texts on various branches of knowledge].

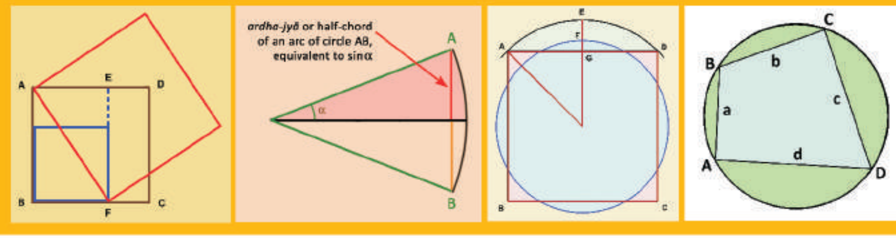
In fact, *jyotiṣa* initially referred to astronomy and mathematics combined; only later did it come to mean astronomy alone (and much later did it include astrology).

First Steps

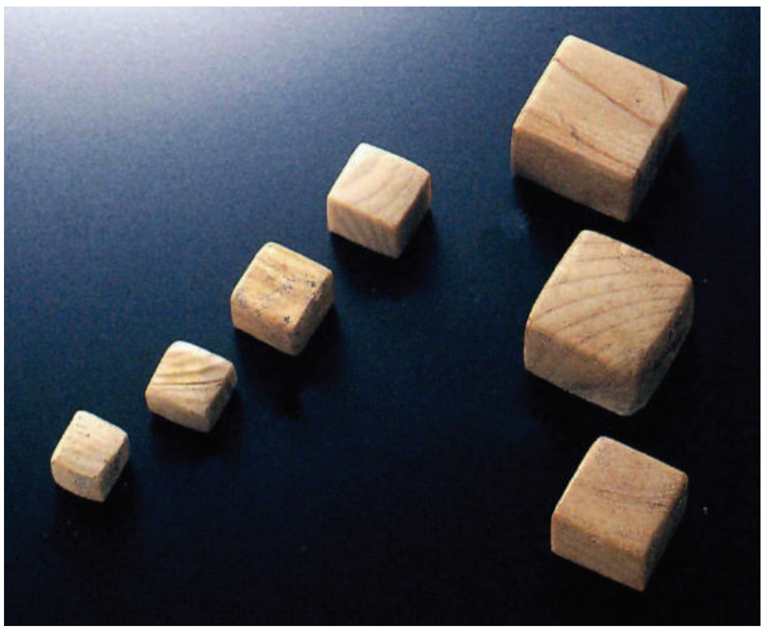
India's first urban development, the Indus or Harappan civilization (2600-1900 BCE), involved a high degree of town planning. A mere glance at the plan of Mohenjo-daro's acropolis (or upper city), Dholavira (in the Rann of Kachchh) or Kalibangan (Rajasthan), reveals fortifications and streets generally aligned to the cardinal directions and exhibiting right angles. Specific proportions in the dimensions of major structures have also been pointed out. All this implies a sound knowledge of basic

How much knowledge of geometry would you need to plan a city?

* The author gratefully acknowledges valuable suggestions for improvement received from Dr. M.D. Srinivas.

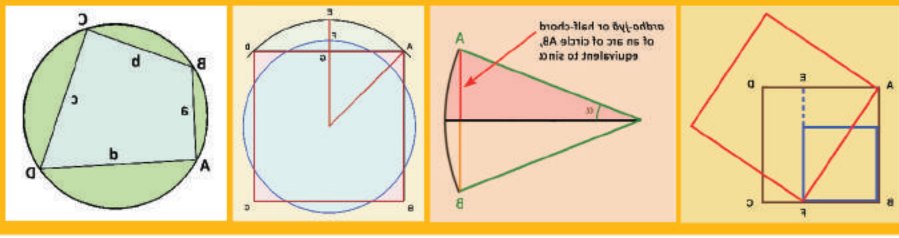


geometric principles and an ability to measure angles, which the discovery of a few cylindrical compasses made of shell, with slits cut every 45°, has confirmed. Besides, for trading purposes the Harappans developed a standardized system of weights in which, initially, each weight was double the preceding one, then, 10, 100 or 1,000 times the value of a smaller weight. This shows that the Harappans could not only multiply a quantity by such factors, but also had an inclination for a decimal system of multiples. However, there is no agreement among scholars regarding the numeral system used by Harappans.



A few Harappan weights made of chert, from Dholavira, Gujarat (Courtesy: ASI)

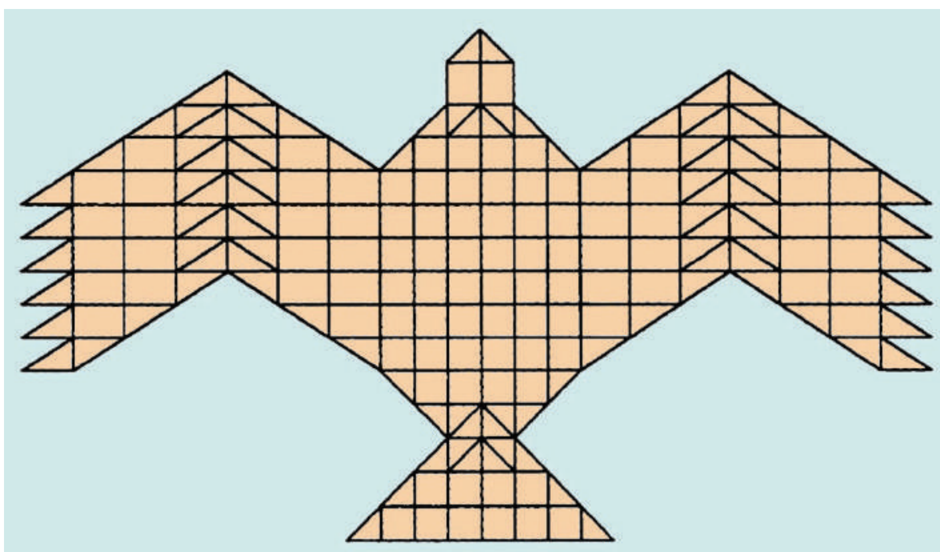
There is no scholarly consensus on the dates of the four Vedas, India's most ancient texts, except that they are over 3,000 years old at the very least. We find in them frequent mentions of numbers by name, in particular multiples of tens, hundreds and thousands, all the way to a million millions in the *Yajur Veda* — a number called *parārdha*. (By comparison, much later, the Greeks named numbers only up to 10,000, which was a 'myriad'; and only in the 13th century CE would the concept of a 'million' be adopted in



Europe.) The *Brāhmanas*, commentaries on the Vedas, knew the four arithmetical operations as well as basic fractions.

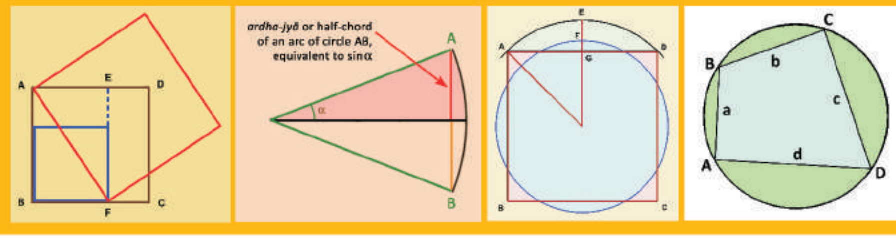
Early Historical Period

The first Indian texts dealing explicitly with mathematics are the *Śulbasūtras*, dated between the 8th and 6th centuries BCE. They were written in Sanskrit in the highly concise *sūtra* style and were, in effect, manuals for the construction of fire altars (called *citis* or *vedis*) intended for specific rituals and made of bricks. The altars often had five layers of 200 bricks each, the lowest layer symbolizing the earth, and the highest, heaven; they were thus symbolic representations of the universe.



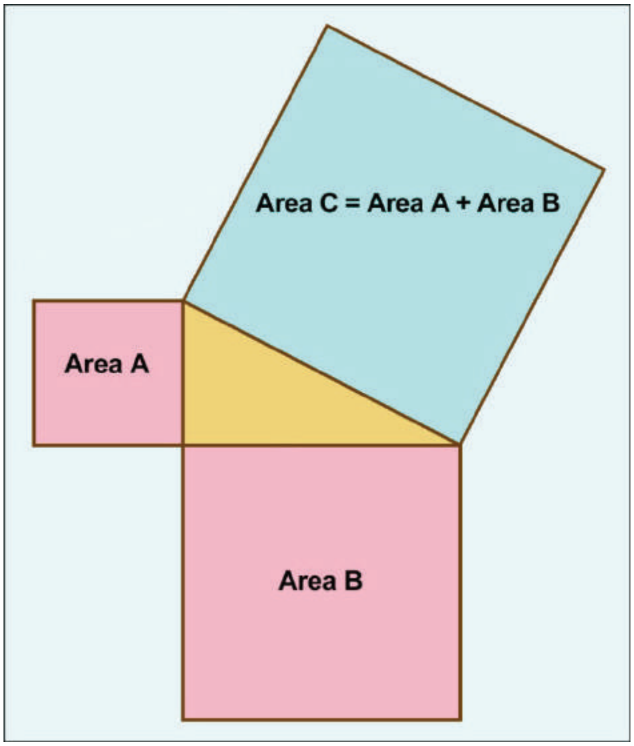
The first layer of one kind of *śyenaciti* or falcon altar described in the *Śulbasūtras,* made of 200 bricks of six shapes or sizes, all of them adding up to a specified total area.

Because their total area needed to be carefully defined and constructed from bricks of specified shapes and size, complex geometrical calculations followed. The *Śulbasūtras*, for instance, are the earliest texts of geometry offering a general statement, in geometric



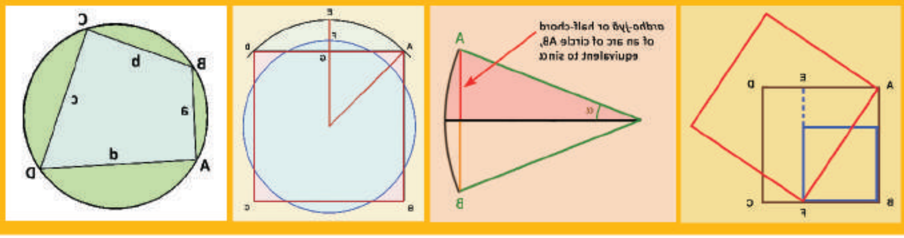
form, of the so-called Pythagoras theorem (which was in fact formulated by Euclid around 300 BCE).

What is meant by 'transcendental' and why should this nature of π preclude exact geometrical solutions to the squaring of a circle?



The geometrical expression of the Pythagoras theorem found in the *Śulbasūtras*.

They spelt out elaborate geometric methods to construct a square resulting from the addition or subtraction of two other squares, or having the same area as a given circle, and vice-versa — the classic problems of the squaring of a circle or the circling of a square (which, because of π's transcendental nature, cannot have exact geometrical solutions, only approximate ones). All these procedures were purely geometrical, but led to interesting corollaries; for instance, √2 was given a rational approximation which is correct to the fifth decimal!



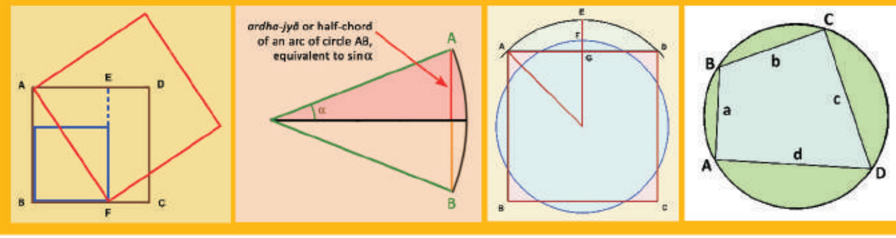
$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{(3)(4)} - \frac{1}{(3)(4)(34)}$$

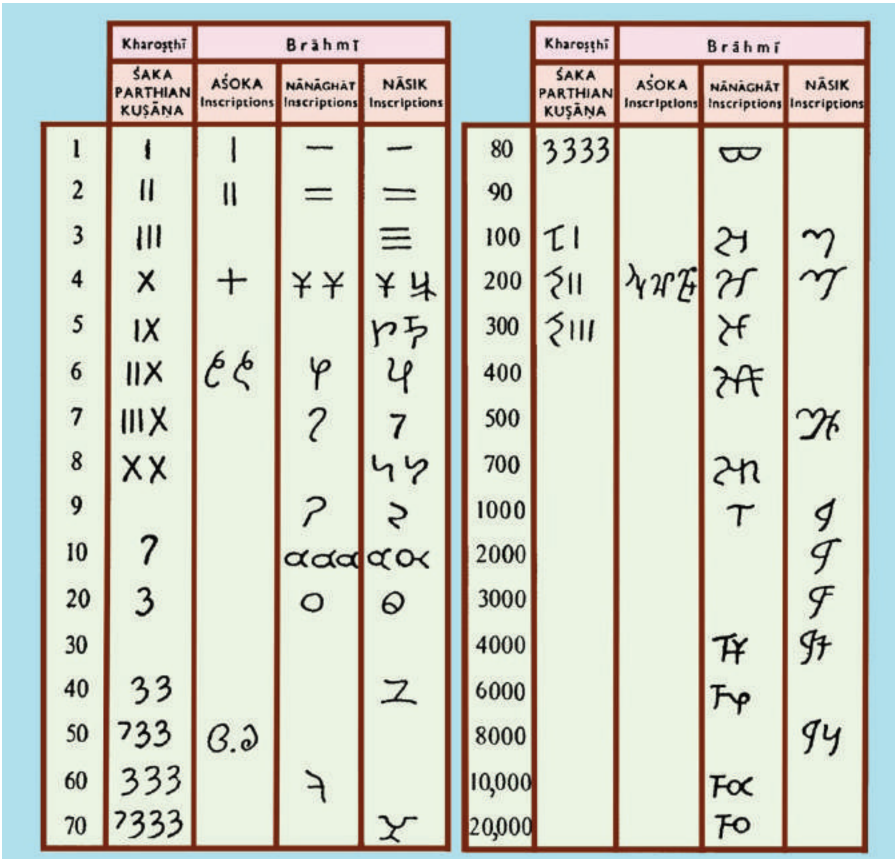
The *Śulbasūtras* also introduced a system of linear units, most of them based on dimensions of the human body; they were later slightly modified and became the traditional units used across India. The chief units were:

- 14 *aṇus* (grain of common millet) = 1 *aṅgula* (a digit)
- 12 *aṅgulas* = 1 *prādeśa* (the span of a hand, later *vitasti*)
- 15 *aṅgulas* = 1 *pada* (or big foot)
- 24 *aṅgulas* = 1 *aratni* (or cubit, later also *hasta*)
- 30 *aṅgulas* = 1 *prakrama* (or step)
- 120 *aṅgulas* = 1 *puruṣa* (or the height of a man with his arm extended over his head)

A few centuries later, Piṅgala's *Chandasūtras*, a text on Sanskrit prosody, made use of a binary system to classify the metres of Vedic hymns, whose syllables may be either light (*laghu*) or heavy (*guru*); rules of calculation were worked out to relate all possible combinations of light and heavy syllables, expressed in binary notation, to numbers in one-to-one relationships, which of course worked both ways. In the course of those calculations, Piṅgala referred to the symbol for *śūnya* or zero.

About the same time, Jaina texts indulged in cosmological speculations involving colossal numbers, and dealt with geometry, combinations and permutations, fractions, square and cube powers; they were the first in India to come up with the notion of an unknown (*yāvat-tāvat*), and introduced a value of π equal to √10, which remained popular in India for quite a few centuries.

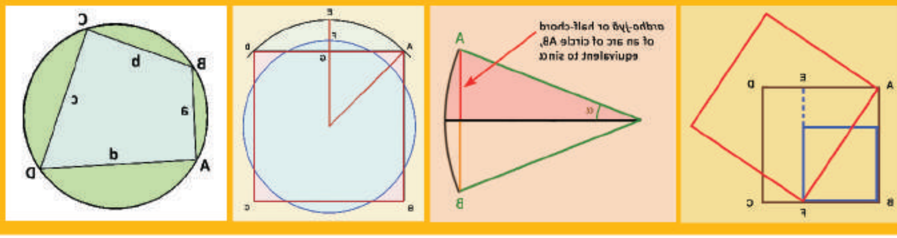


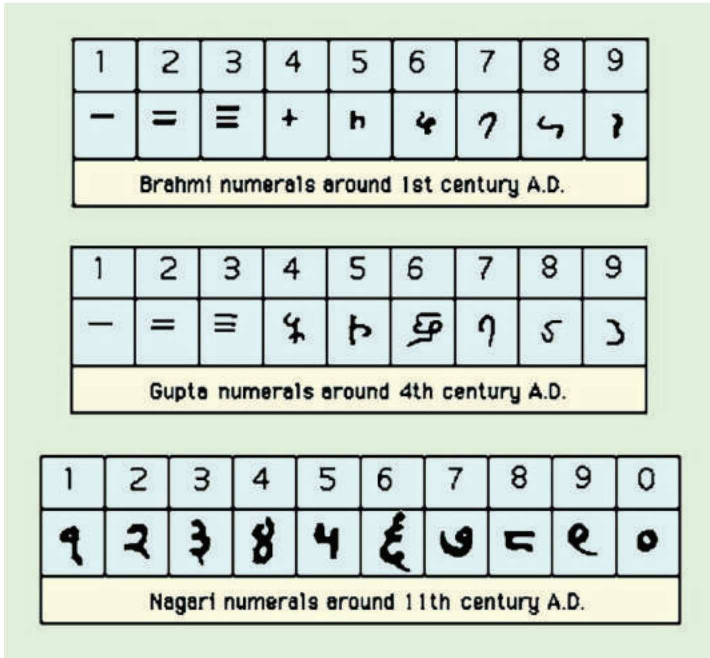


Numerals as they appeared in early inscriptions, from the 3rd century BCE to the 1st century CE. Note that they do not yet follow a decimal positional system; for instance, in the first column, 40 is written as '20, 20', 60 as '20, 20, 20'. (Adapted from INSA)

Notice how, in columns 2 to 4, multiples of hundreds are represented through a single sign. What does this imply?

With the appearance of the Brāhmī script a few centuries BCE, we come across India's first numerals, on Ashoka's edicts in particular, but as yet without any decimal positional value. These numerals will evolve in shape; eventually borrowed by Arabs scholars, they will be transmitted, with further alterations, to Europe and become our modern 'Arabic' numerals.





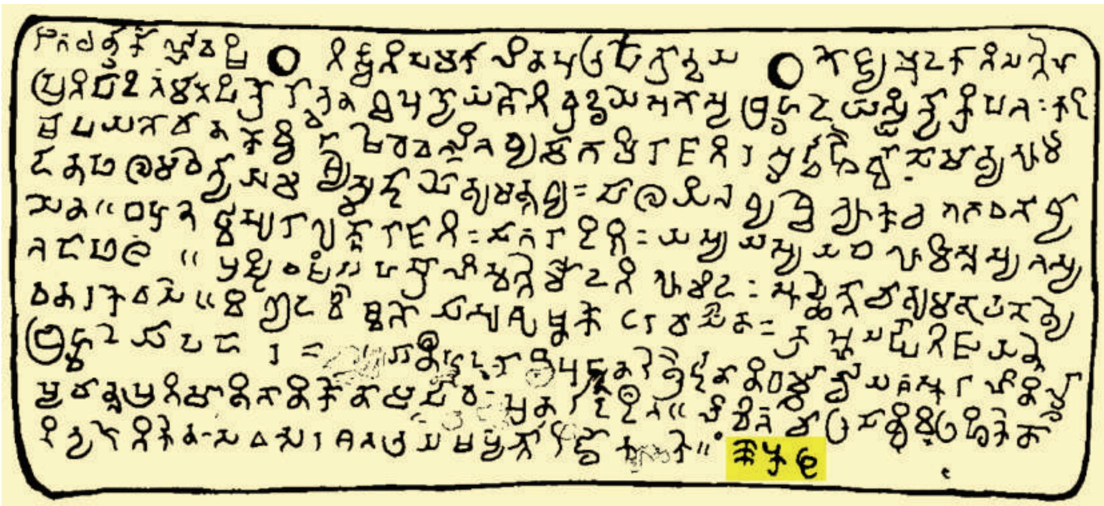
Evolution of Indian numerals, as evidenced by inscriptions. The first script, Brāhmī, was used by Aśoka in his Edicts; the last is an antecedent of the Devanagari script. (Adapted from J.J. O'Connor & E.F. Robertson)

The Classical Period

Together with astronomy, Indian mathematics saw its golden age during India's classical period, beginning more or less with the Gupta age, i.e. from about 400 CE. (See module **Astronomy in India** for a map of Indian astronomers and mathematicians.)

Shortly before that period, the full-fledged place-value system of numeral notation — our 'modern' way of noting numbers, unlike non-positional systems such as those depicted above or Roman numbers — had been worked out, integrating zero with the nine numerals. It is a pity that we shall never know who conceived of it. Amongst the earliest known references to it is a first-century CE work by the Buddhist philosopher Vasumitra, and it is worked out more explicitly in the Jain cosmological work *Lokavibhāga,* written in 458 CE. Soon it was adopted across India, and later taken to

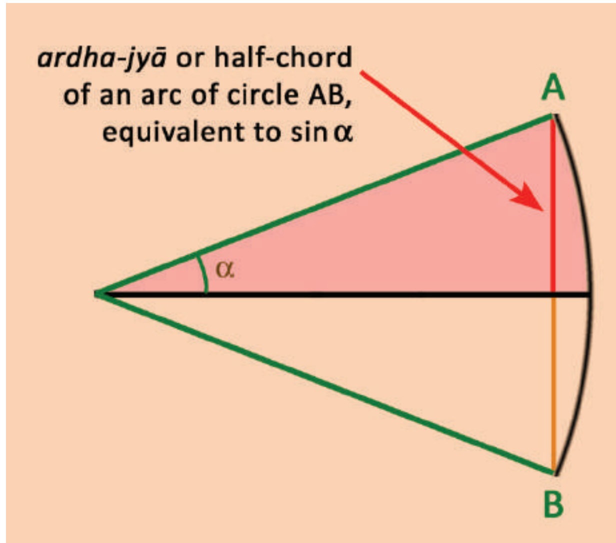
Europe by the Arabs. This was a major landmark in the world history of science, since it permitted rapid developments in mathematics.



One of the first attested inscriptions (from Sankheda, Gujarat) recording a date written with the place-value system of numeral notation. The date (highlighted) reads 346 of a local era, which corresponds to 594 CE. (Adapted from Georges Ifrah)

About 499 CE, living near what is today Patna, Āryabhaṭa I (born 476 CE) authored the *Āryabhaṭīya*, the first extant *siddhānta* (or treatise) attempting a systematic review of the knowledge of mathematics and astronomy prevailing in his days. The text is so concise (just 121 verses) as to be often obscure, but between the 6th and the 16th century, no fewer than twelve major commentaries were authored to explicate and build upon its contents. It was eventually translated into Arabic about 800 CE (under the title *Zīj al-Ārjabhar*), which in turn led to a Latin translation in the 13th century (in which Āryabhaṭa was called 'Ardubarius').

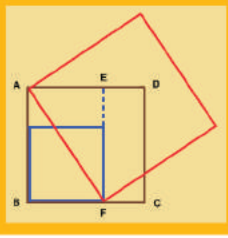
Why should the study of the half-chord of an arc of circle be an advance over that of the full chord?

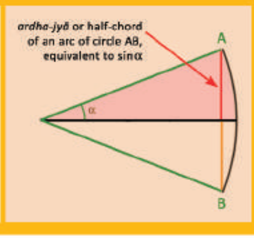


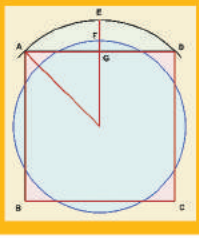
Āryabhaṭa introduced the notion of a half-chord, a substantial advance over Greek trigonometry, which considered the full chord of an arc of circle.

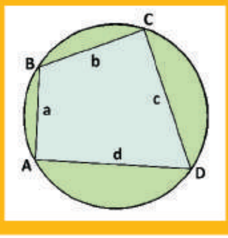
The mathematical content of *Āryabhaṭīya* ranges from a very precise table of sines and an equally precise value for π (3.1416, stated to be 'approximate') to the area of a triangle, the sums of finite arithmetic progressions, algorithms for the extraction of square and cube roots, and an elaborate algorithm called *kuṭṭaka* ('pulverizing') to solve indeterminate equations of the first degree with two unknowns: $ax + c = by$. By 'indeterminate' is meant that solutions should be integers alone, which rules out direct algebraic methods; such equations came up in astronomical problems, for example to calculate a whole number of revolutions of a planet in a given number of years.

It is worth mentioning that despite its great contributions, the *Āryabhaṭīya* is not free of errors: its formulas for the volumes of a pyramid and a sphere were erroneous, and would be later corrected by Brahmagupta and Bhāskarācārya respectively.

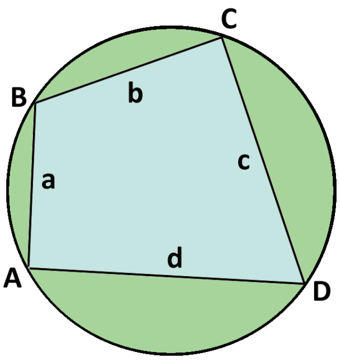








The Classical Period, post-Āryabhaṭa



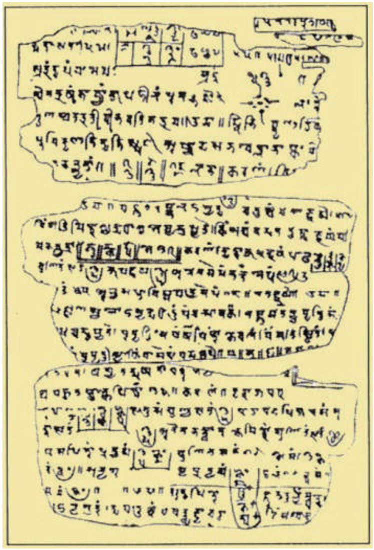
Born in 598 CE, Brahmagupta was an imposing figure, with considerable achievements in mathematics. In his *Brahmasphuta Siddhānta*, he studied cyclic quadrilaterals (i.e., inscribed in a circle) and supplied the formula for their area (a formula rediscovered in 17th-century Europe): if ABCD has sides of lengths a, b, c, and d, and the semi-perimeter is $s = (a + b + c + d)/2$, then the area is given by:

$$\text{Area } ABCD = \sqrt{[(s - a)(s - b)(s - c)(s - a)]}$$

Brahmagupta boldly introduced the notion of negative numbers and ventured to define the mathematical infinite as *khacheda* or 'that which is divided by *kha*', *kha* being one of the many names for zero. He discovered the *bhāvanā* algorithm for integral solutions to second-order indeterminate equations (called *varga prakriti*) of the type $Nx^2 + 1 = y^2$. He was in many ways one of the founders of modern algebra, and his works were translated into Persian and later Latin.

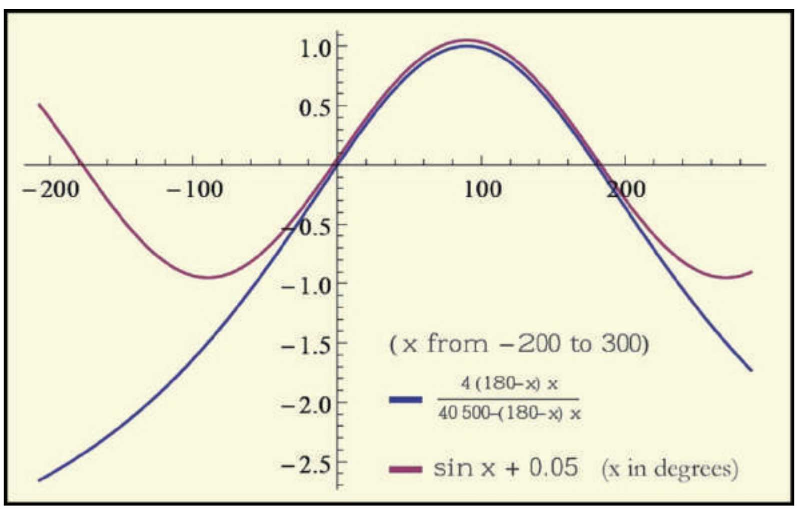
Would Brahmagupta's definition of the mathematical infinite be acceptable to modern mathematics?

Dated around the 7th century, the Bakhshali manuscript, named after the village (now in northern Pakistan) where it was found in 1881 in the form of 70 leaves of birch bark, gives us a rare insight into extensive mathematical calculation techniques of the times, involving in particular fractions, progressions, measures of time, weight and money.



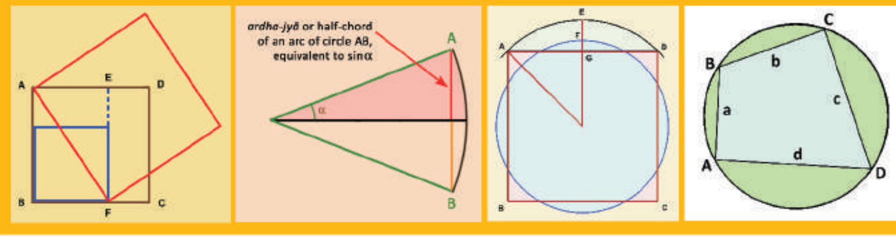
A few leaves from the Bakhshali manuscript (Courtesy: Wikipedia)

Other brilliant mathematicians of the siddhāntic era included Bhāskara I, a contemporary of Brahmagupta, who did pioneering work in trigonometry (proposing a remarkably accurate rational approximation for the sine function), Śrīdhara and Mahāvīra. The last, a Jain scholar who lived in the 9th century in the court of a Rashtrakuta king (in today's Karnataka), authored the first work of mathematics that was not as part of a text on astronomy. In it, Mahāvīra dealt with finite series, expansions of fractions, permutations and combinations (working out, for the first time, some of the standard formulas in the field), linear equations with two unknowns, quadratic equations, and a remarkably close approximation for the circumference of an ellipse, among other important results.



Graph showing the high accuracy of Bhāskara I's rational approximation for the sine function from 0° to 180° (in blue). The sine function (in read) had to be shifted upward by 0.05 to make the two curves distinguishable. (Courtesy: IFIH)

Bhāskara II, often known as Bhāskarācārya, lived in the 12th century. His *Siddhāntaśiromani* (literally, the 'crest jewel of the *siddhāntas*') broke new ground as regards cubic and biquadratic equations. He built upon Brahmagupta's work on indeterminate equations to produce a still more effective algorithm, the *chakravāla* (or



'cyclic method'); with it he showed, for instance, that the smallest integral solutions to $61x^2 + 1 = y^2$ are $x = 226153980$, $y = 1766319049$ (interestingly, five centuries later, the French mathematician Fermat offered the same equation as a challenge to some of his contemporaries). Bhāskarācārya also grasped the notion of integration as a limit of finite sums: by slicing a sphere into ever smaller rings, for instance, he was able to calculate its area and volume. He came close to the modern notion of derivative by discussing the notion of instant speed *(tātkālika gati)* and understood that the derivative of the sine function is proportional to the cosine.

The first part of Bhāskarācārya's *Siddhāntaśiromani* is a collection of mathematical problems called *Līlāvatī,* named after an unknown lady to whom Bhāskara puts problems in an often poetical language. *Līlāvatī* became so popular with students of mathematics across India that four centuries later, Akbar had it translated into Persian by a court poet.

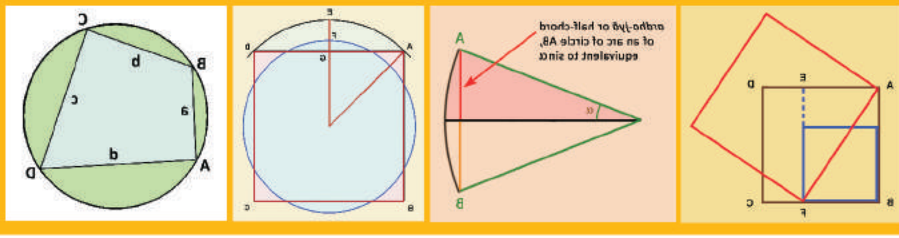
The Kerala School of Mathematics

Along with astronomy, mathematics underwent a revival in the Kerala School, which flourished there from the 14th to the 17th century. Its pioneer, Mādhava (c. 1340–1425), laid some of the foundations of calculus by working out power series expansions for the sine and cosine functions (the so-called Newton series), and by spelling out this fundamental expansion of π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

This is known as the Gregory–Leibniz series, but ought one day to be named after Mādhava. He went on to propose a more rapidly convergent series for π:

$$\pi = \sqrt{12}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots\right)$$



which enabled him to calculate π to 11 correct decimals.

Nīlakaṇṭha Somayāji (c. 1444–1545) and Jyeṣṭhadeva (c. 1500–1600) built on such results and considerably enriched what might be called the Indian foundations of calculus. The latter, for instance, worked out the binomial expansion:

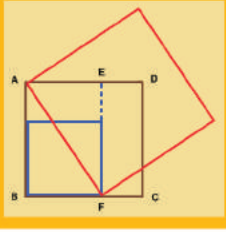
$$(1+x)^{-1} = 1 - x + x^2 + \cdots + (-1)^r x^r + \cdots$$

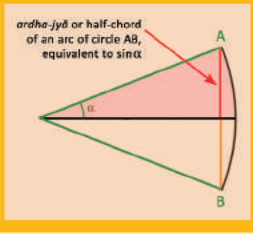
Features of Indian mathematics

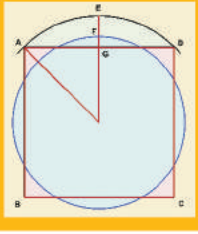
As elsewhere, mathematics in India arose from practical needs: constructing fire altars according to precise specifications, tracking the motion of planets, predicting eclipses, etc. But India's approach remained essentially pragmatic: rather than developing an axiomatic method such as that of the Greek (famously introduced by Euclid for geometry), it focused on obtaining formulas and algorithms that yielded precise and reliable results.

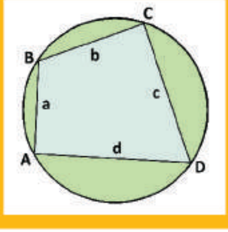
Nevertheless, Indian mathematicians did often provide logically rigorous justifications for their results, especially in the longer texts. Indeed, Bhāskarācārya states that presenting proofs (*upapattis*) is part of the teaching tradition, and Jyeṣṭhadeva devotes considerable space to them in his *Yukti Bhāṣā*. The shorter texts, on the other hand, often dispensed with the development of proofs. In the same spirit, the celebrated S. Ramanujan produced many important theorems but did not take time to supply proofs for them, leaving this for others to do!

Whether those specificities limited the further growth of Indian mathematics is open to debate. Other factors have been discussed by historians of science, such as historical disruptions of centres and networks of learning (especially in north India), limited royal patronage, or the absence of a conquering impulse (which, in Europe, did









fuel the growth of science and technology). Be that as it may, India's contribution in the field was enormous by any standard. Through the Arabs, many Indian inputs, from the decimal place-value system of numeral notation to some of the foundations of algebra and analysis, travelled on to Europe and provided crucial ingredients to the development of modern mathematics.

***

Match the following

Śulbasūtras	*kuṭṭaka*
Āryabhaṭa	expansions of trigonometric functions
Bhāskara I	*Chakravāla*
Brahmagupta	Pythagoras theorem
Bhāskara II	negative numbers
Mādhava	rational approximation for the sine

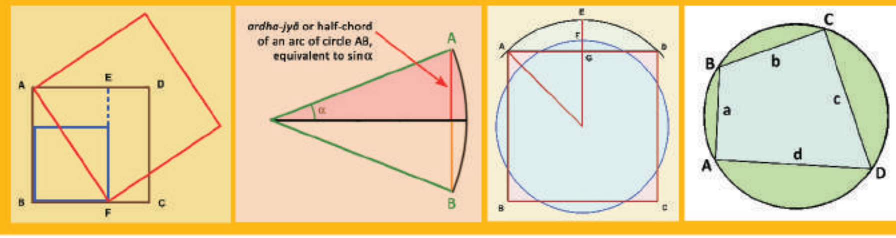
Comprehension questions

1. Write a few sentences on the inception of mathematics in India.
2. How would a rational approximation for the sine function be useful, when tables of sine were already available?
3. The Jain mathematicians used √10 for the value of the ratio of a circle's circumference to its diameter (π). Āryabhaṭa offered a value (62832/2000) which, he said, was 'approximate'. Bhāskarācārya proposed 22/7 for a 'rough approximation', and 3927/1250 for a 'good approximation'. And Mādhava's work on π is summarized above. What conclusions can you draw from these various results?

4. Consider the following statement by the French mathematician Pierre Simon de Laplace in 1814: 'It is to India that we owe the ingenious method of expressing every possible number using a set of ten symbols, each symbol having a positional as well as an absolute value. A profound and important idea, it now appears to us so simple that we fail to appreciate its true merit. But its real simplicity and the way it has facilitated all calculations has placed our arithmetic foremost among useful inventions. We will appreciate the greatness of this invention all the more if we remember that it eluded the genius of the two greatest men of Antiquity, Archimedes and Apollonius.' Discuss this statement and its implications. Why does Laplace find the Indian positional system of numeral notation 'simple'?

Project ideas

- Prepare a PowerPoint presentation on some of the important contributions of the 'siddhāntic' period of mathematics, i.e. from Āryabhaṭa to Bhāskarācārya.
- Prepare a PowerPoint presentation on some of the important contributions of the Kerala School of mathematics.
- Using Internet resources such as the website of University of St. Andrews, Scotland (http://www-history.mcs.st-andrews.ac.uk/history/Indexes/Indians.html and http://www-history.mcs.st-andrews.ac.uk/history/Indexes/HistoryTopics.html), draw a timeline for Indian as well as Babylonian, Greek, Arabic and Chinese mathematics.
- Consider the following four basic operations: 227 + 109; 128 – 77; 56 x 83; 45 ÷ 12. Work them out in full, but with those numbers expressed exclusively with Roman numerals: CCXXVII + CIX, etc. Spell out the rules involved clearly and follow them consistently. State your conclusions.



Exercises*

- Work out the value of √2 given in the *Śulbasūtras*, compare to the true value and calculate the margin of error.
- Āryabhaṭa I expressed the formula for the volume of a sphere thus: 'Half the circumference multiplied by half the diameter is the area of a circle. That area multiplied by its own square root is the exact volume of a sphere' (*Āryabhaṭīya,* 2.7). Show that this works out to $\pi^{3/2} r^3$. Mahāvīra (c. 850 CE) proposed that the volume of a sphere is $9/2\, r^3$. Śrīdhara (c. 900 CE) and Āryabhaṭa II (c. 950 CE) both proposed $38/9\, r^3$. Tabulate these three formulas and calculate their margins of error with respect to the correct formula. Add a column for Bhāskarācārya's formula: '[The sphere's surface] multiplied by its diameter and divided by 6' (*Līlāvatī* 109); work it out and conclude.
- Work out how many terms beyond 1 are required in Mādhava's 'rapidly convergent series' given above to reach 11 correct decimals for π. Use a calculator, but provide an estimate of the time it would have taken you to do the calculations by hand — which is what Mādhava's did!

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3. D.M. Bose, S.N. Sen & B.V. Subbarayappa, eds, *A Concise History of Science in India*, Universities Press, Hyderabad, 2nd edn, 2009

* Twenty more exercises are found in the selections from **Primary Texts** further below.

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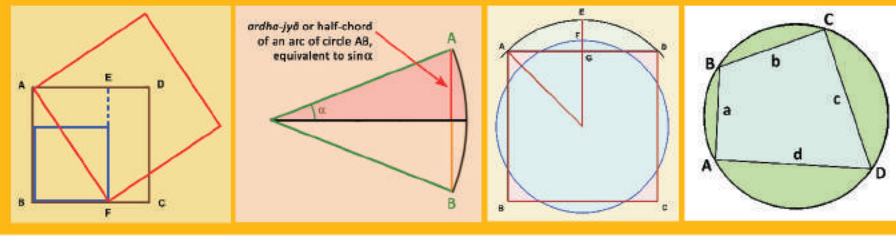
9. George Gheverghese Joseph, *A Passage to Infinity: Medieval Indian Mathematics from Kerala and its Impact*, Sage, New Delhi, 2009
10. S. Parameswaran, *The Golden Age of Indian Mathematics*, Swadeshi Science Movement – Kerala, Kochi, 1998
11. Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, 2009
12. T.R.N. Rao & Subhash Kak, eds, *Computing Science in Ancient India*, Center for Advanced Computer Studies, Louisiana, 1998, and Munshiram Manoharlal, New Delhi, 2000
13. T.A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass, New Delhi, 1999
14. Helaine Selin, & Roddam Narasimha, eds, *Encyclopaedia of Classical Indian Sciences*, Universities Press, Hyderabad, 2007
15. C.S. Seshadri, ed., *Studies in the History of Mathematics,* Hindustan Book Agency, New Delhi, 2010
16. B.S. Yadav & Man Mohan, eds, *Ancient Indian Leaps into Mathematics,* Birkhäuser, Boston, 2011

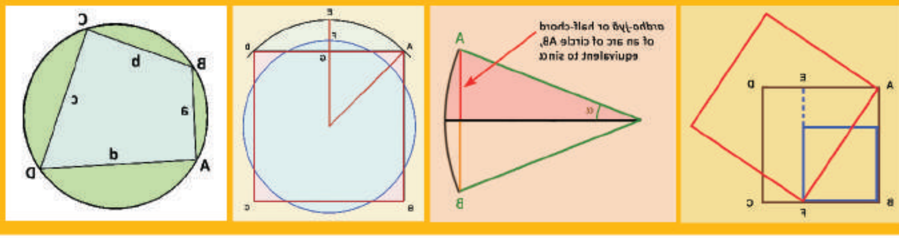
Internet resources (all URLs accessed in June 2012)

- *Indian Journal of History of Science* published by INSA:
  www.insa.nic.in/INSAuth/OurPublications.aspx
- A major resource on Indian mathematicians by J.J. O'Connor & E.F. Robertson:



http://www-history.mcs.st-andrews.ac.uk/history/Indexes/Indians.html

- An overview of Indian mathematics by J.J. O'Connor & E.F. Robertson: http://www-history.mcs.st-and.ac.uk/HistTopics/Indian_mathematics.html
- Indian Mathematics: Redressing the balance, by Ian G. Pearce: http://www-history.mcs.st-andrews.ac.uk/history/Projects/Pearce/index.html
- To download an earlier translation of Āryabhaṭa's *Āryabhaṭīya*: http://archive.org/download/The_Aryabhatiya_of_Aryabhata_Clark_1930/The_Aryabhatiya_of_Aryabhata_Clark_1930.pdf
- A huge resource of primary texts on the history of science, including many Indian texts (with or without translation): www.wilbourhall.org/
- A fascinating film by the BBC, 'The Story of Maths': www.youtube.com/watch?v=i9x4cTJ18is&feature=related
- A resource on the history of mathematics in ancient cultures: http://www-history.mcs.st-andrews.ac.uk/history/Indexes/HistoryTopics.html
- Papers on the history of mathematics: www.journals.elsevier.com/historia-mathematica/open-archive/
- A resource on the history of mathematics, but with very little on India: www.math.tamu.edu/~dallen/masters/hist_frame.htm
- A history of mathematics (limited content, but interesting as it is prepared by students): http://library.thinkquest.org/22584/
- A resource on the history of mathematics, with many links: http://archives.math.utk.edu/topics/history.html
- Useful references on the history of mathematics: http://aleph0.clarku.edu/~djoyce/mathhist/
- More references: http://www.dcs.warwick.ac.uk/bshm/resources.html



Primary Texts on Mathematics in India: A Selection

(All figures courtesy IFIH)

White Yajur-Veda (tr. adapted from R.T.H. Griffith)

> O Agni, may these [sacrificial] bricks be my own milch cows: one, and ten, and ten tens, a hundred, and ten hundreds, a thousand, and ten thousand, and a hundred thousand, and a million, and a hundred millions, and a thousand millions, and a hundred thousand millions, and a million millions. May these bricks be my own milch cows in the world beyond and in this world. (*Yajurveda Vājasaneyisaṃhitā*, 17.2)

Note: The *Yajur-Veda* is one of the four Vedas and exists in two versions (the White and the Black); it is dedicated to the conduct of sacrifices. Here, the priest constructs an altar for a fire sacrifice (Agni is the fire-god) and prays for each brick to become the equivalent of a milk-giving cow, a symbol of wealth (whether material or spiritual).

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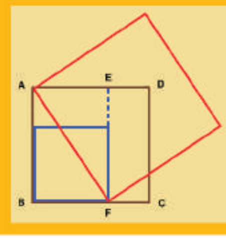
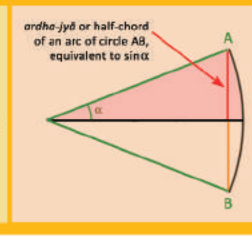
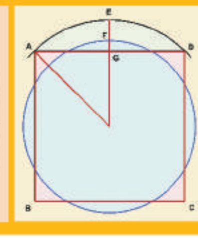
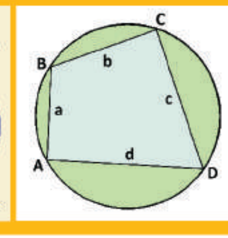
Rāmāyaṇa (tr. Gita Press)

> The wise speak of a hundred thousand multiplied by hundred as a crore, while a lakh [100,000] of crores is called a *śaṅku*. A lakh of *śaṅkus* is known as a *mahāśaṅku*. A lakh of *mahāśaṅkus* is spoken of as a *vṛnda* in this context. A lakh of *vṛndas* is known as a *mahāvṛnda*. A lakh of *mahāvṛndas* is spoken of in this context as a *padma*. A lakh of *padmas* is known as a *mahāpadma.* A lakh of *mahāpadmas* is spoken of in this context as a *kharva*. A lakh of *kharva* is known as a *mahākharva*. A lakh of *mahākharvas* is called a *samudra*. A lakh of *samudras*









is called an *ogha*. A lakh of *oghas* is popularly known as a *mahaugha*. Surrounded according to this computation by a thousand crore and one hundred *śaṅkus* and a thousand *mahāśaṅkus* and likewise by a hundred *vṛndas*, even so by a thousand *mahāvṛndas* and a hundred *padmas*, in the same manner by a thousand *mahāpadmas* and a hundred *kharvas*, nay, by a hundred *samudras* and similarly by a hundred *mahaughas* and by a hundred crore *mahaughas* [of monkey warriors] as well as by the gallant Vibhīṣaṇa and his own ministers, Sugrīva, the ruler of monkeys, is following you for waging war — Sugrīva, who is [thus] surrounded by a huge army and ever endowed with extraordinary might and prowess. Carefully observing, O great king [Rāvaṇa], this army ranged like a blazing planet, a supreme effort may now be put forth so that your victory may be ensured and no discomfiture may follow at the hands of the enemies. (*Yuddhakāṇḍa,* 28.33–42)

Note: The *Rāmāyaṇa* is a famous Indian epic which narrates the abduction of Sītā by the demon-king Rāvaṇa and the resulting war waged against him by Rāma, assisted by Sugrīva, the king of monkeys. In this passage, Rāvaṇa is given a description of Sugrīva's army and its immense numbers of warriors as it prepares to attack Lanka. It is noteworthy that all numbers given are multiples of ten. The total number of warriors adds up to an astronomical number close to 10^{71}, which is clearly intended to be taken metaphorically.

Exercise

- In the above extract, write out the value of each named number. What is the value of the highest named number? Can you justify our statement that 'The total number of warriors adds up to an astronomical number close to 10^{71}'?

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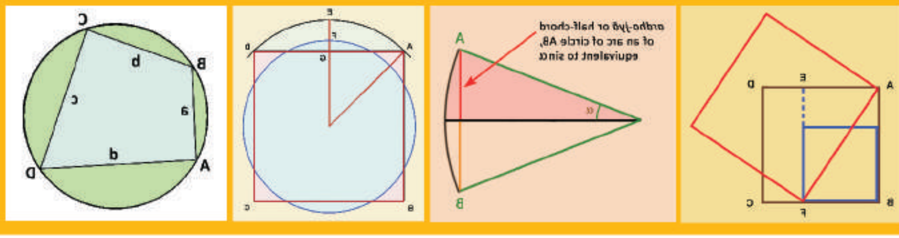
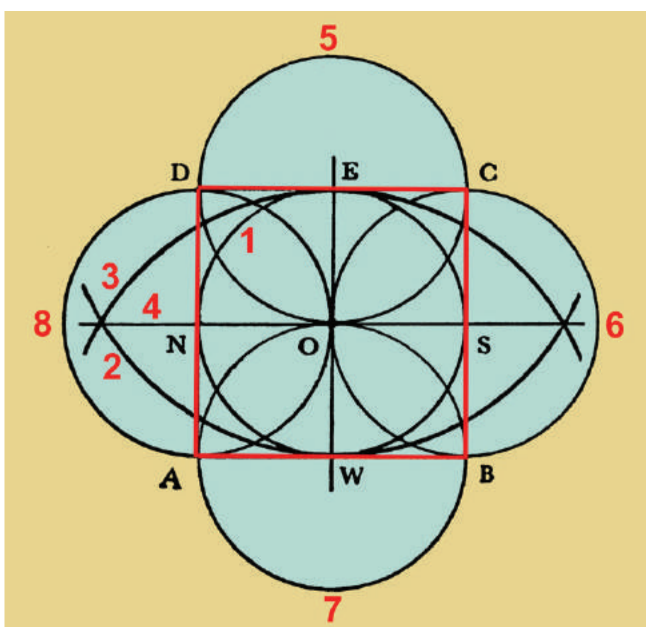
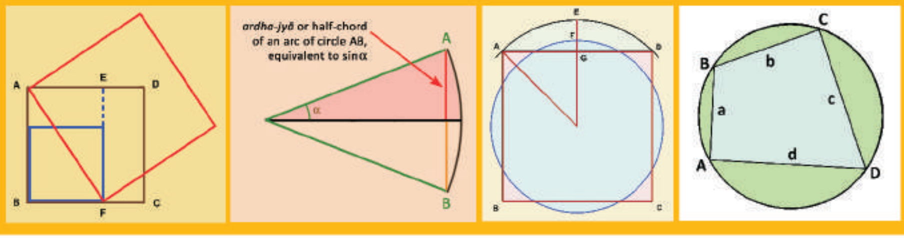


Baudhāyana's *Śulbasūtras* (tr. S.N. Sen & A.K. Bag)

Having desired [to construct] a square, one is to take a cord of length equal to the [side of the] given square, make ties at both ends and mark it at its middle. The [east-west] line [equal to the cord] is drawn and a pole is fixed at its middle. The two ties [of the cord] are fixed in it [pole] and a circle is drawn with the mark [in the middle of the cord]. Two poles are fixed at both ends of the diameter [east-west line]. With one tie fastened to the eastern [pole], a circle is drawn with the other. A similar [circle] about the western [pole]. The second diameter is obtained from the points of intersection of these two [circles]; two poles are fixed at two ends of the diameter [thus obtained]. With two ties fastened to the eastern [pole] a circle is drawn with the mark. The same [is to be done] with respect to the southern, the western and the northern [poles]. The end points of intersection of these [four circles] produce the [required] square. (1.4)





Note: This extract spells out a simple method to draw a square on the ground with nothing more than a rope and two poles or pegs. Given the importance of the square as a basic shape of fire altar, such a construction was fundamental (and a few more methods are proposed by other *Śulbasūtras* authors). Baudhāyana's method can be summarized through the following diagram: axis EW is a given, while axis NS is obtained from the intersections of two larger circles drawn from points E and W; four more circles can now be drawn from the resulting intersections of the two axis with a circle drawn from point O. The intersections of those four circles form the desired square ABCD.

Exercise

- With a one-metre-long rope and two pegs, replicate Baudhāyana's method. Can you propose any other purely method to construct a square with the same apparatus?

Baudhāyana's *Śulbasūtras* (tr. S.N. Sen & A.K. Bag)

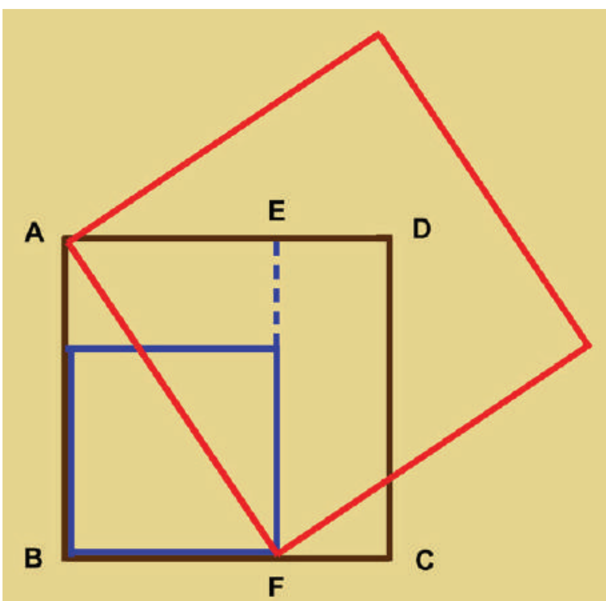
> The areas [of the squares] produced separately by the length and the breadth of a rectangle together equal the area [of the square] produced by the diagonal. This is observed in rectangles having sides 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, 15 and 36. (1.12–13)

Note: The first part of this extract is a geometrical statement of the so-called Pythagoras theorem (see a figure in the **survey text**). The second part lists a few of the so-called 'Pythagoras triplets', i.e. triplets of integers satisfying the Pythagoras theorem, for instance $3^2 + 4^2 = 5^2$, $12^2 + 5^2 = 13^2$. (Here, of course, Baudhāyana takes the result for granted and omits the third term.)

Baudhāyana's *Śulbasūtras* (tr. S.N. Sen & A.K. Bag)

> If it is desired to combine two squares of different measures, a [rectangular] part is cut off from the larger [square] with the side of the smaller; the diagonal of the cut-off [rectangular] part is the side of the combined square.
>
> If it is desired to remove a square from another, a [rectangular] part is cut off from the larger [square] with the side of the smaller one to be removed; the [longer] side of the cut-off [rectangular] part is placed across so as to touch the opposite side; by this contact [the side] is cut off. With the cut-off [part] the difference [of the two squares] is obtained. (2.1–2)

Note: The *Śulbasūtras* deal with transformations of one geometrical figure into another with no change in the figure's area: for instance, a square into a rectangle and vice-versa, of a rectangle into an isosceles trapezium, or a square into a circle and vice-versa (see next extract). Here, Baudhāyana gives a method to geometrically construct a square having an area the sum or difference of the areas of two given squares.



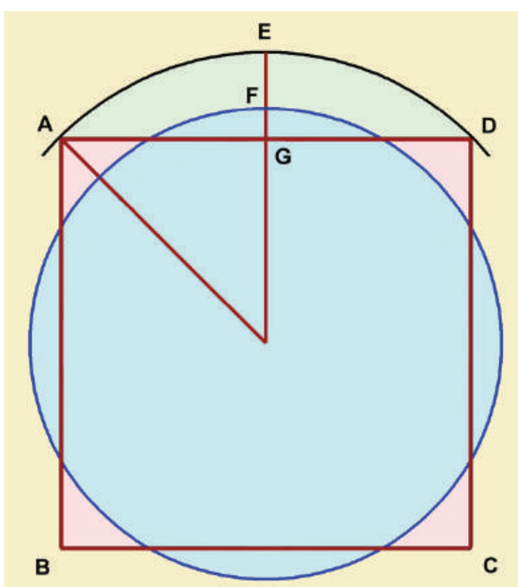
The first case is illustrated in the above figure, in which an application of the Pythagoras theorem to triangle AEF shows that the area of the red square is equal to the sum of the area of square ABCD and that of the blue square.

Exercise

- Work out the geometrical method conveyed in the second extract above (2.2), which deals with subtracting the areas of two squares. Draw a figure explaining the process.

Baudhāyana's *Śulbasūtras* (tr. S.N. Sen & A.K. Bag)

If it is desired to transform a square into a circle [having the same area], [a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square]; with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn. (2.9)



Note: The circling of a square, i.e. producing a circle having the same area as a given square, as well as the reverse problem (see next extract), exercised the best mathematical minds the world over from antiquity to medieval times. Baudhāyana's above method is simple and is summarized through the above figure, in which point F is chosen such that FG = 1/3 GE. The resulting circle (in blue) has nearly the same area as the square ABCD. But how nearly? Calculations show that the resulting circle is too large by about 1.7%.

Exercise

- Justify our statement that the circle resulting from Baudhāyana's method to circle a square is too large by about 1.7%. Can you propose a more accurate geometrical method to construct such a circle?

Baudhāyana's *Śulbasūtras* (tr. S.N. Sen & A.K. Bag)

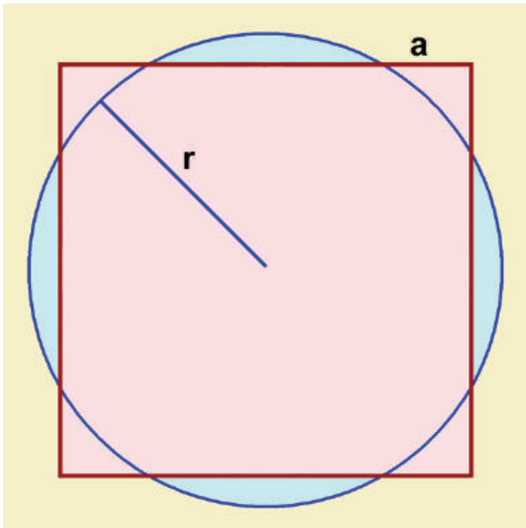
> To transform a circle into a square, the diameter is divided into eight parts; one [such] part after being divided into twenty-nine parts is reduced by twenty-eight of them and further by the sixth [of the part left] less the eighth [of the sixth part].
>
> Alternatively, divide [the diameter] into fifteen parts and reduce it by two of them; this gives the approximate side of the square [desired]. (2.10–11)

Note: Baudhāyana offers two different methods for the reverse problem, i.e. the 'quadrature (or squaring) of a circle'. If the circle's radius is r, its diameter d and the desired square's side a, the first method can be expressed as:

$$a = \frac{7}{8}d + \frac{d}{8} - \left(\frac{28d}{8 \times 29} + \frac{d}{8 \times 29 \times 6} - \frac{d}{8 \times 29 \times 6 \times 8}\right)$$

and the second as:

$$a = d - \frac{2}{15}d = \frac{26}{15}r$$



Calculations show that the resulting square is too small by about 1.7% and 4.4% respectively.

Exercise

- Work out the calculations involved in the above two methods and justify our statement that the resulting square is too small by about 1.7% and 4.4% respectively. Can you propose an approximation of the type $a = \frac{m}{n}r$, where integers *m* and *n* are both less than 25, with *a* within 0.2% of the ideal value?

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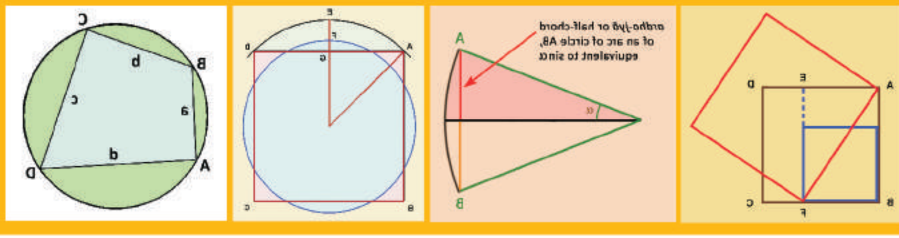


Āryabhaṭa I, *Āryabhaṭīya* (tr. K.S. Shukla)

> 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22 and 7 — these are the Rsine-differences [at intervals of 225 minutes of arc] in terms of minutes of arc. (1.12)

Note: This is a table of sine values from 3.75° to 90°, for every 3.75°. But this table is unusual in several respects for a 21st-century maths student:

5. Āryabhaṭa does not consider our modern sine but, in accordance to the usage of his times, the sine multiplied by the radius *R* of a circle where the angle is considered (see diagram in our **survey text**). This is generally referred to as *R*sine. *R* is given an arbitrary value, which differs from one author to another; Āryabhaṭa adopted 3438’ (taking 360° or 21600’ as the circle’s circumference, and dividing it by 2π to get 3438’). Unlike the non-dimensional sine, *R*sine is a linear dimension. One advantage of the *R*sine is that it can have high values even with small angles.
6. The values given are *incremental,* that is, each value has to be added to all preceding ones in order to get the absolute value; for instance, the *R*sine value given for 7° is 225’ + 224’ = 449’; for 90°, it will be the sum of all the values, i.e. 3438’ (which is the value of *R,* since sin 90° = 1).
7. Āryabhaṭa’s table was actually not expressed with numerals as above, but in a coded language of his own, in which each letter corresponds to a number or to a power of ten. The first few sine values read *makhi, bhakhi, phakhi, dhakhi,* etc.

As calculations show, Āryabhaṭa’s values are highly accurate — to within 0.02%.

Exercise

- Work out in full Āryabhaṭa's table of sines from 3.75° to 90°, comparing with the modern values for every angle, and the error involved. Justify our statement that it never exceeds 0.02%.

Āryabhaṭa I, *Āryabhaṭīya* (tr. K.S. Shukla)

> Having subtracted the greatest possible cube root from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root], divide the second non-cube place [standing on the right of the last cube place] by thrice the square of the cube root [already obtained]; [then] subtract from the first non-cube place [standing on the right of the second non-cube place] the square of the quotient multiplied by thrice the previous [cube root]; and [then subtract] the cube [of quotient] from the cube place [standing on the right of the first non-cube place] [and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there are still digits on the right]. [2.5]

Note: Āryabhaṭa gives here an algorithm for the extraction of a cube root. We reproduce K.S. Shukla's explanation (reformulated by M.S. Sriram):

		c	n'	n	c	n'	n	c	121
		1	7	7	1	5	6	1	(line of cube root)
Subtract 1^3		1							
Divide by 3.1^2	3)	0	7	(2					
		0	6						
			1	7					
Subtract $3.1.2^2$			1	2					
				5	1				
Subtract 2^3				0	8				
Divide by 3.12^2	432)			4	3	5	(1		
				4	3	2			
						3	6		
Subtract $3.12.1^2$						3	6		
							0	1	
Subtract 1^3								1	
								0	

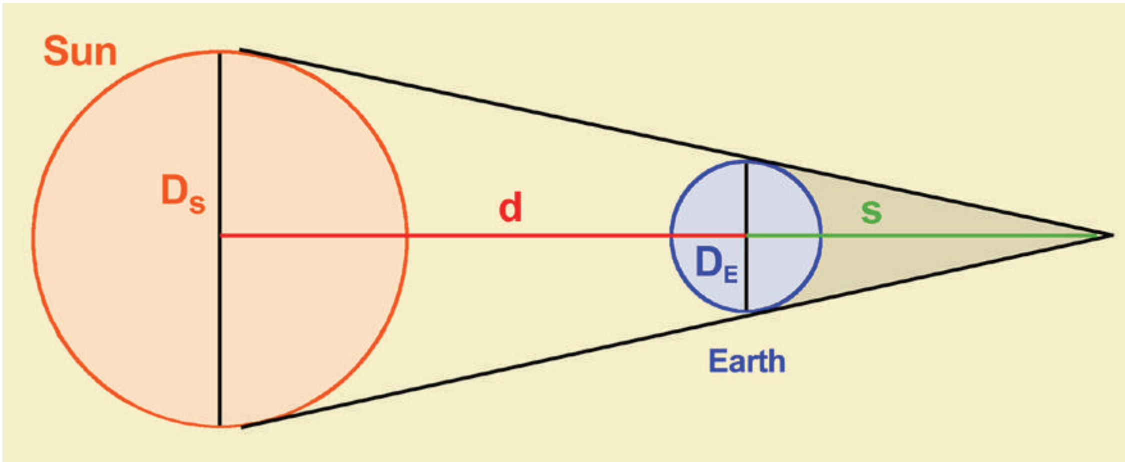
We consider the cube root of 17,71,561 as an example. Beginning from the units place, the notational places are called cube place (*c*), first non-cube place (*n*), second non-cube place(*n*'), cube place (*c*), first non-cube place (*n*), second non-cube place (*n*'), and so on. The process ends and the cube root is 121. The algorithm is obviously based on the algebraic identity: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Exercise

- Following this algorithm, extract the cube root of 970,299.

Āryabhaṭa I, *Āryabhaṭīya* (tr. Kim Plofker)

> Divide the distance between [the centres of] the earth and the sun, multiplied by [the diameter of] the earth, by the difference between [the diameters of] the sun and the earth. The quotient is the length of the earth's shadow [measured] from the [perpendicular] diameter of the earth. (4.39)



Note: Āryabhaṭa gives a simple formula to calculate the length of the shadow cast by the earth; such a calculation is essential to the prediction of a lunar eclipse (when the moon passes in the cone of the earth's shadow), its totality, duration, etc. Let d be the distance between the centres of the sun and the earth, s the distance from the centre of the earth to the tip of its shadow, D_S and D_E the diameters of the sun and the earth, as shown in the above figure.

Āryabhaṭa's formula can be expressed as:

$$s = \frac{dD_E}{D_S - D_E}$$

Exercise

- Prove the above formula.

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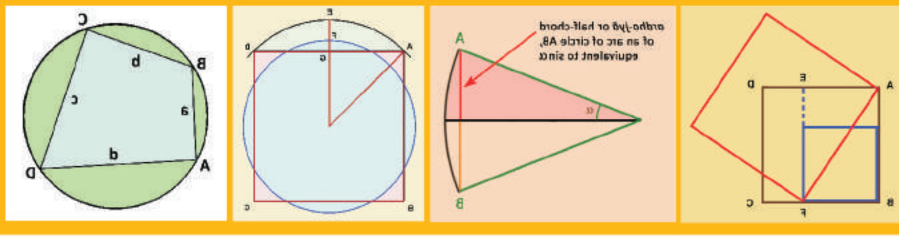

Bhāskara I, *Mahābhāskarīyam* (tr. K.S. Shukla)

> I briefly state the rule [for finding *R*sine values] without making use of the *R*sine differences 225 etc. [as given by Āryabhaṭa, see above]. Subtract the degrees of the *bhuja* or *koṭi* [lateral and vertical sides of a right-angled triangle, i.e. cosine or sine] from the degrees of half a circle [i.e. from 0° to 180°]. Then multiply the remainder by the degrees of the *bhuja* and put down the result at two places. At one place subtract the result from 40,500. By one-fourth of the remainder [thus obtained] divide the result at the other place as multiplied by the *antyaphala* [i.e. the epicyclic radius]. Thus is obtained the entire *bāhuphala* (or *koṭiphala*) for the Sun, Moon or the star-planets. So also are obtained the direct and inverse Rsines. (7.17–19)

Note: Leaving aside the astronomical terms, we can see that Bhāskara proposes the following rational approximation for the *R*sine:

$$R \sin \theta = \frac{4\theta(180° - \theta)R}{40500 - \theta(180° - \theta)}$$

where θ is in degrees. The high accuracy of the formula is illustrated by the curve found in our **survey text**. The maximum absolute error in this range is a tiny 0.0016. The scholar K.S. Shukla provided a geometric 'rationale' for the above approximation.

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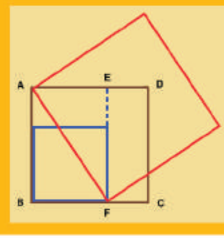
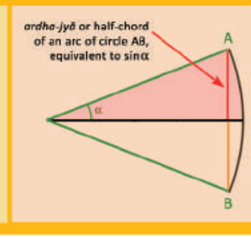
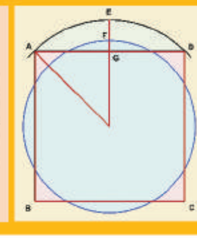
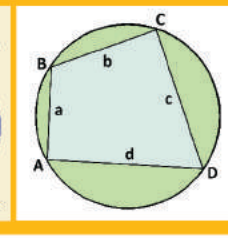
Severus Sebokht, Syria, 662 CE

> I will omit all discussion of the science of the Indians, a people not the same as the Syrians; of their subtle discoveries in astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; and of their









valuable methods of calculation which surpass description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have arrived at the limits of science, [would read the earlier texts], they would perhaps be convinced, even if a little late in the day, that there are others also who know something of value. (From George Gheverghese Joseph, *The Crest of the Peacock*)

Note: Severus Sebokht, a Nestorian bishop from Syria, wrote on geography and astronomy. Piqued by the arrogance of Greek scholars who thought their science was superior to that of other cultures, he authored this well-known praise of the Indian place-value system of numeral notation. It is also a useful chronological marker, since it shows that this system had reached the Mediterranean world by the 7th century CE.

***

Brahmagupta, *Brāhmasphuṭasiddhānta* (tr. Kim Plofker)

[The sum] of two positives is positive, of two negatives negative; of a positive and a negative [the sum] is their difference; if they are equal it is zero. The sum of a negative and zero is negative, [that] of a positive and zero positive, [and that] of two zeros zero.

[If] a smaller [positive] is to be subtracted from a larger positive, [the result] is positive; [if] a smaller negative from a larger negative, [the result] is negative; [if] a larger [negative or positive is to be subtracted] from a smaller [positive or negative, the algebraic sign of] their difference is reversed — negative [becomes] positive and positive negative.

A negative minus zero is negative, a positive [minus zero] positive; zero [minus zero] is zero. When a positive is to be subtracted from a negative or a negative from a positive, then it is to be added.

The product of a negative and a positive is negative, of two negatives positive, and of positives positive; the product of zero and a negative, of zero and a positive, or of two zeros is zero.

A positive divided by a positive or a negative divided by a negative is positive; a zero divided by a zero is zero; a positive divided by a negative is negative; a negative divided by a positive is [also] negative.

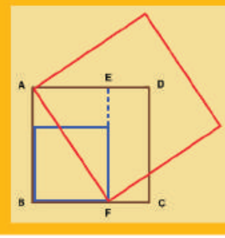
A negative or a positive divided by zero has that [zero] as its divisor, or zero divided by a negative or a positive [has that negative or positive as its divisor]. The square of a negative or of a positive is positive; [the square] of zero is zero. That of which [the square] is the square is [its] square-root.

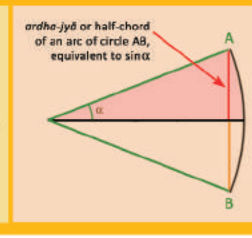
The sum [of two quantities] increased or diminished by [their] difference [and] divided by two is [their] mixture. The difference of [two] squares [of the quantities] divided by the difference [of the quantities themselves] is increased and diminished by the difference [and] divided by two; [this] is the operation of unlikes. (18.30–36)

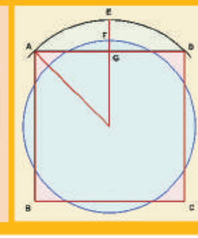
Note: Brahmagupta, who has sometimes been called the ‘father of Indian algebra’, lays down here rules for operations with negative numbers and with zero.

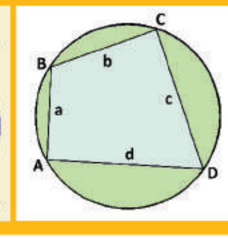
Exercises

- Express all the above rules in algebraic notation. Can you spot any rule that would not be accepted by modern mathematics?









- Remove all the supplementary phrases added by the translator within square brackets; will you be able to make out all meanings? This will give you a feel of the concise style generally adopted by scientific authors of those times (the same applies to Āryabhaṭa's method for cube root extraction above).

Brahmagupta, *Brāhmasphuṭasiddhānta* (tr. Kim Plofker)

The nature of squares:

> [Put down] twice the square-root of a given square multiplied by a multiplier and increased or diminished by an arbitrary [number]. The product of the first [pair], multiplied by the multiplier, with the product of the last [pair], is the last computed.
>
> The sum of the thunderbolt-products is the first. The additive is equal to the product of the additives. The two square-roots, divided by the additive or the subtractive, are the additive *rūpas* [known quantity or constant]. (18.64–65)

Note: The so-called 'square-nature' methods are ways of solving second-degree indeterminate equations. Here, Brahmagupta explains how to find a solution for what is now commonly known as 'Pell's Equation' [$Nx^2 + 1 = y^2$]; we will illustrate the procedure using one of his examples below, namely $83x^2 + 1 = y^2$. The key is to find, for the given 'multiplier' N, a solution (a, b) to an auxiliary equation $Na^2 \pm k = b^2$ where $k \neq 1$, and then manipulate a and b to provide a solution to the original equation. If we take our 'given square' to be 1 and multiply it by the 'multiplier' 83, we want to increase or diminish the result by some quantity to give a perfect square. E.g., $83 \times 1^2 - 2 = 9^2$. After we 'put down twice' the chosen roots, $\begin{vmatrix}1 & 9\\ 1 & 9\end{vmatrix}$, we take the 'sum of the thunderbolt-products' (apparently a technical term for cross-multiplication): $1 \times 9 + 9 \times 1 = 18$. That is the 'first' quantity, and the 'last' is $83 \times 1^2 + 9^2 = 164$. The new 'additive' is the square of the previous one: $2 \times 2 = 4$, giving $83 \times 18^2 + 4 = 164^2$. Then the desired x and y are found by dividing the 'first' and 'last' by the previous 'subtractive': $x = 18/2 = 9$, $y = 164/2 = 82$. The same technique can also be used to form a new solution from two distinct previous solutions, instead of from one solution 'put down twice.' [This note is borrowed from the translator, Kim Plofker. By 'indeterminate equation' is meant an equation for which integral solutions alone are desired. This method of Brahmagupta is called the *bhāvanā.*]

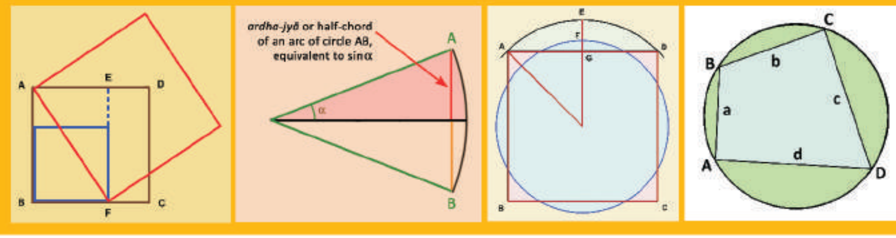
Exercises

- Solve the above second-degree indeterminate equation again, carefully following the step-by-step instructions and comparing them with Brahmagupta's brief explanation.
- By applying the above *bhāvanā,* show that a set of integral solutions to $18x^2 + 1 = y^2$ is (4, 17).
- One of Brahmagupta's corollaries to his *bhāvanā* is that if (a, b) is a solution to $Nx^2 + 1 = y^2$, then $(2ab, b^2 + Na^2)$ is also a solution. Give a proof for this corollary, and use it to produce two more sets of solutions to the above equation.

***

Bhāskarācārya, *Bījagaṇita* (tr. adapted from S.K. Abhyankar)

> Multiply both sides [of an equation] by a known quantity equal to four times the coefficient of the square of the unknown; add to both sides a known quantity equal to the square of the [original] coefficient of the unknown: then [extract] the root. (116)



Note: Bhāskarācārya here spells out a method to solve the equation $ax^2 + bx = c$, where it is tacitly assumed that the coefficients a, b and c are positive; in fact, he attributes this solution to Śrīdhara, an earlier mathematician whose work on algebra is lost.

We first multiply the above equation by $4a$ and then add b^2 to both the sides:

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2,$$

which becomes:

$$(2ax + b)^2 = 4ac + b^2.$$

Taking roots, we obtain:

$$2ax + b = \sqrt{4ac + b^2},$$

from which we get one of the two standard solutions (the other deriving from the negative root of the first member of the above equation). This strategy is extended to methods for solving the cubic and the biquadratic (or fourth-degree) equations.

Bhāskarācārya, *Bījagaṇita* (tr. adapted from S.K. Abhyankar)

> One man says to the other, 'If you will give me 100 rupees, I shall be twice as rich as you are.' The other man says, 'If you give me 10 rupees, I shall be six times as rich as you are.' Tell me the wealth of each of them. (93)
>
> A merchant started with a sum. Entering a city, he paid Rs. 10 as duty. After trading his amount became double. From that he spent Rs. 10 on dinner and left the city after paying Rs. 10 as duty. He went to two other cities; the same was the case in both of them. After coming back his amount had trebled. What was the [initial] sum? (101)

Note: Bhāskarācārya enjoyed mathematical brainteasers, as both his *Bījagaṇita* and his *Līlāvatī* testify (following the example of earlier authors, such as Mahāvīra). They usually involve systems of linear equations, occasionally quadratic ones.

Exercise

- Solve the above two brainteasers.

Bhāskarācārya, *Līlāvatī* (tr. KS Patwardhan et al.)

> Arjuna became furious in the war and, in order to kill Karṇa, picked up some arrows. With half of the arrows, he destroyed all of Karṇa's arrows. He killed all of Karṇa's horses with four times the square root of the arrows. He destroyed the spear with six arrows. He used one arrow each to destroy the top of the chariot, the flag, and the bow of Karṇa. Finally he cut off Karṇa's head with another arrow. How many arrows did Arjuna discharge? (76)

Note: This brainteaser from *Līlāvatī* involves a simple quadratic equation (which assumes that Arjuna discharged all the arrows he had picked up). The equation is deftly woven into a famous episode of the Mahābhārata war.

> A king had a beautiful palace with eight doors. Skilled engineers had constructed four open squares which were highly polished and huge. In order to get fresh air, 1 door, 2 doors, 3 doors, … are opened. How many different types of breeze arrangements are possible?
>
> How many kinds of relishes can be made by using 1, 2, 3, 4, 5 or 6 types from sweet, bitter, astringent, sour, salty and hot substances? (122)

Note: These two examples involve standard combinatorics.

Exercise

- Solve the above three brainteasers.

***

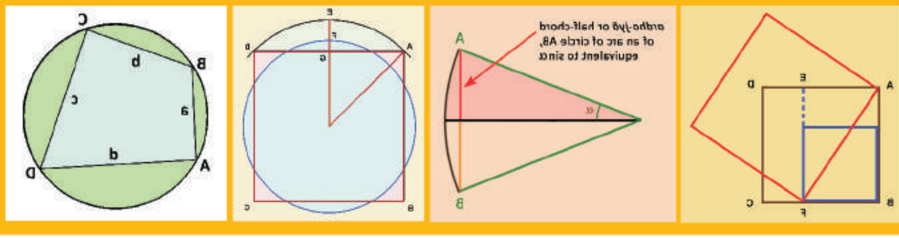
Jyeṣṭhadeva, *Gaṇita-Yukti-Bhāṣā* (tr. K.V. Sarma)

Now, the method to ascertain two numbers if any two of the following five, viz., the sum, difference, product, sum of squares, and difference of squares of the two numbers, are known.

Qn. 1. Here, if the difference of two numbers is added to their sum, the result obtained will be twice the bigger number. Then, if the difference is subtracted from the sum, the result obtained will be twice the smaller number. Then, when the two results, as obtained above, are halved, the two numbers, respectively, will result.

Qn. 2. Now, to ascertain the numbers when their sum and product are known: Here, in accordance with the rationale explained earlier, if four times the product is subtracted from the square of the sum, and the root of the result found, it will be the difference between the numbers. Using this [and the sum of the numbers], the two numbers can be got as explained above.

Qn. 3. Now, [given] the sum and the sum of the squares [of the numbers]: There, when the square of the sum is subtracted from twice the sum of the squares and the root of the result found, it will be the difference between the numbers.



Qn. 4. Then, when the difference between the squares is divided by the sum [of the numbers], the result will be the difference between the numbers, as per the rationale explained earlier.

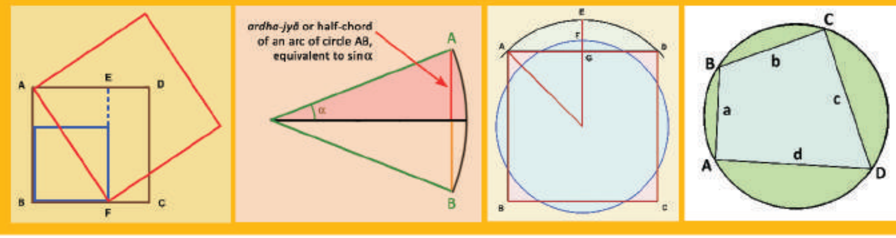
Qn. 5. Then, [given] the difference and the product of the numbers: There, if the product is multiplied by four and the square of the difference added and the root of the result found, it will be the sum of the numbers.

Qn. 6. Then, given the difference and the sum of squares: When the square of the difference is subtracted from double the sum of the squares, and the root of the result found, it will be the sum of the numbers.

Qn. 7. Then, when the difference of the squares is divided by the difference [of the numbers], the result will be the sum of the numbers.

Qn. 8. Then, [given] the product and the sum of the squares [of the numbers]: Here, subtract twice the product from the sum of the squares, and find the root of the result. This will be the difference [between the numbers]. When the product is multiplied by 4 and the square of the difference added, the root of the result is the sum [of the numbers].

Qn. 9. Then, [given] the product and the difference of the squares [of the numbers]: Now, we obtain the squares of the two numbers. Here, the calculations done using the numbers can be done using the squares of the numbers. The distinction here would be that the results will also be in terms of squares. There, when the product is squared, it will be the product of the squares, [since] there is no difference in the result of multiplication when the sequence [of the steps] is altered. Hence, taking that the product and the difference of the squares are known, the sum of the squares can be derived by the same method used for calculating the sum [of two numbers] given their



product and difference. Here, when the square of the product is multiplied by four and added to the square of the difference in the squares, the root of the result will be the sum of the squares. Then placing this sum of the squares in two places, add to one the difference of the squares and subtract it from the other. Then divide both by 2. The results will be the squares of the two numbers.

Qn. 10. Then, the tenth [question] is when the sum of the squares and the difference of the squares are known. This too has been answered above.

These are the ten questions. These have been stated here since they are made use of in several places. Cube roots have no use in planetary computation. Hence they are not stated here. Thus [have been explained] a way of computation.

Note: This is chapter 2 of Jyeṣṭhadeva's *Gaṇita-Yukti-Bhāṣā,* a Malayalam work of the 16^{th} century, divided in two major parts, one on mathematics and the other on astronomy. Here Jyeṣṭhadeva lays down ten commonly used rules of algebra.

Exercise

- How many of these 'ten questions' can you express in algebraic form and prove?

CBSE

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