12. SEQUENCE & SERIES

An arithmetic progression (A.P.) : $a, a + d, a + 2d, \ldots\ldots, a + (n-1)d$ is an A.P.

Let a be the first term and d be the common difference of an A.P., then n^{th} term $= t_n = a + (n-1)d$

The sum of first n terms of are A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]$$

r^{th} term of an A.P. when sum of first r terms is given is $t_r = S_r - S_{r-1}$.

Properties of A.P.

(i) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(ii) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$; five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iii) Sum of the terms of an A.P. equidistant from the beginning & end = sum of first & last term.

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

n – Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in A.P. then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b. $A_1 = a + \frac{b-a}{n+1}$, $A_2 = a + \frac{2(b-a)}{n+1}$, ..., $A_n = a + \frac{n(b-a)}{n+1}$

$\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Geometric Progression: $a, ar, ar^2, ar^3, ar^4, \ldots\ldots$ is a G.P. with a as the first term & r as common ratio.

(i) n^{th} term $= a\,r^{n-1}$

(ii) Sum of the first n terms i.e. $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1}, & r \neq 1 \\ na, & r = 1 \end{cases}$

(iii) Sum of an infinite G.P. when $|r| < 1$ is given by $S_\infty = \frac{a}{1-r}\ (|r| < 1)$.

Geometric Means (Mean Proportional) (G.M.):

If a, b, c > 0 are in G.P., b is the G.M. between a & c, then $b^2 = ac$

n–Geometric Means Between positive number a, b: If a, b are two given numbers & $a, G_1, G_2, \ldots\ldots, G_n$, b are in G.P.. Then $G_1, G_2, G_3, \ldots, G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ..., $G_n = a(b/a)^{n/n+1}$

Harmonic Mean (H.M.):

If a, b, c are in H.P., b is the H.M. between a & c, then $b = \frac{2ac}{a+c}$.

H.M. H of $a_1, a_2, \ldots\ldots a_n$ is given by $\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}\right]$

Relation between means :

$G^2 = AH$, A.M. $\geq$ G.M. $\geq$ H.M. and A.M. = G.M. = H.M. if $a_1 = a_2 = a_3 = ...= a_n$

Important Results

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\, a_r = k \sum_{r=1}^{n} a_r$. (iii) $\sum_{r=1}^{n} k = nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n\,(n+1)}{2}$ (v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2\,(n+1)^2}{4}$

(vii) $2 \sum_{i<j=1}^{n} a_i\, a_j = (a_1 + a_2 + ... + a_n)^2 - (a_1^2 + a_2^2 + ... + a_n^2)$